A Theory of Strong and Weak Scintillations with Applications to Astrophysics

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Lou-Chuang Lee

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Abstract

The propagation of waves in an extended, irregular medium is studied under the "quasi-optics" and the "Markov random process" approximations. Under these assumptions, a Fokker-Planck equation satisfied by the characteristic functional of the random wave field is derived. A complete set of the moment equations with different transverse coordinates and <u>different</u> wavenumbers is then obtained from the characteristic functional. The derivation does <u>not</u> require Gaussian statistics of the random medium and the result can be applied to the <u>time-dependent</u> problem. We then solve the moment equations for the phase correlation function, angular broadening, temporal pulse smearing, intensity correlation function, and the probability distribution of the random waves. The necessary and sufficient conditions for strong scintillation are also given.

We also consider the problem of diffraction of waves by a random, phase-changing screen. The intensity correlation function is solved in the whole <u>Fresnel</u> diffraction region and the temporal pulse broadening function is derived <u>rigorously</u> from the wave equation.

The method of smooth perturbations is applied to interplanetary scintillations. We formulate and calculate the effects of the solar-wind velocity fluctuations on the observed intensity power spectrum and on the ratio of the observed "pattern" velocity and the true velocity of the solar wind in the three-dimensional spherical model. The r.m.s.

solar-wind velocity fluctuations are found to be ~ 200 km/sec in the region about 20 solar radii from the Sun.

We then interpret the observed interstellar scintillation data using the theories derived under the Markov approximation, which are also valid for the strong scintillation. We find that the Kolmogorov power-law spectrum with an outer scale of 10 to 100 pc fits the scintillation data and that the ambient averaged electron density in the interstellar medium is about 0.025 cm^{-3}. It is also found that there exists a region of strong electron density fluctuation with thickness ~ 10 pc and mean electron density ~ 7 cm^{-3} between the PSR 0833-45 pulsar and the earth.

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Chapter 1

Introduction

1. Phenomena in Astrophysical Scintillations

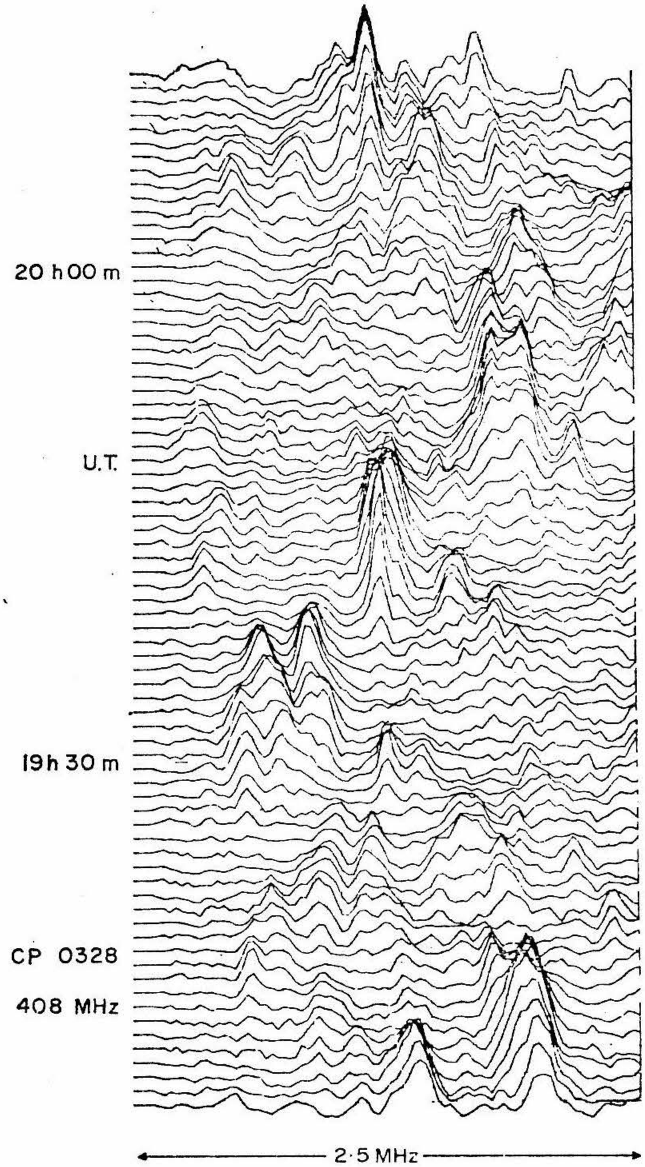
In many situations of astrophysical interest the electromagnetic waves from a radio star propagate through a region of turbulent plasmas and are scattered. The radio waves scattered by the turbulent plasmas are superimposed on the transmitted waves and lead to amplitude and phase fluctuations of the resultant wave field. This produces a wide variety of observed phenomena such as intensity variations, angular broadening and temporal pulse smearing of the radio waves.

The interplanetary medium and interstellar medium are two kinds of medium of astrophysical interest that scatter radio waves. These two media may be characterized respectively by their distances from the earth, $\sim 1.5\times10^{13}$cm and 10^{20}-10^{23}cm; their typical electron density fluctuations, ~ 1 electron/cm^3 and 10^{-2}-10^{-4} electron/cm^3; the size of their irregularities important for the scintillations, $\sim 10^7$cm and 10^{11}-10^{14}cm; and the typical velocities of the plasma media transverse to the line of sight, 350 km/sec, and ~ 50 km/sec. The phenomena of scattering of radio waves by these two media are termed "interplanetary scintillations" (IPS) and "interstellar scintillations" (ISS), respectively.

In interplanetary scintillations, the root mean square of the intensity variation is usually much smaller than the mean intensity and the scintillations are weak. However, interstellar scintillations are strong and the intensity fluctuation is of the same order as mean intensity. The observed data for interplanetary and interstellar scintillations are respectively: transverse scale of intensity fluctuations, 100-200 km and 10^9-10^{10} cm; the correlation time scale of intensity scintillations, ~ 0.5 sec and several minutes; and the characteristic angular broadenings, $\sim 10^{-5}$ radian and $\sim 10^{-7}$ radian.

An example of interstellar scintillations is shown in Figure (1-1) for the CP 0328 pulsar observed by Rickett (1970). This figure is a display of the wave intensity averaged over about 70 pulses as a function of time and frequency. As shown in the figure, the correlation time scale of intensity fluctuations is about 12 minutes and the decorrelation frequency of intensity fluctuations is about 130 kHz.

The objects of this thesis are (a) to develop a theory of wave propagation in a random medium that relates the properties of turbulent plasmas to the observed phenomena of the scattered radio waves, and (b) to study the characteristics of the turbulent plasmas in the interplanetary medium and interstellar medium by utilizing the observed effects.



Spectra from CP 0328 at 408 MHz integrated over about 70 pulses, plotted at successive 50 second intervals. The frequency resolution is about 60 KHz and the spectra include the receiver bandpass, which gives a gradual cut-off at the edges of the diagram.

(from Rickett 1970)

Figure (1-1)

II. Wave Equation in the Plasma Medium

The problems of wave propagation in a random medium can be studied by considering the scattering of waves due to each small volume of irregularities in the medium, summing up all the waves scattered by each irregularity and then calculating the statistical properties of the scattered waves. However, a more convenient way to tackle the problem is first to derive a basic macroscopic equation governing the wave propagation in the random medium and then to solve the basic equation and calculate the statistical properties of the scattered waves. The second method will be used in this thesis. In this section we will derive a basic equation governing the propagation of waves with frequency $\omega \gg \omega_p$, the plasma frequency of the medium, in the plasma medium. This wave equation applies to the propagation of the radio waves in the ionosphere, interplanetary space, or the interstellar medium.

We start from the Maxwell equations

$$\nabla \times \underset{\sim}{E} + \frac{\mu}{c} \frac{\partial \underset{\sim}{H}}{\partial t} = 0 \tag{1-1}$$

$$\nabla \times \underset{\sim}{H} - \frac{\epsilon_o}{c} \frac{\partial \underset{\sim}{E}}{\partial t} = \frac{4\pi \underset{\sim}{J}}{c} \tag{1-2}$$

$$\nabla \cdot \epsilon_o \underset{\sim}{E} = 4\pi\rho_e \tag{1-3}$$

$$\nabla \cdot \mu \underset{\sim}{H} = 0 \tag{1-4}$$

Here $\underset{\sim}{E}$ is the electric field, $\underset{\sim}{H}$ the magnetic field, $\underset{\sim}{J}$ the current density, μ the permeability, ϵ_o the dielectric constant of the medium, and c the speed of light. Gaussian units are used here. In our case, $\mu = \epsilon_o = 1$.

From Eqs. (1-1) and (1-2), we immediately have

$$\nabla \times (\nabla \times \underset{\sim}{E}) + \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \underset{\sim}{J}}{\partial t} = 0 \quad . \tag{1-5}$$

The relation between the current density $\underset{\sim}{J}$ and the electric field $\underset{\sim}{E}$ depends on the properties of the medium. A simple model for calculating the current density $\underset{\sim}{J}$ in a plasma medium is presented as follows. In the presence of electric field $\underset{\sim}{E}$, the electrons will be accelerated by the field and one has, neglecting the collision effect and magnetic force,

$$m \frac{d\underset{\sim}{v}}{dt} = e\, \underset{\sim}{E} \tag{1-6}$$

where $\underset{\sim}{v}$, m and e are the velocity, mass, and charge of an electron respectively. Suppose N_e is the electron density in the plasma medium. Then one has the current density

$$\underset{\sim}{J} = N_e\, e\, \underset{\sim}{v} \quad . \tag{1-7}$$

In Eq. (1-7), we have neglected the ion motion and the electron thermal motions since the mass of an ion is much greater than that of an electron and since the wave velocity we consider is much greater than the electron thermal velocity. From Eqs. (1-6) and (1-7) we have

$$\frac{\partial \underset{\sim}{J}}{\partial t} = \frac{N_e e^2}{m} \underset{\sim}{E} \quad . \tag{1-8}$$

Inserting Eq. (1-8) into Eq. (1-5), one has

$$\nabla \times (\nabla \times \underset{\sim}{E}) + \frac{1}{c^2}\left[\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} + \frac{4\pi N_e e^2 \underset{\sim}{E}}{m}\right] = 0 \quad . \tag{1-9}$$

Take the Fourier transform in time t of Eq. (1-9) and let $\underset{\sim}{f}(\underset{\sim}{r},\omega)e^{-i\omega t}$ be the Fourier component of $\underset{\sim}{E}(\underset{\sim}{r},t)$. Then one has

$$\nabla \times (\nabla\times\underset{\sim}{f}) - \frac{\omega^2}{c^2}\,\varepsilon_\omega(\underset{\sim}{r})\,\underset{\sim}{f}\,(\underset{\sim}{r},\omega) = 0 \tag{1-10}$$

where the refractive index

$$\varepsilon_\omega(\underset{\sim}{r}) \equiv 1 - \frac{4\pi N_e e^2}{m\omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2} \quad . \tag{1-11}$$

Here $\omega_p = \left(\frac{4\pi N_e e^2}{m}\right)^{\frac{1}{2}}$ is the plasma frequency of the medium.

From Eqs. (1-2), (1-3) and (1-8), one also has

$$\nabla \cdot (\varepsilon_\omega(\underset{\sim}{r})\,\underset{\sim}{f}\,(\underset{\sim}{r},\omega)) = 0 \quad . \tag{1-12}$$

Note that Eq. (1-10) is a vector equation. However, using the vector relation $\nabla \times (\nabla\times\underset{\sim}{E}) = -\nabla^2\underset{\sim}{E} + \nabla(\nabla\cdot\underset{\sim}{E})$ and Eq. (1-12), we can write Eq. (1-10) as

$$\nabla^2\underset{\sim}{f}(\underset{\sim}{r},\omega) + \nabla\left(\underset{\sim}{f}\cdot\frac{\nabla\varepsilon_\omega}{\varepsilon_\omega(\underset{\sim}{r})}\right) + \frac{\omega^2}{c^2}\,\varepsilon_\omega(\underset{\sim}{r})\underset{\sim}{f} = 0. \tag{1-13}$$

In our case $N_e(\underset{\sim}{r})$ and $\varepsilon_\omega(\underset{\sim}{r})$ fluctuate irregularly. Let $< >$ denote an average over an ensemble of propagation volumes. Then define

$$\begin{aligned} &< \varepsilon_\omega > = \varepsilon_{\omega_o} \\ &\varepsilon_\omega = \varepsilon_{\omega_o} + \delta\varepsilon_\omega \\ &N_e(\underset{\sim}{r}) = <N_e(\underset{\sim}{r})> + \delta N_e(\underset{\sim}{r}) \\ &\beta(\underset{\sim}{r}) = -4\pi e^2 \delta N_e(\underset{\sim}{r})/mc^2 \end{aligned} \tag{1-14}$$

$$k = \frac{\omega}{c} \quad \epsilon_{\omega_o}$$

and $$\epsilon_k(\underset{\sim}{r}) = \beta(\underset{\sim}{r})/k^2 = \delta\epsilon_\omega/\epsilon_{\omega_o} \ll 1$$

to obtain from Eq. (1-13)

$$\nabla^2 \underset{\sim}{f}(\underset{\sim}{r},\omega) + \nabla(\underset{\sim}{f}\cdot\frac{\nabla\epsilon_k}{1+\epsilon_k}) + k^2(1+\epsilon_k(\underset{\sim}{r}))\ \underset{\sim}{f} = 0 \tag{1-15}$$

where now $\epsilon_k(\underset{\sim}{r})$ is a random variable with zero mean. Note that $\epsilon_k(\underset{\sim}{r})$ is different from $\epsilon_\omega(\underset{\sim}{r})$ and that $\beta(\underset{\sim}{r})$ is a wave-frequency independent random variable.

Now consider the term $\nabla(\frac{\underset{\sim}{f}\cdot\nabla\epsilon_k}{1+\epsilon_k})$. We show that if the smallest scale of the fluctuation in ϵ_k is much larger than the wavelength λ $(= \frac{2\pi}{k})$, this term is negligible (see also Tatarskii 1961, 1971).

Define a scale ℓ_ϵ such that $|\nabla\epsilon_k| \le \epsilon_k/\ell_\epsilon$ and assume $k\ell_\epsilon \gg 1$ (this is equivalently to assuming the smallest scale of ϵ_k to be much larger than λ). Then it is clear that

$$|\ \nabla(\underset{\sim}{f}\cdot\frac{\nabla\epsilon_k}{1+\epsilon_k})| \le |\underset{\sim}{f}|\ \frac{k\epsilon_\omega}{\ell} \tag{1-16}$$

since the smallest variation scale of f is 1/k. The magnitude of this term is much smaller than the magnitude of $k^2\epsilon_k\underset{\sim}{f}$ since $k\ell_\epsilon \gg 1$. And for each component $\Phi_\omega(\underset{\sim}{r})$ of the vector field $\underset{\sim}{f}$, it is a very good approximation to consider only the <u>scalar</u> equation

$$\nabla^2\Phi_\omega(\underset{\sim}{r}) + k^2(1+\epsilon_k(\underset{\sim}{r}))\Phi_\omega(\underset{\sim}{r}) = 0 \quad . \tag{1-17a}$$

Let $\Phi_k(\underset{\sim}{r}) \equiv \Phi_\omega(\underset{\sim}{r})$. Eq. (1-17a) becomes

$$\nabla^2\Phi_k(\underset{\sim}{r}) + k^2(1+\epsilon_k(\underset{\sim}{r}))\ \Phi_k = 0 \tag{1-17b}$$

or,

$$\nabla^2 \Phi_k(\underset{\sim}{r}) + k^2 \left[1 + \frac{\beta(\underset{\sim}{r})}{k^2}\right] \Phi_k = 0 \quad . \tag{1-17c}$$

This equation will reveal the effects of diffraction and refraction, but will not yield information concerning polarization. These effects are small if $k\ell_\varepsilon \gg 1$ and $\varepsilon_k \ll 1$, but should be included in a future, more complete treatment[1]. Eq. (1-17) is the basic equation for the theory of wave propagation in a random plasma medium from which all the statistical properties of the observed random wave may be related to the statistical properties of the medium (in particular, the refractive index $\varepsilon_k(\underset{\sim}{r})$ or $\beta(\underset{\sim}{r})$).

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[1] We also neglect here the effect of Faraday rotation due to any ambient magnetic field. This is justifiable if the difference in Faraday rotation between any optical paths which contribute to the observed intensity is small. Hence even though the Faraday rotation of a typical meter wavelength wave in interstellar space is not small (of order 10^2 radians), the angular spread of the beam is also found below to be small (less than about 10^{-6} radians) and hence the difference in Faraday rotation between contributing ray paths is very small (less than about 3×10^{-3} radians if the magnetic field varies by about 10^{-6} Gauss over a scale of 30 pc). Hence the neglect of Faraday rotation is justifiable for the range of parameters considered here.

III. Power Spectrum of the Turbulent Plasmas

Eq. (1-17) gives the relation between the two random functions $\Phi_k(\underset{\sim}{r})$ and $\varepsilon_k(\underset{\sim}{r})$ (or $\beta(\underset{\sim}{r})$). In order to calculate the statistical properties of the random wave $\Phi_k(\underset{\sim}{r})$, one must know the statistical properties of $\varepsilon_k(\underset{\sim}{r})$. The complete statistical properties of $\varepsilon_k(\underset{\sim}{r})$ are uniquely specified by the joint probability distribution of the $\varepsilon_k(\underset{\sim}{r})$'s, or equivalent by a complete set of the moment functions of $\varepsilon_k(\underset{\sim}{r})$. By definition the first moment of $\varepsilon_k(\underset{\sim}{r})$, $\langle \varepsilon_k \rangle$, is equal to zero. As will be shown later in the thesis, the relevant statistical property of $\varepsilon_k(\underset{\sim}{r})$ that affects the random wave $\Phi_k(\underset{\sim}{r})$ is usually only the second moment (or the two-point correlation) of ε_k, $\langle \varepsilon_k(\underset{\sim}{r}), \varepsilon_k(r') \rangle$.

We will assume in this thesis that the medium is statistically homogeneous, in which case the correlation function $\langle \varepsilon_k(\underset{\sim}{r}) \varepsilon_k(\underset{\sim}{r}') \rangle$ depends only on $|\underset{\sim}{r}-\underset{\sim}{r}'|$. And it is most convenient to specify the spatial power spectrum of the fluctuations, which is related to the correlation function through

$$P_\varepsilon(\underset{\sim}{q}) = \int d^3\underset{\sim}{r} \;\; \langle \varepsilon_k(\underset{\sim}{x}) \varepsilon_k(\underset{\sim}{x}+\underset{\sim}{r}) \rangle \; e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} \quad . \tag{1-18}$$

Note that the mean square fluctuation in ε_k is given by

$$\langle {\varepsilon_k}^2 \rangle = \int d^3\underset{\sim}{q} \;\; P_\varepsilon(q)/(2\pi)^3 \quad . \tag{1-19}$$

A simple and very commonly used form of the power spectrum $P_\varepsilon(q)$ is the Gaussian

$$P_\varepsilon(q) = B \; e^{-q^2/{q_o}^2} \tag{1-20}$$

where $L = 1/q_o$ is the coherence scale of the fluctuations.

However, observations of the solar wind and other turbulent media indicate a more realistic form is given by the modified power-law spectrum defined through

$$P_\varepsilon(q) = \frac{B\, e^{-q^2/q_1^2}}{\left[1+\frac{q^2}{q_o^2}\right]^{\alpha/2}} \qquad (1\text{-}21)$$

with $q_o \ll q_1$. This spectrum is flat for $q < q_o$, is a power law with index $-\alpha$ for $q_o < q < q_1$, and is cut off for $q > q_1$. Here again $L = 1/q_o$ is the coherence scale (or correlation length) and $\ell = 1/q_1$ is termed the inner scale. Usually $3 < \alpha < 4$, and $\alpha = \frac{11}{3}$ corresponds to the Kolmogorov spectrum.

Evidence that the spectrum in the solar wind is essentially of the form in Eq. (1-21) is presented in Jokipii (1973). Prior to about 1970, the published papers on interplanetary scintillation all assumed a Gaussian spectrum for the electron density fluctuation (corresponding to $\varepsilon_k(\underset{\sim}{r})$, or $\beta(\underset{\sim}{r})$), and in many the data were used uncritically to infer a "dominant" density scale, which turned out to be of the order of 100 km. (See, e.g., Hewish & Symond 1969.) This is many orders of magnitude less than the directly observed dominant scale of 10^6 km (Jokipii & Coleman 1968). But the discrepancy went unchallenged until nearly simultaneously Cronyn (1970b), Lovelace et al. (1970), and Jokipii & Hollweg (1970) all converged on this question and pointed out the previous discussions were in error. These authors pointed out that a power-law density spectrum was also consistent with the then-available data and that a dominant scale of 10^6 km or so was consistent with the scintillation data. A more detailed analysis of the problem by Cronyn (1972) and the direct observation of the power spectrum by Unti et al. (1973) also show that the plasma density spectrum is of the form given by Eq. (1-21). Chapter 7 of this thesis shows that the power-law spectrum is also consistent with the interstellar scintillation data. Throughout this thesis we will use both the power-law spectrum and the Gaussian spectrum in calculating the statistical properties of the random wave. The Gaussian spectrum is used mainly for comparison.

Carrying out the integrals in Eq. (1-19) and using the relations in Eq. (1-14), one can write the constant B in Eq. (1-20) or Eq. (1-21), in terms of the mean square electron density fluctuations and one finds that

$$B = 8\pi^{3/2} q_o^{-3} \langle \epsilon_k^2 \rangle = 128\pi^{\frac{7}{2}} \left(\frac{r_e^2}{k^4}\right) q_o^{-3} \langle \delta N_e^2 \rangle \qquad (1\text{-}22)$$

for the Gaussian spectrum in Eq. (1-20), and

$$B = 8\pi^{3/2} q_o^{-3} \langle \epsilon_k^2 \rangle \, \Gamma(\tfrac{\alpha}{2}) / \Gamma(\tfrac{\alpha}{2} - \tfrac{3}{2})$$

$$= 128\pi^{7/2} q_o^{-3} r_e^2 \, k^{-4} \langle \delta N_e^2 \rangle \, \Gamma(\tfrac{\alpha}{2}) / \Gamma(\tfrac{\alpha}{2} - \tfrac{3}{2}) \qquad (1\text{-}23)$$

for power-law spectrum with $\alpha > 3$. Note that $r_e = \left(\frac{e^2}{mc^2}\right)$ is the classical electron radius. In obtaining Eq. (1-23), $q_o \ll q_1$, is assumed.

The following function defined by

$$A_\beta(\underset{\sim}{\rho}) = \frac{1}{2} \int_{-\infty}^{\infty} \langle \beta(z,\underset{\sim}{\rho}_1)\beta(z',\underset{\sim}{\rho}_1+\underset{\sim}{\rho})\rangle dz' \tag{1-24}$$

will be used extensively later in this thesis. By Eqs. (1-14) and (1-18), this function can be written as

$$A_\beta(\underset{\sim}{\rho}) = \frac{k^4}{8\pi^2} \iint_{-\infty}^{\infty} P_\varepsilon(q_z=0, \underset{\sim}{q}_1)\, e^{-i\underset{\sim}{q}_1\cdot\underset{\sim}{\rho}}\, d\underset{\sim}{q}_1 \tag{1-25}$$

where $\underset{\sim}{q}_1 = (q_{x_1}, q_{y_1})$ and $\underset{\sim}{\rho} = (x,y)$. Note that the z-dependence of $A_\beta(\underset{\sim}{\rho})$ is not explicitely expressed for convenience.

For Gaussian spectrum, we have from Eq. (1-20)

$$A_\beta(\underset{\sim}{\rho}) = \frac{Bk^4}{8\pi}\, q_o^2\, e^{-(q_o^2\rho^2)/4} \quad . \tag{1-26}$$

Consider the power-spectrum in Eq. (1-21). For $|\underset{\sim}{\rho}| \gg \ell$, we neglect the effect of the cut-off at $q > q_1$ so that

$$P_\varepsilon(\underset{\sim}{q}) = B\left(1+\frac{q^2}{q_o^2}\right)^{-\alpha/2} \quad .$$

From Eq. (1-24) it follows that

$$A_\beta(\underset{\sim}{\rho}) = \frac{Bk^4\, q_o^2 (q_o\rho)^\mu\, K_\mu(q_o\rho)}{2^{2+\mu}\, \pi\, \Gamma(\mu+1)} \tag{1-27}$$

where $\mu = \frac{\alpha}{2} - 1$, $2\mu + \frac{3}{2} > 0$, and K_μ denotes a modified Bessel function of the second kind. One can further show that for $L \gg \rho \gg \ell$,

$$A_\beta(\underset{\sim}{\rho}) = \frac{Bk^4 q_o^{\;2}}{4\pi(\alpha-2)} \left[1 - \frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \cdot (\frac{q_o \rho}{2})^{\alpha-2}\right] . \tag{1-28}$$

However for $\rho \leq \ell$,

$$A_\beta(\rho) \doteq A_\beta(0) + \frac{A_\beta''(0)}{2} \rho^2 \tag{1-29a}$$

where

$$A_\beta(0) = \frac{Bk^4 q_o^{\;2}}{4\pi(\alpha-2)} \tag{1-29b}$$

and

$$A_\beta''(0) = \frac{d^2 A_\beta(\rho)}{d\rho^2} \Big|_{\rho=0} = \frac{-B}{16\pi} k^4 q_o^{\alpha} q_1^{\;4-\alpha} \Gamma(\frac{4-\alpha}{2}) \quad . \tag{1-29c}$$

We also define for later uses

$$A_\epsilon(\underset{\sim}{\rho}) \equiv A_\beta(\underset{\sim}{\rho})/k^4 \tag{1-30a}$$

and

$$B_\epsilon \equiv Bk^4 \quad . \tag{1-30b}$$

IV. Previous Work on the Theory of Wave Propagation in a Random Medium

Since 1950, many papers and books have been published discussing the problem of wave propagation in a random medium. Due to the complexity of the problem, various methods of approximation were employed to solve the problem. The first approach is the "single-scattering" theory, in which perturbation method is used to solve the stochastic wave equation in Eq. (1-17). (Booker & Gordon 1950, Chernov 1960, Tatarskii 1961, Keller 1962, Hoffman 1964, Budden 1965a, 1965b.) Another approach is the "geometric-optics" theory, in which the propagation of rays is considered and used to calculate the various statistical properties of the random waves, such as the angular distribution of the scattered power, the mean ray displacement, the intensity fluctuation and the pulse profile of the random waves. (Chandrasekhar 1952, Chernov 1960, Tatarskii 1961, Keller 1962, Salpeter 1967, Lovelace 1970 and Williamson 1972.)

However "single-scattering" theory is valid only when the scintillation is weak, i.e., the root mean square intensity fluctuation is much less than the mean intensity. For strong scintillation, the "multiple-scattering" effect is important and must be considered. The range of validity of geometrical optics is also quite limited. The method of geometrical optics breaks down when the interference of the rays cannot be neglected. In the interstellar medium and many other situations the scintillations are strong, and both the "single-scattering" theory and the "geometrical optics" method cannot be applied. Therefore, a theory dealing with the "multiple-scattering" effect is needed.

The angular distribution of the random wave for the case when multiple-scattering must be taken into account has been discussed by Fejer (1953) and Howells (1960). Basic mathematical treatments of multiple scattering have been given by many authors including Foldy (1945), Keller (1964), Twersky (1964), Tatarskii (1964), but much of this work is of a formal nature and is difficult to evaluate in practical cases.

A new method called the "Markov approximation" has been developed recently. This method is valid for most strong scintillation cases, including interstellar scintillations. A set of the moment equations for the random waves with same frequency has been derived under this approximation by many authors (Ho & Beran 1968, Tatarskii 1969, 1971, Beran & Ho 1969, Molyneux 1971, and Brown 1972a). However, only the first and second moment equations were solved. The equation of the fourth-moment, which directly relates to the intensity correlation function, has not completely been solved. Dagkesamanskaya & Shishov (1970) and Brown (1972b) gave numerical solutions of the fourth-moment, but it is hard to draw qualitative properties of the intensity correlation scale from the above numerical calculations. Strohbehn & Wang (1972) and Wang & Strohbehn (1974a, 1974b) calculated the intensity correlation function assuming that the probability function of the random waves is "log-normal" and got "paradox" results.

In this thesis, the fourth-moment will be solved analytically and the probability distribution of the random wave will also be determined. We also extend the "Markov approximation" to time-dependent cases and derive a complete set of the moment equations with <u>different</u> frequencies, which is applied to the calculation of the pulse profile in the interstellar scintillations.

We also mention here the "thin screen diffraction" theory, in which the diffraction of the electromagnetic waves by a "thin" layer of random medium is studied. (Mercier 1962 , Salpeter 1967, Scheuer 1968, Jokipii 1970, Lovelace 1970, Torrieri & Taylor 1971, Taylor 1972, Taylor & Lekhyanada 1973.) However, for strong scintillation, the intensity correlation function still has not been solved in the region of <u>Fresnel</u> diffraction. In Chapter 3 of this thesis, we will solve this problem using a new method.

V. Outline of the Thesis

Chapter 1 is a general introduction to the thesis, in which we also derive the basic stochastic wave equation, discuss the density power spectrum of the plasma medium, and give some comments on the previous work on scintillation theory.

Chapter 2 gives a discussion of various theories on the wave propagation in a random medium. In this chapter we present an exact formulation of the problem and briefly discuss various methods of approximation, including thin phase screen approximation, quasi-optic approximation, Born approximation, method of smooth perturbation, geometric-optics approximation, perturbation of stochastic operator, and the Markov approximation.

Chapter 3 treats the "thin screen diffraction" theory. Previous works on this problem are discussed and the intensity correlation function in the region of Fresnel diffraction is calculated. However, the analytic solution of the pulse broadening within the "thin screen diffraction" theory is given in Chapter 5, where the pulse broadening in a continuous random medium is presented.

In Chapter 4, we derive a complete set of the moment equation of the random wave field with different wave-numbers under the "quasi-optics" and the "Markov random process" approximations. The validity of these two assumptions applied to the interstellar scintillation is also given.

In Chapter 5, we obtain and discuss the phase correlation function, angular broadening, pulse smearing, intensity correlation and the probability distribution function of the random waves from the moment equations derived in Chapter 4.

Chapter 6 is an application of the method of smooth perturbation to the interplanetary scintillation. We formulate and calculate the effects of the solar-wind velocity fluctuations on the observed intensity correlation function and on the ratio of the observed "pattern" velocity and the true velocity of the solar wind in the three-dimensional spherical model.

In Chapter 7 we apply the results of Chapter 5 to interstellar scintillations and find that the power-law spectrum of the interstellar medium also fits the observed data.

Chapter 2

Theories of Wave Propagation in a Random Medium

I. The Wave Equation

We consider the propagation of a monochromatic wave $E_\omega(\underset{\sim}{r},t)$ obeying the scalar wave equation

$$\nabla^2\Phi_\omega(\underset{\sim}{r}) + \frac{\omega^2}{c^2}\,\epsilon_\omega(\underset{\sim}{r})\Phi_\omega(\underset{\sim}{r}) = 0 \qquad (2\text{-}1)$$

with

$$E_\omega(\underset{\sim}{r},t) = \Phi_\omega(\underset{\sim}{r})e^{-i\omega t} \;. \qquad (2\text{-}2)$$

Here we treat $E_\omega(\underset{\sim}{r},t)$ as a complex wave field. The boundary condition for equation (1) is $\Phi_\omega(\underset{\sim}{r})$ on the surface S which encircles the volume we considered. Usually in an actual situation, the surface S is composed of some surface S_1 relatively near the observer and a surface S_2 at infinity. The boundary condition of $\Phi_\omega(\underset{\sim}{r})$ is given on surface S_1 while the Sommerfeld radiation condition is applied on the surface S_2 at infinity.

$\Phi_\omega(\underset{\sim}{r})$ may be regarded as a Fourier component in time of a general wave function. Here $(\frac{\omega}{2\pi})$ is the frequency of the monochromatic wave, c is the speed of light and $\epsilon_\omega(\underset{\sim}{r})$ is the refractive index of the medium in which the wave propagates. Let $\lambda = \frac{2\pi c}{\omega}$ be the wavelength.

The refractive index $\epsilon_\omega(\underset{\sim}{r})$ is taken to be a random function and depends on both the position $\underset{\sim}{r}$ and the wave frequency ω. We will consider in this thesis the propagation of the high frequency waves with $\omega \gg \omega_p$, the plasma frequency of the medium, in the plasma medium. This applies to the propagation of the radio waves in the ionosphere, the interplanetary space or the interstellar medium. If N_e is the electron density, then from Chapter 1 we have by assuming the variation in N_e is small for a distance of the order of the wavelength λ.

$$\epsilon_\omega(\underset{\sim}{r}) = 1 - \frac{\omega_p^2}{\omega^2} \tag{2-3}$$

and

$$\omega_p^2 = \frac{4\pi N_e e^2}{m} \tag{2-4}$$

where m is the mass and e is the charge of an electron.

Now N_e and $\epsilon_\omega(\underset{\sim}{r})$ fluctuate irregularly. Let $< >$ denote an average over an ensemble of propagation volumes. Then using the notations in Eq. (1-14), we have (c.f. Eq. (1-17))

$$\nabla^2 \Phi_k(\underset{\sim}{r}) + k^2 (1 + \frac{\beta(\underset{\sim}{r})}{k^2}) \quad \Phi_k(\underset{\sim}{r}) = 0 \tag{2-5a}$$

or

$$\nabla^2 \Phi_k(\underset{\sim}{r}) + k^2 (1 + \epsilon_k(\underset{\sim}{r}))\ \Phi_k(\underset{\sim}{r}) = 0. \tag{2-5b}$$

Consider an initial plane wave propagating in the + z - direction impinging on our medium at z = 0. It is useful to define

$$\Phi_k(\underset{\sim}{r}) = u(k,\underset{\sim}{r})\ e^{ikz} \tag{2-6}$$

from which we obtain

$$2ik \frac{\partial u(k,\underset{\sim}{r})}{\partial z} + (\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\ u(k,\underset{\sim}{r}) + \beta(\underset{\sim}{r})\ u(k,\underset{\sim}{r}) = 0 \tag{2-7}$$

where $\underset{\sim}{r} = (x,y,z)$. We also define $\underset{\sim}{\rho} = (x,y)$, $s = (\underset{\sim}{\rho},k)$ for later use.

Equation (2-5) is a stochastic equation connecting the refraction index and the wave field $\Phi_k(\underset{\sim}{r})$. The problem at hand is the determination of the probability distribution of Φ_k, or of various statistical properties of Φ_k, such as its expectation value, its variance and its higher moments, in terms of the statistical properties of the refractive index of the medium.

II. Hierachy Equations and Equations for Characteristic Functionals

In this section we will present two formulations for obtaining the statistical properties of the wave field $\Phi_k(\underset{\sim}{r})$ from equation (2-5) without any approximation. One of these two formulations involves a set of equations for the hierarchy of moments of the wave field. The other uses a functional equation for the characteristic (or generating) functional of the wave field.

Consider the wave field $\Phi_k(\underset{\sim}{r})$ satisfying the stochastic equation (2-5b). In order to get an equation for the first moment $\langle\Phi_k(\underset{\sim}{r})\rangle$, we take the ensemble average of equation (2-5b) and find

$$\nabla_{\underset{\sim}{r}}^2\langle\Phi_k(\underset{\sim}{r})\rangle + k^2\langle\Phi_k(\underset{\sim}{r})\rangle + k^2\langle\epsilon_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r})\rangle = 0 \tag{2-8}$$

where the Laplacian operator $\nabla_{\underset{\sim}{r}}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Since the quantity $\langle\epsilon_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r})\rangle$ appears in Eq. (2-8), we must find an equation for this quantity. We multiply equation (2-5) by $\epsilon_k(\underset{\sim}{r}_1)$ and take ensemble average to get

$$\nabla_{\underset{\sim}{r}}^2\langle\Phi_k(\underset{\sim}{r})\epsilon_k(\underset{\sim}{r}_1)\rangle + k^2\langle\Phi_k(\underset{\sim}{r})\epsilon_k(\underset{\sim}{r}_1)\rangle + k^2\langle\epsilon_k(\underset{\sim}{r}_1)\epsilon_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r})\rangle = 0 \tag{2-9}$$

Once the quantity $\langle\Phi_k(\underset{\sim}{r})\epsilon_k(\underset{\sim}{r}_1)\rangle$ is found from Eq. (2-9) we may evaluate it at $\underset{\sim}{r}_1 = \underset{\sim}{r}$ and use it in Eq. (2-8) to compute $\langle\Phi_k(\underset{\sim}{r})\rangle$. Since Eq. (2-9) involves a new moment $\langle \epsilon_k(\underset{\sim}{r}_1)\epsilon_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r})\rangle$ we must obtain another equation for it, etc. Thus we find that the equation for a moment of any order involves moments of higher order. Therefore an infinite system of equations must be considered for the simultaneous determination of all moments. If we want to determine higher moments of $\Phi_k(\underset{\sim}{r})$, we are led to the following infinite set of equations by noting that $\Phi_k(\underset{\sim}{r})$ is treated as a complex quantity

$$(\nabla_{\underset{\sim}{r}}^2+k^2)\langle\Phi_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r}_1)\cdots\Phi_k(\underset{\sim}{r}_i)\Phi_k^*(\underset{\sim}{r}_{i+1})\cdots\Phi_k^*(\underset{\sim}{r}_{i+j})\ \epsilon_k(\underset{\sim}{r}_{i+j+1})\cdots$$

$$\epsilon_k(\underset{\sim}{r}_{i+j+\ell})\rangle + k^2\langle\epsilon_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r})\Phi_k(\underset{\sim}{r}_1)\cdots\epsilon_k(\underset{\sim}{r}_{i+j+\ell})\rangle = 0 \tag{2-10}$$

for $\quad i,j,\ell = 0,1,2,\cdots,$

where * denotes complex conjugate. Keller (1964) derives an equation similar to equation (2-10).

Thus the complete statistical properties of the wave field $\Phi_k(\underset{\sim}{r})$ requires simultaneous solution of all the moments given by the hierarchy Eq. (2-10). This is an impossible job. The same sort of difficulty occurs in statistical mechanics and in plasma kinetic theory. Various approximations made in order to solve the problems will be presented in the next section.

An alternative way to describe the complete statistical properties of a random field is to introduce the "characteristic functional", which generates the infinite set of moments appearing in Eq. (2-10). It was introduced into statistical mechanics in 1947 by Bogolyubov (1959) and into the theory of turbulence in 1952 by Hopf (1952), and was first used to describe the random wave by Keller (1964).

The characteristic functional Ψ can be defined as

$$\Psi(\nu,\nu^*,\eta) = \langle \exp\, i\, R \rangle \equiv \langle \exp\, i \left\{ [\nu,\Phi_k] + [\nu^*,\Phi_k^*] + [\eta,\epsilon_k] \right\} \rangle \tag{2-11}$$

where
$$[A,B] \equiv \int_V A(\underset{\sim}{r})B(\underset{\sim}{r})\, d^3\underset{\sim}{r} \tag{2-12}$$

and
$$R = [\nu,\Phi_k] + [\nu^*,\Phi_k^*] + [\eta,\epsilon_k] \quad . \tag{2-13}$$

Here V is the whole space bounded by the surface S where the boundary conditions for $\Phi_k(\underset{\sim}{r})$ are given. We note that $\Phi_k(\underset{\sim}{r})$ is treated as a complex variable, and an alternative way to define Ψ is to let

$$R = [\nu_1,\mathrm{Re}\Phi_k] + [\nu_2,\mathrm{Im}\Phi_k] + [\eta,\epsilon_k] \tag{2-14}$$

where ν_1 and ν_2 are real independent variables, and where $\mathrm{Re}\Phi_k$ and $\mathrm{Im}\,\Phi_k$ denote the real and the imaginary parts of Φ_k respectively. We find

$$\nu = \frac{1}{2}(\nu_1 - i\nu_2) \text{ and } \nu^* = \frac{1}{2}(\nu_1 + i\nu_2).$$

Mathematically it is more convenient to treat ν and ν^* as independent variables, therefore we will use the form of Ψ given by Eq. (2-11).

Functional differentiation (Volterra 1930, Beran 1968) of Eq. (2-11) yields

$$\frac{\delta^{j+m+n}\Psi(\nu,\nu^*,\eta)}{\delta\nu(\underset{\sim}{r}_1)\cdots\delta\nu(\underset{\sim}{r}_j)\delta\nu^*(\underset{\sim}{r}_{j+1})\cdots\delta\nu^*(\underset{\sim}{r}_{j+m})\delta\eta(\underset{\sim}{r}_{j+m+1})\cdots\delta\eta(\underset{\sim}{r}_{j+m+n})}$$

$$= (i)^{j+m+n}\langle \Phi_k(\underset{\sim}{r}_1)\cdots\Phi_k(\underset{\sim}{r}_j)\Phi_k^*(\underset{\sim}{r}_{j+1})\cdots\Phi_k^*(\underset{\sim}{r}_{j+m})\epsilon_k(\underset{\sim}{r}_{j+m+1})\cdots\epsilon_k(\underset{\sim}{r}_{j+m+n})e^{iR}\rangle \tag{2-15}$$

Evaluation of Eq. (2-15) at $\nu(\underset{\sim}{r}) \equiv 0, \eta(\underset{\sim}{r}) \equiv 0$ leads to the moments in Eq. (2-10) and we have

$$\left[\frac{\delta^{j+m+n}\Psi(\nu,\nu^*,\eta)}{\delta\nu(\underset{\sim}{r}_1)\cdots\delta\nu(\underset{\sim}{r}_j)\,\delta\nu^*(r_{j+1})\cdots\delta\eta(\underset{\sim}{r}_{j+m+n})}\right]_{\nu=\nu^*=\eta=0}$$

$$= \langle\Phi_k(\underset{\sim}{r}_1)\cdots\Phi_k(\underset{\sim}{r}_j)\Phi^*(\underset{\sim}{r}_{j+1})\cdots\Phi^*(\underset{\sim}{r}_{j+m})\varepsilon_k(\underset{\sim}{r}_{j+m+1})\cdots\varepsilon_k(\underset{\sim}{r}_{j+m+n})\rangle$$

$$\equiv \Gamma_{j,m,n} \tag{2-16}$$

Conversely the characteristic functional Ψ can be expressed as a functional Taylor series in term of the moments $\Gamma_{j,m,n}$

$$\Psi(\nu,r^*,\eta) = \sum_{j,m,n} \frac{(i)^{j+m+n}}{j!m!n!} \int \Gamma_{j,m,n}\nu(\underset{\sim}{r}_1)\cdots\nu(\underset{\sim}{r}_j)\nu^*(\underset{\sim}{r}_{j+1})\cdots\nu^*(\underset{\sim}{r}_{j+m})$$

$$\eta(\underset{\sim}{r}_{j+m+1})\cdots\eta(\underset{\sim}{r}_{j+m+n})\; d\underset{\sim}{r}_1\cdots d\underset{\sim}{r}_{j+m+n} \quad . \tag{2-17}$$

From Eq. (2-5), one can deduce the following equation satisfied by the functional $\Psi(\nu,\nu^*,\eta)$

$$(\nabla_{\underset{\sim}{r}}^{\;2} + k^2)\;\frac{\delta\Psi}{\delta\nu(\underset{\sim}{r})} + \frac{\delta^2\Psi}{\delta\nu(\underset{\sim}{r})\delta\eta(\underset{\sim}{r})} = 0 \quad . \tag{2-18}$$

We note that with the Taylor expansion given in Eq. (2-17), Eq. (2-18) yields the hierarchy equations in Eq. (2-10). In fact, Eq. (2-18) can be considered as a compact form for the hierarchy equations in Eq. (2-10). We also note that Eq. (2-18) can be considered as an equation for $\frac{\delta\Psi}{\delta\nu(\underset{\sim}{r})}$ since Ψ does not occur. The boundary conditions for Ψ can be obtained from the wave properties on the surface S.

III. Methods of Approximation

Since the exact solution of the moment equations derived in the last section is impossible, methods of approximation must be employed in order to solve the problem. Each method of approximation has a certain range of validity.

Before discussing various methods of approximation, we will write the wave Eq. (2-5) in the following three forms:

$$\nabla^2\Phi_k(\underset{\sim}{r}) + k^2(1+\epsilon_k(\underset{\sim}{r}))\,\Phi_k(\underset{\sim}{r}) = 0 \tag{2-19a}$$

$$2ik\,\frac{\partial u(k,\underset{\sim}{r})}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(k,\underset{\sim}{r}) + k^2\epsilon_k(\underset{\sim}{r})u(k,\underset{\sim}{r}) = 0 \tag{2-19b}$$

and

$$2ik\,\frac{\partial\varphi}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi + (\underset{\sim}{\nabla}\cdot\varphi)^2 + k^2\epsilon_k(\underset{\sim}{r}) = 0 \tag{2-19c}$$

where in (2-19c) we define φ as

$$\Phi_k(\underset{\sim}{r}) \equiv e^{ikz}u(k,\underset{\sim}{r}) \equiv e^{ikz}e^{\varphi(k,\underset{\sim}{r})} \quad . \tag{2-20}$$

In this thesis, we are interested in plane waves propagating initially in the + z direction with $E(\underset{\sim}{r},t) = e^{i(kz-\omega t)}$.

Below we discuss various methods of approximation.

(A) Thin Phase Screen Approximation

If the terms $\nabla^2\varphi$ and $(\nabla\cdot\varphi)^2$ in Eq. (2-19c) are neglected, then one gets

$$2ik\,\frac{\partial\varphi(k,\underset{\sim}{r})}{\partial z} + k^2\epsilon_k(\underset{\sim}{r}) = 0 \tag{2-21}$$

from which one immediately obtains

$$\varphi(k,\underset{\sim}{r}) = \int_0^z \frac{i}{2} k\, \epsilon_k(x,y,z')dz' \tag{2-22}$$

assuming at $z = 0$, $\varphi(k,\underset{\sim}{r}) = 0$. Note that (φ/i) is the random phase of the wave field $\Phi_k(\underset{\sim}{r})$. Equation (2-21) is valid only when the scattering is weak and when the diffraction effect is small (very thin region of turbulence). This approximation is very crude.

Note that the "thin screen" approximation given in Eq. (2-21) is different from the usual "thin screen diffraction theory" (Mercier 1962, Salpeter 1967). In the "thin screen diffraction theory", the random medium is assumed to be concentrated in a "thin slab". Inside the "thin slab", the "thin screen" approximation in Eq. (2-21) is applied, while beyond the slab diffraction theory is used for the propagating wave. This problem is discussed in detail in Chapter 3.

(B) Quasi-Optic Approximation

Consider the initial wave propagating in the + z direction. When the smallest scale of the fluctuating medium is much larger than the wavelength λ of the propagating wave, the scattering angle is small. In this case, one can neglect the term $(\frac{\partial^2}{\partial z^2}u)$ in Eq. (2-19b) compared to the term $(ik\frac{\partial}{\partial z} u)$, and one obtains

$$2ik\frac{\partial}{\partial z}u(k,\underset{\sim}{r}) + \nabla_{\underset{\sim}{\rho}}^2 u(k,\underset{\sim}{r}) + k^2\epsilon_k(\underset{\sim}{r})u(k,\underset{\sim}{r}) = 0. \tag{2-23}$$

This is called the "quasi-optic" approximation or "parabolic-equation" approximation since Eq. (2-23) is parabolic.

Insight into the parabolic approximation can be obtained as follows. Note that if we define the operator L as,

$$L = 2ik\frac{\partial}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right), \tag{2-24}$$

then Eq. (2-19b) becomes

$$L\, u(k,\underset{\sim}{r}) = -\,k^2 \varepsilon_k(\underset{\sim}{r}) u(k,\underset{\sim}{r})$$

from which we get

$$\begin{aligned} u(k,\underset{\sim}{r}) &= -\,L^{-1} k^2 \varepsilon_k(\underset{\sim}{r}) u(k,\underset{\sim}{r}) \\ &= \int G(\underset{\sim}{r},\underset{\sim}{r}')[-k^2 \varepsilon_k(\underset{\sim}{r}') u(k,\underset{\sim}{r}')]\, d\underset{\sim}{r}' \end{aligned} \tag{2-25}$$

where L^{-1} is the inverse operator of L and the Green's function G

$$G(\underset{\sim}{r},\underset{\sim}{r}') = e^{-ik(z-z')} \frac{-e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|} \tag{2-26}$$

Physically, the value of $u(k,\underset{\sim}{r})$ in Eq. (2-25) can only be appreciably affected by the inhomegeneities included in a cone with vertex at the observation point ($\underset{\sim}{r}$), with axis directed towards the wave source, and with angular aperture θ, which is assumed to be small. In most of this region

$$|z-z'| \gg \sqrt{(y-y')^2 + (x-x')^2}\,.$$

Therefore we expand $|\underset{\sim}{r}-\underset{\sim}{r}'|$ as,

$$|\underset{\sim}{r}-\underset{\sim}{r}'| = (z-z')\sqrt{1+\frac{(x-x')^2+(y-y')^2}{(z-z')^2}}$$

$$= (z-z') + \frac{|\underset{\sim}{\rho} - \underset{\sim}{\rho}|^2}{2(z-z')} + \cdots. \tag{2-27}$$

Using this expression in $e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}$, and retaining only the first term of the expansion in the denominator of Eq. (2-26), we obtain the approximate formula for $G(\underset{\sim}{r},\underset{\sim}{r}')$

$$G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{-e^{\frac{ik|\underset{\sim}{\rho}-\underset{\sim}{\rho}'|^2}{2(z-z')}}}{4\pi(z-z')} \tag{2-28}$$

which is the <u>exact</u> Green's function of Eq. (2-23). Thus the "quasi-optics" approximation is equivalent to the approximation of the Green's function in Eq. (2-28).

In what follows in this thesis, the "quasi-optics" approximation will always be assumed. However, Eq. (2-23) is still not easy to solve, hence further assumptions must be made to solve the problem.

(C) Born Approximation

If the fluctuating part $\epsilon_k(\underset{\sim}{r})$ of the refractive index is small, and the fluctuation of the wave is small compared to the unperturbed wave, then one can expand $u(k,\underset{\sim}{r})$ as a perturbation series

$$u = u_o + u_1 + \cdots \tag{2-29}$$

from which we have

$$(2ik\frac{\partial}{\partial z} + \nabla^2)\, u_o = 0 \tag{2-30}$$

and

$$(2ik\frac{\partial}{\partial z} + \nabla^2)\ u_1 + k^2\varepsilon_k(\underset{\sim}{r})\ u_o = 0\ . \qquad (2\text{-}31)$$

We normalized u_o such that

$$u_o(\underset{\sim}{r}) = u_o(z=0,\underset{\sim}{\rho}) = 1 \qquad (2\text{-}32)$$

This is equivalent to plane wave of unit amplitude being incident from $-\infty$ on a medium beginning at z = 0. Then Eq. (2-31) gives

$$(2ik\frac{\partial}{\partial z} + \nabla^2)u_1 + k^2\varepsilon_k(\underset{\sim}{r}) = 0 \quad . \qquad (2\text{-}33)$$

Note that Eq. (2-33) is valid only when

$$\left|\frac{u_1}{u_o}\right| \ll 1 \quad . \qquad (2\text{-}34)$$

Define the amplitude A and phase S of the perturbed wave u as

$$u = A\ e^{iS} \qquad (2\text{-}35)$$

where A and S are real quantities. Then we have to first order,

$$\log u = \log A + iS = \log (u_o+u_1) = \log u_o + \log (1+\frac{u_1}{u_o}).$$

Since $|u_1/u_o| \ll 1$, then

$$\log (1+\frac{u_1}{u_o}) \approx \frac{u_1}{u_o}$$

Thus

$$\log A + iS = \log u_o + \frac{u_1}{u_o}$$

Let $u_o = A_o e^{iS_o}$, where A_o is the unperturbed amplitude and S_o the phase.

We have

$$\log\left(\frac{A}{A_o}\right) \equiv \chi = \mathrm{Re}\left(\frac{u_1}{u_o}\right) \tag{2-36}$$

and

$$S - S_o = S_1 = \mathrm{Im}\left(\frac{u_1}{u_o}\right) \tag{2-37}$$

Thus we see that Born approximation is valid only if $|\chi| \ll 1$ and $|S_1| \ll 1$. In addition to the fluctuation in amplitude being small, the fluctuation in phase must be small.

(D) Method of Smooth Perturbation (MSP)

If one uses Eq. (2-19c) instead of Eq. (2-19b) and neglects the non-linear term $(\nabla\cdot\varphi)^2$, one gets under the quasi-optic approximation,

$$2ik\frac{\partial\varphi}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi + k^2\epsilon_k(\underset{\sim}{r}) = 0 \quad . \tag{2-38}$$

Equation (2-38) is called the method of Smooth Perturbation (MSP). This method was first used by Rytov (1937) and Tatarskii (1961) gives a detailed discussion. Since the quantities in Eq. (2-19c) involve only the derivative of φ instead of φ itself, the linearization of Eq. (2-19c) is valid if the <u>derivative of φ</u> is small.

Again define the amplitude A and phase S as in Eq. (2-35)

$$u = e^{\varphi} = A\,e^{iS} \tag{2-39a}$$

It is easy to show that

$$\begin{cases} \chi \equiv \log A = \mathrm{Re}\,\varphi \\ S = \mathrm{Im}\,\varphi \quad . \end{cases} \tag{2-39b}$$

Since the smallness of φ is not required, the fluctuation on S is not limited to be small. Tatarskii (1971) has discussed the validity of MSP and found that MSP is valid if

$$\langle \chi^2 \rangle \ll 1 \tag{2-40}$$

$\langle \chi^2 \rangle$ can be related to the intensity fluctuation by noting that $\chi = \log A = \frac{1}{2} \log I$.

We note that Eqs. (2-33) and (2-38) are identical if we put $u_1 = \varphi$. If $|\varphi| \ll 1$, $u = e^{\varphi} = 1 + \varphi + \varphi^2 + \cdots = 1 + u_1 + u_2 + \cdots$. Thus $\varphi = u_1$ in the case $|\varphi| \ll 1$, and the Born approximation and MSP are equivalent. Clearly MSP has a broader range of validity then the Born approximation.

We will discuss MSP in detail when we apply this approximation to the interplanetary scintillation in Chapter 6.

(E) Geometrical Optics

When the diffraction term ($\nabla^2\varphi$) in Eq. (2-19c) is small, we negelct this term and obtain the equation of geometrical optics,

$$2ik\frac{\partial\varphi}{\partial z} + (\underset{\sim}{\nabla}\cdot\varphi)^2 + k^2\varepsilon_k(\underset{\sim}{r}) = 0 \tag{2-41}$$

Define

$$i\,\Theta \equiv ikz + \Phi \tag{2-42}$$

and

$$n^2(k,\underset{\sim}{r}) = 1 + \varepsilon_k(\underset{\sim}{r}) \tag{2-43}$$

where Θ can be regarded as the generalized phase of the wave and $n(k,\underset{\sim}{r})$ the index of refraction. From Eq. (2-41), we obtain the following Eikonal equation in geometrical optics

$$(\underset{\sim}{\nabla}\,\Theta)^2 - n^2k^2 = 0 \tag{2-44}$$

From Eq. (2-41), one obtains the following "ray equation"

$$\frac{d}{ds}\left(n \frac{d\underset{\sim}{r}}{ds}\right) = \underset{\sim}{\nabla} \cdot n \tag{2-45}$$

where the position $\underset{\sim}{r}$ of a light ray is parametrized by s, which is the wave path, or $ds = \sqrt{|d\underset{\sim}{r}|^2}$. The ray equation is equivalent to the famous "Fermat's Principle", which states the following quantity I is stationary along the ray path,

$$I = \int_{\text{ray path}} \frac{\sqrt{\left(\frac{d\underset{\sim}{r}}{ds}\right)^2} \cdot n(k, \underset{\sim}{r})}{c} \, ds \tag{2-46}$$

Thus Eqs. (2-41), (2-44), (2-45) and $\delta I = 0$ are all equivalent.

The ray equation has been applied to calculate many quantities for random wave propagation, such as the angular distribution of rays, the mean ray displacement, the intensity fluctuation of the random wave and the phase fluctuation of wave and the pulse broadening due to the random medium. (Chandrasekhar 1952, Chernov 1960, Tatarskii 1961, Keller 1962, Salpeter 1967, Hollweg 1970 Lovelace 1970, and Williamson 1972.)

However, the range of validity of geometrical optics is quite limited. Let the scale of the fluctuation medium be L. The angle θ due to diffraction of the medium will be of the order of $\frac{\lambda}{L}$. Also let the transverse characteristic scale of the rays be a. Then when the wave propagates a distance z, with

$$\theta z > a \tag{2-47}$$

two neighbor rays will intersect. Thus geometrical optics breaks down at $z > (\frac{aL}{\lambda})$. In Chapter 5 we will find $a \sim (L/\Phi_o)$ where Φ_o is the root mean square phase fluctuation of the wave. Geometrical optics is valid only when

$$z \lesssim \frac{L^2}{\lambda\Phi_o} \tag{2-48}$$

The condition in Eq. (2-48) for the validity of geometrical optics is the same as that obtained by Salpeter (1967).

(F) Perturbation of Stochastic Operator (PSO)

All methods of approximations discussed above involve approximations made on the wave equation, which must be valid all the way through the volume in which the random wave propagates if the methods of approximations are applicable. The criterion for the validity of approximations is not easily met when the propagation distance is large, or when the scattering is strong, or both. This is true especially for the thin screen approximation, geometric optics and linearization approximations (i.e. Born approximation and MSP).

In this and the next sections, we will present two methods of approximation, in which the approximations are made locally. If the approximations are valid in each step, then one can integrate the results of each step and get the solution of the problem considered. In this section, a method proposed by Keller (1962,1964) will be discussed. We call this method "perturbation of stochastic operator" (PSO). The Markov random process approximation, which is a special case of "perturbation of stochastic operator", will be given in the next section.

Following Keller, we consider the following linear stochastic equation for Φ,

$$M(\alpha)\Phi = g \tag{2-49}$$

where $M(\alpha)$ is a linear stochastic operator depending upon a random variable α and g is a given function independent of α. The solution of equation (2-49) can be written as

$$\Phi = M^{-1}(\alpha)g \tag{2-50}$$

Then take ensemble average of (2-50) ,

$$\langle\Phi\rangle = \langle M^{-1}(\alpha)\rangle g \tag{2-51}$$

from which we multiply $\langle M^{-1}\rangle^{-1}$ to obtain

$$\langle M^{-1}\rangle^{-1}\langle\Phi\rangle = g \ . \tag{2-52}$$

This is an exact equation satisfied by $\langle\Phi\rangle$, although in this form it is not yet useful.

To make Eq. (2-54) useful we assume that M is the sum of a non-random operator L_o and a small random operator δL_1, where δ is a small parameter. Thus we write

$$M = L_o + \delta L_1 . \tag{2-53}$$

From Eq. (2-53), we have

$$M^{-1} = (L_o(1+\delta L_o^{-1}L_1))^{-1} = (1+\delta L_o^{-1}L_1)^{-1}L_o^{-1} .$$

Therefore by expansion,

$$\langle M^{-1}\rangle = \sum_{n=0}^{\infty} \langle(-\delta L_o^{-1}L_1)^n\rangle L_o^{-1} ,$$

from which we obtain

$$\langle M^{-1}\rangle^{-1} = L_o+\delta\langle L_1\rangle + \delta^2 [\langle L_1\rangle L_o^{-1}\langle L_1\rangle - \langle L_1 L_o^{-1} L_1\rangle] + O(\delta^3) . \qquad (2\text{-}54)$$

Assume $\langle L_1\rangle = 0$ without any loss of generality. Eq. (2-54) then gives

$$L_o\langle\Phi\rangle - \delta^2 \langle L_1 L_o^{-1} L_1\rangle\langle\Phi\rangle = g + O(\delta^3) \qquad (2\text{-}55)$$

Now back to our problem, we compare Eq. (2-19a) with Eqs. (2-49) and (2-53)

$$L_o = \nabla^2+k^2 \qquad (2\text{-}56a)$$

$$\delta L_1 = k^2\epsilon_k(\underset{\sim}{r}) \qquad (2\text{-}56b)$$

and $g \equiv 0.$

Also

$$L_o^{-1}(\underset{\sim}{r})f(\underset{\sim}{r}) = \int_V G(\underset{\sim}{r},\underset{\sim}{r}')f(\underset{\sim}{r}')d\underset{\sim}{r}' \qquad (2\text{-}57)$$

where

$$G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{-e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi\ |\underset{\sim}{r}-\underset{\sim}{r}'|} . \qquad (2\text{-}58)$$

Thus Eq. (2-55) gives to second order of δ,

$$(\nabla^2+k^2)\langle\Phi_k(\underset{\sim}{r})\rangle - k^4 \int_V d^3\underset{\sim}{r}'\ G(\underset{\sim}{r},\underset{\sim}{r}')\langle\epsilon_k(\underset{\sim}{r})\epsilon_k(\underset{\sim}{r}')\rangle\langle\Phi(\underset{\sim}{r}')\rangle = 0 \qquad (2\text{-}59)$$

Eq. (2-59) is an integral equation for $\langle\Phi_k(\underset{\sim}{r})\rangle$ and can be solved for $\langle\Phi(\underset{\sim}{r})\rangle$ if $\langle\epsilon_k(\underset{\sim}{r})\epsilon_k(\underset{\sim}{r}')\rangle$ is known. The same technique can be used to derive an equation for a higher moment. I call this method "perturbation of stochastic operator" (PSO).

The assumption made by Keller that δ is small is not stated precisely. As we can see clearly, the magnitude of the second term in equation (2-59) depends not only on the magnitude of $\epsilon_k(\underset{\sim}{r})$, but also on the correlation scale of $\epsilon_k(\underset{\sim}{r})$. Thus the smallness of $\epsilon_k(\underset{\sim}{r})$ is not the criterion of the validity of the expansion of the stochastic operator. The criterion can roughly be stated as "the change of the wave field due to the scattering of the random medium within the scale size of the medium is small". This criterion is a local property as we pointed out at the beginning of this section.

If in particular, the correlation scale of $\epsilon_k(\underset{\sim}{r})$ approaches zero, then Eq. (2-59) is exact. This is just the Markov random process approximation in method (G). (In fact, if the "quasi-optic" approximation is used, only the correlation scale in the propagation direction must be assumed zero, for the Markov approximation to be valid.)

(G) Method of Markov Random Process Approximation and Quasi-Optic Approximation (MQA)

Mathematically, if the following two assumptions are made

(i) Quasi-optic approximation is valid, i.e. equation (2-23) is used.

(ii) The correlation scale of $\epsilon_k(\underset{\sim}{r})$ or $\beta(\underset{\sim}{r})$ in the z-direction is zero, i.e.

$$\langle \beta(z,\underset{\sim}{\rho})\ \beta(z',\underset{\sim}{\rho}')\rangle = 2\ \delta(z-z')\ A_\beta(\underset{\sim}{\rho}-\underset{\sim}{\rho}') \tag{2-60}$$

then a complete set of the moment equations of the random field $u(k,\underset{\sim}{r})$ without coupling between different moments can be derived. This is worked out and discussed completely in Chapter 4. The assumption made in Eq. (2-60) is called the Markov random process approximation.

We will derive the equation for the first moment under the above two assumptions using the result of Keller's method.

Under the assumption (i), we have

$$L_o = (2ik\frac{\partial}{\partial z} + \nabla_{\underset{\sim}{\rho}}^2), \quad \delta L_1 = k^2 \epsilon_k(\underset{\sim}{r}) \tag{2-61a}$$

and

$$G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{-1}{4\pi|z-z'|} \times e^{\frac{ik|\underset{\sim}{\rho}-\underset{\sim}{\rho}|^2}{2|z-z'|}} \quad . \tag{2-61b}$$

Note that

$$\lim_{z'\to z} G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{-i}{4k} \delta(\underset{\sim}{\rho}-\underset{\sim}{\rho}') \quad . \tag{2-62}$$

We then have from Eqs. (2-60), (2-61), (2-62) and (2-59)

$$[2ik\frac{\partial}{\partial z} + \nabla_{\underset{\sim}{\rho}}^2 + \frac{i}{2k} A_\beta(0)]\langle u(k,z,\underset{\sim}{\rho})\rangle = 0 \tag{2-63}$$

which is exactly the same as we derived in Chapter 4. The higher order term can be shown to be zero if $\epsilon_k(\underset{\sim}{r})$ is a Gaussian variable or the assumption made in Eq. (4-21b) is valid.

Note that MQA method can be applied to strong scintillation cases where the scintillation index $m_z^2 \sim 1$. The scintillation index m_z is defined to be the ratio between the root mean square intensity fluctuation and the mean intensity of the random wave.

Chapter 3

General Thin Screen Diffraction Theory

I. Introduction

Before considering the full problem of propagation of waves in a random medium, consider the diffraction by a random, phase-changing screen. The problem of diffraction of electromagnetic wave by a layer of random medium has been studied. (Mercier 1962, Salpeter 1967, Jokipii 1970, Lovelace 1970, Torrieri & Taylor 1971, Taylor 1972, Taylor & Lekhyanada 1973.) They used a "thin, phase-changing screen approximation", in which the random phase fluctuations of the wave are produced by the random medium and the intensity fluctuations inside the medium are neglected. This is a reasonable approximation if the medium is "thin". Suppose the complex amplitude of the signal at the plane z = 0, where the screen is located, is $E(x,y,z = 0)\, e^{-i\omega t}$. Under the "thin screen approximation", we have

$$E(x,y,z = 0) = A(x,y,z = 0)\, e^{i\Phi(x,y)} \tag{3-1}$$

where $A(x,y,z = 0) = 1$, and $\Phi(x,y)$ is the phase fluctuations at the plane z = 0. A schematic sketch of the problem is presented in Figure (3-1). Let D be the thickness of the thin slab. From Eq. (2-22) we have

$$\Phi(x,y) = \int_{-D}^{0} k\varepsilon_k(x,y,z')dz' \tag{3-2}$$

from which we have $\langle \Phi(x,y) \rangle = 0$. One defines the two-point correlation function

$$\Phi_o^{\,2}\, P_\Phi(\underset{\sim}{\rho}) = \langle\, \Phi(\underset{\sim}{\rho}_1)\Phi(\underset{\sim}{\rho}_1+\underset{\sim}{\rho})\,\rangle \tag{3-3a}$$

Figure (3-1)

A schematic sketch of the thin screen diffraction problem. The random medium is confined to a thin layer of thickness D (from $z = -D$ to $z = 0$). The plane wave $e^{i(kz-\omega t)}$ hits the "thin screen" from the $-z$ direction. After passing the screen, the phase of the wave is randomized and is characterized by the function $\Phi(x,y)$.

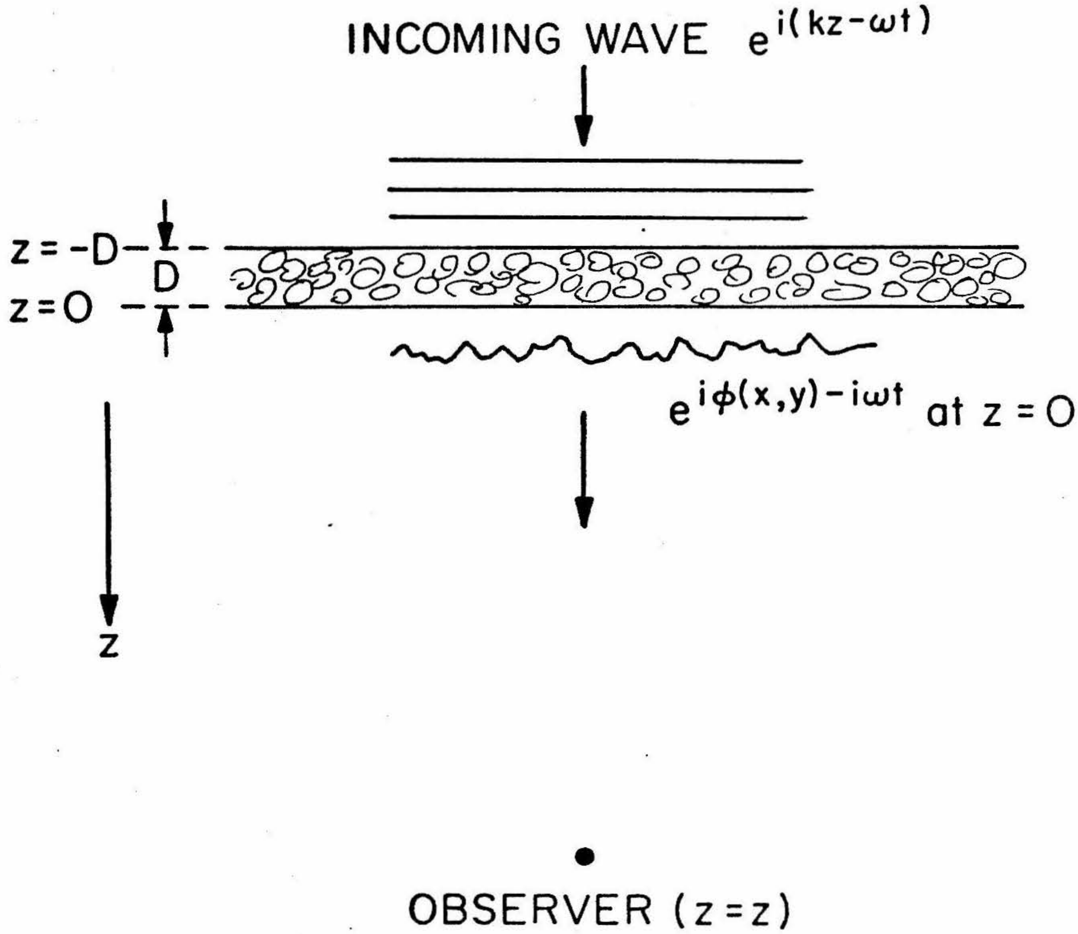
INCOMING WAVE $e^{i(kz-\omega t)}$
z = -D
D
z = O
$e^{i\phi(x,y)-i\omega t}$ at z = O
z
OBSERVER (z = z)

Figure (3-1)

with $P_{\Phi}(0) = 1$. Here $P_{\Phi}(\underset{\sim}{\rho})$ is the normalized phase correlation function and Φ_o is the root-mean-square phase fluctuation. In equation (3-3a) we have assumed that the two-point correlation depends only on the distance between the two points in the initial plane z = 0. If the thickness D is greater than the correlation scale of the medium in the z-direction, then by Eqs. (1-14), (1-24), (3-2) and (3-4), one has

$$\Phi_o{}^2 P_{\Phi}(\underset{\sim}{\rho}) = \frac{D}{2k^2} A_\beta(\underset{\sim}{\rho}) \qquad (3\text{-}3b)$$

$$\Phi_o{}^2 = \frac{D}{2k^2} A_\beta(0) \qquad (3\text{-}3c)$$

and

$$P_{\Phi}(\rho) = A_\beta(\rho)/A_\beta(0) \quad . \qquad (3\text{-}3d)$$

The intensity fluctuations of the electromagnetic wave are then built up by interference as the wave propagates to the observers. The object of the diffraction theory is to calculate the intensity fluctuation at a distance from the screen. Assuming the radiation conditions hold at large distances, the Helmholtz formula gives without approximation the complex amplitude E(x,y,z) in front of the screen (z > 0)

$$E(x,y,z) = -\frac{1}{4\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \left(\frac{e^{ikr}}{r}\right) \frac{\partial E(x',y',z=0)}{\partial z} - E(x',y',z=0)\frac{\partial}{\partial z}\left(\frac{e^{ikr}}{r}\right) \right\} dx'dy' \qquad (3\text{-}4)$$

where

$$r = \left[(x-x')^2+(y-y')^2+(z-0)^2\right]^{\frac{1}{2}} .$$

Also, let

$$E(x,y,z) = u(x,y,z)e^{ikz} .$$

Assuming $kz \gg 1$, one can use the stationary phase approximation, putting (c.f. Eq. (2-27)

$$kr = kz + \frac{k}{2z}\left\{(x-x')^2+(y-y')^2\right\} . \tag{3-5}$$

Then using the Kirchhoff approximation $\frac{\partial E(x,y,z=0)}{\partial z} = ikE(x,y,0)$, one has

$$u(x,y,z) = \frac{-ik}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x',y',z=0)\,\exp\left\{\frac{ik}{2z}\,[x-x')^2+(y-y')^2]\right\} dx'dy' . \tag{3-6}$$

The approximation made in Eq. (3-5) is called Fresnel approximation (Born & Wolf, 1959) or quasi-optics approximation (Chapter 2, Section III). A sufficient condition for the validity of Eq. (3-5) can be obtained by noting the higher order term in the expansion of (kr) be smaller than 1. Thus we have

$$kz\left[\frac{(x-x')^2+(y-y')^2}{z^2}\right]^2 \ll 1.$$

Noting that for the important region of the integrals in Eq.(3-6)

$$|x-x'|,\ |y-y'| \lesssim z\,\langle\theta^2\rangle^{\frac{1}{2}}$$

we have

$$z \ll \frac{1}{k\langle\theta^2\rangle^2} . \tag{3-7}$$

In interstellar scintillation, using $k = 10^{-1}\text{cm}^{-1}$ and the observed $\langle\theta^2\rangle \sim 10^{-14}$ Eq. (3-7) becomes $z \ll 10^{27}$cm, which is true for all cases.

When the distance z is at such a position where the approximation made in Eq. (3-5) or Eq. (3-6) is accurate, the observer is said to be in the region of <u>Fresnel diffraction</u> (Born & Wolf, 1959). Note that the criterion for the validity of Fresnel approximation in Eq. (3-7) is different from that of classical diffraction theory.

Assuming $\Phi(x,y)$ is a Gaussian random variable and using Eqs. (3-1), (3-3) and (3-6), one finds

$$\langle I(x,y,z)\rangle = \langle E(x,y,z)\ E^{*}(x,y,z)\rangle = 1 \qquad (3\text{-}8)$$

and

$$\langle I(\underset{\sim}{\rho}_1,z)I(\underset{\sim}{\rho}_1+\underset{\sim}{\rho},z)\rangle \equiv M_z^{\ 2}(\underset{\sim}{\rho}) + 1$$

$$= \int\cdots\int d\underset{\sim}{\rho}_1 d\underset{\sim}{\rho}_2 d\underset{\sim}{\rho}_3 d\underset{\sim}{\rho}_4 \left(\frac{k}{2\pi z}\right)^4 \exp\left[-\frac{ik}{2z}\left(\underset{\sim}{\rho}_1^{\ 2}-\underset{\sim}{\rho}_2^{\ 2}+\underset{\sim}{\rho}_3^{\ 2}-\underset{\sim}{\rho}_4^{\ 2}\right)\right] \times f \qquad (3\text{-}9)$$

where I is the intensity of the wave and

$$f \equiv \langle E(\underset{\sim}{\rho}_1-\underset{\sim}{\rho},z=0)\ E^{*}(\underset{\sim}{\rho}_2-\underset{\sim}{\rho},z=0)\ E(\underset{\sim}{\rho}_3,z=0)\ E^{*}(\underset{\sim}{\rho}_4,z=0)\rangle$$

$$= \exp\left[-\Phi_o^{\ 2}\left\{2-P_{\Phi}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2) - P_{\Phi}(\underset{\sim}{\rho}_3-\underset{\sim}{\rho}_4) + P_{\Phi}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}-\underset{\sim}{\rho}_3)\right.\right.$$

$$\left.\left. + P_{\Phi}(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}-\underset{\sim}{\rho}_4) - P_{\Phi}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}-\underset{\sim}{\rho}_4) - P_{\Phi}(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}-\underset{\sim}{\rho}_3)\right\}\right]. \qquad (3\text{-}10)$$

The purpose of the diffraction theory is to carry out the integration appearing in Eq. (3-9) for the intensity correlation function $M_z^{\ 2}(\underset{\sim}{\rho})$. The scintillation index m_z is defined as

$$m_z = M_z(0),$$

which is the ratio of root mean square intensity fluctuation and mean intensity.

It is hard to carry out the integration appearing in Eq. (3-9). Various asymptotic forms of the intensity correlation function have been obtained (Mercier 1962, Salpeter 1967, Jokipii 1970). Mercier (1962) has obtained an asymptotic formula for the intensity fluctuation at a great distance from the random screen. Under the assumption that the correlation function

$P_{\Phi}(\rho)$ in Eq. (3-3) is Gaussian, Mercier found

$$M_z^{\,2}(\rho) = \exp\left\{-\,2\Phi_o^{\,2}\,[1-P_{\Phi}(\rho)]\right\} - \exp\,[-2\Phi_o^{\,2}] \qquad (3\text{-}11)$$

if $z \gg \frac{L^2}{\lambda}$ where L is the correlation scale of $P_{\Phi}(\rho)$. When $z \gg \frac{L^2}{\lambda}$, the observer is said to be in the region of Fraunhofer diffraction.

Salpeter (1967) found a formula for $M_z(\rho)$ for the case $\Phi_o^{\,2} \ll 1$. The results are expressed in terms of $\hat{M}_z^{\,2}(q)$ and $\hat{P}_{\Phi}(q)$ which are the Fourier transform of $M_z^2(\rho)$ and $P_{\Phi}(\rho)$, respectively,

$$\hat{M}_z^2(q) = (\tfrac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\,\rho\; e^{-iq\cdot\rho}\, M_z^{\,2}(\rho) \qquad (3\text{-}12)$$

and

$$\hat{P}_{\Phi}(q) = (\tfrac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\rho\; e^{-iq\cdot\rho}\, P_{\Phi}(\rho) \quad . \qquad (3\text{-}13)$$

One finds that

$$\hat{M}_z^{\,2}(q) = 4\,\sin^2\,(\frac{q^2 z}{2k})\;\Phi_o^{\,2}\;\hat{P}_{\Phi}(q) \quad . \qquad (3\text{-}14)$$

Lovelace (1970) presented a heuristic physical model considering the propagation of each ray to calculate the intensity correlation. However, his result is not rigorous.

Jokopii (1970) found that for $\hat{P}_{\Phi}(q)$ with a power-law spectrum, Eq. (3-14) can be valid even when $\Phi_o^{\,2} \gg 1$. He found the sufficient conditions for the validity of Eq. (3-14) as

(a) $z \lesssim kL\ell$ where L and ℓ are the outer scale and inner scale of the phase correlation function $P_{\Phi}(\rho)$, respectively.

$$\text{(b)}\ \Phi_2^{\ 2} \equiv \int_{|\underset{\sim}{q}| \gtrsim q_*} \hat{P}_\Phi(\underset{\sim}{q})\ d\underset{\sim}{q} \ll 1 \text{ where}$$

$$q_* \equiv \left(\frac{k}{z}\right)^{\frac{1}{2}} \tag{3-15}$$

and

$$\text{(c)}\ z \lesssim \frac{kL^2}{\Phi_o} \quad .$$

Condition (b) is equivalent to $m_z \lesssim 1$ (weak scintillation).

Furthermore, one can easily show that for $P_\Phi(\underset{\sim}{\rho})$ with a power law spectrum, Eq. (3-11) of Mercier's result also holds if $z \gg \frac{L^2}{\lambda}$ where L is the outer scale of $P_\Phi(\underset{\sim}{\rho})$.

Salpeter (1967) shows that in the case of Gaussian spectrum of $\hat{P}_\Phi(\underset{\sim}{q})$, geometric-optics applies when

$$z \ll \frac{L^2}{\lambda} \text{ and } z \ll \frac{L^2}{\lambda\Phi_o} \equiv \ell_o \text{ (focal length)} \tag{3-16}$$

ℓ_o is called the focal length. From geometric-optics, one obtains

$$\hat{M}_z^{\ 2}(\underset{\sim}{q}) = \Phi_o^{\ 2}\left(\frac{q^2 z}{k}\right)^2 \hat{P}_\Phi(\underset{\sim}{q}) \tag{3-17a}$$

or

$$M_z^{\ 2}(\underset{\sim}{\rho}) = \left(\frac{z\Phi_o}{k}\right)^2 \nabla^4\ P_\Phi(\underset{\sim}{\rho}). \tag{3-17b}$$

It can be shown from Eq. (3-17) that the scintillation index $m_z \ll 1$ when the conditions in Eq. (3-16) are applied.

Thus in conclusion, we find that the problem of strong scintillations in the whole Fresnel diffraction region has not been solved. In this chapter we, instead of carrying out the integral in Eq. (3-9), will use a new method to obtain the intensity correlation function $M_z^{\,2}(\rho)$.

In the interstellar scintillation, the scintillation is strong and one has that for Gaussian spectrum $L \approx 10^{14-15}$cm (Salpeter 1969, Scheuer 1968) and for power-law spectrum, the outer scale $L \gtrsim 10^{17-18}$cm. Therefore, for $z \approx 10^{21-24}$cm and $\lambda \approx 10^{2}$cm, one has

$$z \ll \frac{L^2}{\lambda} \quad .$$

Thus the scintillations are always <u>not</u> in the Fraunhofer diffraction region and Mercier's result of Eq. (3-11) cannot be applied.

II. New Method

A new method of solving the diffraction problem may be developed by noting that the wave function u(x,y,z) in Eq. (3-6) satisfies the following differential equation,

$$\frac{\partial u(z,\underset{\sim}{\rho})}{\partial z} = \frac{i}{2k} \nabla_{\underset{\sim}{\rho}}^{2} u(z,\underset{\sim}{\rho}) \tag{3-18}$$

where $\underset{\sim}{\rho} \equiv (x,y)$ and $\nabla_{\underset{\sim}{\rho}}^{2} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Define

$$\Gamma_2(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2) \equiv \langle u(z,\underset{\sim}{\rho}_1)\ u^*(z,\rho_2)\rangle \tag{3-19}$$

and

$$\Gamma_4(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_3,\underset{\sim}{\rho}_4) \equiv \langle u(z,\underset{\sim}{\rho}_1)u^*(z,\underset{\sim}{\rho}_2)u(z,\underset{\sim}{\rho}_3)u^*(z,\underset{\sim}{\rho}_4)\rangle \ . \tag{3-20}$$

We note that

$$\langle I(z,\underset{\sim}{\rho})\rangle = \Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}) \tag{3-21}$$

and

$$\langle I(z,\underset{\sim}{\rho})I(z,\underset{\sim}{\rho}')\rangle = \Gamma_4(z,\underset{\sim}{\rho},\underset{\sim}{\rho},\underset{\sim}{\rho}',\underset{\sim}{\rho}') \tag{3-22}$$

$$= M_z^{\ 2}(\underset{\sim}{\rho}-\underset{\sim}{\rho}') + 1 \ .$$

From Eq. (3-18), we can obtain a differential equation for $\Gamma_2(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)$. We take the complex conjugate of Eq. (3-18) and obtain

$$\frac{\partial u^*(z,\underset{\sim}{\rho}')}{\partial z} = \frac{-i}{2k} \nabla_{\underset{\sim}{\rho}'}^{2} \overset{*}{u}(z,\underset{\sim}{\rho}') \ . \tag{3-18'}$$

Multiply Eq. (3-18) by $u^*(z,\underset{\sim}{\rho}')$ and Eq. (3-18') by $u(z,\underset{\sim}{\rho})$, and then take the ensemble average of the sum to get

$$\frac{\partial}{\partial z} \Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}') = \frac{i}{2k} (\nabla_{\rho}^{\ 2} - \nabla_{\underset{\sim}{\rho}'}^{\ 2}) \Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}') \quad . \tag{3-23}$$

Eq. (3-23) is the differential equation for the second moment $\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}')$. Similarly one can obtain the following differential equation for the fourth moment $\Gamma_{\Phi}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_3,\underset{\sim}{\rho}_4)$,

$$\frac{\partial}{\partial z} \Gamma_4(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_3,\underset{\sim}{\rho}_4) = \frac{i}{2k} (\nabla_{\underset{\sim}{\rho}_1}^{\ 2} - \nabla_{\underset{\sim}{\rho}_2}^{\ 2} + \nabla_{\underset{\sim}{\rho}_3}^{\ 2} - \nabla_{\underset{\sim}{\rho}_4}^{\ 2}) \Gamma_4 \quad . \tag{3-24}$$

Eqs. (3-23) and (3-24) can be greatly simplified if the initial condition is invariant relative to shifts and rotations in the initial plane which we assume to be true in our problem. Thus, $\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}')$ depends only on $|\underset{\sim}{\rho}-\underset{\sim}{\rho}'|$ and $(\nabla_{\underset{\sim}{\rho}}^{\ 2} - \nabla_{\underset{\sim}{\rho}'}^{\ 2})\Gamma_2 = 0$. One obtains from (3-23)

$$\frac{\partial}{\partial z} \Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}') = 0 \tag{3-25}$$

or

$$\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}') = \Gamma(0,\underset{\sim}{\rho},\underset{\sim}{\rho}') \quad . \tag{3-26}$$

In particular, $\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}) = \langle I(z)\rangle = \langle I(0)\rangle = 1 \quad .$ (3-26')

For $\Gamma_4(z,\underset{\sim}{\rho}_1\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_3,\underset{\sim}{\rho}_4)$, we change the variables $\rho_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_3$ and $\underset{\sim}{\rho}_4$ to the new variables $\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta,\underset{\sim}{\rho}_\gamma$ and $\underset{\sim}{\rho}_\delta$ by defining

$$\begin{aligned} \underset{\sim}{\rho}_\alpha &= \underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2 \\ \underset{\sim}{\rho}_\beta &= \underset{\sim}{\rho}_2-\underset{\sim}{\rho}_3 \\ \underset{\sim}{\rho}_\gamma &= \underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2+\underset{\sim}{\rho}_3-\underset{\sim}{\rho}_4 \\ \text{and} \qquad \underset{\sim}{\rho}_\delta &= \frac{1}{4}(\underset{\sim}{\rho}_1+\underset{\sim}{\rho}_2+\underset{\sim}{\rho}_3+\underset{\sim}{\rho}_4) \quad . \end{aligned} \tag{3-27}$$

Then Eq. (3-24) becomes

$$\frac{\partial\Gamma_4}{\partial z} = \frac{i}{2k}\left[2\nabla_{\rho_\alpha}\cdot\nabla_{\rho_\beta} + \nabla_{\rho_\delta}\cdot(\nabla_{\rho_\alpha} - \nabla_{\rho_\beta} + 2\nabla_{\rho_\gamma})\right]\Gamma_4(z,\rho_\alpha,\rho_\beta,\rho_\gamma,\rho_\delta) \quad . \qquad (3\text{-}28)$$

Since Γ_4 is invariant to the shift and rotations in the plane z = constant, the function Γ_4 is independent of ρ_δ. Thus $\nabla_{\rho_\delta}\Gamma_4 = 0$ and

$$\frac{\partial\Gamma_4}{\partial z} = \frac{i}{k}(\nabla_{\rho_\alpha}\cdot\nabla_{\rho_\beta})\,\Gamma_4(z,\rho_\alpha,\rho_\beta,\rho_\gamma) \quad . \qquad (3\text{-}29)$$

The fact that Eq. (3-29) does not contain any differential with respect to the variable ρ_γ allows us to simplify this equation further by equating ρ_γ to zero. In effect, this means that we consider the case in which the four points ρ_i (i=1,4) are located at the corners of a parallelogram. With $\rho_\gamma = 0$ in mind, we will write Eq. (3-29) as

$$\frac{\partial\Gamma_4(z,\rho_\alpha,\rho_\beta)}{\partial z} = \frac{i}{k}(\nabla_{\rho_\alpha}\cdot\nabla_{\rho_\beta})\,\Gamma_4(z,\rho_\alpha,\rho_\beta) \qquad (3\text{-}29')$$

where the dependence of Γ_4 on the variable ρ_γ has been dropped. We note that the correlation function $M_z^2(\rho)$ is related to Γ_4 by

$$1 + M_z^2(\rho) = \Gamma_4(z,\rho,0) = \Gamma_4(z,0,\rho) \quad . \qquad (3\text{-}30)$$

Our new method to solve the diffraction problem is to solve Eq.(3-29') instead of carrying out the integral in Eq. (3-9).

The initial condition for $\Gamma_4(z,\rho_\alpha,\rho_\beta)$ can be obtained as

$$\Gamma_4(z,\rho_\alpha,\rho_\beta) = \langle u(\rho_1,z=0)u^*(\rho_2,z=0)u(\rho_3,z=0)u^*(\rho_4,z=0)\rangle$$

$$= \exp\left\{-\Phi_o^{\,2}[2-2P_\Phi(\rho_\alpha)-2P_\Phi(\rho_\beta)+P_\Phi(\rho_\alpha+\rho_\beta)+P_\Phi(\rho_\alpha-\rho_\beta)]\right\} \qquad (3\text{-}31)$$

The appropriate boundary condition is

$$\Gamma_4(z,\rho_\alpha,\rho_\beta)\Big|_{\rho_{\alpha,\beta}=\infty} = [\Gamma_2(z,\rho,\rho_{\beta,\alpha}+\rho)]^2 \qquad (3\text{-}32)$$

where the notation $\rho_{\beta,\alpha}$, $\rho_{\alpha,\beta}$ means that when ρ_α is infinite, ρ_β appears on the right side of the equation and vice versa. Eq. (3-32) means that when $\rho_\alpha = \infty$, the two pairs $(u(\rho_1,z),u(\rho_4,z))$ and $(u(\rho_2,z),u(\rho_3,z))$ are uncorrelated, and similarly for $\rho_\beta = \infty$. From Eq. (3-26) we have

$$\Gamma_2(z,\rho,\rho_\alpha+\rho) = \Gamma_2(0,\rho,\rho_\alpha+\rho)$$

$$= \exp\left\{-\Phi_o^{\,2}[1-P_\Phi(\rho_\alpha)]\right\} . \qquad (3\text{-}33)$$

Thus

$$\Gamma_4(z,\rho_\alpha,\rho_\beta)\Big|_{\rho_{\alpha,\beta}=\infty} = \exp\left\{-2\Phi_o^{\,2}[1-P_\Phi(\rho_{\beta,\alpha})]\right\} . \qquad (3\text{-}34)$$

In the next subsections we will present the analytic and the numerical result of the integration of Eq. (3-29) with the initial condition (3-31) and boundary condition (3-34).

III. Numerical Results

In this section, we will solve Eq. (3-29) numerically for two cases, namely, (a) the phase correlation function $P_{\Phi}(\underset{\sim}{\rho})$ being Gaussian, and (b) the phase correlation function having a Kolmogorov spectrum. For simplicity of the calculation, we will consider only a "one-dimensional phase screen" at z = 0 (Salpeter 1967), so that the transverse variables $\underset{\sim}{\rho}_{\alpha}$, $\underset{\sim}{\rho}_{\beta}$ in Eq. (3-29) reduce to x_{α}, x_{β} and Eq. (3-29') becomes

$$\frac{\partial \Gamma_4}{\partial z} = \frac{i}{k} \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} \Gamma_4(z, x_{\alpha}, x_{\beta}) \quad . \tag{3-35}$$

(A) Gaussian Spectrum

For the Gaussian spectrum in Eq. (1-26)

$$\Phi_o^2 P_{\Phi}(x) = \Phi_o^2 \exp\left(- \frac{q_o^2 x^2}{4} \right) \tag{3-36}$$

where $L = q_o^{-1}$ is the correlation scale of the phase function and Φ_o^2 is the mean square of the phase fluctuation. From Eq. (3-11), we have in the Fraunhofer diffraction region, the intensity correlation scale $x_* = \frac{\sqrt{2}L}{\Phi_o}$ for $\Phi_o \gtrsim 1$. For this we will introduce the following dimensionless variables.

$$\zeta = z/(k x_*^2) \tag{3-37a}$$

$$\xi_{\alpha} = x_{\alpha}/x_* \tag{3-37b}$$

$$\xi_{\beta} = x_{\beta}/x_* \quad . \tag{3-37c}$$

Then Eq. (3-35) becomes

$$\frac{\partial \Gamma_4}{\partial \zeta}(\zeta, \xi_\alpha, \xi_\beta) = i \frac{\partial^2 \Gamma_4}{\partial \xi_\alpha \partial \xi_\beta}(\zeta, \xi_\alpha, \xi_\beta) \quad . \tag{3-38}$$

We also note that the boundary condition (3-34) for large Φ_o becomes

$$\Gamma_4(\zeta, \xi_\alpha, \xi_\beta)\Big|_{\xi_{\alpha,\beta} = \infty} = \exp\left\{- \xi^2_{\beta,\alpha}\right\} \quad . \tag{3-39}$$

In practice, of course, we cannot apply the boundary condition given in Eq. (3-34) as it stands because this would require an infinite number of mesh points. We will truncate the ξ_α, ξ_β at appropriately large values of ξ_α, ξ_β. The truncated boundary, $\xi_{B.C.}$ (or $x_{B.C.}$) must satisfy the following conditions:

(i) Since the correlation scale of $P_\Phi(\underset{\sim}{\rho})$ is L, we must have

$$\rho_{B.C.} > L \ (\text{or } \zeta_{B.C.} > \Phi_o) \quad . \tag{3-40}$$

(ii) Since the mean scattering angle θ_o for such a screen is

$$\theta_o = \frac{\Phi_o}{2kL}$$

(see Chapter 5) and the transverse spreading of the ray after propagating a distance z is z θ_o, we must have

$$x_{B.C.} > z\ \theta_o = \frac{z\Phi_o}{2kL}$$

or

$$\xi_{B.C.} > \zeta \ . \tag{3-41}$$

In the Fraunhofer diffraction region $\zeta > \Phi_o{}^2$.

The numerical results for $\Phi_o = 1,2,5$ are shown in Figures (3-2),(3-3),(3-4) and (3-5). We find that in the Fresnel region

(a) For $\Phi_o = 5$, strong scintillation is developed somewhere between $\zeta = 1$ and $\zeta = \Phi_o$. At $\zeta = 1$, the scintillation is still weak. And the peak of scintillation index occurs near z = focal length ($\zeta = \Phi_o$).

(b) For $\Phi_o = 1$, the results are about the same as those predicted by weak scintillation theory (Eq.(3-14)) when the scintillation is weak.

(c) When the scintillation is strong, the correlation scale of intensity fluctuation is about the same as that in the Fraunhofer region. The numerical result here is consistent with the asymptotic analytic solution in the next subsection.

B. Kolmogorov Spectrum

For Kolmogorov Spectrum,

$$\Phi_o^2 P_{\Phi}(\rho) = \frac{D}{2k^2} A_\beta(\rho) \tag{3-42}$$

where $A_\beta(\rho)$ is given in Eq. (1-28) or Eq. (1-29) with $\alpha = \frac{11}{3}$. For one-dimensional phase screen, we have

$$\Phi_o^2 P_{\Phi}(x) = \frac{D}{2k^2} A_\beta(x) \tag{3-43}$$

and

$$\Phi_o^2 = \frac{D}{2k^2} A_\beta(0) \quad . \tag{3-44}$$

Introducing the following dimensionless variables,

$$\begin{cases} \xi_\alpha = x_\alpha / x_* & \text{(3-45a)} \\ \xi_\beta = x_\beta / x_* & \text{(3-45b)} \\ \zeta = z/(k x_*^2) & \text{(3-45c)} \end{cases}$$

Figure Captions

Figure (3-2). This figure shows the numerical results of the scintillation index $m_z^2 = M_z^2(0)$ as a function of the normalized propagating distance ζ for the medium with a Gaussian spectrum. In curve (1), $\Phi_o = 1$; in curve (2), $\Phi_o = 2$; and in curve (3), $\Phi_o = 5$. Φ_o is the root mean square of phase fluctuation.

Figure (3-3). For Gaussian spectrum with $\Phi_o = 1$, the intensity correlation function $M_z^2(\xi)$ is plotted as a function of the normalized transverse coordinate ξ for various values of the normalized distance ζ.

Figure (3-4). As in Figure (3-3) with $\Phi_o = 2$.

Figure (3-5). As in Figure (3-3) with $\Phi_o = 5$.

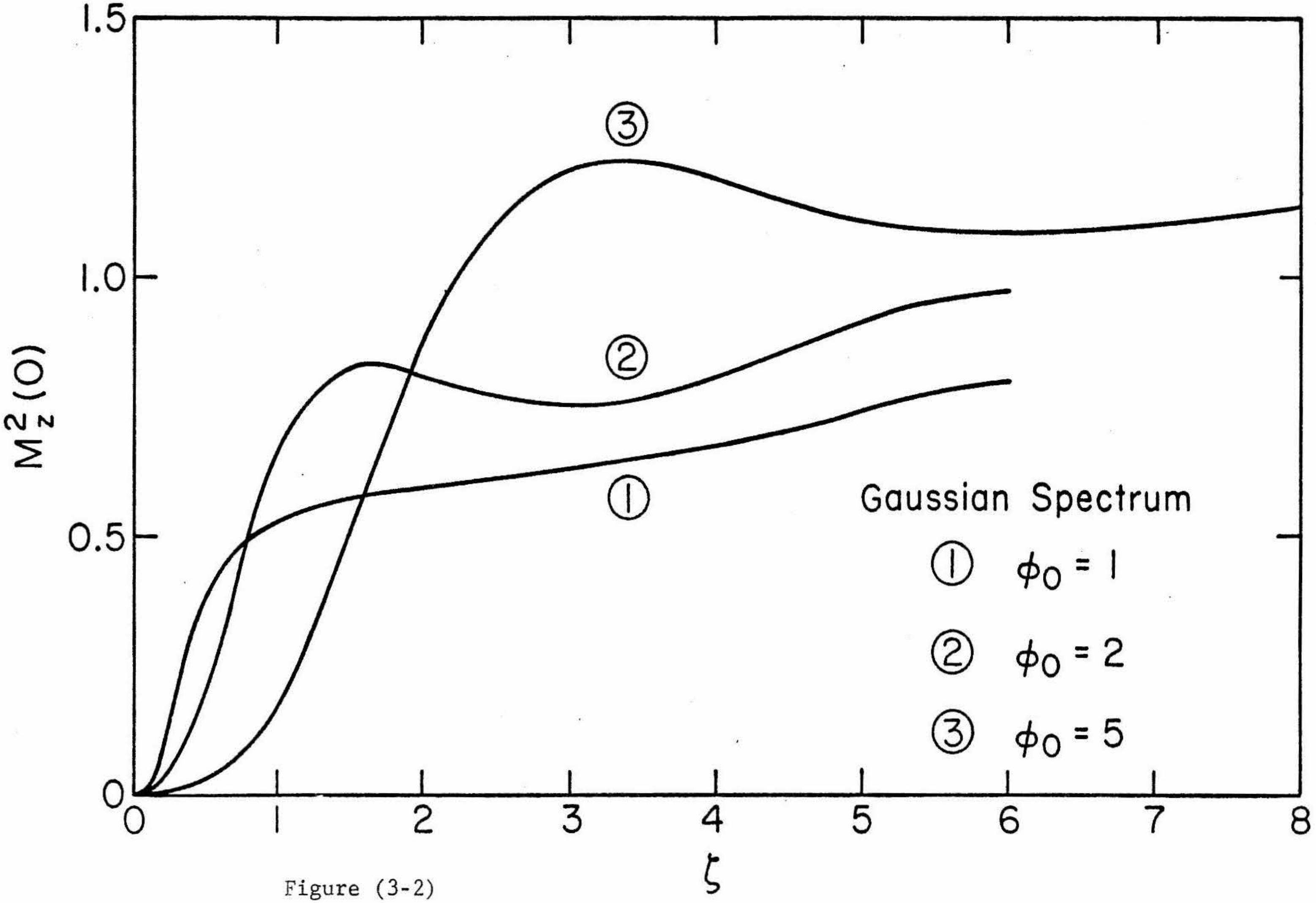
1.5
1.0
0.5
0
$M_z^2(0)$
0
1
2
3
4
5
6
7
8
ζ
③
②
①
Gaussian Spectrum
① $\phi_0 = 1$
② $\phi_0 = 2$
③ $\phi_0 = 5$

Figure (3-2)

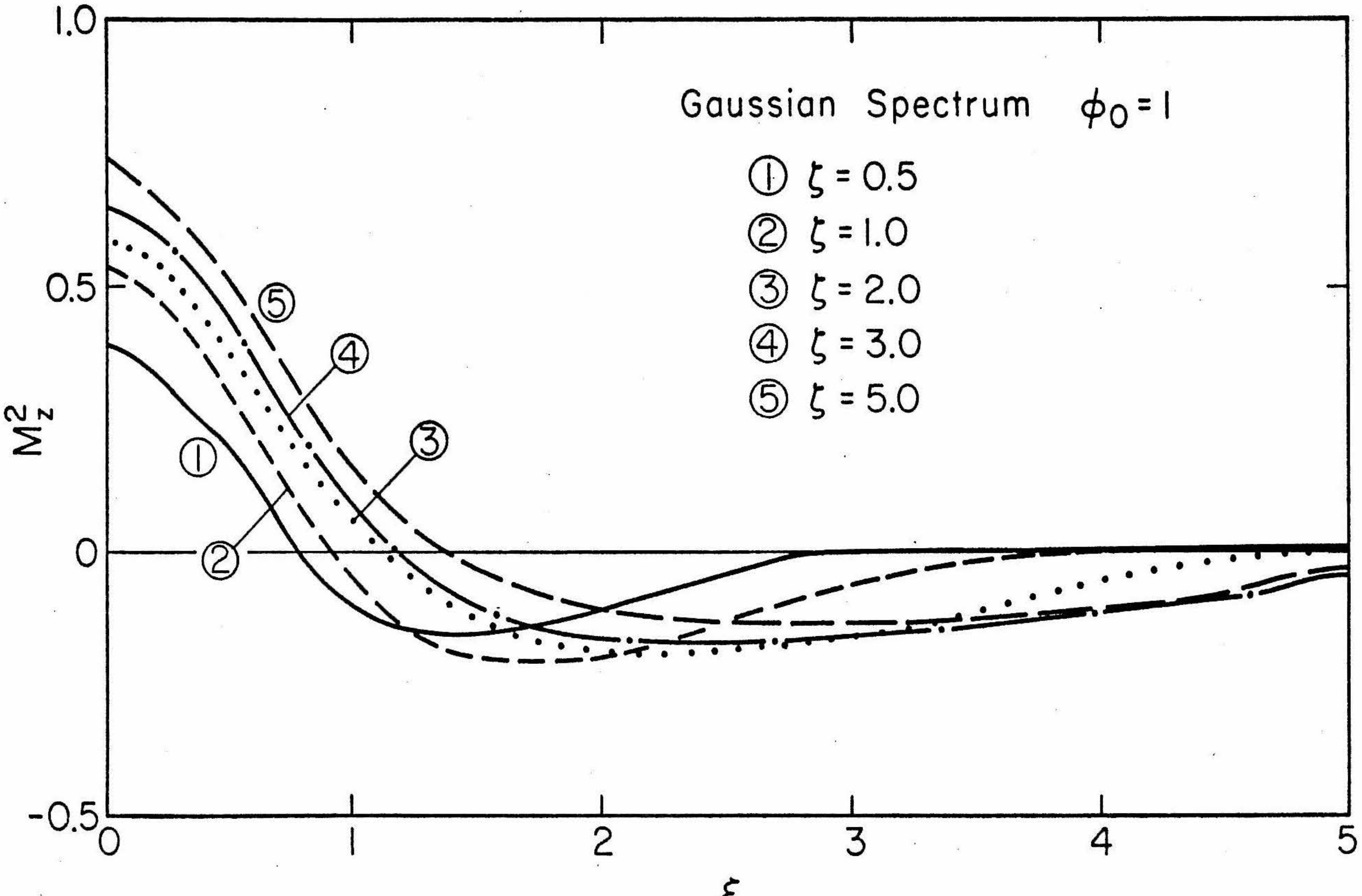
Gaussian Spectrum $\phi_0 = 1$
① $\zeta = 0.5$
② $\zeta = 1.0$
③ $\zeta = 2.0$
④ $\zeta = 3.0$
⑤ $\zeta = 5.0$
M_z^2
1.0
0.5
0
-0.5
0
1
2
3
4
5
ξ

Figure (3-3)

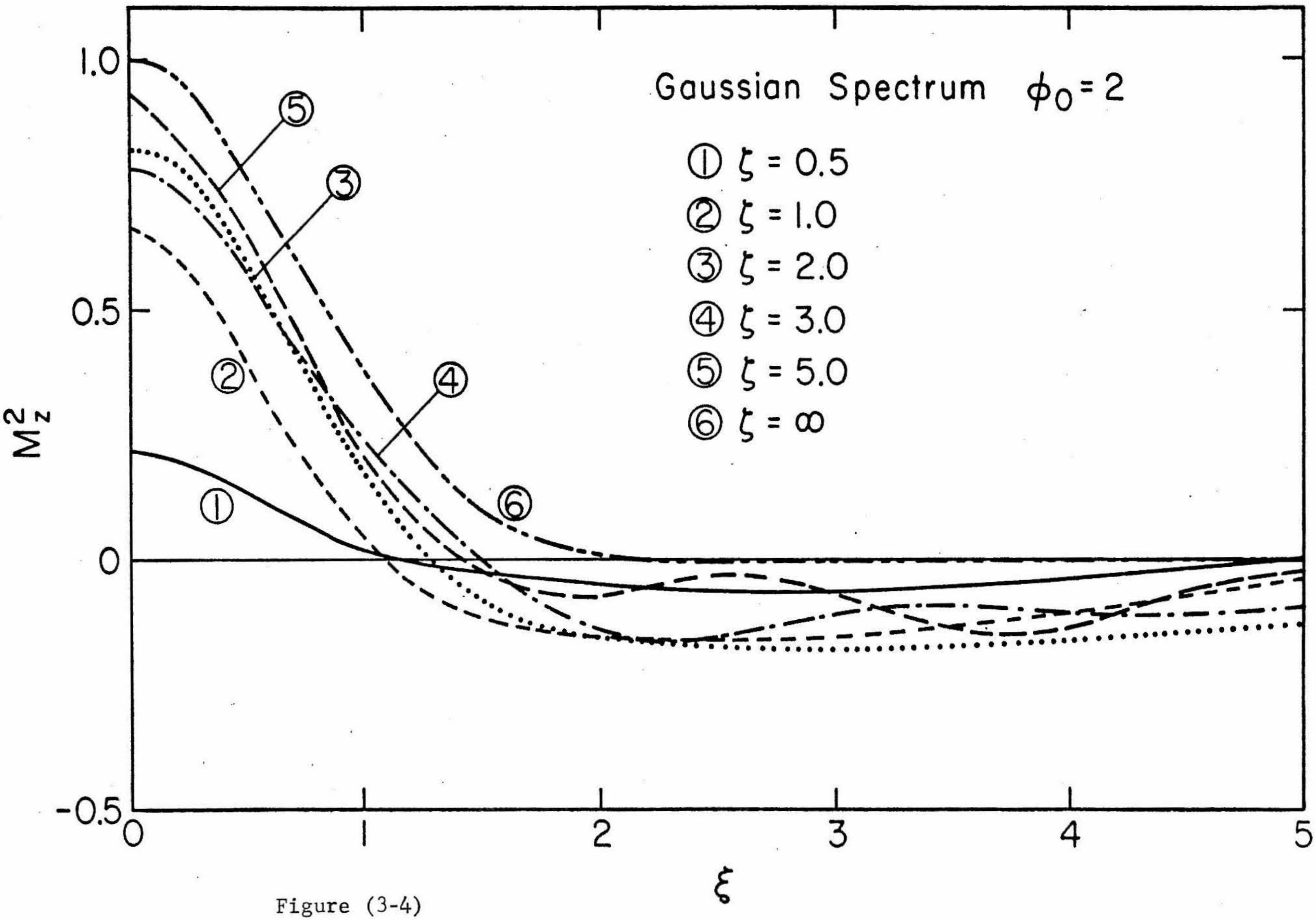
Gaussian Spectrum $\phi_0 = 2$
① ζ = 0.5
② ζ = 1.0
③ ζ = 2.0
④ ζ = 3.0
⑤ ζ = 5.0
⑥ ζ = ∞
1.0
0.5
0
-0.5
0
1
2
3
4
5
M_z^2
ξ

Figure (3-4)

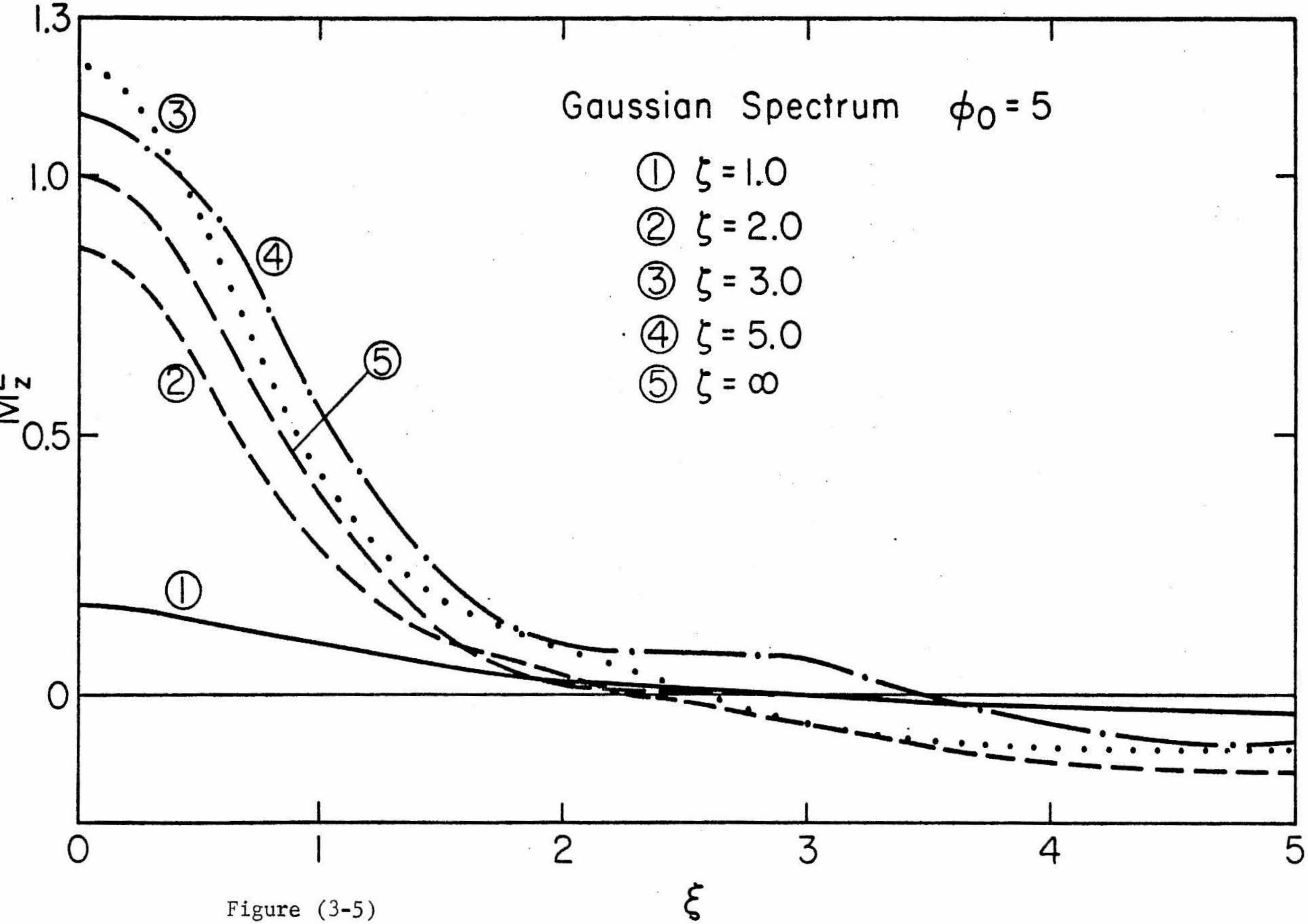
Gaussian Spectrum $\phi_0 = 5$
① $\zeta = 1.0$
② $\zeta = 2.0$
③ $\zeta = 3.0$
④ $\zeta = 5.0$
⑤ $\zeta = \infty$
M_z^2
ξ
1.3
1.0
0.5
0
0
1
2
3
4
5

Figure (3-5)

where $x_* = L(\Phi_o^{\ 2}/0.6)^{-0.6}$, we have

$$\frac{\partial\Gamma_4(\zeta,\xi_\alpha,\xi_\beta)}{\partial_\zeta} = i\,\frac{\partial^2\Gamma_4}{\partial\xi_\alpha\partial\xi_\beta} \tag{3-46}$$

and

$$\Phi_o^{\ 2}\ P_\Phi(x) = 0.6\gamma_o^{-5/3}A(x)/A(0) \tag{3-47}$$

where $\xi = x/x_*$ and $\gamma_o = q_o x_* = (\Phi_o^{\ 2}/0.6)^{-0.6}$.

Define

$$D(\rho) \equiv P_\Phi(0) - P_\Phi(\rho). \tag{3-48}$$

For $L \gg x \gg \ell$, we have from Eq. (1-28)

$$D(\xi) \to 1.115\ |\xi|^{5/3}. \tag{3-49}$$

In the Fraunhofer diffraction region, we have

$$M_z^{\ 2}(\underset{\sim}{\rho}) = \exp\left[-\ 2\Phi_o^{\ 2}D(\rho)\right]$$

For $L \gg x \gg \ell$,

$$M_z^{\ 2}(\rho) \approx \exp\left[-\ 2.230|\xi|^{5/3}\right] \quad . \tag{3-50}$$

So the characteristic scale of the intensity correlation function in Fraunhofer region is about x_*.

In the case $L \gg x$, the boundary condition (34) becomes

$$\Gamma_4(\zeta,\xi_\alpha,\xi_\beta)\Big|_{\xi_{\alpha,\beta}\, =\, \infty} = \exp\left\{-2.30\ \xi_{\beta,\alpha}^{\ \ 5/3}\right\} \tag{3-51}$$

and the initial condition (31) becomes

$$\Gamma_4(\zeta=0,\xi_\alpha,\xi_\beta) = \exp\left\{-\ 1.115\ [2|\xi_\alpha|^{5/3}+2|\xi_\beta|^{5/3}-|\xi_\alpha+\xi_\beta|^{5/3}-|\xi_\alpha-\xi_\beta|^{5/3}]\right\}$$

(3-52)

Again the boundary condition (51) is truncated at appropriately large values of ξ_α,ξ_β. The results for $\Phi_o = 1,5,\infty$ are shown in Figures (3-6),(3-7),(3-8), and (3-9).

From the Figures (3-6,7,8,9) we find that in Fresnel regions,

(a) For $\Phi_o \geq 1$, the scintillation becomes strong and the scintillation index is about 1.0 as $\zeta \gtrsim 1$.

(b) When the scintillation is weak ($m_z \lesssim\!\!\!< 1$), the correlation scale x_I of intensity fluctuation increases as the propagation distance z increases. This is consistent with that predicted by Eq. (3-14).

(c) The intensity correlation scale x_I for strong scintillation is x_*, which is the same as the scale in the Fraunhofer region. Therefore the analytic solution of Mercier (1962) in the Fraunhofer region can also approximately to be applied to Fresnel diffraction region. This is also consistent with the asymptotic analytic solution in the next subsection.

Figure Captions

Figure (3-6). The scintillation index $m_z^2 = M_z^2(0)$ is plotted as a function of the normalized propagating distance ζ for the medium with a Kolmogorov spectrum. In curve (1), $\Phi_o = 1$; in curve (2), $\Phi_o = 5$; and in curve (3), $\Phi_o = \infty$. Φ_o is the root mean square of the phase fluctuation.

Figure (3-7). For Kolmogorov spectrum with $\Phi_o = 1$, the intensity correlation function $M_z^2(\xi)$ is plotted as a function of the normalized transverse coordinate ξ for various values of the normalized distance ζ.

Figure (3-8). As in Figure (3-7) with $\Phi_o = 5$.

Figure (3-9). As in Figure (3-7) with $\Phi_o = \infty$.

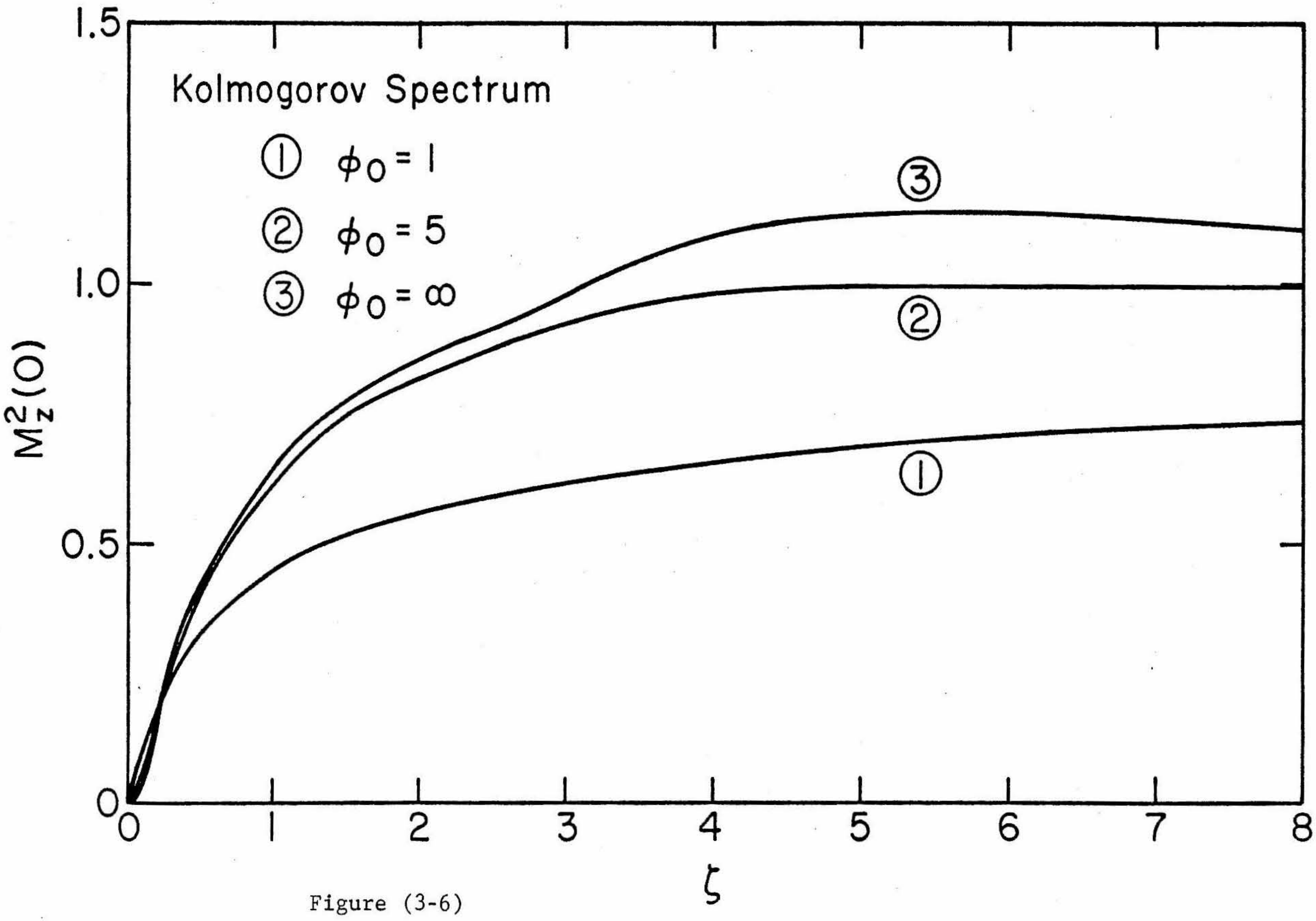
Kolmogorov Spectrum
① $\phi_0 = 1$
② $\phi_0 = 5$
③ $\phi_0 = \infty$
③
②
①
$M_z^2(0)$
1.5
1.0
0.5
0
0
1
2
3
4
5
6
7
8
ζ

Figure (3-6)

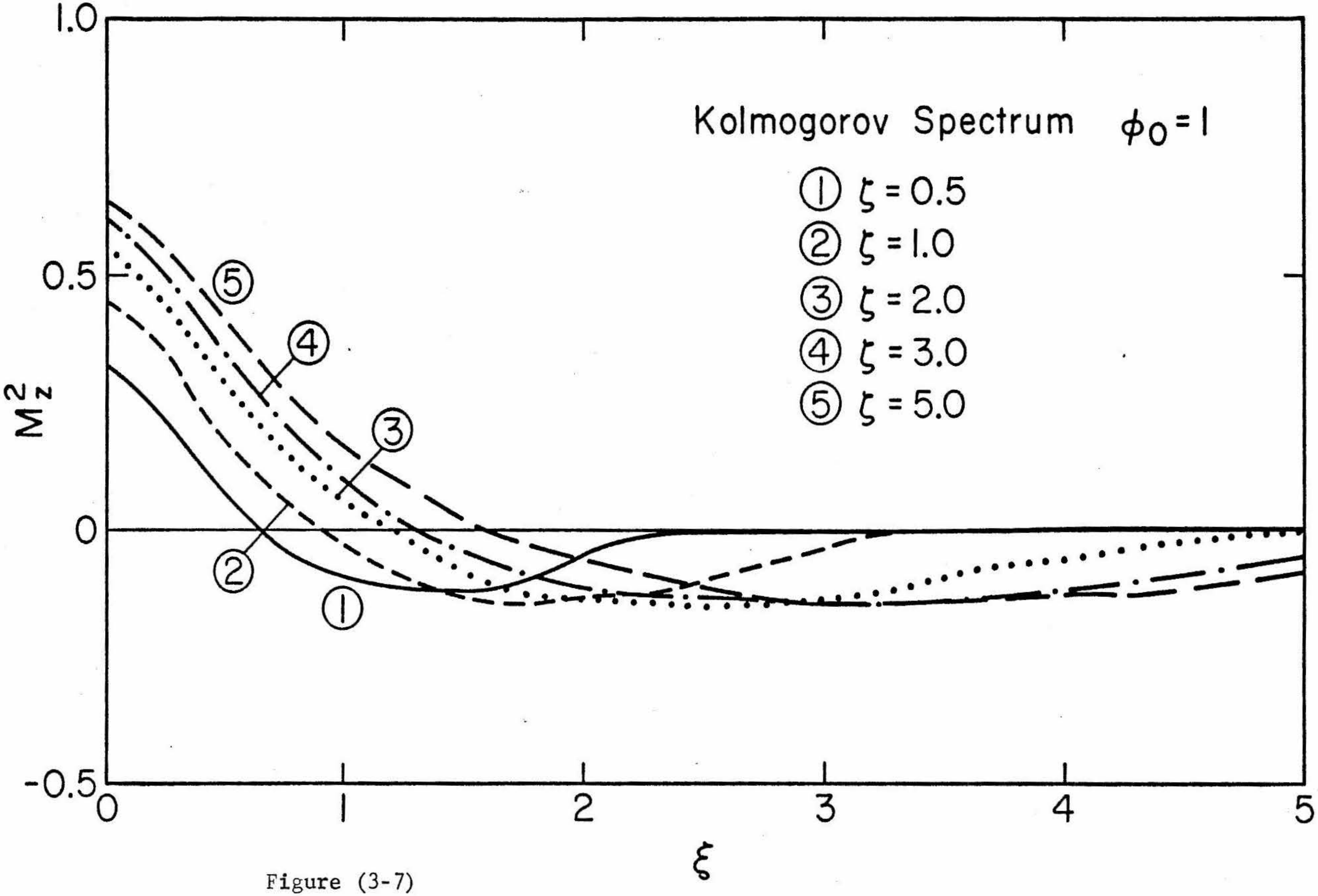
Kolmogorov Spectrum φ0=1
① ζ = 0.5
② ζ = 1.0
③ ζ = 2.0
④ ζ = 3.0
⑤ ζ = 5.0
1.0
0.5
0
-0.5
M2z
0
1
2
3
4
5
ξ
①
②
③
④
⑤

Figure (3-7)

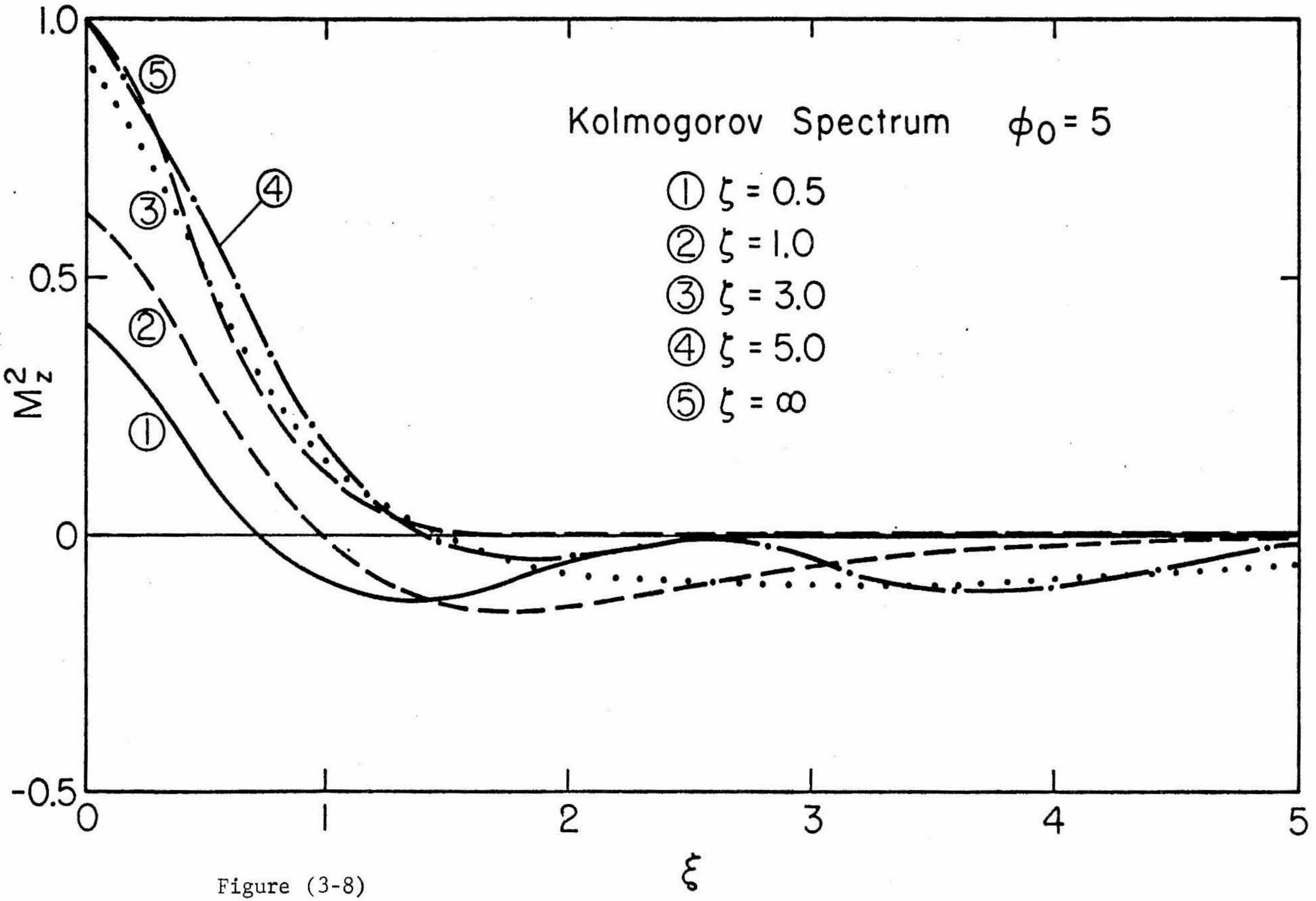
Kolmogorov Spectrum $\phi_0 = 5$
① $\zeta = 0.5$
② $\zeta = 1.0$
③ $\zeta = 3.0$
④ $\zeta = 5.0$
⑤ $\zeta = \infty$
M_z^2
ξ
1.0
0.5
0
-0.5
0
1
2
3
4
5

Figure (3-8)

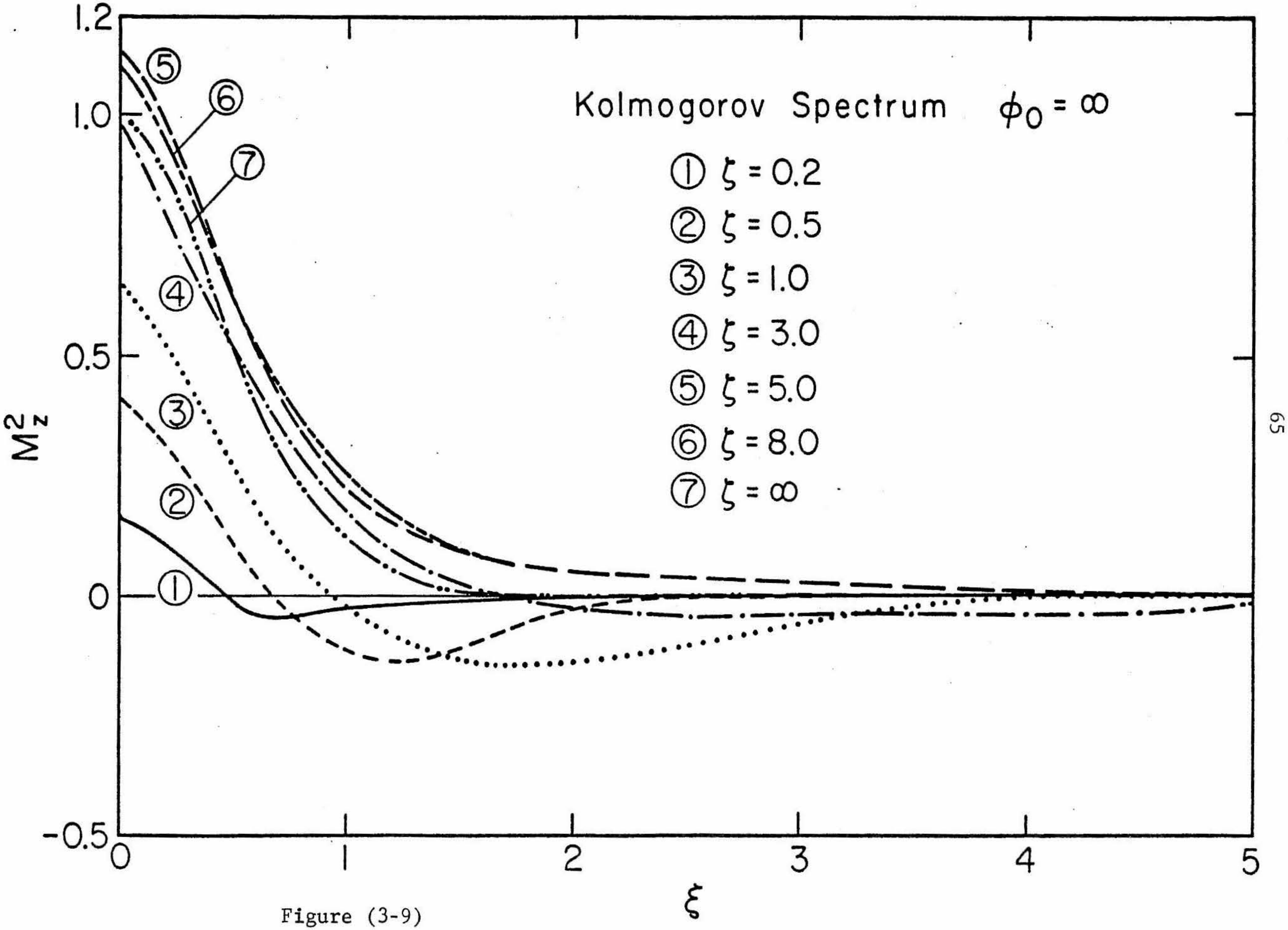
Kolmogorov Spectrum $\phi_0 = \infty$
① $\zeta = 0.2$
② $\zeta = 0.5$
③ $\zeta = 1.0$
④ $\zeta = 3.0$
⑤ $\zeta = 5.0$
⑥ $\zeta = 8.0$
⑦ $\zeta = \infty$
M_z^2
ξ
1.2
1.0
0.5
0
-0.5
0
1
2
3
4
5

Figure (3-9)

IV. Analytic Solutions

An asymptotic analytic solution of Eq. (3-29') can be obtained by noting that Eq (3-29') is a diffusion-like equation for $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$. The solution of Γ_4 in Eq. (3-29') can be written immediately in terms of the inital value $\Gamma_4(z=0,\underset{\sim}{\rho}_\alpha\cdot\underset{\sim}{\rho}_\beta)$ as

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \left(\frac{k}{z}\right)^2 \iint \Gamma_4(z=0,\underset{\sim}{\rho}'_\alpha,\underset{\sim}{\rho}'_\beta)\, e^{-i\left(\frac{k}{z}\right)(\underset{\sim}{\rho}_\alpha-\underset{\sim}{\rho}'_\alpha)\cdot(\underset{\sim}{\rho}_\beta-\underset{\sim}{\rho}'_\beta)} \times d\underset{\sim}{\rho}'_\alpha\, d\underset{\sim}{\rho}'_\beta \tag{3-53}$$

Eq. (3-53) shows that for $z > 0$, the values of $\Gamma_4(z=0,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ are re-distributed among different transverse coordinates. The mechanism of re-distribution (or "diffusion") is continued until $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ reaches a steady-state[1]. For steady-state, we have from Eq. (3-29')

$$\frac{\partial\Gamma_4}{\partial z} = \frac{i}{k}\nabla_{\underset{\sim}{\rho}_\alpha}\cdot\nabla_{\underset{\sim}{\rho}_\beta}\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = 0 \tag{3-54}$$

from which we find that for large z, Γ_4 must be of the following form

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = f_1(\underset{\sim}{\rho}_\alpha) + f_2(\underset{\sim}{\rho}_\beta) + c_o \tag{3-55}$$

---

[1] One can easily show the existence of a steady-state for $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ at large z by the same argument as in Mercier (1962) in obtaining the asymptotic form for $\langle R^2R'^2\rangle/(AA^*)^2$, which corresponds to $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta=0)$ in this thesis.

where f_1, f_2 are arbitrary functions of $\underset{\sim}{\rho}_\alpha$ and $\underset{\sim}{\rho}_\beta$ respectively and c_o is a constant. By applying the boundary conditions in Eq. (3-32), one easily finds that from Eq. (3-55),

$$\Gamma_4\ (z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = |\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}_\alpha+\underset{\sim}{\rho})|^2+|\Gamma_2(z,\underset{\sim}{\rho},\underset{\sim}{\rho}_\beta+\underset{\sim}{\rho})|^2 - |\Gamma_2(z,\underset{\sim}{0},\underset{\sim}{\rho}\rightarrow\infty)|^2 \quad . \tag{3-56}$$

By Eq. (3-33), Eq. (3-56) can be written as

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \exp\left\{-2\Phi_o^{\,2}[1-P_\Phi(\underset{\sim}{\rho}_\alpha)]\right\} + \exp\left\{-2\Phi_o^{\,2}[1-P_\Phi(\underset{\sim}{\rho}_\beta)]\right\} - \exp\left\{-2\Phi_o^{\,2}\right\} \quad . \tag{3-57}$$

Combining Eqs. (3-30) and (3-57), we have the intensity correlation function

$$M_z^{\,2}(\underset{\sim}{\rho}) = \exp\left\{-2\Phi_o^{\,2}[1-P_\Phi(\underset{\sim}{\rho})]\right\} - \exp\left[-2\Phi_o^{\,2}\right] \tag{3-58}$$

for large z. Note that Eq. (3-58) is the same as Eq. (3-11) derived by Mercier (1962).

Next we determine how large the propagating distance z must be in order that Eq. (3-57) is a valid solution for Γ_4. Suppose the transverse characteristic scale of $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ with respect to $\underset{\sim}{\rho}_\alpha$ (or $\underset{\sim}{\rho}_\beta$) is ρ_c. Then from Eq. (3-29'), we find that the "transient scale" z_c of the propagating distance z is given by

$$z_c \simeq k\,\rho_c^{\,2}. \tag{3-59}$$

$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ reaches the steady-state solution in Eq. (3-57) when $z > z_c$. The transverse characteristic scale ρ_c of $\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ can be determined from the initial condition for Γ_4 in Eq. (3-31). We consider two cases.

Case 1. $\Phi_o^2 < 1$

For $\Phi_o^2 < 1$, it is easy to show that from Eq. (3-31) the characteristic scale ρ_c of Γ_4 is equal to L, the correlation scale of the phase function with a Gaussian spectrum or a Kolmogorov spectrum. Thus

$$z_c = kL^2, \text{ for } \Phi_o^2 < 1 \; . \tag{3-60}$$

Case 2. $\Phi_o^2 > 1$

We first consider the phase function with a Gaussian spectrum. From Eqs. (1-26) and (3-3), we have

$$\Phi_o^2 P_\Phi(\rho) = \Phi_o^2 \, e^{-\frac{\rho^2 q_o^2}{4}} \tag{3-61a}$$

and

$$\Phi_o^2 = \frac{DA_\beta(0)}{2k^2} \; . \tag{3-61b}$$

For $(\frac{\rho q_o}{2}) < 1$, $P_\Phi(\rho)$ can be expanded as

$$P_\Phi(\rho) = 1 - \frac{\rho^2 q_o^2}{4} + \frac{\rho^4 q_o^4}{32} + \cdots \, . \tag{3-62}$$

Using Eq. (3-62), we can write $\Gamma_4(z=0,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta)$ in Eq. (3-31) as

$$\Gamma_4(z=0,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \exp\left\{-\frac{\Phi_o^2 q_o^2}{32}\left[|\underset{\sim}{\rho}_\alpha+\underset{\sim}{\rho}_\beta|^4+|\underset{\sim}{\rho}_\alpha-\underset{\sim}{\rho}_\beta|^4-2|\underset{\sim}{\rho}_\alpha|^4-2|\rho_\beta|^4\right]\right\} \tag{3-63}$$

from which we find $\rho_c = L/\sqrt{\Phi_o}$. Thus

$$z_c \simeq kL^2/\Phi_o = \ell_o . \tag{3-64}$$

Here ℓ_o is the focal length in Eq. (3-16).

Similarly, we find from Eq. (3-52) that for Kolmogorov spectrum,

$$\rho_c \simeq x_* = [\frac{5}{3} \Phi_o^2 q_o^{5/3}]^{-\frac{3}{5}} \approx L\Phi_o^{-1.2} \tag{3-65a}$$

and

$$z_c = kL^2 \Phi_o^{-2.4} \quad . \tag{3-65b}$$

From the above discussions, we find that the intensity correlation function $M_z^2(\underset{\sim}{\rho})$ given by Eq. (3-11) or Eq. (3-58) is valid only in the Fraunhofer diffraction region for $\Phi_o^2 < 1$. However for $\Phi_o^2 > 1$, the $M_z^2(\underset{\sim}{\rho})$ in Eq. (3-11) or Eq. (3-58) is valid not only in the Fraunhofer region, but also in the Fresnel region as can be seen from Eqs. (3-64) and (3-65).

Finally we will show that for $\Phi_o^2 > 1$ the condition that

$$z > z_c \tag{3-66}$$

for the validity of Eq. (3-58) is also the criterion for strong scintillation ($m_z^2 \simeq 1$). From Eq. (3-58), we have the scintillation index m_z^2

$$m_z^2 = 1 - \exp[- 2\Phi_o^2] \approx 1 \text{ for } \Phi_o^2 > 1. \tag{3-67}$$

Thus the scintillation is strong for $\Phi_o^2 > 1$ and $z > z_c$. For $z < z_c$, Eq. (3-16) is satisfied for the phase function with a Gaussian spectrum and Eq. (3-17') can be used to calculate the intensity correlation, from which we find

$$m_z^2 = M_z^2(\underset{\sim}{\rho}=0) \simeq (\frac{z\Phi_o}{k})^2 \cdot \frac{1}{L^4} \approx (\frac{z}{z_c})^2 < 1. \tag{3-68}$$

Similarly for Kolmogorov spectrum, when $z < z_c$, Eq. (3-15) is valid and Eq.(-14) can be applied. One then finds for $z \ll z_c$

$$m_z^2 \ll 1 \quad . \tag{3-69}$$

Combining Eqs. (3-67), (3-68) and (3-69), we conclude that the criterion for strong scintillation is that

$$\text{(i)} \quad \Phi_o^2 > 1 \tag{3-70a}$$

and

$$\text{(ii)} \quad z > z_c = \begin{cases} kL^2/\Phi_o, \text{for Gaussian spectrum} \\ kL^2/\Phi_o^{12/5}, \text{ for Kolmogorov spectrum} \end{cases} \tag{3-70b}$$

When $\Phi_o^2 < 1$ and/or $z < z_c$, the scintillation is weak. The solution for strong scintillation is given by Eq. (3-58) while Eq. (3-14) or Eq. (3-17) is the solution for weak scintillation ($m_z \ll 1$). We also note the numerical solutions in the last subsection are in good agreement with the analytic asymptotic solution here.

Chapter 4

The Markov Random Process Approximation (A)

As mentioned in Chapter 1, most of the theories of wave propagation in a random medium are based either on the "single-scattering" theory (e.g. Born approximation, and M.S.P.), or on the geometric-optics approximation. (Booker & Gordon 1950, Chandrasekhar 1952, Chernov 1960, Tatarskii 1961, Keller 1962, Hoffman 1964, Budden 1965a, 1965b, Salpeter 1967, Lovelace 1970, and Williamson 1972.) However, single-scattering theory is valid only when the scintillation is weak or the scintillation index $m_z \ll 1$ and geometric-optics breaks down when the interference of the rays cannot be neglected. For strong scintillation where $m_z^2 \doteq 1$, the multiple-scattering effect is important and neither the single-scattering theory nor the geometric-optics approximation can be applied. A theory dealing with the strong scintillations is needed.

In this and next chapters, we will develop a theory, which is valid for strong scintillations, under the "Markov random process" and the "quasi-optics" approximations. In this chapter, a complete set of the moment equations of the random wave fields with different frequencies is derived and the validity of the two approximations applied to the interstellar scintillation is discussed. In Chapter 5, we apply the moment equations to solve the phase correlation function, angular broadening,[1] pulse smearing, intensity correlation and the probability distribution function of the random waves.

Most of the text of this chapter is a published paper (Lee, 1974), and for clarity the paper is presented in its entirety. In this paper, $A(\underset{\sim}{\rho})$ corresponds to $A_\beta(\underset{\sim}{\rho})$ in other places of this thesis.

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[1] Discussions of the validity of the Markov and quasi-optic approximations, the phase correlation function and the angular broadening in this and the following chapters come mostly from Lee & Jokipii (to be published in the March 15th, 1975 issue of the Astrophysical Journal).

Wave propagation in a random medium: A complete set of the moment equations with different wavenumbers

L. C. Lee

Downs Laboratory of Physics, California Institute of Technology, Pasadena, California 91109
(Received 7 March 1974)

Propagation of waves in a random medium is studied under the "quasioptics" and the "Markov random process" approximations. Under these assumptions, a Fokker–Planck equation satisfied by the characteristic functional of the random wave field is derived. A complete set of the moment equations with different transverse coordinates and *different* wavenumbers is then obtained from the Fokker–Planck equation of the characteristic functional. The applications of our results to the pulse smearing of the pulsar signal and the frequency correlation function of the wave intensity in interstellar scintillation are briefly discussed.

I. INTRODUCTION

Phenomena such as the twinkling of starlight and the ionospheric, interplanetary, and interstellar radio wave scintillations involve the propagation of an electromagnetic wave in a random medium. A complete statistical description of the wave field requires the solution of all moments of the wave field with different positions and different wavenumbers.

A complete set of the moment equations of the wave field with different transverse coordinates but the *same* wavenumbers has been derived under the "quasioptics" and the "Markov random process" approximations,[1,2] which can be applied to both weak and strong scatterings. However, such a set of the moment equations with the *same* wavenumbers is not sufficient to describe all the statistical properties of the random wave field. Some observed quantities in interstellar scintillations, such as the pulse smearing and the correlation function of the intensity fluctuation with different wavenumbers,[3–5] need the solution of the moment equations with different wavenumbers. It is the purpose of this paper to derive a complete set of the moment equations with different transverse positions and *different* wavenumbers under the quasioptics and the Markov random process approximations. The results reduce to those of Tatarskii[1,2] in the case of the *same* wavenumbers. It is noted that the method of the derivation used here is new, and simpler than that by Tatarskii.[1,2]

It is the idea of Hopf[6] to introduce the "characteristic functional" as an alternative way to describe the complete statistical properties of a random field. In Sec. II, we will derive a Fokker–Planck equation for the characteristic functional of the random electromagnetic field. In Sec. III, a complete set of the moment equations will be derived from the Fokker–Planck equation satisfied by the characteristic functional. Some applications of the results will be briefly discussed in Sec. IV.

II. FOKKER–PLANCK EQUATION FOR THE CHARACTERISTIC FUNCTIONAL OF THE WAVE FIELD

We consider the propagation of a monochromatic wave $E_\omega(\mathbf{r}, t)$ obeying the scalar wave equation

$$\nabla^2\Phi_\omega(\mathbf{r}) + (\omega^2/c^2)\epsilon_\omega(\mathbf{r})\Phi_\omega(\mathbf{r}) = 0, \tag{1}$$

where

$$E_\omega(\mathbf{r}, t) = \Phi_\omega(\mathbf{r})e^{-i\omega t}. \tag{2}$$

$\Phi_\omega(\mathbf{r})$ may be regarded as a Fourier component in time of a general wavefunction. Here $(\omega/2\pi)$ is the frequency of the monochromatic wave, c is the speed of light, and $\epsilon_\omega(\mathbf{r})$ is the refractive index of the medium in which the wave propagates.

The refractive index $\epsilon_\omega(\mathbf{r})$ is a random function and depends on both the position $\mathbf{r}$ and the wave frequency ω. As an example, we will consider in this paper the propagation of the high frequency waves with $\omega \gg \omega_p$, the plasma frequency of the medium, in the plasma medium. This applies to the propagation of the radio waves in the ionosphere, the interplanetary space, or the interstellar medium. If N_e is the electron density, then we have

$$\epsilon_\omega(\mathbf{r}) = 1 - \omega_p^2/\omega^2 \tag{3}$$

and

$$\omega_p^2 = 4\pi N_e e^2/m, \tag{4}$$

where m is the mass and e is the charge of an electron.

Now N_e and $\epsilon_\omega(\mathbf{r})$ fluctuate irregularly. Let $\langle\ \rangle$ denote an average over an ensemble of propagation volumes. Then define

$$\langle\epsilon_\omega(\mathbf{r})\rangle = \epsilon_{\omega 0}(\mathbf{r}),$$

$$N_e(\mathbf{r}) = \langle N_e(\mathbf{r})\rangle + \delta N_e(\mathbf{r}),$$

$$\beta(\mathbf{r}) = -4\pi c^2\delta N_e(\mathbf{r})/mc^2.$$

We have

$$\nabla^2\Phi_\omega(\mathbf{r}) + k^2[1 + \beta(\mathbf{r})/k^2]\Phi_\omega(\mathbf{r}) = 0, \tag{5}$$

where now $\beta(\mathbf{r})$ is a wave-frequency independent random variable with zero mean and where the wavenumber $k = (\omega/c)\sqrt{\epsilon_{\omega 0}}$.

It is useful to define

$$\Phi_\omega(\mathbf{r}) = u(k, \mathbf{r})e^{ikz}, \tag{6}$$

from which we obtain

$$2ik\frac{\partial u(k,\mathbf{r})}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(k,\mathbf{r}) + \beta(\mathbf{r})u(k,\mathbf{r}) = 0. \tag{7}$$

Let

$$\mathbf{r} = (z, \rho), \quad \rho = (x, y), \quad \text{and} \quad s = (\rho, k).$$

In order to proceed further, we will make two assumptions about the wave equation and the properties of the medium.

First, we assume that the term $\partial^2 u/\partial z^2$ in Eq. (7) can be neglected. This is called the "quasioptics" approximation or "parabolic" approximation. Physically this assumption is equivalent to neglect the reflected wave since the equation has been reduced to one with a first-order derivative in z from the one with a second-order derivative. Thus we have

$$\frac{\partial}{\partial z}u(z,\rho,k)+\frac{1}{2ik}\nabla_\rho^2 u(z,\rho,k)+\frac{1}{2ik}\beta(z,\rho)u(z,\rho,k)=0, \quad (8)$$

where

$$\nabla_\rho^2=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

Second, we assume that $\beta(z,\rho)$ is delta-correlated in z direction. This is called the Markov random process approximation. As we can see later, this is equivalent to assume that the correlation scale of $\beta(z,\rho)$ in z direction is much less than the correlation scale of the wave field u in z direction. We then have

$$\langle\beta(z,\rho)\beta(z',\rho')\rangle=2\delta(z-z')A(\rho-\rho') \quad (9a)$$

and

$$A(\rho-\rho')=\int_{-\infty}^{z}\langle\beta(z,\rho)\beta(z',\rho')\rangle\,dz'. \quad (9b)$$

Note that the z dependence of $A(\rho)$ is not explicitly expressed for convenience.

The validity of the above two assumptions has been discussed.[2,7] We will only note that the "quasioptics" approximation and the "Markov" approximation can be applied in the strong scattering cases.

It is known that the probability distribution function at time t of a random variable $x(t)$ that satisfies a differential equation of the first order in time with a delta-correlated external random force satisfies the Fokker–Planck equation. In our case, z plays the role of time. However, for a fixed value of z, the random field $u(z,\rho,k)$ does not have just a discrete value but has an infinite number of values and is a function of ρ and k. It is the idea of Hopf[6] to introduce a characteristic functional Ψ to describe the statistical properties of a random field. One defines the characteristic functional as

$$\begin{aligned}\Psi(z,\nu,\nu^*)&=\langle\exp(iR_z)\rangle\\&=\langle\exp\{i\int\int[u(z,\rho,k)\nu(\rho,k)\\&\quad+u^*(z,\rho,k)\nu^*(\rho,k)]\,d^2\rho\,dk\}\rangle,\end{aligned} \quad (10)$$

where * denotes complex conjugate and the range of integration is over all the allowed values of ρ and k. Here ν and ν^* are treated as independent functions of ρ and k.

It is the purpose of this section to derive a Fokker–Planck equation for the characteristic functional Ψ defined above. Tatarskii[1] derived an equation for the characteristic functional with constant wavenumber k. It is noted that we treat in Eq. (10) the wavenumber k as a variable.

Using $s=(\rho,k)$, we write Eq. (10) as

$$\Psi(z,\nu,\nu^*)=\langle\exp\{i\int[u(z,s)\nu(s)+u^*(z,s)\nu^*(s)]\,ds\}\rangle. \quad (10')$$

We differentiate Eq. (10) with respect to z and obtain by Eq. (8)

$$\begin{aligned}\frac{\partial}{\partial z}\Psi(z,\nu,\nu^*)&=\Big\langle\exp(iR_z)i\int\Big[\Big(\frac{-1}{2ik}\Big)[\nabla_\rho^2u(z,s)+\beta(z,\rho)u(z,s)]\nu(s)\\&\quad+\Big(\frac{1}{2ik}\Big)[\nabla_\rho^2u^*(z,s)+\beta(z,\rho)u^*(z,s)]\nu^*(s)\Big]\,ds\Big\rangle.\end{aligned} \quad (11)$$

First we calculate the terms $\langle\exp(iR_z)\nabla_\rho^2u(z,s)\rangle$ and $\langle\exp(iR_z)\nabla_\rho^2u^*(z,s)\rangle$ in Eq. (11). From Eq. (10), we have

$$\frac{\delta\Psi(z,\nu,\nu^*)}{d\nu(s)}=i\langle u(z,s)\exp(iR_z)\rangle \quad (12a)$$

and

$$\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu^*(s)}=i\langle u^*(z,s)\exp(iR_z)\rangle. \quad (12b)$$

The operators $\delta/\delta\nu(s)$ and $\delta/\delta\nu^*(s)$ denote functional derivatives.[6,8] Operating ∇_ρ^2 on Eqs. (12a) and (12b), we have respectively

$$\langle\nabla_\rho^2u(z,s)\exp(iR_z)\rangle=\frac{1}{i}\nabla_\rho^2\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu(s)} \quad (13a)$$

and

$$\langle\nabla_\rho^2u^*(z,s)\exp(iR_z)\rangle=\frac{1}{i}\nabla_\rho^2\frac{\delta\Psi(z,\nu,\nu^*)}{\delta\nu^*(s)}\ . \quad (13b)$$

Next we consider the other terms in Eq. (11), namely, $\langle\exp(iR_z)\beta(z,\rho)u(z,s)\rangle$ and $\langle\exp(iR_z)\beta(z,\rho)u^*(z,s)\rangle$. We define

$$g(\nu,\nu^*,z,s)=\langle\exp(iR_z)\beta(z,\rho)\rangle. \quad (14)$$

Expand $\exp(iR_z)$ in power series as follows:

$$\exp(iR_z)=\sum_{m=0}^{\infty}\frac{1}{m!}\{i\int[u(z,s)\nu(s)+u^*(z,s)\nu^*(s)]\,ds\}^m. \quad (15)$$

Then we have

$$\begin{aligned}g(\nu,\nu^*,z,s)&=\sum_{m=0}^{\infty}\frac{i^m}{m!}\langle[\int(u_1\nu_1+u_1^*\nu_1^*)\,ds_1]\\&\quad\times[\int(u_2\nu_2+u_2^*\nu_2^*)\,ds_2]\cdots[\int(u_m\nu_m+u_m^*\nu_m^*)\,ds_m]\beta(z,\rho)\rangle,\end{aligned} \quad (16)$$

where we define $s_i=(\rho_i,k_i)$, $\nu_i=\nu(s_i)$, $u_i=u(z,s_i)$, and etc. for $i=1,2,3,\cdots$. In the expansion of Eq. (16), the existence of moments of all orders is assumed.

Consider now the term in Eq. (16) like $\langle u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_m^{\alpha_m}\beta\rangle$, where $u_i^{\alpha_i}$ denotes either u_i or u_i^*. From Eq. (8), we may write $u(z,s)$ as

$$u(z,s)=u(0,s)+\frac{i}{2}\int_0^z\frac{1}{k}\times[\nabla_\rho^2u(z',s)+\beta(z',\rho)u(z',s)]\,dz'. \quad (17)$$

Note that $u(z,s)$ does not depend on $\beta(z',s)$ for $z'>z$. Let Δz be an increment in z, which is larger than the correlation scale of $\beta(z,\rho)$ in z direction, and write

$$u(z,s)=u(z-\Delta z,s)+\frac{i}{2}\int_{z-\Delta z}^{z}\frac{i}{k}\times[\nabla_\rho^2u(z',s)+\beta(z',\rho)u(z',s)]\,dz', \quad (18)$$

where $u(z-\Delta z, s)$ has no correlation with $\beta(z,\rho)$. Suppose Δz is small, and expand $u(z,s)$ as

$$u(z,s)=u(z-\Delta z,s)+\frac{i}{2}\left(\frac{\Delta z}{k}\right)\nabla_\rho^2 u(z-\Delta z,s)$$
$$+\frac{i}{2}\frac{u(z-\Delta z,s)}{k}\int_{z-\Delta z}^{z}\beta(z',\rho)\,dz'+O(\Delta^2 z). \qquad (19)$$

Under the Markov approximation, the correlation scale of $\beta(z,\rho)$ in z direction is zero. Therefore, we let $\Delta z \to 0$. We note that

$$\lim_{\Delta z\to 0} u(z-\Delta z,s)=u(z,s) \qquad (20)$$

and

$$\langle\beta(z,\rho')\int_{z-\Delta z}^{z}\beta(z',\rho)\,dz'\rangle=A(\rho-\rho'). \qquad (21a)$$

For higher moments such as

$$T_i=\langle\beta(z,\rho)\int_{z-\Delta z}^{z}\beta(z_1,\rho_1)\,dz_1\cdots$$
$$\times\int_{z-\Delta z}^{z}\beta(z_i,\rho_i)\,dz_i\rangle,\quad i\geq 2,$$

we will assume as in the derivation of ordinary Fokker–Planck equation[9]

$$\lim_{\Delta z\to 0}T_i=0,\quad i\geq 2. \qquad (21b)$$

This assumption can be satisfied if the random function $\beta(z,\rho)$ has a Gaussian, or normal statistics. However, the assumption made in (21b) is more general and does not require the Gaussian statistics of $\beta(z,\rho)$ in general.

It follows directly from Eqs. (16), (19), (20), (21a), and (21b) that, as $\Delta z\to 0$,

$$\langle u(z,s_1)\beta(z,\rho)\rangle=(i/2k_1)\langle u(z,s_1)\rangle A(\rho-\rho_1)$$

and, in general,

$$\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_m\nu_m+u_m^*\nu_m^*)\beta(z,\rho)\rangle$$
$$=\sum_{j=1}^{m}A(\rho-\rho_j)\left(\frac{i}{2k_j}\right)\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots$$
$$\times(u_{j-1}\nu_{j-1}+u_{j-1}^*\nu_{j-1}^*)(u_j\nu_j-u_j^*\nu_j^*)$$
$$\times(u_{j+1}\nu_{j+1}+u_{j+1}^*\nu_{j+1}^*)\cdots(u_m\nu_m+u_m^*\nu_m^*)\rangle \qquad (22)$$

by noting that $\langle u(z-\Delta z,s)\beta(z,\rho')\rangle=0$. Other than the assumption made in (21b), Eq. (22) is exact under the delta-correlation assumption. But we see that really we only require the existence of an intermediate scale Δz which is larger than the coherence scale of $\beta(z,\rho)$ but smaller than the scale of variation of u such that $u(z-\Delta z,s)\approx u(z,s)$. The existence of the intermediate scale and Eq. (21) are the essence of the Fokker-Planck equation.

Substituting Eq. (22) into (16) and noting that all the s_i's are dummy variables, we then have

$$g(\nu,\nu^*,z,s)$$
$$=\sum_{m=1}^{\infty}\left(\frac{i^m}{m!}\right)\left(\frac{i}{2k_m}\right)(m)\int\cdots\int A(\rho-\rho_m)\langle(u_1\nu_1+u_1^*\nu_1^*)\cdots$$
$$\times(u_{m-1}\nu_{m-1}+u_{m-1}^*\nu_{m-1}^*)(u_m\nu_m-u_m^*\nu_m^*)\rangle\,ds_1\cdots ds_m. \qquad (23)$$

We can also write Eq. (23) as

$$g(\nu,\nu^*,z,s)$$
$$=\sum_{m=1}^{\infty}\frac{i^{m-1}}{(m-1)!}\left(-\frac{1}{2}\right)\int\cdots\int\frac{1}{k'}A(\rho-\rho')\langle(u'\nu'-u^{*\prime}\nu^{*\prime})$$
$$\times(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_{m-1}\nu_{m-1}+u_{m-1}^*\nu_{m-1}^*)\rangle\,ds'ds_1\cdots ds_{m-1}. \qquad (24)$$

where $s'=(\rho',k')$, $u'=u(z,s')$, and $\nu'=\nu(s')$.

Setting $m-1=n$, we have

$$g(\nu,\nu^*,z,s)$$
$$=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}\left(\frac{-1}{2}\right)\int\cdots\int\frac{i}{k'}A(\rho-\rho')\langle(u'\nu'-u^{*\prime}\nu^{*\prime})$$
$$\times(u_1\nu_1+u_1^*\nu_1^*)\cdots(u_n\nu_n+u_n^*\nu_n^*)\rangle\,ds'ds_1\cdots ds_n. \qquad (25)$$

From Eq. (25) it is easy to show

$$g(\nu,\nu^*,z,s)=\left(\frac{-1}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')[\nu(s')\langle u(z,s')\exp(iR_z)\rangle$$
$$-\nu^*(s')\langle u^*(z,s')\exp(iR_z)\rangle]. \qquad (25')$$

By Eqs. (12a) and (12b), we write g as

$$g(\nu,\nu^*,z,s)$$
$$=\left(\frac{i}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')\left(\nu(s')\frac{\delta\Psi}{\delta\nu(s')}-\nu^*(s')\frac{\delta\Psi}{\delta\nu^*(s')}\right). \qquad (26)$$

Define the operator $\hat{M}(s)$ as

$$\hat{M}(s)=\nu(s)\frac{\delta}{\delta\nu(s)}-\nu^*(s)\frac{\delta}{\delta\nu^*(s)}. \qquad (27)$$

We then have

$$g(\nu,\nu^*,z,s)=\left(\frac{i}{2}\right)\int\frac{ds'}{k'}A(\rho-\rho')\hat{M}(s')\Psi(z,\nu,\nu^*). \qquad (28)$$

We also note that

$$\langle\beta(z,\rho)u(z,s)\exp(iR_z)\rangle=\frac{1}{i}\frac{\delta g(\nu,\nu^*,z,s)}{\delta\nu(s)} \qquad (29a)$$

and

$$\langle\beta(z,\rho)u^*(z,s)\exp(iR_z)\rangle=\frac{1}{i}\frac{\delta g(\nu,\nu^*,z,s)}{\delta\nu^*(s)}. \qquad (29b)$$

By Eqs. (11), (13a), (13b), (28), (29a), and (29b), we obtain

$$\frac{\delta\Psi(z,\nu,\nu^*)}{\delta z}=\left(\frac{i}{2}\right)\int\frac{ds}{k}\left(\nu(s)\nabla_\rho^2\frac{\delta\Psi}{\delta\nu(s)}-\nu^*(s)\nabla_\rho^2\frac{\delta\Psi}{\delta\nu^*(s)}\right)$$
$$-\frac{1}{4}\iint\frac{ds\,ds'}{kk'}A(\rho-\rho')\hat{M}(s)\hat{M}(s')\Psi. \qquad (30)$$

This is the Fokker–Planck equation for the characteristic functional Ψ of the random electromagnetic field $u(z,\rho,k)$. Since the characteristic functional is the Fourier transform of the probability functional, Eq. (30) is in fact the Fourier transform of the Fokker-Planck equation. Our technique used here can also be applied to the derivation of the Fokker-Planck equation for the ordinary characteristic function of a random function $x(t)$.

III. MOMENT EQUATIONS

We want to derive a complete set of moment equations in this section. First, we expand $\Psi(z,\nu,\nu^*)$ as a power series

$$\Psi(z,\nu,\nu^*)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}\left(\int u(z,s)\nu(z,s)\,ds\right)^m \times\left(\int u^*(z,s')\nu^*(z,s')\,ds'\right)^n$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}K_{m,n}(z,\nu,\nu^*), \tag{31}$$

where

$$K_{m,n}(z,\nu,\nu^*)=\int\cdots\int\Gamma_{m,n}(z,s_1,\cdots,s_m;s_1',\cdots,s_n') \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n'$$

and

$$\Gamma_{m,n}(z,s_1,\cdots,s_m;s_1',\cdots,s_n')=\langle u_1u_2\cdots u_m u_1^{*\prime}\cdots u_n^{*\prime}\rangle. \tag{32}$$

$\Gamma_{m,n}$ is the m-nth moment of the random field $u(z,s)$. The object of this section is to derive a differential equation satisfied by $\Gamma_{m,n}$.

We note that, for any function $f(s)$ of s, we have

$$\int f(s)\nu(s)\frac{\delta}{\delta\nu(s)}K_{m,n}(z,\nu,\nu^*)\,ds$$
$$=\int\cdots\int\sum_{i=1}^{m}\Gamma_{m,n}(z,s,\cdots,s_m;s_1',\cdots,s_n')f(s_i) \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n' \tag{33a}$$

and

$$\int f(s)\nu^*(s)\frac{\delta}{\delta\nu^*(s)}K_{m,n}(z,\nu,\nu^*)\,ds$$
$$=\int\cdots\int\sum_{i=1}^{n}\Gamma_{m,n}(z,s_1\cdots s_m;s_1'\cdots s_n')f(s_i') \times\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}\,ds_1\cdots ds_m ds_1'\cdots ds_n'. \tag{33b}$$

From (33a) and (33b) we obtain

$$-\frac{1}{4}\int\int\frac{ds\,ds'}{kk'}A(\rho-\rho')\hat{M}(s)\hat{M}(s')K_{m,n}(z,\nu,\nu^*)$$
$$=-\frac{1}{4}\int\cdots\int\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{A(\rho_i-\rho_j)}{k_ik_j}-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{[A(\rho_i-\rho_j')+A(\rho_j'-\rho_i)]}{k_ik_j'}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\right)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n'. \tag{34}$$

We also note that

$$\int\nu(s)\nabla_\rho^2\frac{\delta K_{m,n}}{\delta\nu(s)}\,ds =\int\cdots\int(\nabla_1^2+\nabla_2^2+\cdots+\nabla_m^2)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n' \tag{35a}$$

and

$$\int\nu^*(s)\nabla_\rho^2\frac{\delta K_{m,n}}{\delta\nu^*(s)}\,ds =\int\cdots\int(\nabla_1'^2+\nabla_2'^2+\cdots+\nabla_n'^2)\Gamma_{m,n}\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime} \times ds_1\cdots ds_m ds_1'\cdots ds_n', \tag{35b}$$

where $\nabla_j^2=\nabla_{\rho j}^2$ and $\nabla_j'^2=\nabla_{\rho j'}^2$.

By Eqs. (31), (34), (35a), and (35b), we can write Eq. (30) as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{i^{m+n}}{m!n!}\int\cdots\int\Bigg[\frac{\partial\Gamma_{m,n}}{\partial z}-\frac{i}{2}\left(\frac{\nabla_1^2}{k_1}+\cdots+\frac{\nabla_m^2}{k_m}-\frac{\nabla_1'^2}{k_1'}-\cdots-\frac{\nabla_n'^2}{k_n'}\right)\Gamma_{m,n}+\frac{1}{4}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{A(\rho_i-\rho_j)}{k_ik_j}$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(A(\rho_i-\rho_j')+A(\rho_j'-\rho_i))}{k_ik_j'}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\Bigg)\Gamma_{m,n}\Bigg]\nu_1\cdots\nu_m\nu_1^{*\prime}\cdots\nu_n^{*\prime}ds_1\cdots ds_m ds_1'\cdots ds_n'=0. \tag{36}$$

Since $\nu(s)$ and $\nu^*(s)$ are arbitrarily defined, the quantity inside the bracket in Eq. (36) must be zero. We have then the following differential equation for the moment function $\Gamma_{m,n}$:

$$\frac{\partial\Gamma_{m,n}}{\partial z}(z,s_1\cdots s_m,s_1'\cdots s_n')=\frac{i}{2}\left[\frac{\nabla_1^2}{k_1}+\cdots+\frac{\nabla_m^2}{k_m}-\frac{\nabla_1'^2}{k_1'}-\cdots-\frac{\nabla_n'^2}{k_n'}\right]\Gamma_{m,n}$$
$$-\frac{1}{4}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{A(\rho_i-\rho_j)}{k_ik_j}-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{[A(\rho_i-\rho_j')+A(\rho_j'-\rho_i)]}{k_ik_j'}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A(\rho_i'-\rho_j')}{k_i'k_j'}\right)\Gamma_{m,n}. \tag{37}$$

It is noted that we can also derive the moment equation (37) directly from the wave equation (8), using the same technique in obtaining Eq. (22). Equation (37) thus gives us a complete set of the moment equations of the random wave field with different transverse coordinates and *different* wavenumbers.

IV. APPLICATIONS

First we note that we have derived a complete set of the moment equations with different transverse coordinates and *different* wavenumbers for the high-frequency waves propagating in a plasma medium. However, we can easily extend the argument to the other cases when the index of refraction $\epsilon_\omega(\mathbf{r})$ has a different frequency dependence.

Next we consider some applications.

A. Identical wavenumbers

When all the wavenumbers are identical, Eq. (37) becomes

$$\frac{\partial \Gamma_{m,n}}{\partial z}(z, \rho_1 \cdots \rho_m, \rho_1' \cdots \rho_n')$$

$$= \frac{i}{2k}(\nabla_1^2 + \cdots + \nabla_m^2 - \nabla_1'^2 - \cdots - \nabla_n'^2)\Gamma_{m,n} - \frac{1}{4k^2}$$

$$\times\left(\sum_{i=1}^{m}\sum_{j=1}^{n} A(\rho_i - \rho_j)\right.$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}[A(\rho_i - \rho_j') + A(\rho_j' - \rho_i)]$$

$$\left. + \sum_{i=1}^{n}\sum_{j=1}^{n} A(\rho_i' - \rho_j')\right)\Gamma_{m,n}, \tag{38}$$

which is identical to that obtained by Tatarskii.[2] However, the derivation by Tatarskii requires that the refraction index fluctuations possess Gaussian statistics while we do not require the assumption of Gaussian statistics in our derivation in general.

B. $\Gamma_{1,1}\ (z, s_1, s_2)$

When $m=1$, and $n=1$, Eq. (37) gives

$$\frac{\partial \Gamma_{1,1}}{\partial z}(z, \rho_1, k_1, \rho_2, k_2)$$

$$= \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right)\Gamma_{1,1} - \frac{1}{4}\left[\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right)A(0)\right.$$

$$\left. - \frac{2A(\rho_1 - \rho_2)}{k_1 k_2}\right]\Gamma_{1,1}, \tag{39}$$

where $\Gamma_{1,1}(z, \rho_1, k_1, \rho_2, k_2) = \langle u(z, \rho_1, k_1)u^*(z, \rho_2, k_2)\rangle$.

Equation (39) can be used to calculate the mean intensity profile $\langle I(\mathbf{r}, t)\rangle$ at position $\mathbf{r}$. Consider the random wave observed by a detector with a bandwidth function $f_B(k)$. Then we have the total observed wave amplitude $h(z, \rho, t)$ at position z, ρ and time t

$$h(z, \rho, t) = \int_{-\infty}^{\infty} u(z, \rho, k) f_B(k) \exp\{i[kz - \omega(k)t]\}\, dk. \tag{40}$$

The average intensity profile is then

$$\langle I(\mathbf{r}, t)\rangle = \langle h(z, \rho, t)h^*(z, \rho, t)\rangle$$

$$= \int\int_{-\infty}^{\infty} \langle u(z, \rho, k_1)u^*(z, \rho, k_2)\rangle f_B(k_1) f_B(k_2)$$

$$\times \exp\{i[k_1 z - \omega(k_1)t]\}\exp\{-i[k_2 z - \omega(k_2)t]\}\, dk_1 dk_2. \tag{41}$$

Thus $\Gamma_{1,1}$ is related to the average intensity profile $\langle I(\mathbf{r}, t)\rangle$ by Eq. (41). Equations (39) and (41) have been applied to calculate the pulse profile of pulsar in interstellar scintillation. The details will be given in a later paper.

C. $\Gamma_{2,2}$

When $m=2$, and $n=2$, Eq. (39) becomes

$$\frac{\partial}{\partial z}\Gamma_{2,2}(z, s_1, s_2, s_3, s_4)$$

$$= \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} + \frac{\nabla_2^2}{k_2} - \frac{\nabla_3^2}{k_3} - \frac{\nabla_4^2}{k_4}\right)\Gamma_{2,2}$$

$$- \frac{1}{4}\left[\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} + \frac{1}{k_4^2}\right)A(0) + 2\frac{A(\rho_1 - \rho_2)}{k_1 k_2}\right.$$

$$+ 2\frac{A(\rho_3 - \rho_4)}{k_3 k_4} - 2\frac{A(\rho_1 - \rho_3)}{k_1 k_3} - 2\frac{A(\rho_1 - \rho_4)}{k_1 k_4}$$

$$\left. - 2\frac{A(\rho_2 - \rho_3)}{k_2 k_3} - 2\frac{A(\rho_2 - \rho_4)}{k_2 k_4}\right]\Gamma_{2,2}, \tag{42}$$

where

$$\Gamma_{2,2}(z, s_1, s_2, s_3, s_4) = \langle u(z, s_1)u(z, s_2)u^*(z, s_3)u^*(z, s_4)\rangle. \tag{43}$$

If one sets $s_3 = s_1$, $s_4 = s_2$, and $\rho_1 = \rho_2$, then

$$\Gamma_{2,2}(z, s_1, s_2, s_3, s_4)$$

$$= \langle |u(z, \rho_1, k_1)|^2 |u(z, \rho_1, k_2)|^2\rangle = \langle I(z, \rho_1, k_1),$$

$$I(z, \rho_1, k_2)\rangle = P_I(k_1 - k_2). \tag{44}$$

Here I is the intensity and P_I is the correlation function of intensity at different frequencies. Thus $\Gamma_{2,2}$ gives in this special case the intensity correlation function $P_I(k_1 - k_2)$ at a given observation point with different wavenumbers. The intensity correlation function has been measured in interstellar scintillations,[3–5,10] and Eq. (42) provides a theoretical base of interpretation.

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V. Second Moment and Others

For the moment equations with same wavenumbers in Eq. (4-38), we will consider in particular $\Gamma_{1,0}$, $\Gamma_{1,1}$, $\Gamma_{2,0}$ and $\Gamma_{2,2}$ which satisfy, respectively, the following equations

$$\frac{\partial\Gamma_{1,0}(z,\underset{\sim}{\rho})}{\partial z} = \frac{i}{2k}\nabla_{\underset{\sim}{\rho}}^{2}\Gamma_{1,0} - \frac{A_{\beta}(0)}{4k^{2}}\Gamma_{1,0} \tag{4-45}$$

$$\frac{\partial\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)}{\partial z} = \frac{i}{2k}(\nabla_1^{2}-\nabla_2^{2})\Gamma_{1,1} - \frac{1}{2k^{2}}[A_{\beta}(0)-A_{\beta}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)\Gamma_{1,1} \tag{4-46}$$

$$\frac{\partial\Gamma_{2,0}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)}{\partial z} = \frac{i}{2k}(\nabla_1^{2}+\nabla_2^{2})\Gamma_{2,0} - \frac{1}{2k^{2}}[A_{\beta}(0)+A_{\beta}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)]\Gamma_{2,0} \tag{4-47}$$

and

$$\frac{\partial\Gamma_{2,2}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_1',\underset{\sim}{\rho}_2')}{\partial z} = \frac{i}{2k}[\nabla_1^{2}+\nabla_2^{2}-\nabla_1'^{2}-\nabla_2'^{2}]\Gamma_{2,2} - \frac{\Gamma_{2,2}}{2k^{2}}[2A_{\beta}(0) +$$

$$A_{\beta}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)+A_{\beta}(\underset{\sim}{\rho}_1'-\underset{\sim}{\rho}_2')-A_{\beta}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_1')-A_{\beta}(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2')-A_{\beta}(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}_1')-A_{\beta}(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}_2')] \quad . \tag{4-48}$$

We assume throughout this thesis that ε_k of the random medium is statistically homogeneous in the transverse plane and that u is homogeneous in the initial plane. Therefore $\Gamma_{1,0}(z,\underset{\sim}{\rho})$ does not depend on $\underset{\sim}{\rho}$ and $\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)$ depends only on z and the difference of the arguments $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$, $|\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2|$. Thus we have $\nabla_{\underset{\sim}{\rho}}^{2}\Gamma_{1,0}(z,\underset{\sim}{\rho}) = 0$ and $(\nabla_1^{2}-\nabla_2^{2})\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2) = 0$. Eqs. (4-45) and (4-46) can be written respectively as

$$\frac{\partial}{\partial z} \Gamma_{1,0}(z,\underset{\sim}{\rho}) = -\frac{A_\beta(0)}{4k^2} \Gamma_{1,0} \tag{4-49}$$

and

$$\frac{\partial}{\partial z} \Gamma_{1,1}(z,\underset{\sim}{\rho}) = -\frac{1}{2k^2} [A_\beta(0)-A_\beta(\underset{\sim}{\rho})] \Gamma_{1,1}(z,\underset{\sim}{\rho}) \tag{4-50}$$

where in Eq. (4-49), $\underset{\sim}{\rho}=\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2$ and $\Gamma_{1,1}(z,\underset{\sim}{\rho}) = \Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)$. Since we have normalized our incident wave such that $u(z=0,\underset{\sim}{\rho})= 1$, we have immediately upon integrating Eqs. (4-49) and (4-50) the general results

$$\Gamma_{1,0}(z,\underset{\sim}{\rho}) = e^{-\frac{zA_\beta(0)}{4k^2}} \tag{4-51}$$

and

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}) = e^{-\frac{z}{2k^2}[A_\beta(0)-A_\beta(\underset{\sim}{\rho})]} \quad . \tag{4-52}$$

$\Gamma_{1,0}(z,\underset{\sim}{\rho}) = \langle u(z,\underset{\sim}{\rho})\rangle$ is called the "mean wave" or the "coherence wave", which describes the coherent part of the random wave. Eq. (4-51) shows that the "mean wave" decays as wave propagates in the random medium. $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ is called the "mutual coherence function" and is the quantity that describes the loss of coherence of an initial coherent wave propagating in a random medium. $\Gamma_{1,1}(z,0) = \langle I(z,\underset{\sim}{\rho})\rangle$ is the mean intensity observed at z and from Eq. (4-52), $\langle I(z,\underset{\sim}{\rho})\rangle = 1$ for all z. $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ is directly related to the angular spectrum of the random waves, which will be discussed in detail in Chapter 5. Whitman & Beran (1970) have solved $\Gamma_{1,1}$ for the beam spread of laser light propagating in a random medium where u is initially <u>inhomogeneous</u> and in Eq. (4-45), $(\nabla_1{}^2-\nabla_2{}^2)\Gamma_{1,1} \neq 0$.

Eq. (4-52) is very similar to the expression for the angular spectrum discussed by Fejer (1953), Bramley (1954) and Ratcliffe(1956), who use "angular spectrum method" which can only be applied for the homogeneous cases. Eq. (4-45) is the general equation describing the second moment $\Gamma_{1,1}$ for both homogeneous and inhomogeneous cases.

$\Gamma_{2,0}$ and $\Gamma_{2,2}$ will be discussed in Chapter 5. We note that $\Gamma_{2,2}$ is directly related to the <u>intensity</u> correlation function of the random wave.

VI. Validity of the Markov and Quasi-optics Approximation

The limits of validity of the Markov and parabolic approximation for $\Gamma_{1,1}$ are discussed thoroughly by Tatarskii (1971) and we simply quote his results here. He shows that incorporating the term $\partial^2/\partial z^2$ which is neglected in the parabolic approximation leads to a second-order approximation

$$\Gamma_{1,1}^{(2)} = \Gamma_{1,1}^{(1)} \left\{1 + \frac{k^2 A_\varepsilon(0)}{16\pi} z^2 \left[2\frac{\pi k^2 z}{3}\left(\frac{\partial A_\varepsilon(\rho)}{\partial \rho}\right)^2 + 2\pi \nabla_\perp^2 A_\varepsilon(\rho)\right]\right\} . \qquad (4\text{-}53)$$

Clearly, for $\Gamma_{1,1}^{(1)}$ to be a good approximation, the second term in curly brackets must be small compared with unity. That is, we require

$$\left| \frac{k^2 A_\varepsilon(0)}{16\pi} z^2 \left[\frac{2\pi k^2 z}{3} \left(\frac{\partial A_\varepsilon}{\partial \rho}\right)^2 + 2\pi \nabla_\perp^2 A_\varepsilon \right] \right| \ll 1. \qquad (4\text{-}53')$$

Tatarskii derives this condition assuming Gaussian refractive index fluctuations, but it is readily generalized to non-Gaussian statistics by the techniques used in Section II of this chapter.

Tatarskii also derives the criterion for assuming that the coherence length of the fluctuations is short (the Markov approximation) and finds that it is effectively the same as that given in Eq. (4-53').

In what follows, representative calculation will be presented for both types of spectra given in Chapter 1, Section III. In order to apply the above results we must first ascertain the limits on the parameters for which the correlation terms in Eq. (4-53) are small. Clearly this will not be satisfied for z large enough. In addition, the condition also depends on the value of ρ, and may break down for large enough ρ.

For the Gaussian spectrum given in Eq. (1-21) we have

$$A_\epsilon(\underset{\sim}{\rho}) = \frac{B}{8\pi} q_o^2 e^{-(\rho^2 q_o^2)/4} , \tag{4-54}$$

in which case we require

$$\frac{Bk^2 q_o^2}{128\pi^2} z^2 \left| \left[\frac{2\pi k^2 z}{3} \left(\frac{B}{16\pi} \rho q_o^4 e^{-\rho^2 q_o^2/4} \right)^2 + \frac{Bq_o^4}{8} \left(-2 + \frac{q_o^2 \rho^2}{2} \right) e^{-\rho^2 q_o^2/4} \right] \right| \ll 1 \quad . \tag{4-55}$$

For the power law spectrum in Eq. (1-21) the expression for $A(\underset{\sim}{\rho})$ cannot be evaluated exactly. However, it will be sufficient to consider only intermediate values of ρ, $L \gg \rho \gg \ell$ or $q_o \ll \frac{1}{\rho} \ll q_1$, in which case one may show that

$$A(\underset{\sim}{\rho}) = \frac{B}{4\pi} q_o^2 \frac{1}{\alpha-2} \left[1 - \frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\alpha/_2)} \left(\frac{q_o \rho}{2} \right)^{\alpha-2} \right] , \tag{4-56}$$

(c.f. Eq. (1-28)) so that our condition (4-53')becomes

$$\frac{Bk^2 q_o^2}{64\pi^2} \frac{1}{\alpha-2} z^2 \left\{ \frac{2\pi k^2 z}{3} \left[\frac{Bq_o^\alpha}{2^\alpha \pi} \frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \rho^{\alpha-3} \right]^2 + \frac{Bq_o^\alpha \Gamma(2-\frac{\alpha}{2})}{2^{\alpha-1}\Gamma(\frac{\alpha}{2})} (\alpha-2) \rho^{\alpha-4} \right\} \ll 1. \tag{4-57}$$

Now, it is desirable to remove the dependence on ρ from Eqs. (4-55) and (4-57) in order to find the maximum value of z, the propagation distance, for which these equations are valid. First we note from Eq. (4-51) that the transverse coherence scale of $\Gamma_{1,1}$ may be much smaller than the coherence scale of ε. It is also clear that we are not interested in ρ much larger than the transverse scale of $\Gamma_{1,1}$. Hence we may evaluate this scale and substitute it in Eqs. (4-55) and (4-57) to obtain equations giving the allowed value of z in terms of B, q_o and α.

From Eq. (4-51), we see that the transverse scale of $\Gamma_{1,1}$ is simply that value of ρ for which

$$\frac{zk^2}{2} \left[A_\varepsilon(0) - A_\varepsilon(\underset{\sim}{\rho})\right] \simeq 1 \quad . \tag{4-58}$$

For our Gaussian spectrum this becomes for the cases of interest where $\rho \ll \frac{1}{q_o}$,

$$\frac{Bzk^2}{64\pi} \left(\rho^2 q_o^{\ 4}\right) \simeq 1. \tag{4-59}$$

Defining ρ_c as the transverse scale, we have

$$\rho_c = \frac{1}{q_o^{\ 2}} \left(\frac{64\pi}{Bzk^2}\right)^{1/2} \quad . \tag{4-60}$$

Substituting this for ρ in Eq. (4-55) and remembering that $q_o \rho_c \ll 1$, we have

$$\frac{Bk^2 q_o^{\ 2}}{128\pi^2} z^2 \left| \left[\frac{2\pi k^2 z}{3} \left(\frac{B}{16\pi} \cdot q_o^{\ 2} \left(\frac{64\pi}{Bzk^2} \right)^{\frac{1}{2}} \right)^2 - \frac{Bq_o^{\ 4}}{2} \right] \right| \ll 1 \ . \tag{4-61}$$

This limit on z is illustrated in Figure (4-1) for typical values of the parameters.

Proceeding similarly for the power law spectrum, we obtain (if $\rho_c > \ell$)

$$\frac{zk^2}{2} \cdot \frac{Bq_o^{\;2}}{4\pi} \quad \frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})(\alpha-2)} \quad \left(\frac{q_o\rho_c}{2}\right)^{\alpha-2} = 1 \tag{4-62}$$

or

$$\rho_c = \frac{2}{q_o}\left[\frac{8\pi}{Bzk^2 q_o^{\;2}} \quad \frac{\Gamma(\frac{\alpha}{2})(\alpha-2)}{\Gamma(2-\frac{\alpha}{2})}\right]^{\frac{1}{\alpha-2}} \quad . \tag{4-63}$$

In this case our condition on z becomes

$$\frac{Bk^2 q_o^{\;2}}{64(\alpha-2)\pi^2} z^2 \left\{ \frac{2\pi k^2 z}{3} \quad \frac{B^2 q_o^{\;2\alpha}}{2^{2\alpha}\pi^2}\left(\frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})}\right)^2 \left[\frac{8\pi}{Bzk^2 q_o^{\;2}} \quad \frac{\Gamma(\frac{\alpha}{2})(\alpha-2)}{\Gamma(2-\frac{\alpha}{2})}\right]^{(\frac{2\alpha-6}{\alpha-2})} \cdot \left(\frac{2}{q_o}\right)^{(2\alpha-6)} \right.$$

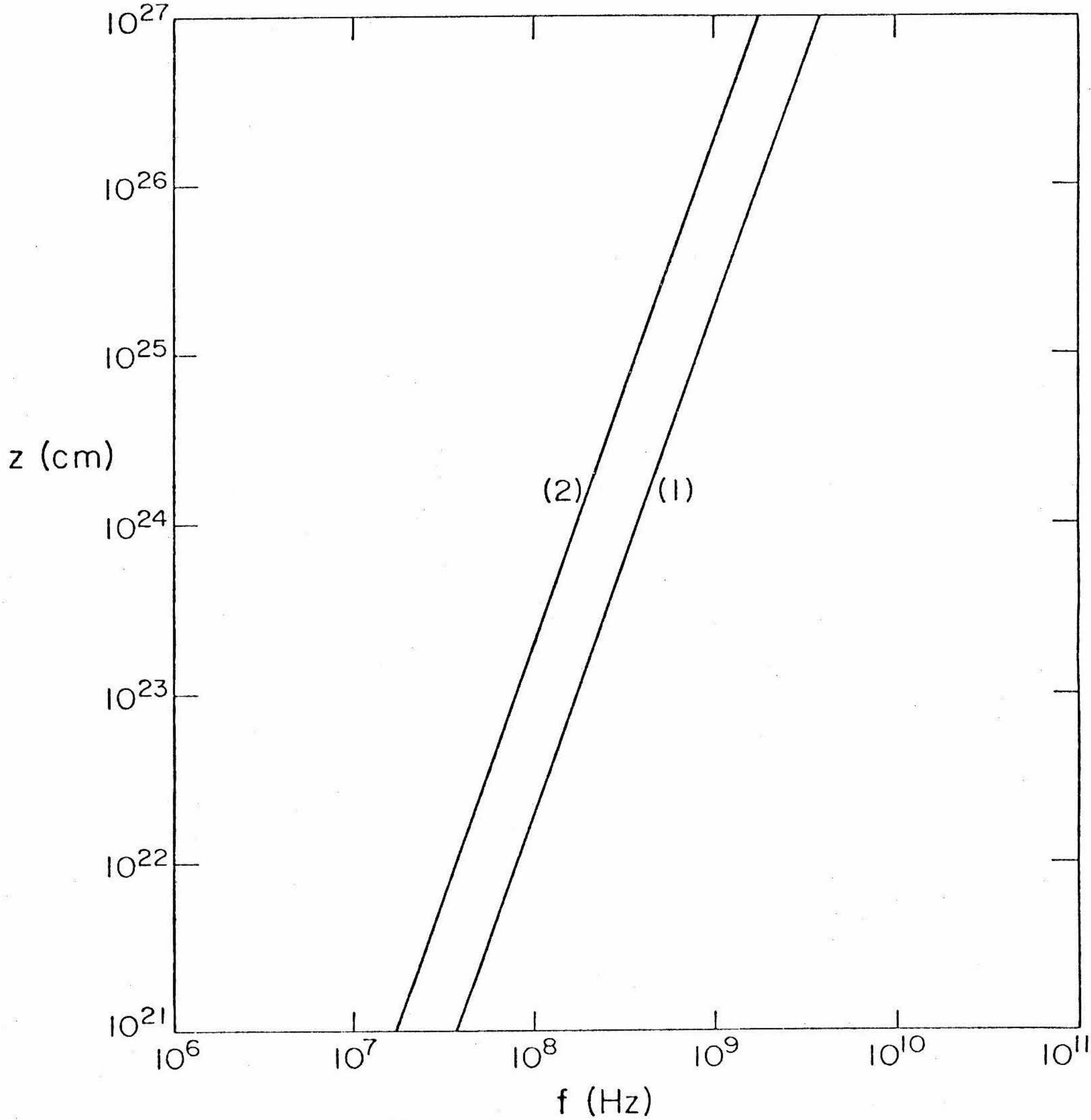
$$\left. + \frac{Bq_o^{\;\alpha}\Gamma(2-\frac{\alpha}{2})}{2^{\alpha-1}\Gamma(\frac{\alpha}{2})} \quad (\alpha-2) \quad \cdot \ell^{\alpha-4} \right\} \ll 1 \tag{4-64}$$

where we have noted that the second term on the left is always less than its value where $\rho = \ell$, the inner scale. The limitation on z, as a function of $L = \frac{1}{q_o}$, provided by Eq. (4-64) for a power-law spectrum is plotted in Figure (4-2) for $\alpha = {}^{11}/_{3}$, corresponding to the Kolmogorov spectrum, for parameters of interest for the interstellar medium. It is readily seen that if $\langle\delta n_e^{\;2}\rangle \sim 10^{-4}\text{cm}^{-6}$ and $\epsilon > 10^8$ hz, one may apply our equation at typical galactic distances.

Figure Captions

Figure (4-1). Upper limits on z for the validity of the Markov approximation, for the Gaussian refractive index spectrum given by Eq. (1-20). The maximum value of z is plotted as a function of electromagnetic wave frequency f. Curve (1) is for $<\delta N_e^2> = 10^{-4}cm^{-6}$ and curve (2) is for $<\delta N_e^2> = 10^{-5}cm^{-6}$. Note that the value of $z \propto (<\delta N_e^2>)^{-1} f^3$.

Figure (4-2). As in Figure (4-1) for the power-law refractive index spectrum given in Eq. (1-21) with the Kolmogorov value for $\alpha = 11/3$. The maximum value of z is plotted as a function of outer scale L for various values of $<\delta N_e^2>$ and f. Curve (1) is for $<\delta N_e^2> = 10^{-4}cm^{-6}$ and $f = 10^8$ Hz, curve (2): $<\delta N_e^2> = 10^{-5}cm^{-6}$ and $f = 10^8$ Hz, curve (3): $<\delta N_e^2> = 10^{-4}cm^{-6}$ and $f = 3x10^8$ Hz and curve (4): $<\delta N_e^2> = 10^{-5}cm^{-6}$ and $f = 3x10^8$ Hz. Note that the value of z scales $\propto (<\delta N_e^2>)^{-1} L^{-2/11} f^{32/11}$.



10^{27}
10^{26}
10^{25}
10^{24}
10^{23}
10^{22}
10^{21}
z (cm)
(2)
(1)
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}
10^{11}
f (Hz)

Figure (4-1)

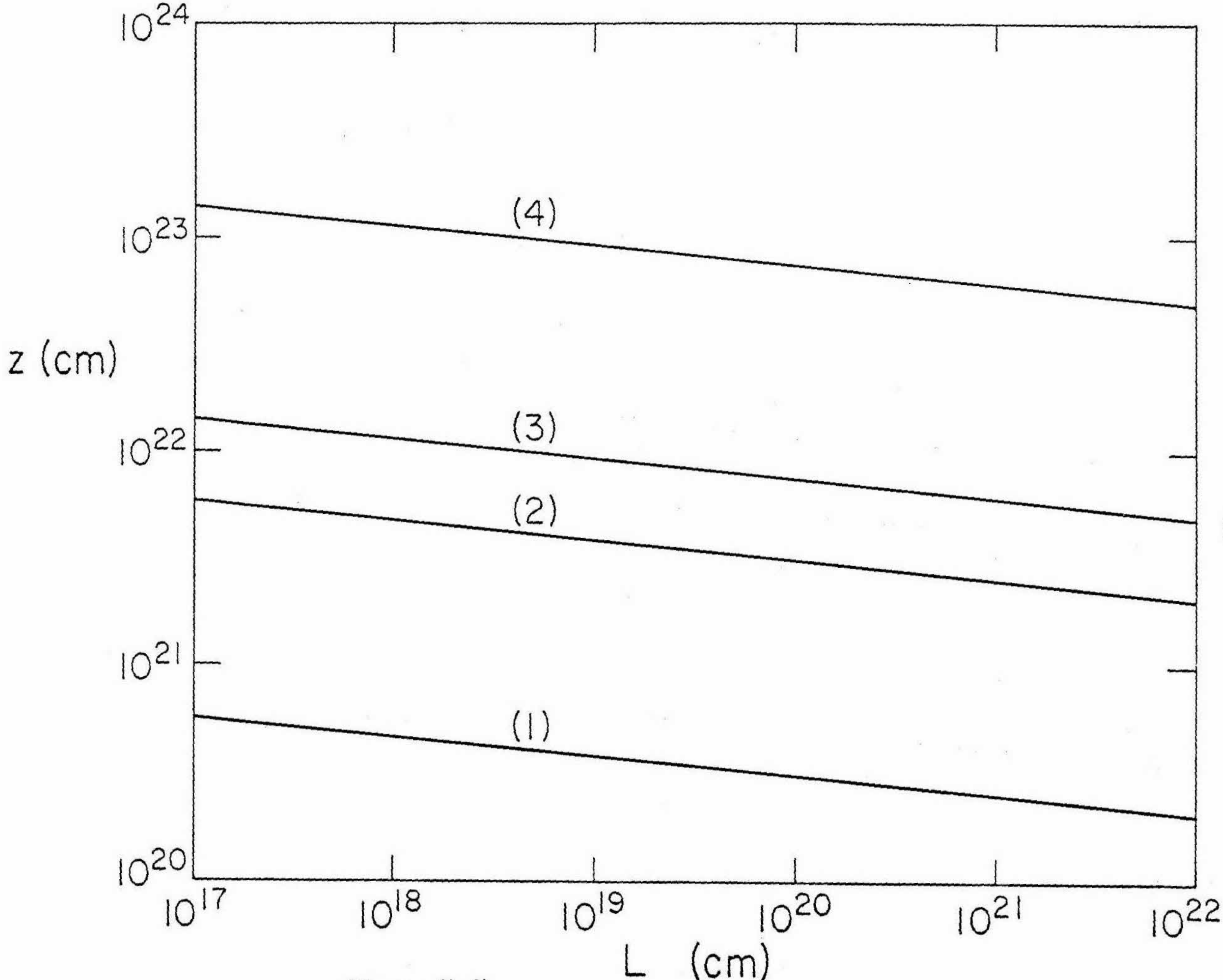

10^{24}
10^{23}
10^{22}
10^{21}
10^{20}
z (cm)
(4)
(3)
(2)
(1)
10^{17}
10^{18}
10^{19}
10^{20}
10^{21}
10^{22}
L (cm)

Figure (4-2)

Chapter 5

The Markov Random Process Approximation (B)

I. Introduction

In Chapter 4, we have derived a complete set of the moment equations of the random waves with different frequencies under the Markov and the quasi-optics approximations, which are valid for both strong scintillations ($m_z \simeq 1$) and weak scintillations ($m_z \ll 1$). In this chapter, we will apply the moment equations to study the various statistical properties of the waves propagating in a thick random medium. The multiple-scattering effect, which is important for strong scintillations and for a thick medium, is included in our treatment here.

In Section II of this chapter, we discuss the angular distribution of the scattered wave and calculate the root mean square scattering angle and the characteristic scattering angle. Section III gives the spectrum of phase fluctuations. In Section IV, we calculate and discuss the temporal smearing of pulses propagating in the turbulent plasmas. In Section V, the spatial intensity correlation function is calculated. In Section VI, we discuss the probability distribution of the random wave. Finally in Section VII, we calculate the intensity correlation function of the waves observed at the same point but with different frequency-bands, and determine the decorrelation bandwidth of the random waves.

II. Angular Power Spectrum

In this section, we will derive an expression for the angular broadening of waves scattered by a random medium. Before proceeding we recall the result that the angular distribution of radiation is essentially the Fourier transform of the $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ in Eq. (4-46) with respect to the transverse coordinate $\underset{\sim}{\rho}$ (Booker & Clemmow 1950, Booker, Ratcliffe & Shinn 1950, Ratcliffe 1956). Suppose we regard our wave field as being a superposition of plane waves which are propagating at a small angle relative to the z axis. For these waves, define the angles $\theta_x = k_x/k$, $\theta_y = k_y/k$ and $\theta = \sqrt{\theta_x^2 + \theta_y^2}$. It may be shown that for waves with random phases, the distribution of intensity over angle, $\psi(\theta_x, \theta_y)$ is related to $\Gamma_{1,1}$ (z,x,y) by the relationship

$$\psi(\theta_x,\theta_y) = \frac{k^2}{4\pi^2} \iint \Gamma_{1,1}(z,x,y) \exp\left[-ik(x\theta_x + y\theta_y)\right] dxdy \qquad (5\text{-}1)$$

with the inverse relationship

$$\Gamma_{1,1}(z,x,y) = \int d\theta_x \int d\theta_y \psi(\theta_x,\theta_y) \exp\left[+ik(x\theta_x + y\theta_y)\right] . \qquad (5\text{-}2)$$

$\psi(\theta_x,\theta_y)$ is commonly called the "angular power spectrum", and gives the intensity as a function of angle. Thus the average total intensity at any given point is

$$\langle I \rangle = \int d\theta_x \int d\theta_y \psi(\theta_x,\theta_y) \qquad (5\text{-}3)$$

and the mean square angular deviation is

$$\langle\theta^2\rangle = \frac{1}{\langle I\rangle} \int d\theta_x \int d\theta_y [\theta_x{}^2 + \theta_y{}^2] \, \psi\, (\theta_x,\theta_y). \tag{5-4}$$

Using the expression in Eq. (4-52) for $\Gamma_{1,1}$ we have immediately for an extended, homogeneous medium,

$$\psi(\theta_x,\theta_y) = \frac{k^2}{4\pi^2} \int d^2\underset{\sim}{\rho}\ \exp\left\{ - \frac{z}{2k^2}[A_\beta(0) - A_\beta(\underset{\sim}{\rho})] - ik(\underset{\sim}{\theta}\cdot\underset{\sim}{\rho}) \right\} \tag{5-5}$$

and, after some manipulation

$$\langle\theta^2\rangle = \frac{z}{2k^4} \nabla_\perp^2 A_\beta(\underset{\sim}{\rho}) \Big|_{\text{at } \underset{\sim}{\rho}=0} \quad . \tag{5-6}$$

Note that the mean square angle $\langle\theta^2\rangle$ is always proportional to λ^4 (or k^{-4}).

We now consider the functional form of the angular spectrum for our two representative refractive index power spectra. We note that the angular spectrum as given in Eq. (5-5) is <u>not</u> in general a Gaussian. However, as z becomes large it asymptotically approaches a Gaussian and we now derive the conditions under which this occurs.

It is most convenient to discuss the two angles θ_x and θ_y separately (for isotropic spectra the distributions of θ_x and θ_y are equal). It is clear from Eq. (5-5) that the moments of θ_x are given by

$$\begin{aligned}\langle\theta_x{}^{2n}\rangle &= \iint \theta_x{}^{2n}\, \psi(\theta_x,\theta_y) d\theta_x d\theta_y \\ &= \left(\frac{i}{k}\right)^{2n} \frac{\partial^{2n}}{\partial x^{2n}} \left[\left\{ \exp\left(- \frac{z}{2k^2} [A_\beta(0) - A_\beta(\underset{\sim}{\rho})] \right) \right\} \right]_{\underset{\sim}{\rho}=0}\end{aligned} \tag{5-7}$$

with an analogous expression for $\langle\theta_y^{2n}\rangle$. Consider now the second and fourth moments.

$$\langle\theta_x^{2}\rangle = -\frac{z}{2k^4}\left(\frac{\partial^2 A_\beta}{\partial x^2}\right)_{\underset{\sim}{\rho}=0}$$

$$= \frac{z}{16\pi^2}\iint dq_x dq_y d_x^{\;2} P_\varepsilon{}_{(q_z=0,q_x,q_y)} \tag{5-8}$$

and

$$\langle\theta_x^{4}\rangle = \frac{z}{2k^6}\left(\frac{\partial^4 A_\beta}{\partial x^4}\right)_{\underset{\sim}{\rho}=0} + 3(\langle\theta_x^{2}\rangle)^2$$

$$= \frac{z}{16\pi^2 k^2}\iint dq_x dq_y q_x^{\;4} P_\varepsilon(q_z=0,q_x,q_y)+3(\langle\theta_x^{2}\rangle)^2. \tag{5-9}$$

Now, if the distribution of θ_x is a Gaussian it is necessary that the moments satisfy the relation

$$\langle\theta_x^{2n}\rangle = \frac{(2n)!}{2^n n!}\;(\langle\theta_x^{2}\rangle)^n. \tag{5-10}$$

It is apparent from Eqs. (5-9) and (5-10) that a necessary condition for $\langle\theta_x^{4}\rangle$ to satisfy the Gaussian relation (5-10) is that the first term on the right in Eq. (5-9) be much smaller than the second term on the right. Thus we require

$$z \gg \frac{16\pi^2}{3k^2}\;\frac{\iint dq_x dq_y q_x^{\;4} P_\varepsilon(q_z=0,q_x,q_y)}{\left[\iint dq_x dq_y q_x^{\;2} P_\varepsilon(q_z=0,q_x,q_y)\right]^2} \tag{5-11}$$

in order that the fourth moment satisfies the Gaussian condition. The corresponding conditions for the higher moments of θ_x are essentially equivalent to (5-10) for Gaussian spectrum and power law spectrum with $0 < \alpha < 4$. So we take Eq. (5-10) as the condition for $\Psi(\theta_x, \theta_y)$ to be approximately Gaussian.

This condition can be written explicitly for our representative refractive index spectra given in Eqs. (1-20) and (1-21). For the Gaussian spectrum we have

$$z \gg \frac{16\pi}{k^2 B q_o^2} \tag{5-12}$$

and for the power law spectrum

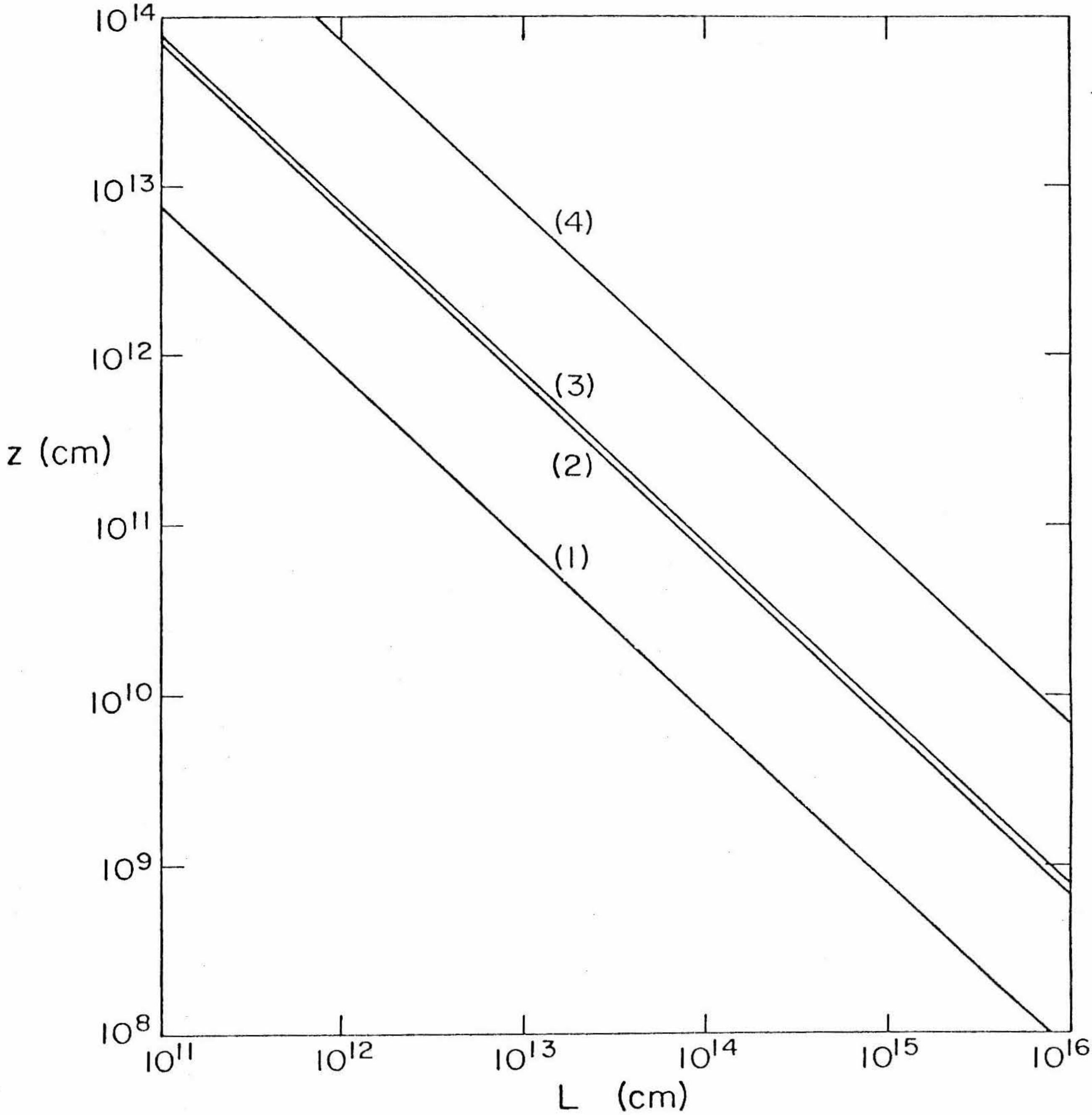
$$z \gg \frac{8\pi}{k^2 B} \frac{q_1^{\alpha-2}}{q_o^{\alpha}} \frac{\Gamma(3-\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} \quad . \tag{5-13}$$

The conditions (5-12) and (5-13) are plotted for various values of the parameters in Figures (5-1a,b). It is clear that rather extreme values of the parameters are required for $\Psi(\theta_x, \theta_y)$ to be Gaussian, and that for a power law spectrum we do not expect a Gaussian angular distribution for reasonable values of the parameters.

Figure Captions

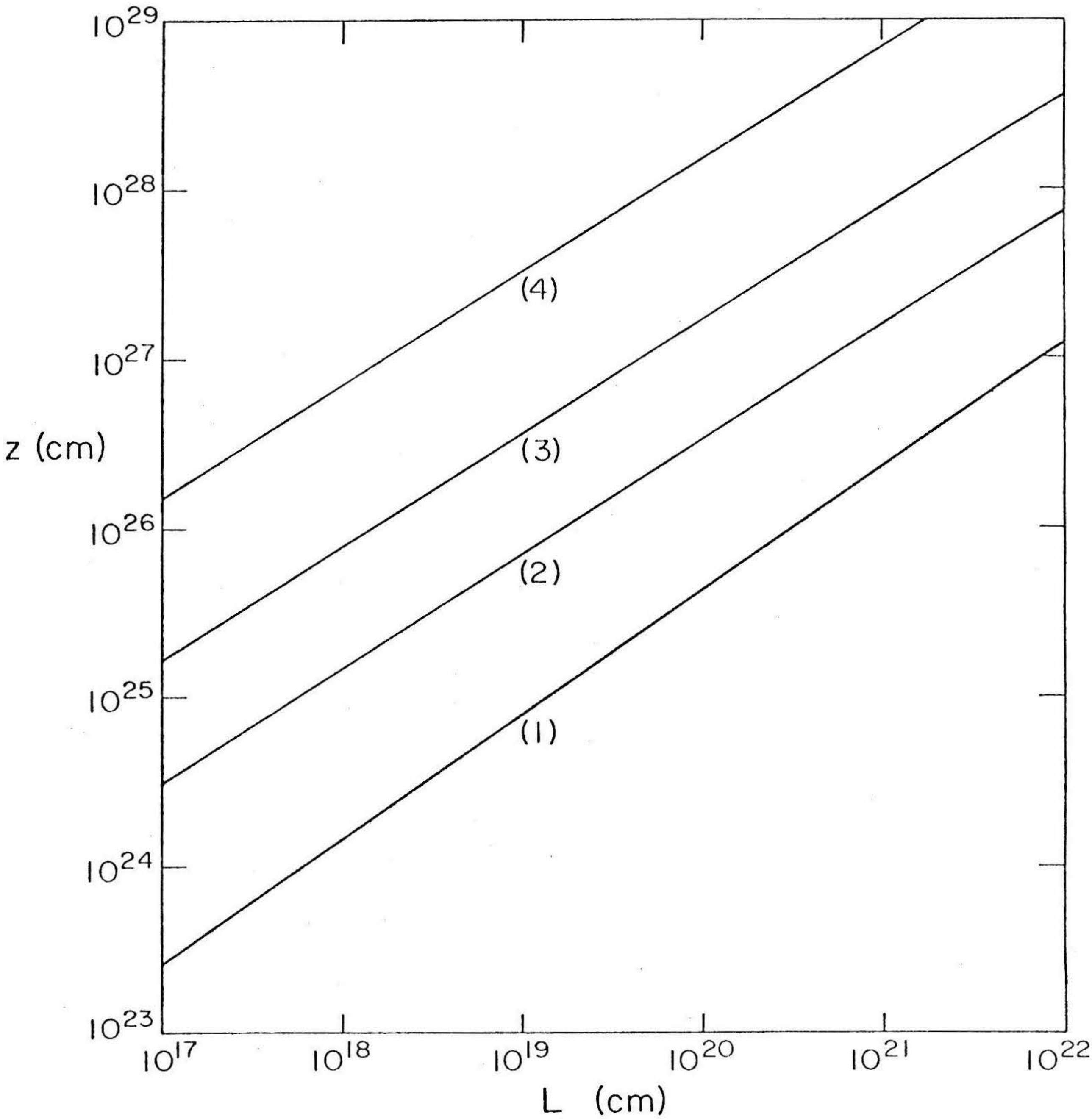
Figure (5-1a). The value of z, beyond which the angular spectrum is approximately Gaussian, plotted versus the correlation scale L, for the Gaussian refractive index spectrum in Eq. (1-20) curve (1) is for $\langle \delta N_e^2 \rangle = 10^{-4} cm^{-6}$ and $f = 10^8$ Hz; curve (2): $\langle \delta N_e^2 \rangle = 10^{-4} cm^{-6}$ and $f = 3 \times 10^8$ Hz, curve (3): $\langle \delta N_e^2 \rangle = 10^{-5} cm^{-6}$ and $f = 10^8$ Hz, and curve (4): $\langle \delta N_e^2 \rangle = 10^{-5} cm^{-6}$ and $f = 3 \times 10^8$ Hz. The value of z scales $\propto L^{-1}(\langle \delta N_e^2 \rangle)^{-1} f^2$.

Figure (5-1b). As in Figure (5-1a) for the Kolmogorov power-law refractive index spectrum. In all curves $\langle \delta N_e^2 \rangle = 10^{-4} cm^{-6}$. Here the result is sensitive to the inner scale $\ell = q_1^{-1}$. In curve (1), $f = 10^8$ Hz and $\ell = 10^6$cm; curve (2): $f = 3 \times 10^8$ Hz and $\ell = 10^6$cm, curve (3): $f = 10^8$ Hz and $\ell = 10^5$cm and curve (4): $f = 3 \times 10^8$ Hz and $\ell = 10^5$cm. Note that z scales $\propto (\langle \delta N_e^2 \rangle)^{-1} f^2 \ell^{-5/3} L^{+2/3}$.



10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
z (cm)
(4)
(3)
(2)
(1)
10^{11}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
L (cm)

Figure (5-1a)



10^{29}
10^{28}
10^{27}
z (cm)
10^{26}
10^{25}
10^{24}
10^{23}
(4)
(3)
(2)
(1)
10^{17}
10^{18}
10^{19}
10^{20}
10^{21}
10^{22}
L (cm)

Figure (5-1b)

We now compute the mean angular broadening as a function of the parameters for our representative spectra. This is represented by $\langle \theta_x^2 \rangle + \langle \theta_y^2 \rangle = \langle \theta^2 \rangle$. We have for the Gaussian spectrum

$$\langle \theta^2 \rangle = \frac{zBq_o^4}{16\pi} \tag{5-14}$$

and for our power law (if $\alpha > 4$ and $q_1 \ll q_o$)

$$\langle \theta^2 \rangle = \frac{zB}{16\pi} q_o^{\alpha} q_1^{4-\alpha} \Gamma(2-\tfrac{\alpha}{2}) \quad . \tag{5-15}$$

Again, the value of $\langle \theta^2 \rangle^{1/2}$ for various values of the parameters is illustrated in Figures (5-2a,b).

Finally, we note that observations of angular broadening usually refer to a 1/e power angle, or some other angle characteristic of the fall-off of the angular spectrum with increasing $|\theta|$. We define a characteristic angle θ_c such that if ρ_c is the characteristic scale of $\Gamma_{1,1}$ (see Eq. (4-58) and following discussion) then

$$\theta_c = (k\rho_c)^{-1} \quad . \tag{5-16}$$

It is readily shown that for a Gaussian spectrum, $\theta_c \sim \langle \theta^2 \rangle^{1/2}$, but that for a power-law spectrum, θ_c may differ considerably from $\langle \theta^2 \rangle^{1/2}$. Using Eqs. (4-63) and (5-16) we have for a power law

$$\theta_c = \frac{q_o}{2k} \left[\frac{Bzk^2 q_o^2}{8\pi(\alpha-2)} \frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(\alpha/2)} \right]^{\frac{1}{\alpha-2}} \quad . \tag{5-17}$$

Note that θ_c for a power law is independent of q_1 and is proportional to $\lambda^{(\frac{\alpha}{\alpha-2})}$ in contrast to $\langle\theta^2\rangle$. The value of θ_c is illustrated in Figure (5-3) for Kolmogorov spectrum.

Figure (5-2a). The root-mean-square scattering angle $\langle \theta^2 \rangle^{1/2}$ plotted versus the correlation scale L for a Gaussian refractive index spectrum. In both cases $\langle \delta N_e{}^2 \rangle = 10^{-4} cm^{-6}$ and $z = 10^3$ PC. In curve (1) $f = 10^8$ Hz and in curve (2) $f = 3 \times 10^8$ Hz. The value of $\langle \theta^2 \rangle^{1/2}$ scales $\propto (\langle \delta N_e{}^2 \rangle)^{1/2} z^{1/2} L^{-1/2} f^{-2}$.

Figure (5-2b). As in Figure (5-2a) for a Kolmogorov power-law refractive index spectrum. In curve (1) $f = 3 \times 10^8$ Hz and the inner scale $\ell = 10^6$cm, curve (2): $f = 10^8$ Hz and $\ell = 10^6$cm and curve (3): $f = 10^8$ Hz and $\ell = 10^5$cm. In all cases $\langle \delta N_e{}^2 \rangle = 10^{-4} cm^{-6}$ and $z = 10^3$ PC. The value of $\langle \theta^2 \rangle^{1/2}$ scales $\propto (\langle \delta N_e{}^2 \rangle)^{1/2} z^{1/2} L^{-1/3} \ell^{-1/6} f^{-2}$.

Figure (5-3). Characteristic half-power angle θ_c plotted versus correlation length L for a Kolmogorov power-law refractive index spectrum. In curve (1) $f = 10^8$ Hz and in curve (2) $f = 3 \times 10^8$ Hz. In both cases $z = 10^3$ PC and $\langle \delta N_e{}^2 \rangle = 10^{-4} cm^{-6}$. The value of θ_c scales $\propto z^{3/5} \langle \delta N_e{}^2 \rangle^{3/5} f^{-11/5} L^{-2/5}$.

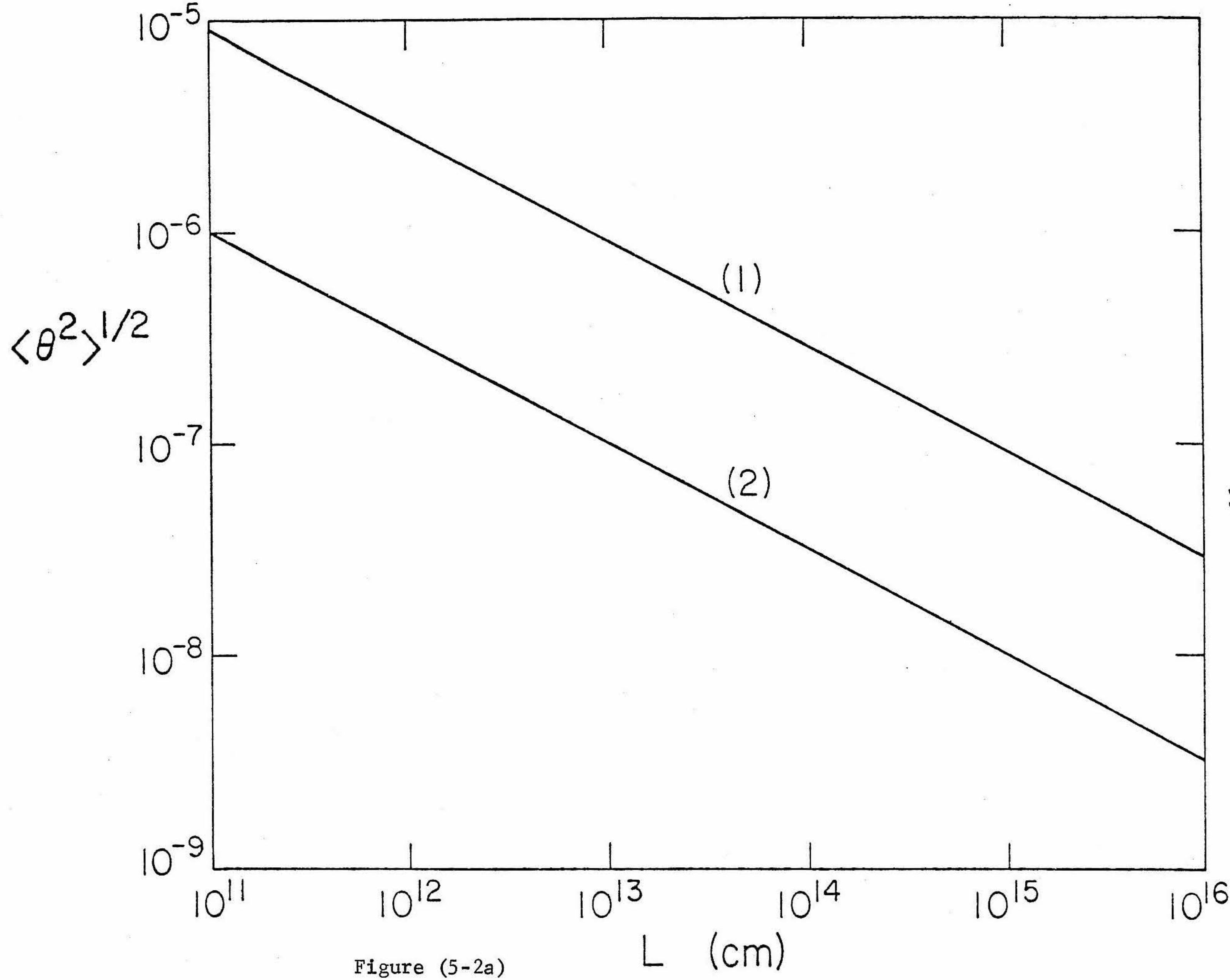

10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
$\langle\theta^2\rangle^{1/2}$
(1)
(2)
10^{11}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
L (cm)

Figure (5-2a)

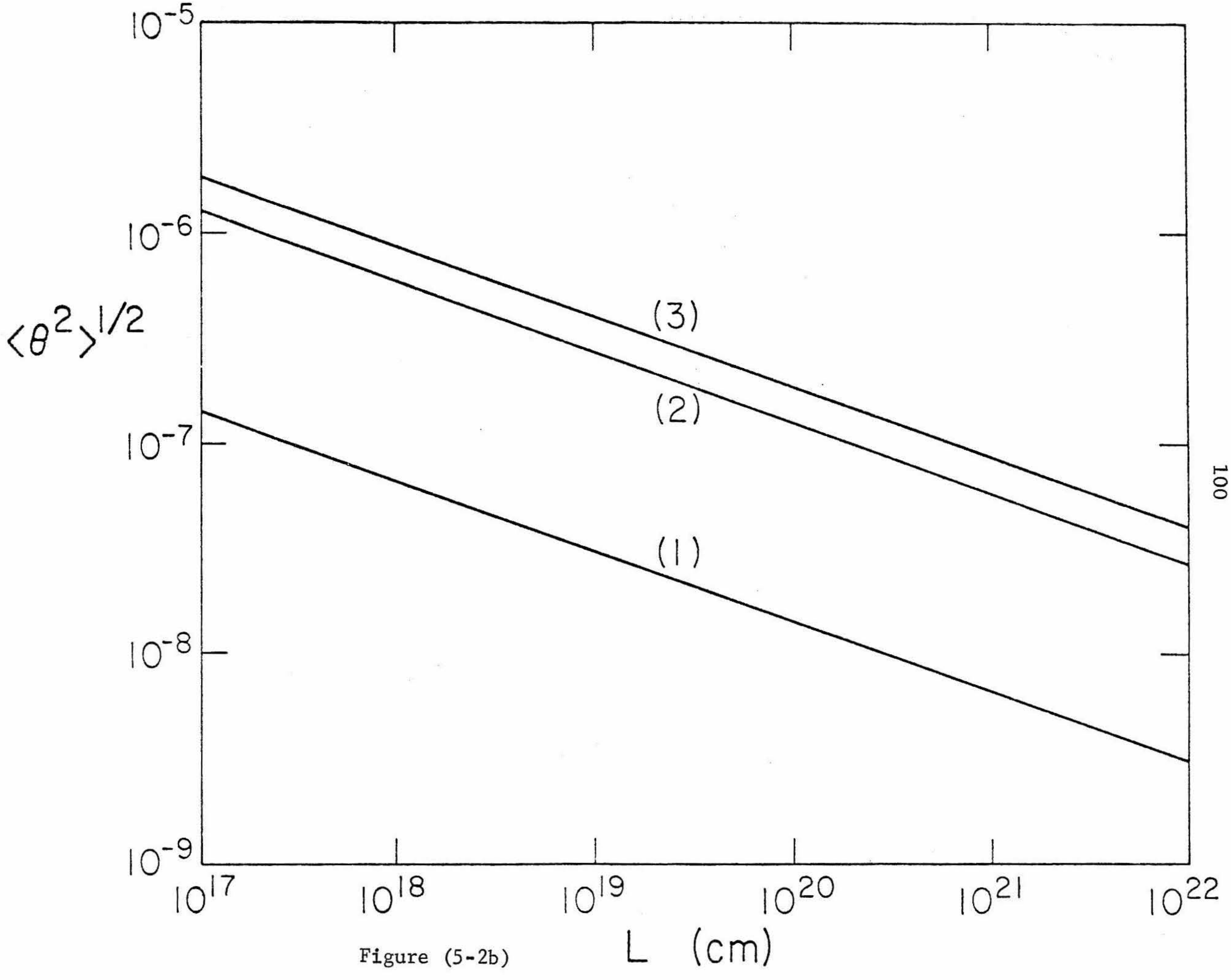


Figure (5-2b)

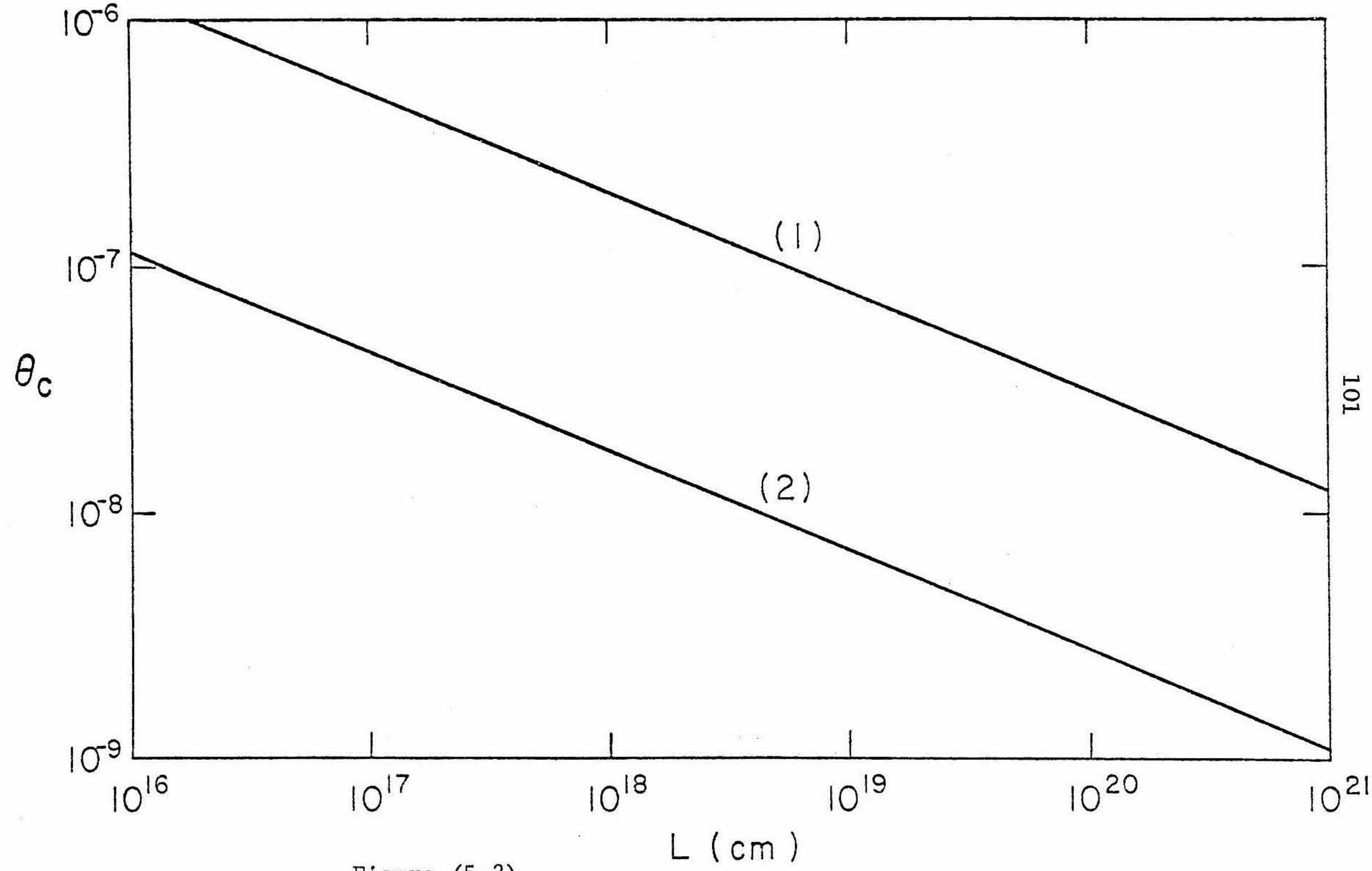


Figure (5-3)

III. Phase Fluctuations Induced by the Medium

An interesting and useful result concerning the phase fluctuations induced by the medium may be derived using the results for $\Gamma_{1,1}$ derived in Eq. (4-52). We express the wave function u $(z,\underset{\sim}{\rho})$ in the form

$$u(z,\underset{\sim}{\rho}) = A(z,\underset{\sim}{\rho}) \exp [iS(z,\underset{\sim}{\rho})] \qquad (5\text{-}18)$$

(c.f. Eq. (2-39)) where the (non-negative) amplitude A and the phase S are real functions of z and $\underset{\sim}{\rho}$.

If the medium is thick enough that the phase fluctuation induced is large compared with unity (strong scattering) then it is expected that the correlation between the amplitude and phase is negligible. Then we may express $\Gamma_{1,1}$ $(z,\underset{\sim}{\rho})$ in the form

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}) = \langle A(z,0)\, A(z,\underset{\sim}{\rho})\rangle \left\langle \exp \left\{i \left[S(z,\underset{\sim}{0}) - S(z,\underset{\sim}{\rho})\right]\right\} \right\rangle . \qquad (5\text{-}19)$$

Since $\Gamma_{1,1}$ $(z,\underset{\sim}{0}) = 1$, we have also

$$\langle A(z,0)\, A(z,0)\rangle = \langle A(z,\underset{\sim}{\rho})\, A(z,\underset{\sim}{\rho})\rangle = 1. \qquad (5\text{-}20)$$

It will be shown in Section V that in strong scintillations the fluctuation in intensity is approximately equal to the mean, $(m_z^{\ 2} \sim 1)$, so that

$$\langle A^4(z,0)\rangle = 1 + m_z^{\ 2} \simeq 2 \quad . \qquad (5\text{-}21)$$

Further, the phase S is the result of a large number of independent, random increments, so that by the central limit theorem S $(z,\underset{\sim}{\rho})$ has a Gaussian probability distribution. Hence if the fluctuation of S about its mean $\langle S\rangle$ is denoted by δS, we have

$$\langle \exp(iS) \rangle = \exp(i\langle S \rangle) \exp\left(-\frac{1}{2}\langle \delta S^2 \rangle\right). \tag{5-22}$$

Now define the correlation function of the phase function S as

$$C_S(z,\underset{\sim}{\rho}) = \langle \delta S(z,\underset{\sim}{\rho}_1)\ \delta S(z,\underset{\sim}{\rho}_1 + \underset{\sim}{\rho}) \rangle . \tag{5-23}$$

Eqs. (5-19), (5-22) and (5-23) yield immediately

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}) = \langle A(z,0)\ A(z,\underset{\sim}{\rho}) \rangle \exp\left[C_S(z,\underset{\sim}{\rho}) - C_S(z,0)\right] . \tag{5-24}$$

From our solution for $\Gamma_{1,1}$ in Eq. (4-52), we then have

$$C_S(z,0) - C_S(z,\underset{\sim}{\rho}) = \frac{z}{2k^2}\left[A_\beta(0) - A_\beta(\underset{\sim}{\rho})\right] + \ln\left[\langle A(z,0)A(z,\underset{\sim}{\rho}) \rangle\right] \tag{5-25}$$

or

$$C_S(z,\underset{\sim}{\rho}) = \frac{z}{2k^2} A_\beta(\underset{\sim}{\rho}) - \ln\left[\frac{\langle A(z,0)A(z,\underset{\sim}{\rho}) \rangle}{\langle A(z,0)A(z,\infty) \rangle}\right] , \tag{5-26}$$

where we have made use of the fact that $C_S(z,\infty) = 0$. Eq. (5-26) is quite similar to earlier results obtained by Bramley (1954) and Ratcliffe (1956) who used a thin screen approximation to obtain the phase fluctuations. (c.f. Eqs. (2-22), (3-3).) Little and Matheson (1973) used a similar expression. Indeed, if the second term on the right in Eq. (5-26) were zero, we would have precisely the previous result for the phase.

In fact, this term is <u>not</u> in general zero. However, some general consideration suggests that it may be quite small for strong scintillations. This may be seen as follows. We note from Eq. (5-20) and the fact that A is non-negative, that

$$0 \le \langle A(z,0)\ A(z,\underset{\sim}{\rho}) \rangle \le 1. \tag{5-27}$$

Now for ($\underset{\sim}{\rho}$) small, we have seen in Eq. (5-20) that ln $[\langle A(z,0)A(z,\underset{\sim}{\rho})\rangle]$ approaches zero. For ($\underset{\sim}{\rho}$) large, this term is

$$\ln[\langle A(z,0)\ A(z,\underset{\sim}{\rho})\rangle] \simeq 2\ \ln\ \langle A(z,0)\rangle \quad . \tag{5-28}$$

But the probability distribution of A, P(A), must satisfy the constraints

$$\int_0^\infty P(A)dA = 1 \tag{5-29a}$$

$$\int_0^\infty A^2 P(A)dA = 1 \tag{5-29b}$$

$$\int_0^\infty A^4 P(A)dA = 1 + m_z^{\ 2} \simeq 2 \quad . \tag{5-29c}$$

It is possible to demonstrate that for any P(A) such that Eq. (5-29) is satisfied, $\langle A(z,0)\rangle \geq (1+m_z^{\ 2})^{1/2} \simeq .707$ for $m_z^{\ 2} \simeq 1$ and that probably $\langle A(z,0)\rangle$ is even closer to unity. A proof of this lower bound on $\langle A\rangle$ is given in Appendix A. For example, if P(A) were log normal $\langle A\rangle \simeq .915$ and if A had a Rayleigh distribution, $\langle A\rangle \simeq .89$. Hence,

$$C_S(z,0) = \frac{z}{2k^2} A_\beta(0) - d \tag{5-30}$$

where $0 \leq d \leq .69$. Since in most cases of interest the quantity $(z/2k^2)A_\beta(0)$ is very large, Eq. (5-30) neglecting d is in fact a very good approximation for $C_S(z,0)$. For the same reason, we expect that $C_S(z,\underset{\sim}{\rho})$ is also approximately given by $(z/2k^2)A_\beta(\underset{\sim}{\rho})$, although we have been unable to find a rigorous proof.

It is clear from the above that identifying $(z/2k^2)A_\beta(\underset{\sim}{\rho})$ with the phase correlation function is not always correct, but that within the Markov approximation Eq. (4-52) for $\Gamma_{1,1}$ is exact. It appears to be an accident that setting $C_S(z,\underset{\sim}{\rho}) = (z/2k^2)A_\beta(\underset{\sim}{\rho})$, and then neglecting the fluctuations in A entirely, leads to the correct expression for $\Gamma_{1,1}(z,\underset{\sim}{\rho})$.

IV. Pulse Broadening

(§1). Introduction

It has been observed that the pulses received from pulsars are temporally broadened at low frequencies. (Staelin & Sutton 1970; Rankin et al. 1970; Counselman & Rankin 1971; Ables et al. 1970; Lang 1971; Sutton 1971.) The theoretical works about pulse broadening have been based on "thin screen" model (Scheuer 1968; Cronyn 1970; Lovelace 1970) and on ray optics (Williamson 1972; Mathews & Jokipii 1972). The thin screen approximation is not necessarily realistic in interstellar scintillation since the fluctuating electron density may not be confined to a small region between source and observers, and the ray optics approximation is not justified in strong scintillation. Furthermore the calculations of the pulse broadening within the thin screen model by Scheuer (1968), Cronyn (1970) and Lovelace (1970) are not rigorous. The estimations by Scheuer (1968) and Cronyn (1970) are based on ray optics. Lovelace (1970) calculates the pulse broadening function using a heuristic physical model in which the propagation of rays is also considered.

In this section, we will consider the broadening of pulses propagating in a <u>thick</u> medium and through a <u>thin</u> phase-changing screen. First we will calculate the pulse broadening for the waves propagating in a <u>thick</u> medium in which the multiple-scattering effect is important. Then we will derive the pulse broadening function <u>rigorously</u> from the wave equation for the <u>thin</u> screen case.

(§2). Some General Consideration

Suppose the source emits a pulse in the initial plane z = 0. Let $h(z=0,\underset{\sim}{\rho},t)$ be the wave function in the plane z = 0. As the wave propagates through the turbulent medium, the wave function becomes $h(z,\underset{\sim}{\rho},t)$ in the observing plane z = z. We write $h(z=0,\underset{\sim}{\rho},t)$ and $h(z,\underset{\sim}{\rho},t)$ in terms of their Fourier components in time, $\Phi_\omega(z=0,\underset{\sim}{\rho})$ and $\Phi_\omega(z,\underset{\sim}{\rho})$ respectively. We have then

$$h(z=0,\underset{\sim}{\rho},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\omega(z=0,\underset{\sim}{\rho})\, e^{-i\omega t}\, d\omega \tag{5-31}$$

and

$$h(z,\underset{\sim}{\rho},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\omega(z,\underset{\sim}{\rho})\, e^{-i\omega t}\, d\omega \quad . \tag{5-32}$$

We note that each Fourier component $\Phi_\omega(z,\underset{\sim}{\rho})$ propagates independently and is described by Eq. (1-17) or (2-1).

The quantity $h(z,\underset{\sim}{\rho},t)$ will be observed only if the receiver has an infinite bandwidth. In practice, the receiver has a finite bandwidth and the "observed" wave function is given by

$$h_B(z,\underset{\sim}{\rho},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\omega(z,\underset{\sim}{\rho})\, f_B(\omega)\, e^{-i\omega t}\, d\omega \tag{5-33}$$

where $f_B(\omega)$ is the bandpass function of the receiver. In what follows we assume that $f_B(\omega)$ can be characterized by a mean frequency ω_o and a bandwidth 2Δ, and that it can be written in the form

$$f_B(\omega) = \exp\left\{-\frac{(\omega-\omega_o)^2}{2\Delta^2}\right\} \quad . \tag{5-34}$$

We suppose that the initial wave function is a delta function,[1]

$$h(z=0,\underset{\sim}{\rho},t) = \delta(t) \quad . \tag{5-35}$$

Then we have $\Phi_\omega(z=0,\underset{\sim}{\rho}) = 1$. We note that the argument below can be easily generalized to cases with any other pulse shape.

Following Eq. (4-6), we write

$$\Phi_\omega(z,\underset{\sim}{\rho}) = u(k,z,\underset{\sim}{\rho})\ e^{ikz} \tag{5-36}$$

where $u(k,z,\underset{\sim}{\rho})$ is the fluctuating part of $\Phi_\omega(z,\underset{\sim}{\rho})$.

Suppose first that the medium is free space, then $u(k,z,\underset{\sim}{\rho}) = 1$ and $k = \omega/c$. We have

$$h(z,\underset{\sim}{\rho},t) = \delta(\frac{z}{c} - t) \tag{5-37a}$$

and

$$h_B(z,\underset{\sim}{\rho},t) = \frac{\Delta}{2\pi} \exp\left\{ - \frac{(t-\frac{z}{c})^2 \Delta^2}{2} \right\} \tag{5-37b}$$

as expected. The finite width $(\frac{1}{\Delta})$ of h_B is just an example of the uncertainty principle.

Now consider the case where the medium through which the wave propagates is a plasma with a uniform electron density N_e. The dispersion relation is

$$k(\omega) = \frac{\omega}{c} (1-\frac{\omega_p^2}{\omega^2})^{1/2} \tag{5-38}$$

where ω_p is given by Eq. (4-4). If the bandwidth Δ is narrow, we may write

$$k(\omega) = k(\omega_o) + (\omega-\omega_o)(\frac{dk}{d\omega})_{\omega_o} + \frac{1}{2} (\omega-\omega_o)^2 (\frac{d^2k}{d\omega^2})_{\omega_o} + \cdots \quad . \tag{5-39}$$

---

[1] In fact it is the intensity that has the delta function. But the difference is just a matter of normalization.

Assuming that the third and high-order terms in Eq. (5-39) are small and can be neglected, we find from Eq. (5-33) that for $h(z=0,t)=\delta(t)$,

$$I_B(z,\underset{\sim}{\rho},t) = |\, h_B(z,\underset{\sim}{\rho},t)\,|^2$$

$$= \frac{\Delta}{2\pi}(1+R^2\Delta^2)^{-\frac{1}{2}} \cdot \exp\left\{ - \frac{(t-\frac{z}{v_g})^2\Delta^2}{(1+R^2\Delta^4)} \right\} \qquad (5\text{-}40)$$

where $v_g = \frac{1}{(\frac{dk}{d\omega})_{\omega_o}}$ is the group velocity and $R = z\left|(\frac{d^2k}{d\omega^2})_{\omega_o}\right|$. It is readily demonstrated that the neglect of the third order term in Eq. (5-40) for the expansion of exp(ikz) requires

$$\left|\frac{1}{6}\Delta^3 z(\frac{d^3k}{d\omega^3})_{\omega_o}\right| \simeq \frac{z\Delta^3\omega_p^2}{2\omega_o^4 c} \ll 1 \qquad (5\text{-}41)$$

which is generally well satisfied in astrophysical problems.

With the dispersion given by Eq. (5-38) and the fact that $\omega_p^2 \ll \omega^2$, we have

$$v_g = 1/(\frac{dk}{d\omega})_{\omega_o} = c\sqrt{1-\frac{\omega_p^2}{\omega_o^2}} \simeq c(1-\frac{\omega_p^2}{2\omega_o^2}) \qquad (5\text{-}42a)$$

and

$$R \simeq \frac{z\omega_p^2}{c\omega_o^3} \; . \qquad (5\text{-}42b)$$

From Eq. (5-41), we find that

(a) The reception time $(\frac{z}{v_g})$ of pulse from pulsar depends on the observing frequency ω_o. If t_1 and t_2 denote the reception times of a pulse at frequencies $(\frac{\omega_1}{2\pi})$ and $(\frac{\omega_2}{2\pi})$ respectively, then

$$t_1 - t_2 = \left(\frac{z\omega_p^2}{2c}\right)\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) \quad . \tag{5-43}$$

This effect has been observed (Counselman & Rankin, 1971).

(b) Even in the absence of scattering, a pulse is broadened upon passing through a dispersive medium such as a plasma. The characteristic time for the broadened pulse is $t_1 = \sqrt{1+R^2\Delta^4}/\Delta$. In cases of interest in the interplanetary and interstellar gas it appears certain that the broadening indicated in Eq. (5-40) is not large enough to be interesting.

We next go on to the cases of present interest in which the electron density (and hence the index of refraction) varies randomly. In this case $\epsilon_k(z,\underset{\sim}{\rho})$ also varies randomly and it is most useful to consider the average of the various quantities over an ensemble of propagation volumes. Let $\langle\ \rangle$ denote an ensemble average. We have from Eqs. (5-33) and (5-36),

$$\begin{aligned}\langle I_B(z,\underset{\sim}{\rho},t)\rangle &= \langle h_B(z,\underset{\sim}{\rho},t)\, h_B^*(z,\underset{\sim}{\rho},t)\rangle \\ &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \langle u(k,z,\underset{\sim}{\rho})u^*(k',z,\underset{\sim}{\rho})\rangle\, f_B(\omega) f_B(\omega') \\ &\quad \times e^{\,i[kz-\omega t] - i[k'z-\omega' t]}\, d\omega\, d\omega' \end{aligned} \tag{5-44}$$

where

$$k = k(\omega) \text{ and } k' = k(\omega').$$

Now we have to calculate the frequency correlation function $\langle u(k,z,\underset{\sim}{\rho})\ u^*(k'.z,\underset{\sim}{\rho})\rangle$. This correlation depends on both ω (or ω') and the differences $(\omega-\omega')$ in general. But if the bandwidth of the receiver is narrow, then the dependence of this correlation function on ω within the bandwidth can be neglected and one has immediately

$$\langle I_B(z,\underset{\sim}{\rho},t)\rangle = P_1(z,t) \bigstar P_2(z,\underset{\sim}{\rho},t) \tag{5-45}$$

where $\bigstar$ represents the convolution of P_1 and P_2 with respect to t, and where

$$P_1(z,t) = \frac{\Delta}{2\pi\sqrt{1+R^2\Delta^4}} \exp\left\{-\frac{(t-\frac{z}{v_g})^2\Delta^2}{(1+R^2\Delta^2)}\right\} \tag{5-46}$$

and

$$P_2(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds \langle u(k(\omega),z,\underset{\sim}{\rho})\ u^*(k(\omega+s),z,\underset{\sim}{\rho})\rangle\ e^{-is(t-\frac{z}{v_g})} . \tag{5-47}$$

Note that $P_1(z,t)$ is identical to I_B in Eq. (5-40) and represents the pulse profile due to dispersion. P_2 is then the pulse profile due to the random index of refraction. Eq. (5-47) can be written in terms of k

$$P_2(z,t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} dk_1 \langle u(k,z,\underset{\sim}{\rho})u^*(k+k_1,z,\underset{\sim}{\rho})\rangle\ e^{-ick_1(t-\frac{z}{v_g})}\, dk_1 \tag{5-48}$$

where $k = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} \approx \frac{\omega}{c}$ is used in the transformation from ω to k. In order to proceed further, we must determine $\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2)$ $\equiv \langle u(k_1,z,\underset{\sim}{\rho}_1)u^*(k_2.z,\underset{\sim}{\rho}_2)\rangle$, which satisfies the differential Eq. (4-39).

<u>(§3).</u> Calculation of $\langle u(k_1,z,\underset{\sim}{\rho})u^*(k_2,z,\underset{\sim}{\rho})\rangle$

From Eq. (4-39), we have for $\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2)$

$$\frac{\partial\Gamma_{1,1}}{\partial z} = \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right)\Gamma_{1,1} - \frac{1}{4}\left[\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)A_\beta(0) - \frac{2A_\beta(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)}{k_1k_2}\right]\Gamma_{1,1} . \tag{5-49}$$

For a statistically homogeneous medium, $\Gamma_{1,1}$ depends on $\rho = |\underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_2|$ instead of $\underset{\sim}{\rho}_1$ and $\underset{\sim}{\rho}_2$, and one has $\nabla_1^2 = \nabla_2^2 = \nabla_{\underset{\sim}{\rho}}^2$. Eq. (5-49) becomes

$$\frac{\partial \Gamma_{1,1}(z,\underset{\sim}{\rho},k_1,k_2)}{\partial z} = \frac{i}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\nabla_{\underset{\sim}{\rho}}^2 \Gamma_{1,1} - \frac{1}{4}\left[\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right)A_\beta(0) - \frac{2A_\beta(\underset{\sim}{\rho})}{k_1 k_2}\right]\Gamma_{1,1}. \tag{5-49'}$$

where $\Gamma_{1,1}(z,\underset{\sim}{\rho},k_1,k_2) = \Gamma_{1,1}(z,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2)$ and $\underset{\sim}{\rho} = \underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_2$.

Define
$$k_1 = k + \frac{\Delta k}{2} \tag{5-50a}$$
and
$$k_2 = k - \frac{\Delta k}{2} \quad . \tag{5-50b}$$

Also write

$$\Gamma(z,\underset{\sim}{\rho},\Delta k) \equiv \Gamma_k(z,\underset{\sim}{\rho},\Delta k) \equiv \Gamma_{1,1}(z,\underset{\sim}{\rho},k_1,k_2) \quad . \tag{5-51}$$

Expanding k_1 and k_2 in Eq. (5-49') in terms of k and Δk, we have

$$\frac{\partial}{\partial z}\Gamma(z,\underset{\sim}{\rho},\Delta k) + \frac{i}{2}\frac{\Delta k}{k^2}\Delta_{\underset{\sim}{\rho}}^2\Gamma + \frac{1}{2k^4}\left(k^2 + \frac{\Delta k^2}{4}\right)\left[A_\beta(0) - A_\beta(0)\right]\Gamma$$
$$+ \frac{1}{4}\frac{\Delta k^2}{k^4}A_\beta(0)\Gamma + 0\,(\Delta k^3) = 0 \tag{5-52}$$

We assume Δk is small and $|\Delta k| \ll k$ and neglect terms of order Δk^3. In addition we put $(k^2 + \frac{\Delta k^2}{4}) \simeq k^2$ in the third term of Eq. (5-52) since $|\frac{\Delta k}{k}| \ll 1$ as assumed. We must keep the fourth term which is proportional to Δk^2, because at $\underset{\sim}{\rho} = 0$ the third term vanishes. Note that if we neglect the diffraction term, $(\frac{i\Delta k}{2k^2})\ \Delta_{\rho}^{\ 2}\ \Gamma$, one obtains

$$\Gamma_R(z,\rho=0,\Delta k) = \exp\left\{-\frac{\Delta k^2}{4k^4} A_\beta(0)z\right\} \tag{5-53}$$

which can be termed the "pure refraction" effect and gives the effect of the differing transit times due to the varying index of refraction. In this case, Eq. (5-53) gives the spread due to this effect as

$$P_R(z,t) = \frac{2ck^2}{(2\pi A_\beta(0))^{1/2}} \exp\left\{-\frac{2(t-\frac{z}{v_g})^2 k^4 c^2}{A_\beta(0)z}\right\}, \tag{5-54}$$

which is a symmetric Gaussian with width

$$t_R = \frac{1}{k^2 c}\left(\frac{A_\beta(0)z}{2}\right)^{1/2} . \tag{5-55}$$

The effect of diffraction can be found by defining a new $\Gamma_D(z,\rho,\Delta k)$ by

$$\Gamma(z,\rho,\Delta k) = \Gamma_D(z,\rho,\Delta k)\ \Gamma_R(z,\rho=0,\Delta k) . \tag{5-56}$$

The equation for Γ_D is then obtained by substituting Eq. (5-56) into Eq. (5-54). One obtains

$$\frac{\partial\Gamma_D}{\partial z} + \frac{i\Delta k}{2k^2}\nabla_{\rho}^{\ 2}\Gamma_D + \left(\frac{1}{2k^2}\right)[A_\beta(0) - A_\beta(\rho)]\ \Gamma_D = 0 \tag{5-57}$$

with its associated

$$P_D(z,t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} \Gamma_D(z,\Delta k,\rho=0)\ e^{-i\Delta kc(t-\frac{z}{v_g})}\ d(\Delta k) . \tag{5-58}$$

$P_D(z,t)$ gives the pulse profile due to the pure "diffraction" term. We have, finally, the convolution

$$P_2(z,t) = P_R(z,t) \bigstar P_D(z,t) \quad . \tag{5-59}$$

The equation for Γ_D cannot be solved analytically, and we defer its discussion until later.

We note that from Eqs. (5-45) and (5-59), the time dependence of $\langle I_B \rangle$ is given by the convolution of three functions, each one of which is related to a specific physical effect. This simplifies the discussion considerably, and if (as is usually the case) one effect gives a much larger spread than the others, the temporal profile is dominated by the P function for that effect. Thus, in most cases of interest, the convolution need never be carried out. This is particularly valuable since the functional form for Γ_D is not simple and carrying out the full convolution would be difficult.

<u>(§4)</u>. Discussion of $\Gamma_D(z,\underset{\sim}{\rho},t)$

We will find that the dominant effect on the observed temporal pulse smearing is given by the pure diffractive effect, represented by Γ_D and P_D. Unfortunately, as has already been noted, the equation for Γ_D cannot be solved analytically. It is possible, however, to develop scaling laws which permit a discussion of how the pulse smearing scales with frequency, etc. Let the transverse characteristic scale for Γ_D be ρ_c. The development depends critically on the behavior of $A_\beta(\rho)$ for $\rho < \rho_c$, and this is a strong function of the power spectrum of the refractive-index irregularities. In all cases of interest we may write for $\rho < \rho_c$,

$$D_\beta(\underset{\sim}{\rho}) = A_\beta(0) - A_\beta(\underset{\sim}{\rho}) \approx \beta_o \rho^\nu \tag{5-60}$$

where $\nu \leq 2$. If the spectrum at large wavenumber falls off more rapidly than q^{-4}, (this includes Gaussian spectrum) then the value of ν is the same and equals to 2 in all cases. However, if the spectrum is less steep than q^{-4}, (and this includes the Kolmogorov spectrum $q^{-11/3}$) then ν is less than two. If the spectrum is given as $q^{-\alpha}$ with $2 < \alpha < 4$, we have

$$D_\beta(\underset{\sim}{\rho}) = \beta_o \ \rho^{(\alpha-2)} \quad . \tag{5-61}$$

Thus if for $\rho < \rho_c$, $D_\beta(\rho)$ is given by Eq. (5-60), we may write Eq. (5-57) as

$$\frac{\partial \Gamma_D}{\partial z} + \frac{i\Delta k}{2k^2} \Delta_{\underset{\sim}{\rho}}^{\ 2} \Gamma_D + \frac{\beta_o \rho^\nu}{2k^2} \Gamma_D = 0 \quad . \tag{5-62}$$

For $\Delta k \neq 0$, we introduce the dimensionless variables η and ξ through

$$z = \frac{2k^2}{\Delta k} \left(\frac{\Delta k}{\beta_o}\right)^{\frac{2}{2+\nu}} \eta \tag{5-63}$$

$$\rho = \left(\frac{\Delta k}{\beta_o}\right)^{\frac{1}{2+\nu}} \xi \tag{5-64}$$

in which case the equation for Γ_D becomes

$$\frac{\partial \Gamma_D}{\partial \eta} + i \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \xi^2}\right] \Gamma_D + \xi^\nu \Gamma_D = 0 \quad . \tag{5-65}$$

This equation must be solved only once for each value of the power law index ν, since the dependence of Γ_D at $\underset{\sim}{\rho} = 0$ on both Δk and z have been collapsed into one variable η. Techniques for numerical solution for Eq. (5-65) are discussed in Appendix (B).

Before discussing the shape of the smearing introduced by the diffraction (given by $P_D(z,t)$) we note that from the computer results in Appendix (B) we see that for $\Delta k \neq 0$, the characteristic scale of Γ_D as a function of η for $\underset{\sim}{\rho} = 0$ is at $\eta \approx 0(1)$. Hence from Eq. (5-65) we see that for a given z, the characteristic value of Δk, Δk_c is given by Eq. (5-63) with $\eta \approx 1$. Thus we have

$$z \simeq \frac{2k^2}{\Delta k_c} \left(\frac{\Delta k_c}{\beta_o}\right)^{\frac{2}{2+\nu}}$$

or

$$\Delta k_c \approx \beta_o^{-\frac{2}{\nu}} \, k^{\frac{2(\nu+2)}{\nu}} \left(\frac{z}{2}\right)^{-\left(\frac{\nu+2}{\nu}\right)} \quad . \tag{5-66}$$

Eq. (5-58) and the nature of Fourier transform then tell us immediately that the characteristic time for smearing of the pulse due to diffraction is

$$t_D \approx \frac{1}{c\Delta k_c} \approx \frac{\beta_o^{\frac{2}{\nu}}}{c} \, k^{-2\left(\frac{\nu+2}{\nu}\right)} \left(\frac{z}{2}\right)^{\left(\frac{\nu+2}{\nu}\right)} \quad . \tag{5-67}$$

Since β_o is independent of the wave number k, we have from Eq. (5-67) that t_D is proportional to $\lambda^{(\frac{4}{\nu} + 2)}$. For Gaussian spectrum, $\nu = 2$ and $t_D \propto \lambda^4$. It can be shown that for both Gaussian and power-law spectra,

$$t_D = z\theta_c^{\,2}/2c \tag{5-67'}$$

Figure Captions

Figure (5-4a). This figure shows the numerical values of the function Γ_D as a function of the frequency difference $\Delta\omega$ for the Gaussian refractive index spectrum in Eq. (1-20). Note that the subscript D for Γ_D is omitted in the figure, and that the abscissa is $(\Delta\omega/\omega_c)^{\frac{1}{2}}$, where $\omega_c = ck_c = c\beta_o^{-1}k^4(\frac{z}{2})^{-2}$. Re($\Gamma$) and Im($\Gamma$) are respectively the real and the imaginary parts of Γ_D.

Figure (5-4b). The pulse profile $P_D(t_*)$ due to the "diffraction effect" is plotted as a function of the normalized time $t_* = (t - \frac{z}{v_g})/t_c$ for the medium with a Gaussian spectrum. The scale for P_D is arbitrary.

Figure (5-5a). As in Figure (5-4a) for a Kolmogorov spectrum in Eq. (1-21). In this figure $\omega_c = ck_c = c\beta_o^{-1.2}k^{4.4}(\frac{z}{2})^{-2.2}$.

Figure (5-5b). As in Figure (5-4b) for the medium with a Kolmogorov spectrum.

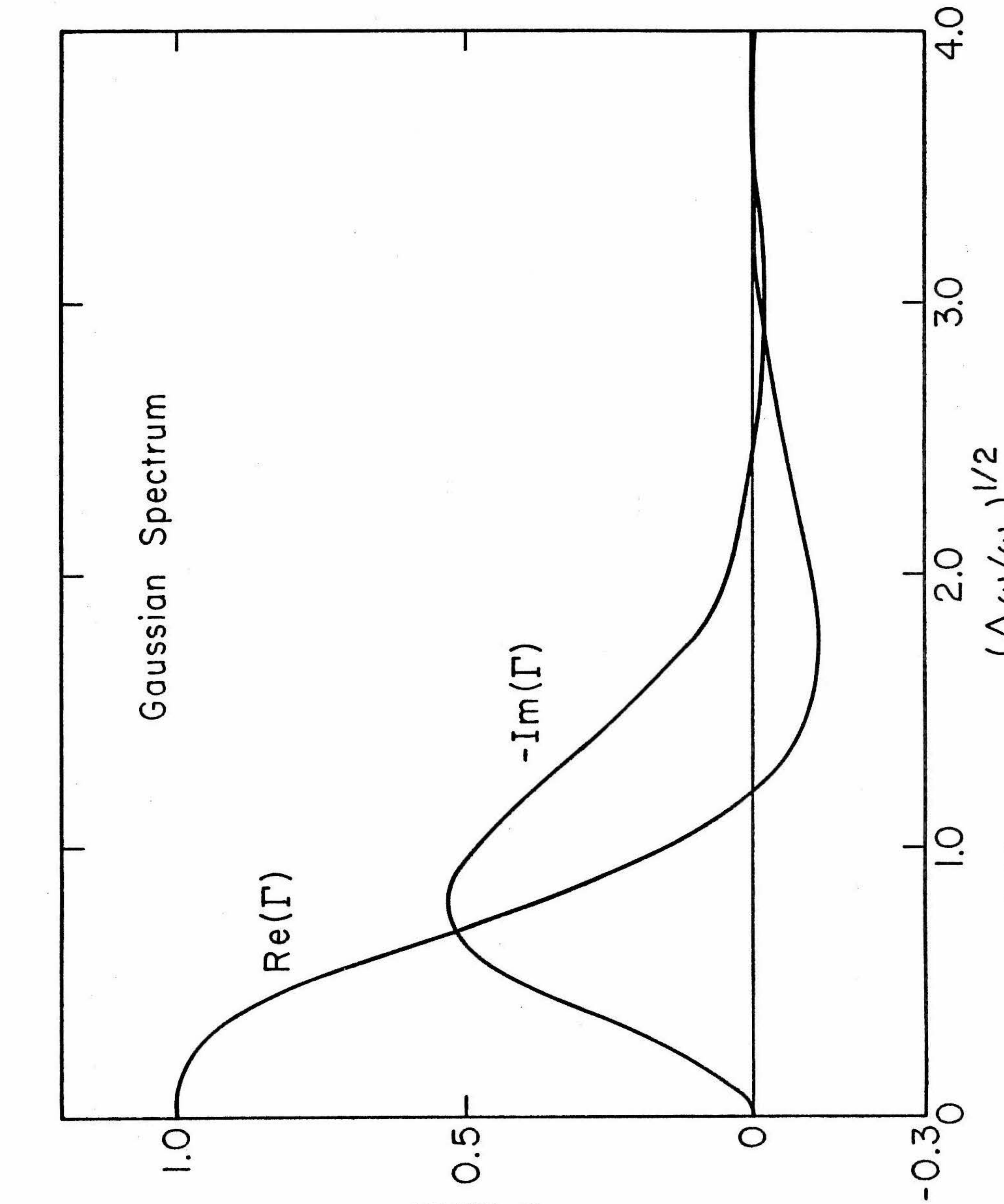

Gaussian Spectrum
Re(Γ)
-Im(Γ)
Γ(Δω)
1.0
0.5
0
-0.3
0
1.0
2.0
3.0
4.0
$(\Delta\omega/\omega_c)^{1/2}$

Figure (5-4a)

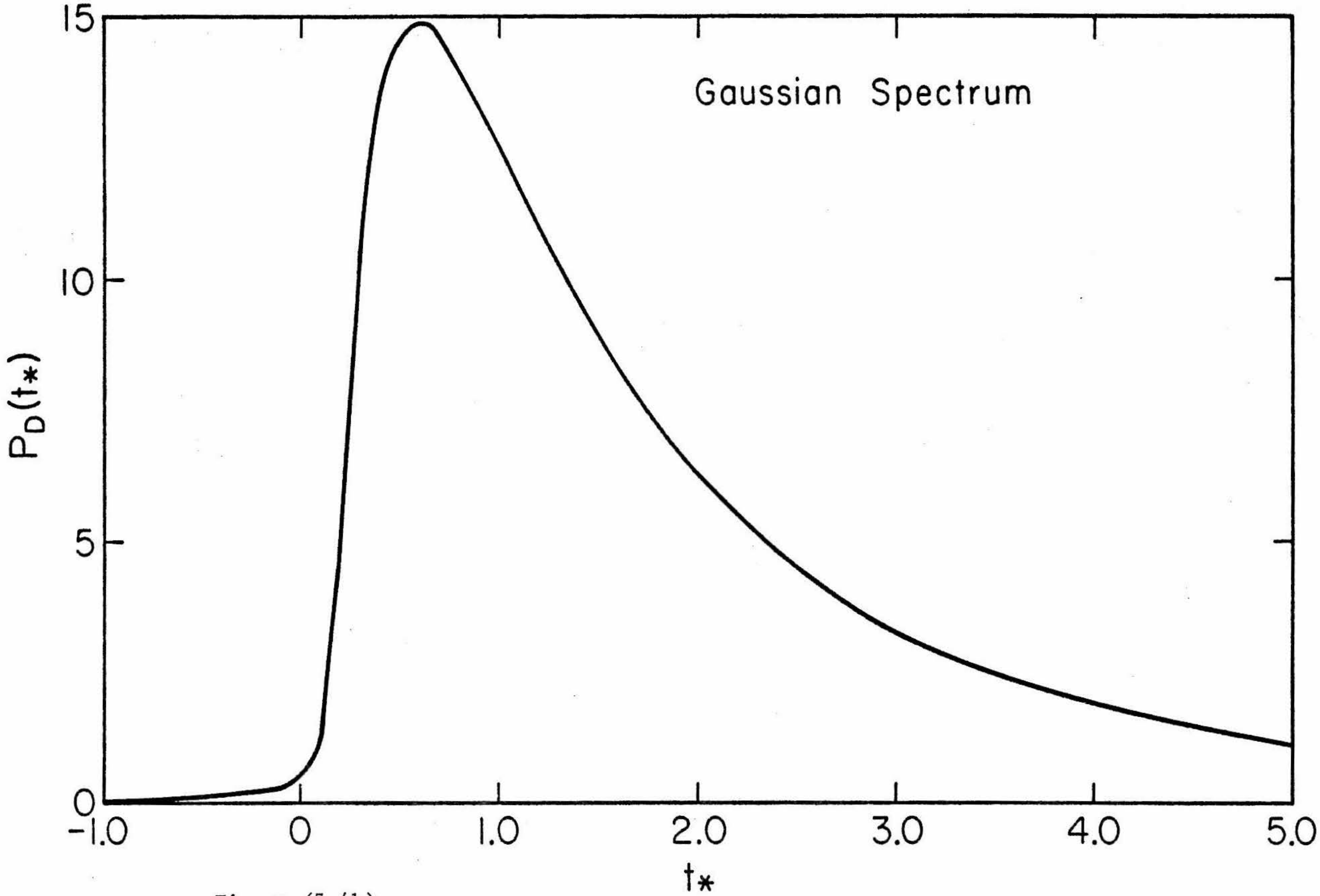
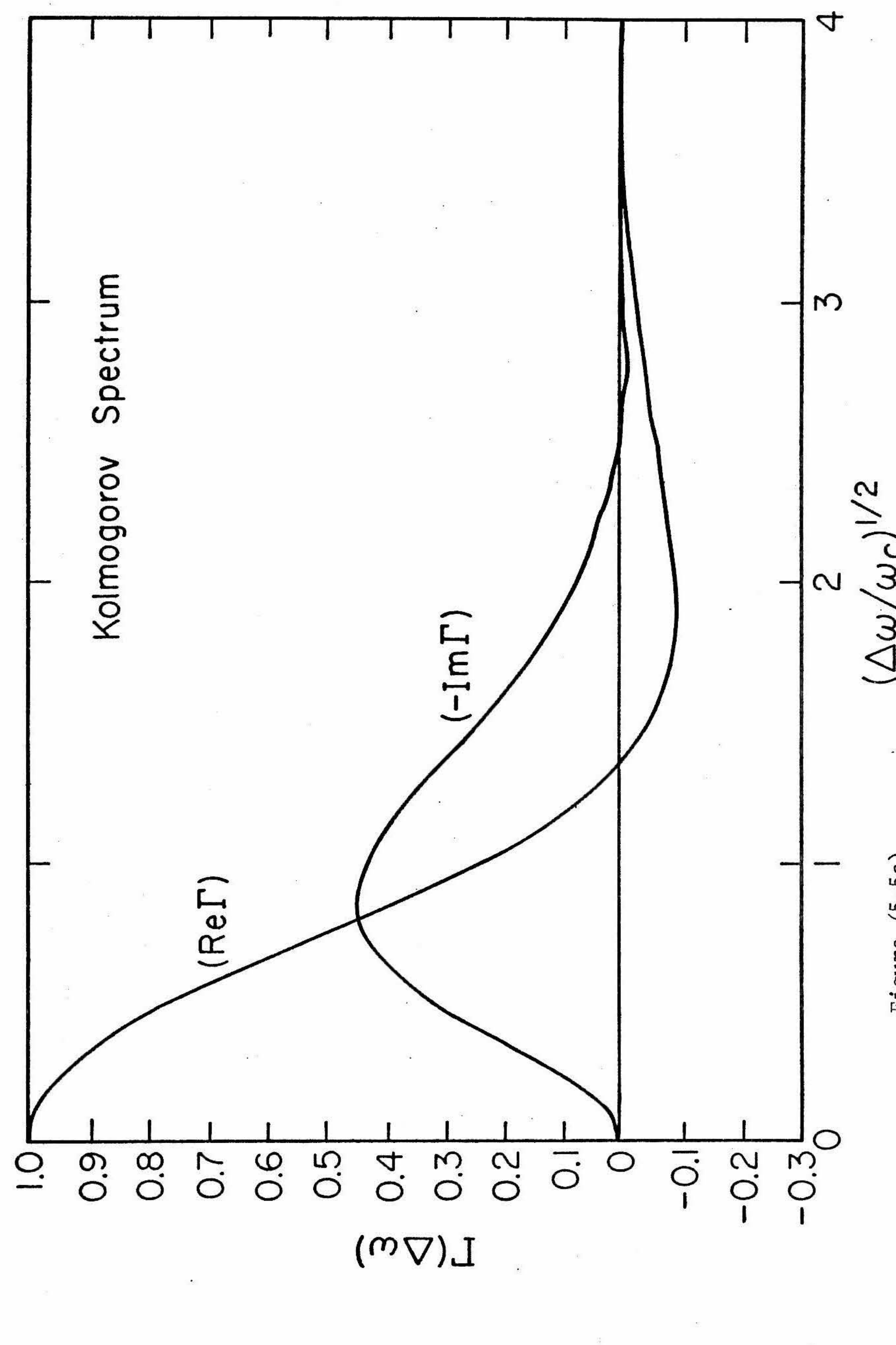


Figure (5-4b)



Kolmogorov Spectrum
(ReΓ)
(-ImΓ)
Γ(Δω)
$(\Delta\omega/\omega_c)^{1/2}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
1
2
3
4

Figure (5-5a)

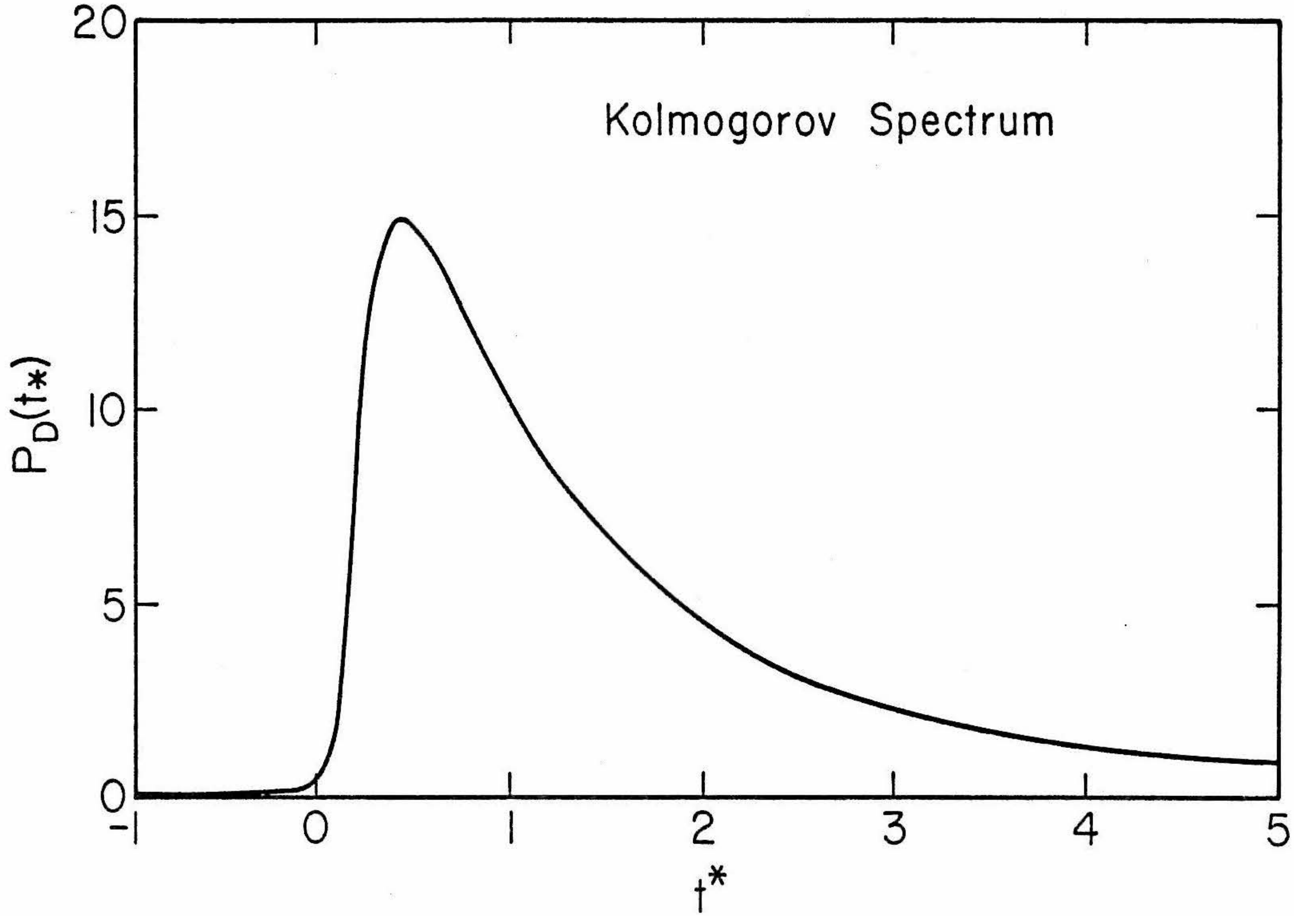


Figure (5-5b)

where θ_c is given by Eq. (5-16).

We thus have three characteristic broadening times given by Eqs. (5-46), (5-55) and (5-67), which represent the various physical effects. In any given observational situation, the longest time will dominate the pulse profile.

Finally, we discuss the effect of these various effects on the pulse shape. The dispersion effect (Eq. (5-46)) and the pure refraction effect (Eq. (5-55)) produce symmetrical Gaussian pulses centered on the mean time of propagation of the pulse. The diffraction effect must be computed numerically and is not a Gaussian function. The results for a Gaussian spectrum (or any spectrum with index $\nu = 2$) are shown in Figure (5-4a,b) and the results for a Kolmogorov power law spectrum ($\alpha = \frac{11}{3}$, $\nu = 5/3$) are given in Figure (5-5a,b). The curves are quite similar and suggest that the pulse shape is not sensitive to the precise form of the power spectrum. The shape is surprisingly similar to that derived by Williamson (1972) on the basis of a statistical, geometrical optics calculation (compare, e.g., with his figure (9)).

§(5). Pulse Broadening in the Thin Screen Approximation

Scheuer (1968), Lovelace (1970) and Cronyn (1970) have calculated the pulse broadening for waves propagating through a thin phase-changing screen. However, their calculations are not rigorous. Scheuer (1968) and Cronyn (1970) estimated the pulse broadening by considering the propagation of rays. Lovelace(1970) derived the pulse broadening function using a heuristic physical model in which the propagation of wave rays is considered.

But ray optics breaks down for strong scintillation ($m_z \approx 1$). In this section we will calculate the pulse broadening function rigorously from the wave equation. We also find a "pure refraction effect" on the pulse broadening, which cannot be found and is neglected in the calculations by the authors mentioned above.

Figure (3-1) is a schematic sketch for the thin screen model. The random medium is concentrated in a thin screen with thickness D at $z = 0$. The observer is located at z. We want to calculate the average pulse profile $\langle I_B(z,\underset{\sim}{\rho},t)\rangle$ observed at z. Again we have from Eq. (5-45) that

$$\langle I_B(z,\underset{\sim}{\rho},t)\rangle = P_1(z,t) \bigstar P_2(z,t) \qquad (5\text{-}68)$$

where $\bigstar$ represents the convolution of P_1 and P_2 with respect to t, and where $P_1(z,t)$, representing the effects of instrument and dispersion, is given by Eq. (5-46) and

$$P_2(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \, \langle u(k(\omega),z,\underset{\sim}{\rho}) u^*(k(\omega+s),z,\underset{\sim}{\rho})\rangle e^{-is(t-\frac{z}{v_g})} \, . \qquad (5\text{-}69)$$

Again let

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2) \equiv \langle u(z,k_1,\underset{\sim}{\rho}_1) u^*(z,k_2,\underset{\sim}{\rho}_2)\rangle \quad . \qquad (5\text{-}70)$$

$\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2)$ is governed by Eq. (5-49). However, within the thin screen with thickness D, the diffraction effect can be neglected and we have from Eq. (5-49)

$$\Gamma_{1,1}(z=0,\underset{\sim}{\rho}_1,k_1,\underset{\sim}{\rho}_2,k_2) = \exp\left[-\frac{1}{4} \left\{ \left(\frac{1}{k_1^{\,2}} + \frac{1}{k_2^{\,2}}\right) A_\beta(0) - \frac{2A_\beta(\underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_2)}{k_1 k_2} \right\} \right] . \qquad (5\text{-}71)$$

For homogeneous case, $\Gamma_{1,1}$ depends on $|\rho_1-\rho_2|$ instead of ρ_1 and ρ_2 and Eq. (5-71) becomes as in Eq. (5-49')

$$\Gamma_{1,1}(z=0,\rho,k_1,k_2) = \exp\left[-\frac{1}{4}\left\{ \left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)A_\beta(0) - \frac{2A_\beta(\rho)}{k_1k_2} \right\} \right]. \quad (5\text{-}72)$$

In the free space between z = 0 and z = z, we have from Eq. (5-49')

$$\frac{\partial}{\partial z}\Gamma_{1,1}(z,\rho,k_1,k_2) = \frac{i}{2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)\nabla_\rho^2\,\Gamma_{1,1} \,. \quad (5\text{-}73)$$

From Eq. (5-73), we have immediately

$$\Gamma_{1,1}(z,\rho=0,k_1,k_2) = \frac{1}{(2\pi)^2}\iint e^{-\frac{iqz}{2}\left(\frac{1}{k_1}-\frac{1}{k_2}\right)}\Gamma_{1,1}(z=0,\rho',k_1,k_2) \times e^{-iq\cdot\rho'}\,d\rho'\,dq \quad (5\text{-}74)$$

where $\Gamma_{1,1}(z=0,\rho,k_1,k_2)$ is given by Eq. (5-72).

Again define $k_1 = k + \frac{\Delta k}{2}$ and $k_2 = k - \frac{\Delta k}{2}$. Then one has

$$\Gamma_{1,1}(z=0,\rho,k_1,k_2) = e^{-\left\{\frac{\Delta k^2}{4k^2}DA_\beta(0)\right\}} \times e^{-\left\{\frac{D}{2k^2}\left(1+\frac{\Delta k^2}{4k^2}\right)[A_\beta(0)-A_\beta(\rho)]\right\}} \quad (5\text{-}75)$$

and

$$\Gamma_{1,1}(z,\rho=0,k_1,k_2) = \frac{1}{(2\pi)^2}\iint e^{+\left[\frac{i\Delta k q^2 z}{2k^2} - iq\cdot\rho'\right]}\Gamma_{1,1}(z=0,\rho',k_1,k_2)\,d\rho'\,dq \quad (5\text{-}76)$$

to the order of $(\Delta k)^2$. For $\frac{\Delta k^2}{k^2} \ll 1$ as assumed the case, we set

$(1 + \frac{\Delta k^2}{4k^2}) = 1$ in Eq. (5-75), while we keep the factor $e^{-\frac{\Delta k^2}{4k^4} DA_\beta(0)}$ since at $\underset{\sim}{\rho} = 0$, $A(0)-A(\rho) = 0$ and the second factor in the right-hand side of Eq. (5-75) equals to 1. Then we have

$$\Gamma_{1,1}(z=0,\underset{\sim}{\rho},k_1,k_2) = e^{-\left\{\frac{\Delta k^2}{4k^2} DA_\beta(0)\right\}} \times e^{-\left\{\frac{D}{2k^2}[A_\beta(0)-A_\beta(\rho)]\right\}} . \quad (5\text{-}77)$$

Combining Eqs. (5-69), (5-76) and (5-77), it is easy to show that

$$P_2(z,t) = P_R(z,t) \bigstar P_D(z,t) \quad (5\text{-}78)$$

where

$$P_R(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} cd(\Delta k)\, e^{-\frac{\Delta k^2}{4k^4} DA_\beta(0)} \times e^{-ic\Delta k(t-\frac{z}{v_g})}$$

$$= \frac{2ck^2}{(2\pi A_\beta(0)D)^{1/2}} \exp\left\{ -\frac{2c^2k^4(t-\frac{z}{v_g})^2}{A_\beta(0)D} \right\} \quad (5\text{-}79)$$

and

$$P_D(z,t) = \frac{1}{(2\pi)^3}\iiint cd(\Delta k)d\underset{\sim}{\rho}'d\underset{\sim}{q}\, e^{-[ic\Delta k(t-\frac{z}{v_g}) - \frac{i\Delta k q^2 z}{2k^2} + i\underset{\sim}{q}\cdot\underset{\sim}{\rho}']}$$

$$\times e^{-\frac{D}{2k^2}[A_\beta(0)-A_\beta(\rho')]} . \quad (5\text{-}80)$$

We note that the second moment with same wavenumber k, $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ given in Eq. (4-50), is invariant in the free space from z = 0 to all z > 0, and one has

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}) = e^{-\frac{D}{2k^2}[A_\beta(0)-A_\beta(\rho)]} \quad . \tag{5-81}$$

By setting $\underset{\sim}{q} = k\underset{\sim}{\theta}$ in Eq. (5-80), we have

$$P_D(z,t) = \iint \delta(t-\frac{z}{v_g} - \frac{z\theta^2}{2c}) \times \Psi\ (\underset{\sim}{\theta})d\underset{\sim}{\theta}^2 \tag{5-82}$$

where $\theta = |\underset{\sim}{\theta}|$ and

$$\Psi(\underset{\sim}{\theta}) = \frac{k^2}{4\pi^2}\iint d\underset{\sim}{\rho}\quad e^{-ik\underset{\sim}{\theta}\cdot\underset{\sim}{\rho}}\ \Gamma_{1,1}(z,\underset{\sim}{\rho}) \tag{5-83}$$

is angular distribution function of the random wave observed at z, (c.f. Eq. (5-1)). Since in our case, $\Psi(\underset{\sim}{\theta}) = \Psi(\theta)$, Eq. (5-82) can be written as

$$P_D(z,t) = \begin{cases} (\frac{2\pi c}{z})\ \Psi\left(\theta = \left\{\dfrac{2c(t-\frac{z}{v_g})}{z}\right\}^{1/2}\right), & \text{for } (t-\frac{z}{v_g}) \geq 0 \\ 0\quad , & \text{for } (t-\frac{z}{v_g}) < 0 \quad . \end{cases} \tag{5-84}$$

Note that from Eq. (5-82), we have

$$\int_{-\infty}^{\infty} P_D(z,t)dt = \iint \Psi(\underset{\sim}{\theta})d\underset{\sim}{\theta}^2 = 1 \quad . \tag{5-85}$$

Thus $P_D(z,t)$ is normalized.

Eq. (5-84) shows that if $\Psi(\theta)$ has a characteristic angle θ_c, then the characteristic time scale t_c of P_D is

$$t_c = \frac{z\theta_c^2}{2c} \qquad . \qquad (5\text{-}86)$$

The fact that $P_D(z,t) = 0$ for $(t-\frac{z}{v_g}) < 0$ indicates that all the scattered rays are delayed. Since $\Psi(\theta)$ usually has a peak at $\theta = 0$, $P_D(z,t)$ will also have a peak at $t' \equiv (t-\frac{z}{v_g}) = 0$. If $\Psi(\theta)$ is a Gaussian function, then $P_D(z,t)$ is an exponential function for $t' \geq 0$. Section II of this chapter shows that in the interstellar scintillation with a Gaussian spectrum, $\Psi(\theta)$ is Gaussian, but with a power-law-spectrum, $\Psi(\theta)$ is not Gaussian for reasonable values of the parameters in the interstellar medium.

In conclusion, we compare the results of pulse broadening for both scattering all the way (thick medium) case and thin screen case. The dispersion and "pure refraction" effects are the same in both cases. For the "diffraction" effect, the pulse shapes in these two cases are different. In the thin screen case, the peak of $P_D(z,t)$ is at $t' = t - \frac{z}{v_g} = 0$, while $P_D(z,t)$ has its peak at $t' \approx t_c > o$ for thick medium case.

V. Spatial Intensity Correlation Function

§(1) Introduction

In order to calculate the intensity correlation function, one has to solve Eq. (4-48) for the 4th moment $\Gamma_{2,2}$. However, no analytic solution for Eq. (4-48) has been given since the derivation of this differential equation. Tatarskii (1971) gives an analytic solution for $\Gamma_{2,2}$ under the "single-scattering approximation", but his results are not satisfactory since his result for correlation scale does not agree with the interstellar scintillation data. Dagkesamanskaga & Shishov (1970) have obtained a numerical solution of Eq. (4-48) for an initially plane wave propagating in a random medium which has a Gaussian correlation function. Brown (1972b) solved numerically this equation for a "two dimensional" medium with Kolmogorov spectrum. Since the propagating distance for numerical calculations is limited due to the accumulation of numerical errors, it is hard to draw qualitative properties of the intensity correlation scale from the numerical calculations mentioned above.

In this section, an analytic asymptotic solution of $\Gamma_{2,2}$ for strong scintillation will be presented and the properties of the spatial intensity-correlation function obtained from $\Gamma_{2,2}$ will also be discussed.

§(2) Analytic Asymptotic Solution for $\Gamma_{2,2}$

From Eq. (4-48), we have for $\Gamma_{2,2}$

$$\frac{\partial\Gamma_{2,2}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_1',\underset{\sim}{\rho}_2')}{\partial z} = \frac{i}{2k}\left[\nabla_1{}^2+\nabla_2{}^2-\nabla_1'{}^2-\nabla_1'{}^2\right]\Gamma_{2,2}-\frac{1}{2k^2}\Gamma_{2,2}\ \times$$

$$[2A_\beta(0)+A_\beta(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)+A_\beta(\underset{\sim}{\rho}_1'-\underset{\sim}{\rho}_2') - A_\beta(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_1')-A_\beta(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2')-A_\beta(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}_1')-A_\beta(\underset{\sim}{\rho}_2-\underset{\sim}{\rho}_2')]$$

(5-87)

where

$$\Gamma_{2,2}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_1',\underset{\sim}{\rho}_2') \equiv \langle u(z,\underset{\sim}{\rho}_1)u^*(z,\underset{\sim}{\rho}_1')u(z,\underset{\sim}{\rho}_2)u^*(z,\underset{\sim}{\rho}_2')\rangle \quad . \qquad (5\text{-}88)$$

Suppose that the random medium is statistically homogeneous in the transverse direction and that the problem has a homogeneous initial condition; then Eq. (5-87) can be reduced to (c.f. Eq. (3-29'))

$$\frac{\partial}{\partial z}\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \frac{i}{k}\nabla_{\underset{\sim}{\rho}_\alpha}\cdot\nabla_{\underset{\sim}{\rho}_\beta}\Gamma_4-\frac{1}{2k^2}[2A_\beta(0)-2A_\beta(\underset{\sim}{\rho}_\alpha)-2A_\beta(\underset{\sim}{\rho}_\beta)+A_\beta(\underset{\sim}{\rho}_\alpha+\underset{\sim}{\rho}_\beta)$$

$$+ A_\beta(\underset{\sim}{\rho}_\alpha-\underset{\sim}{\rho}_\beta)]\ \Gamma_4 \qquad (5\text{-}89)$$

where

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \langle u(z,\underset{\sim}{\rho}_o)u^*(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\alpha)u(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\alpha+\underset{\sim}{\rho}_\beta)u^*(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\beta)\rangle \quad . \qquad (5\text{-}90)$$

We note that for $\underset{\sim}{\rho}_\beta = 0$ (or $\underset{\sim}{\rho}_\alpha = 0$),

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta=0) = \langle I(z,\underset{\sim}{\rho}_o)I(z\ \underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\alpha)\rangle = P_I(z,\underset{\sim}{\rho}_\alpha)+\langle I\rangle^2 \qquad (5\text{-}91)$$

where $P_I(z,\underset{\sim}{\rho}_\alpha)$ is the spatial intensity correlation function. Thus the solution of Eq. (5-89) for Γ_4 gives immediately the intensity correlation function $P_I(z,\underset{\sim}{\rho})$.

We normalize the wave function such that at $z = 0$, $u = 1$. Then we have the following initial condition for Γ_4

$$\text{at } z = 0,\ \Gamma_4(z=0,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = 1 \quad . \tag{5-92}$$

The boundary condition for large $|\underset{\sim}{\rho}_\beta|$ can be obtained by noting that for large $|\underset{\sim}{\rho}_\beta|$, Eq. (5-90) becomes

$$\Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = \langle u(z,\underset{\sim}{\rho}_o)u^*(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\alpha)\rangle\langle u(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\alpha+\underset{\sim}{\rho}_\beta)u^*(z,\underset{\sim}{\rho}_o+\underset{\sim}{\rho}_\beta)\rangle$$

$$= |\Gamma_{1,1}(z,\underset{\sim}{\rho}_\alpha)|^2$$

where $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ is the second moment given by Eq. (5-58). Thus we have

$$\text{at } |\underset{\sim}{\rho}_\beta| \to \infty, \quad \Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = |\Gamma_{1,1}(z,\underset{\sim}{\rho}_\alpha)|^2 \quad . \tag{5-93a}$$

Similarly, we have

$$\text{at } |\underset{\sim}{\rho}_\alpha| \to \infty,\ \Gamma_4(z,\underset{\sim}{\rho}_\alpha,\underset{\sim}{\rho}_\beta) = |\Gamma_{1,1}(z,\underset{\sim}{\rho}_\beta)|^2 \ . \tag{5-93b}$$

From Eq. (4-52), we have

$$\Gamma_{1,1}(z,\underset{\sim}{\rho}) = e^{-\frac{z}{2k^2}[A_\beta(0)-A_\beta(\underset{\sim}{\rho})]} \quad . \tag{5-94}$$

Let the correlation scale of $A_\beta(\underset{\sim}{\rho})$ be L. (c.f. Eqs. (1-20) and (1-21).)

We define

$$\beta_A = \frac{1}{2k^2} A_\beta(0)$$

$$\gamma = \beta_A kL^2$$

$$\eta = \beta_A z \qquad (5\text{-}95)$$

$$\underset{\sim}{\rho}_\alpha = L\underset{\sim}{\alpha}$$

$$\underset{\sim}{\rho}_\beta = L\underset{\sim}{\beta}$$

and $H(\underset{\sim}{\rho}) \equiv A_\beta(\underset{\sim}{\rho})/A_\beta(0)$.

Then Eq. (5-89) can be written as

$$\frac{\partial}{\partial\eta} \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{i}{\gamma} \nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} \Gamma_4 - f(\underset{\sim}{\alpha},\underset{\sim}{\beta})\Gamma_4 \qquad (5\text{-}96)$$

where

$$f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \equiv 2 - 2\hat{H}(\underset{\sim}{\alpha}) - 2\hat{H}(\underset{\sim}{\beta}) + \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta}) + \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta}) \qquad (5\text{-}97)$$

and $\hat{H}(\alpha) \equiv H(\underset{\sim}{\rho}_\alpha)$ and etc.. We call γ the "diffraction parameter" since it determines the importance of the diffraction term $(\frac{i}{\gamma})\nabla_\alpha \cdot \nabla_\beta \Gamma_4$ for unit propagating distance ($\Delta\eta = 1$). For $\gamma \gg 1$, the effect of the diffraction term is small for $\Delta\eta = 1$, while for $\gamma \lesssim 1$, the effect is large. The initial condition is

$$\Gamma_4(\eta=0,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 1 \quad . \qquad (5\text{-}98)$$

The boundary conditions in Eq. (5-93) become

$$\text{at } |\underset{\sim}{\alpha}| \to \infty, \; \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta}) = e^{-2\eta[1-\hat{H}(\underset{\sim}{\beta})]} \qquad (5\text{-}99a)$$

and

$$\text{at } |\underset{\sim}{\beta}| \to \infty,\ \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\alpha}) = e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} \quad . \tag{5-99b}$$

We now consider the properties of the function $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ in Eq. (5-97). We find that $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of the order of 1 everywhere except near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$. We have

$$\text{at } \underset{\sim}{\alpha} = 0 \text{ and /or } \underset{\sim}{\beta} = 0,\ f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0,$$

and

$$\text{at } |\underset{\sim}{\alpha}| \gg 1,\ |\underset{\sim}{\beta}| \gtrsim 1,$$

$$f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \approx 2 + \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta}) \simeq \begin{cases} 2 & , \text{ for } |\underset{\sim}{\alpha}-\underset{\sim}{\beta}| \gtrsim 1 \\ 3 & , \text{ for } |\underset{\sim}{\alpha}-\underset{\sim}{\beta}| \lesssim 1. \end{cases}$$

The numerical calculation of $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, for both Gaussian and Kolmogorov spectra, also shows that $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of order of 1, except near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$. Note that in all cases $f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \geq 0$.

We next discuss the properties of Eq. (5-89) for Γ_4. The term $(-f(\underset{\sim}{\alpha},\underset{\sim}{\beta})\Gamma_4)$ in the right-hand side of Eq. (5-89) tends to make the function Γ_4 decay. Without the diffraction term $(\frac{i}{\gamma}\nabla_{\underset{\sim}{\alpha}}\cdot\nabla_{\underset{\sim}{\beta}}\Gamma_4)$, we have

$$\frac{\partial}{\partial\eta}\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = -f(\underset{\sim}{\alpha},\underset{\sim}{\beta})\Gamma_4 \tag{5-100}$$

or

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_4(\eta=0,\underset{\sim}{\alpha},\underset{\sim}{\beta})\ e^{-\eta f(\underset{\sim}{\alpha},\underset{\sim}{\beta})} \quad . \tag{5-100'}$$

From the properties of $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ discussed above, we see that Γ_4 will decay to about e^{-1} of its initial value when $\Delta\eta = 1$ for all points $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$, except near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$.

The diffraction term $(\frac{i}{\gamma} \nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} \Gamma_4)$ in Eq. (5-89) tends to diffuse the value of $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ among different transverse coordinates $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. Without the decay term $(-f\Gamma_4)$, Eq. (5-89) can be written as

$$\frac{\partial}{\partial\eta} \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{i}{\gamma} \nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \quad . \tag{5-101}$$

The solution of the above equation can be written immediately as

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{\gamma^2}{\eta^2} \iint \Gamma_4(\eta{=}0,\underset{\sim}{\alpha}',\underset{\sim}{\beta}') \, e^{i\frac{\gamma}{\eta}(\underset{\sim}{\alpha}-\underset{\sim}{\alpha}')\cdot(\underset{\sim}{\beta}-\underset{\sim}{\beta}')} \, d\underset{\sim}{\alpha}' d\underset{\sim}{\beta}' \quad . \tag{5-102}$$

Thus at $\eta > 0$, the values of $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ are re-distributed as described by the above equation. For example, if we set $\Gamma_4(\eta{=}0,\underset{\sim}{\alpha},\underset{\sim}{\beta})=\delta(\underset{\sim}{\alpha})\,\delta(\underset{\sim}{\beta})$, then we have

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{\gamma^2}{\eta^2} \, e^{i(\frac{\gamma}{\eta})(\underset{\sim}{\alpha}\cdot\underset{\sim}{\beta})} \quad . \tag{5-102'}$$

However, if $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of the following form

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = f_1(\eta,\underset{\sim}{\alpha}) + f_2(\eta,\underset{\sim}{\beta}) + c_o(\eta) \tag{5-103}$$

where f_1, f_2 are arbitrary functions of $(\eta,\underset{\sim}{\alpha})$ and $(\eta,\underset{\sim}{\beta})$, respectively, and $c_o(\eta)$ is a function of η, then the diffraction term $(\frac{i}{\gamma} \nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} \Gamma_4)$ is zero and $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is not re-distributed as wave propagates. In this case,

$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ keeps the same value for all η if the decay term is neglected. We also note that Eq. (5-103) is the only form of Γ_4 for which the diffraction term is zero.

Physically, Eq. (5-96) for Γ_4 can be considered as the combination of Eq. (5-100) and Eq. (5-101). Consider the following "multiple-thin-screen" model for the wave propagation in the random medium as shown in Figure (5-6). One divides the random medium between the incoming wave and the observer into N thin layers where the i-th layer has thickness Δz_i for i = 1,2,...,N. In the "multiple-thin-screen" model, the total random medium in the i-th layer is considered to be concentrated in the "thin screen" between z_i^- and z_i^+ where $(z_i^+ - z_i^-) \ll \Delta z_i$. Within the "thin-screen" between z_i^- and z_i^+, the diffraction effect can be neglected and Γ_4 is described by Eq. (5-100). Let the normalized quantities η_i^-, η_i^+ and $\Delta\eta_i$ correspond to z_i^-, z_i^+ and Δz_i respectively. We then have

$$\Gamma_4(\eta_i^+,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_4(\eta_i^-,\underset{\sim}{\alpha},\underset{\sim}{\beta})\, e^{-\Delta\eta_i f(\underset{\sim}{\alpha},\underset{\sim}{\beta})} \quad . \tag{5-104}$$

Outside the "thin screen" is a free space without random medium and for η between η_i^+ and η_i^- , $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is described by Eq. (5-101). Thus in the "multiple-thin-screen" model, Γ_4 is alternatively described by Eq. (5-100) and Eq. (5-101), which give respectively the decay effect and the diffraction effect. In the limit that the thickness Δz of each layer is very small, the "multiple-thin-screen" model must correspond to the actual situation where the random medium is evenly distributed over the whole layer, since in the "multiple-thin-screen" model the position of the random medium within each layer is just slightly shifted from that in the actual situation.

Figure (5-6)

In the "multiple-thin-screen" model, the total random medium is divided into N thin layers where the i-th layer has thickness Δz_i for $i = 1, 2, \ldots, N$. The random medium of the i-th layer is concentrated in the "thin screen" between z_i^- and z_i^+ where $(z_i^+ - z_i^-) \ll \Delta z_i$. Between two thin screens is a free space.

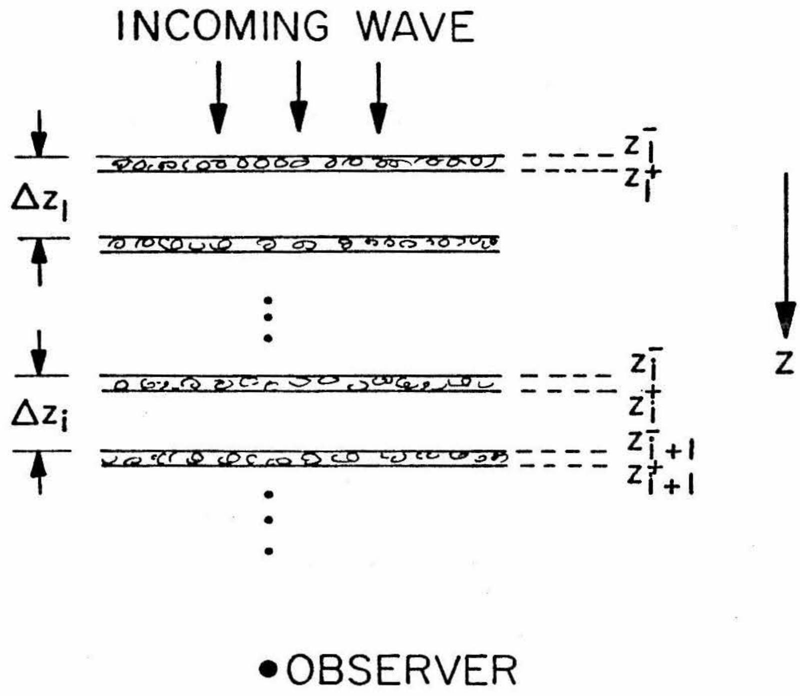


Figure (5-6)

From the above considerations, we find that due to the decay term, $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ will decay to zero when $\eta \gg 1$, for all $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ except near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$. However due to the diffraction effect the values of $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$ will be re-distributed to all other points except if $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of the form given by Eq. (5-103). Once the values of $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ near $\underset{\sim}{\alpha} = 0$ and/or $\underset{\sim}{\beta} = 0$ are distributed to all other points $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ where $f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \neq 0$, $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ will again decay to zero because of the decay term $(-f\Gamma_4)$. Thus we see that for large η, $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ will decay to zero except if Γ_4 is of the form given by Eq. (5-103). Therefore for large η, we can write $\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ as the following form,

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = f_1(\eta,\underset{\sim}{\alpha}) + f_2(\eta,\underset{\sim}{\beta}) + c_o(\eta) \quad . \qquad (5\text{-}103')$$

In order to satisfy the boundary conditions in Eqs. (5-99a) and (5-99b), we must have

$$\text{as } |\underset{\sim}{\alpha}| \to \infty,\ \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta}) = f_1(\eta,\underset{\sim}{\alpha} = \infty) + f_2(\eta,\underset{\sim}{\beta}) + c_o(\eta).$$

Therefore,

$$f_2(\eta,\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta}) - f_1(\eta,\underset{\sim}{\alpha} = \infty) - c_o(\eta)$$

and Γ_4 can be written as

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta}) + f_1(\eta,\underset{\sim}{\alpha}) - f_1(\eta,\underset{\sim}{\alpha} = \infty) \quad .$$

From Eq. (5-99b), we have

$$\text{as } |\underset{\sim}{\beta}| \to \infty,\ \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\alpha}) = \Gamma_{1,1}{}^2(\eta,\infty) + f_1(\eta,\underset{\sim}{\alpha}) - f_1(\eta,\infty) \ ,$$

from which we obtain the following expression

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\alpha}) + \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta}) - \Gamma_{1,1}{}^2(\eta,\infty) \ . \tag{5-105}$$

By Eqs. (5-99a,b), we write the above equation as

$$\Gamma_4(\eta,\alpha,\beta) = e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} + e^{-2\eta[1-\hat{H}(\underset{\sim}{\beta})]} - e^{-2\eta} \ . \tag{5-105'}$$

Eq. (5-105) or (5-105') is the asymptotic form for Γ_4 at large η.

In Appendix (C), we will show that the asymptotic solution for Γ_4 in Eq. (5-105) satisfies the partial differential equation in Eq. (5-96) to terms of order $(\frac{1}{\eta})$, and that the error of the asymptotic solution for Γ_4 is of the order of $(\frac{1}{\eta})$.

Finally, we determine how large η must be in order that Eq. (1-105) is a valid solution for Γ_4. We consider two cases

(i) When $\gamma \lesssim 1$, we see that the diffraction term is effective to diffuse the function Γ_4 as η propagates a unit distance. We expect that Eq. (5-105) is valid for $\eta \gtrsim 1$.

(ii) When $\gamma > 1$, the effect of the diffraction term $(\frac{i}{\gamma} \nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} \Gamma_4)$ is still small as η propagates a unit distance and the redistribution of Γ_4 is not large enough to make Eq. (5-105) a valid solution for η near or just greater than 1. Consider the case that the random medium has a Kolmogorov spectrum, with the correlation scale L and the inner scale ℓ. For $\ell \ll |\underset{\sim}{\rho}_\alpha|$ (and $|\underset{\sim}{\rho}_\beta|) \ll L$, the function $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ can be written as

$$f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \approx 1.86 \times [2|\underset{\sim}{\alpha}|^{5/3} + 2|\underset{\sim}{\beta}|^{5/3} - |\underset{\sim}{\alpha}+\underset{\sim}{\beta}|^{5/3} - |\underset{\sim}{\alpha}-\underset{\sim}{\beta}|^{5/3}] \tag{5-106}$$

Define

$$\begin{aligned} \eta &= a_1 t \\ \underset{\sim}{\alpha} &= b_1 \underset{\sim}{s}_\alpha \\ \underset{\sim}{\beta} &= b_1 \underset{\sim}{s}_\beta \end{aligned} \tag{5-107}$$

where $a_1 \equiv \gamma^{5/11}$ and $b_1 \equiv \gamma^{-3/11}$. Then we can write Eq. (5-96) as

$$\frac{\partial}{\partial t}\Gamma_4(t,\underset{\sim}{s}_\alpha,\underset{\sim}{s}_\beta) = i\,\nabla_{\underset{\sim}{s}_\alpha}\cdot\nabla_{\underset{\sim}{s}_\beta}\Gamma_4 - 1.86[2|\underset{\sim}{s}_\alpha|^{5/3}+2|\underset{\sim}{s}_\beta|^{5/3}-|\underset{\sim}{s}_\alpha+\underset{\sim}{s}_\beta|^{5/3}-|\underset{\sim}{s}_\alpha-\underset{\sim}{s}_\beta|^{5/3}]$$

$$\times\,\Gamma_4 \qquad (5\text{-}108)$$

with $\Gamma_4(t=0,\underset{\sim}{s}_\alpha,\underset{\sim}{s}_\beta)=1$. Note that the γ in Eq. (5-96) no longer appears in Eq. (5-108). We expect that for $t \gtrsim 1$, the diffraction effect will be large enough to make Eq. (5-105) a good approximation for Γ_4.

From Eqs. (5-105) and (5-91), we have for intensity correlation function,

$$P_I(z,\underset{\sim}{\rho}) = \Gamma_{1,1}^{\;2}(z,\underset{\sim}{\rho}) - \bar{u}^4(z) \qquad (5\text{-}109)$$

by noting $\langle I\rangle = 1$ and $\bar{u}^2 = \Gamma_{1,1}(z,\underset{\sim}{\rho}\to\infty)$. For $\eta \gg 1$,

$$P_I(z,\underset{\sim}{\rho}) \approx \Gamma_{1,1}^{\;2}(z,\underset{\sim}{\rho}) \quad . \qquad (5\text{-}110)$$

From the above discussions, we conclude that Eq. (5-109) is a valid solution for the medium with a Kolmogorov spectrum when

$$\text{(i)}\quad \eta = \frac{zA_\beta(0)}{2k^2} \geq 1, \text{ for } \gamma = \frac{A_\beta(0)}{2kq_o^{\;2}} \lesssim 1 \qquad (5\text{-}111a)$$

and

$$\text{(ii)}\quad \eta = \frac{zA_\beta(0)}{2k^2} > \gamma^{5/11} = \left(\frac{A_\beta(0)}{2kq_o^{\;2}}\right)^{5/11}, \text{ for } \gamma = \frac{A_\beta(0)}{2kq_o^{\;2}} \qquad (5\text{-}111b)$$

Similarly, for a Gaussian spectrum in Eq. (1-20), the conditions for Eq. (5-109) to be valid are

$$\text{(i)} \quad \eta = \frac{zA_\beta(0)}{2k^2} \gtrsim 1, \text{ for } \gamma = \frac{A_\beta(0)}{2kq_o^2} \lesssim 1, \qquad \text{(5-112a)}$$

and

$$\text{(ii)} \quad \eta = \frac{zA_\beta(0)}{2k^2} > \gamma^{\frac{2}{3}} = \left(\frac{A_\beta(0)}{2kq_o^2}\right)^{\frac{2}{3}}, \text{ for } \gamma = \frac{A_\beta(0)}{2kq_o^2} > 1. \qquad \text{(5-112b)}$$

When the conditions in Eq. (5-111) for Kolmogorov spectrum, or in Eq. (5-112) for Gaussian spectrum, are satisfied, the scintillation is strong and the scintillation index $m_z^2 = P_I(z,\underset{\sim}{\rho}=0)=1-e^{-2\eta} \simeq 1$.

§(3). Discussion and the Properties of $P_I(z,\underset{\sim}{\rho})$

In the last subsection, we found that when the conditions in Eq. (5-111) for Kolmogorov spectrum, or in Eq. (5-112) for Gaussian spectrum are satisfied, the spatial intensity correlation function $P_I(z,\underset{\sim}{\rho})$ is given by Eq. (5-109) and the scintillation is strong ($m_z^2 \simeq 1$). However we still do not know whether the conditions in Eq. (5-111) (or in Eq. (5-112)) are the necessary conditions for strong scintillations. The method of Smooth Perturbation (MSP) (Tatarskii 1961, 1971) gives the weak scintillation result, which is valid when the mean square logarithmic amplitude $\bar{\chi}^2$ (or m_z^2) is smaller than unity. From the results of Tatarskii (1961, 1971), one finds that $\bar{\chi}^2$ is smaller than 1 when

(i) $\eta \leq 1$, for $\gamma \leq 1$ (5-113a)

and (ii) $\eta \lesssim \gamma^{(5/11)}$, for $\gamma \gtrsim 1$ (5-113b)

for Kolmogorov spectrum, and when

(i) $\eta \leq 1$, for $\gamma \leq 1$ (5-114a)

and (ii) $\eta \leq \gamma^{2/3}$, for $\gamma \geq 1$, (5-114b)

for Gaussian spectrum.

Comparing Eqs. (5-111), (5-112), (5-113) and (5-114), one finds that the conditions in Eq. (5-111) for Kolmogorov spectrum, or in Eq. (5-112) for Gaussian spectrum, are both the <u>necessary</u> and the <u>sufficient</u> conditions for strong scintillation. We conclude that from Eq. (5-109), for <u>strong</u> scintillation, the intensity correlation function <u>is</u>

$$P_I(z,\underset{\sim}{\rho}) = \exp\left[-\frac{z}{k^2}\left(A_\beta(0)-A_\beta(\underset{\sim}{\rho})\right)\right] - \exp\left[-\frac{A_\beta(0)z}{k^2}\right]. \quad (5\text{-}115)$$

The characteristic scale of $P_I(z,\underset{\sim}{\rho})$, $\rho_{c.s.}$, can be obtained by noting that for Kolmogorov spectrum, when $\ell < \rho < L$,

$$P_I(z,\underset{\sim}{\rho}) = \exp\left[-\frac{1.86A_\beta(0)z}{k^2}(\rho q_o)^{5/3}\right] \equiv \exp\left[-\left(\frac{\rho}{\rho_{c.s}}\right)^{5/3}\right] \quad (5\text{-}116)$$

and for Gaussian spectrum, when $\rho < L = \frac{1}{q_o}$,

$$P_I(z,\underset{\sim}{\rho}) = \exp\left[-\frac{A_\beta(0)z}{k^2}\left(\frac{\rho q_o}{2}\right)^2\right] \equiv \exp\left[-\left(\frac{\rho}{\rho_{c.s}}\right)^2\right]. \quad (5\text{-}117)$$

Thus for strong scintillation the correlation scale is

$$\rho_{c.s}(z) = [1.86zA_\beta(0)]^{-0.6} q_o^{-1} k^{1.2} \tag{5-118}$$

for Kolmogorov spectrum, and is

$$\rho_{c.s}(z) = 2\,[zA_\beta(0)]^{-0.5} q_o^{-1} k \tag{5-119}$$

for a Gaussian spectrum. Hence $\rho_{c.s}(z)$ is proportional to $k^{1.2}$ for a Kolmogorov spectrum and proportional to k for the Gaussian spectrum, in the strong scintillation region.

For the Kolmogorov spectrum, the correlation scale $\rho_{c.s}$ as a function of z is plotted in Figure (5-7). In the weak scintillation region we have $\rho_{c.s}(z) = \sqrt{2\pi z/k}$ for the case where $\gamma > 1$ or $\sqrt{2\pi z/k} < L$ (Tatarskii 1961). The scale $\rho_{c.s}$ for weak scintillation is plotted on curve (1) of Figure (5-7) where

$$\rho_{c.s.}(z) = \sqrt{2\pi z/k} \quad . \tag{5-120a}$$

Curve (2) shows the correlation scale for strong scintillation where

$$\rho_{c.s.}(z) = \left(\frac{1.86zA_\beta(0)}{k^2}\right)^{-0.6} \times q_o^{-1} \quad . \tag{5-120b}$$

The intersection of the two curves is at $P(z_*, \rho_*)$ where

Figure (5-7)

For $\gamma > 1$, the intensity correlation scale $\rho_{c.s.}$ as a function of z is shown in solid lines. $\rho_{c.s.}(z) = 2\pi z/k$ is shown in curve (1), and $\rho_{c.s.}(z) = [\frac{1.86 z A_{\beta}(0)}{k^2}]^{-0.6} \times q_o^{-1}$ is shown in curve (2). $P(z_*, \rho_*)$ is the point where curve (1) and curve (2) intersect. This curve is plotted for the medium with a Kolmogorov spectrum.

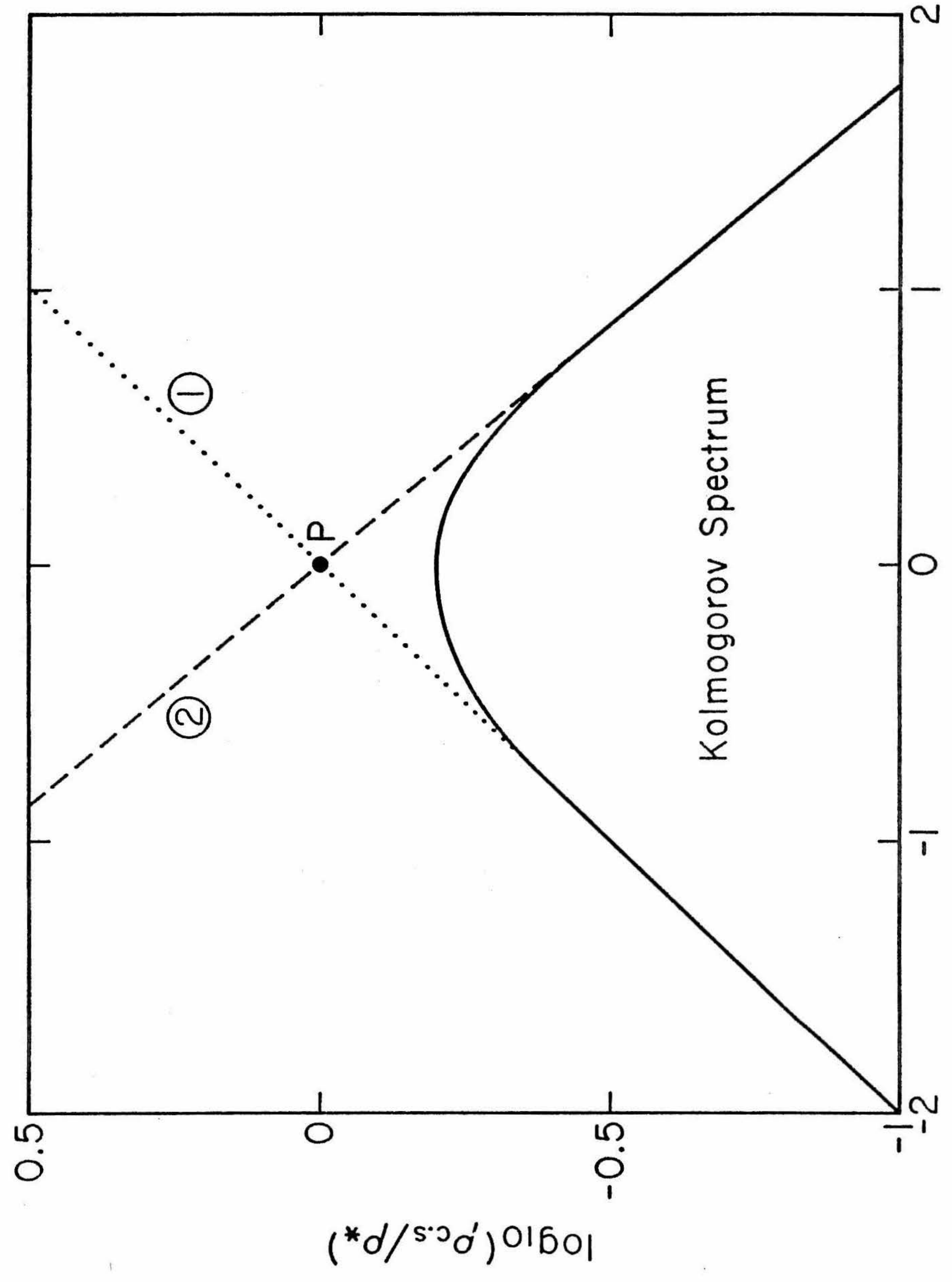

0.5
0
-0.5
-1
$\log_{10}(\rho_{c.s}/\rho_*)$
-2
-1
0
1
2
$\log_{10}(z/z_*)$
P
①
②
Kolmogorov Spectrum

Figure (5-7)

$z_* = (\frac{k}{2\pi})^{5/11} [\frac{k^2}{1.86A_\beta(0)}]^{\frac{6}{11}} q_o^{-10/11}$ and $\rho_* = (\frac{2\pi z_*}{k})^{\frac{1}{2}}$. We note that at $z = z_*$, where $\rho_{c.s.}(z)$ has its peak value, $\eta \simeq \gamma^{5/11}$ and that near $z = z_*$, the intensity correlation scale $\rho_{c.s.}$ makes the transition from curve (1) to curve (2).

Eq. (5-108) is the same normalized equation used by Brown (1972), who employed numerical method to solve the 4-th moment equation. The results of Brown show that as $t \gtrsim 1$, the scintillation is strong and $m_z^2 \simeq 1$, which is consistent with our result here. Concerning the correlation scale, Brown finds that near $t \simeq 1$ (from $t = 0.5$ to $t \simeq 5$), the correlation scale is relatively insensitive to the propagating distance z, which is consistent with the result shown in Figure (5-7) since near $t = 1$ (corresponding to $z = z^*$), the slope of the correlation scale is nearly zero.

Finally we note that our result about the intensity correlation function is consistent with that derived by Yura (1974), who generalized Tatarskii's geometric-optics model to include both diffraction and the loss of spatial coherence of wave as it propagates through the turbulent medium.

VI. The Probability Distribution of the Random Wave

§(1) Introduction

As pointed out in Chapter 4, a complete statistical description of the random wave requires the solution of all moments of the wave field. However if the additional assumption is made that probability distribution of the wave field is known, then usually only some few lowest-order moments of the wave field are sufficient to describe all the statistical properties of the wave field. For example, if the random wave has a log-normal probability distribution (Strohbehn & Wang, 1972) or a joint-normal distribution (Uscinskii 1968a,b) (detailed definitions are given in subsection §(2)), then only the moments $\Gamma_{1,0}$, $\Gamma_{1,1}$, and $\Gamma_{2,0}$ of Eq. (4-37) are sufficient to describe all the statistical properties of the random field. All the higher moments of the random wave can be written in terms of these three moments.

In this section, we will discuss, in particular, two kinds of probability distribution, namely, the log-normal distribution and the joint-normal distribution. Log-normal distribution is of interest because in the weak scattering cases the probability is log-normal (Tatarskii 1971, Young 1971). However Rice (1954) shows that the probability distribution of noise plus sine wave has a Rice distribution, which is a particular case of joint-normal distribution. Mercier (1962) shows that the probability distribution in the Fraunhofer diffraction region, of a wave passing through a random phase-modulation screen (see also Chapter 3) also has a Rice-distribution.

Under the log-normal assumption, Strohbehn & Wang (1972, 1974a, 1974b) have derived a relation between the intensity correlation function and the lowest moments $\Gamma_{1,0}$, $\Gamma_{1,1}$ and $\Gamma_{2,0}$, and have calculated the intensity correlation function when the diffraction parameter γ (see Section V) is small. In this section we will calculate, under the log-normal assumption, the intensity correlation function for both small and large values of the diffraction parameter γ and will give a more complete discussion of the log-normal assumption.

In this section, we will also give a relation between intensity correlation function and the three lower moments $\Gamma_{1,0}$, $\Gamma_{1,1}$ and $\Gamma_{2,0}$ under joint-normal distribution and calculate the intensity correlation function through these three lower moments. Our results are compared to those of Uscinskii (1968a, 1968b), who calculated the intensity correlation through complicated physical arguments under the joint-normal assumption.

Finally we will show that the joint-normal distribution is valid for strong scintillation in predicting the spatial intensity correlation function.

§(2) Relation Between Intensity Correlation Function and Lower Moments

(a) Log-normal Distribution

The random wave $u(z,k,\underset{\sim}{\rho})$ can be written as

$$u(z,k,\underset{\sim}{\rho}) \equiv u(z,s) \equiv e^{\varphi(z,s)} = e^{\chi+iS} = A\, e^{iS} \qquad (5\text{-}121)$$

(see Chapter 2, Eq. (38')) where $s = (k,\underset{\sim}{\rho})$, A is amplitude of the wave and S is the phase measured relative to the free space value, kz. Let

$$\chi = \langle\chi\rangle + \chi_1, \quad S = \langle S\rangle + S_1 \quad . \tag{5-122}$$

Under the assumption that u has a log-normal probability distribution, φ is a Gaussian variable and therefore χ_1 and S_1 are joint Gaussian random variables with zero mean. If y is a Gaussian random variable with zero mean, then it is easy to show that

$$\langle e^y\rangle = e^{\frac{1}{2}\langle y^2\rangle} \quad . \tag{5-123}$$

Define the correlation functions between χ and S as

$$P_\chi(z,s_1,s_2) = \langle\chi_1(z,s_1)\chi_1(z,s_2)\rangle \tag{5-124a}$$

$$P_S(z,s_1,s_2) = \langle S_1(z,s_1)S_1(z,s_2)\rangle \tag{5-124b}$$

and

$$P_{\chi S}(z,s_1,s_2) = \langle\chi_1(z,s_1)S_1(z,s_2)\rangle \quad . \tag{5-124c}$$

Using the formula in Eq. (5-123), one can express all the moments of u, $\Gamma_{m,n}$, in terms of $\langle\chi\rangle$, $\langle S\rangle$, P_χ, P_S and $P_{\chi S}$. The first three moments, namely,

$$\Gamma_{1,0}(z,s) = \langle u(z,s)\rangle = \overline{u} \tag{5-125a}$$

$$\Gamma_{1,1}(z,s_1,s_2) = \langle u(z,s_1)u^*(z,s_2)\rangle \tag{5-125b}$$

and

$$\Gamma_{2,0}(z,s_1,s_2) = \langle u(z,s_1)u(z,s_2)\rangle \quad , \tag{5-125c}$$

determine the values of $\langle\chi\rangle$, $\langle S\rangle$, P_χ, P_S and $P_{\chi S}$. It is easy to show that (Strohbehn & Wang, 1972)

$$\langle \chi(z,s) \rangle = \text{Re}\ [2\ \ell n\ \langle u,(z,s) \rangle - \frac{1}{2}\ \ell n\ \Gamma_{2,0}(z,s,s)] \qquad (5\text{-}126a)$$

$$\langle S(z,s) \rangle = \text{Im}\ [2\ \ell n\ \langle u(z,s) \rangle - \frac{1}{2}\ \ell n\ \Gamma_{2,0}(z,s,s)] \qquad (5\text{-}126b)$$

$$P_\chi(z,s_1,s_2) = \frac{1}{2}\ \text{Re}\ [\ell n\Gamma_{1,1}(z,s_1,s_2) + \ell n\Gamma_{2,0}(z,s_1,s_2) - 2\ \ell n\langle u(z,s_1) \rangle$$

$$- 2\ \ell n\langle u(z,s_2) \rangle] \qquad (5\text{-}126c)$$

$$P_S(z,s_1,s_2) = \frac{1}{2}\ \text{Re}\ [\ell n\Gamma_{1,1}(z,s_1,s_2) - \ell n\Gamma_{2,0}(z,s_1,s_2)] \qquad (5\text{-}126d)$$

and

$$P_{\chi S}(z,s_1,s_2) = \frac{1}{2}\ [\ell n\Gamma_{1,1}(z,s_1,s_2) + \ell n\Gamma_{2,0}(z,s_1,s_2) - 2\ \ell n\langle u(z,s_1) \rangle] \qquad (5\text{-}126e)$$

where Re and Im denote, respectively, the real part and the imaginary part of the quantity following. Once the above five quantities in Eq. (5-126) are known, all the higher moments can be determined from these five quantities. For example, the normalized spatial intensity correlation function P_I is given as follows,

$$P_I(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2) \equiv \frac{\langle I(z,\underset{\sim}{\rho}_1)I(z,\underset{\sim}{\rho}_2) \rangle - \langle I(z,\underset{\sim}{\rho}_1) \rangle \langle I(z,\underset{\sim}{\rho}_2) \rangle}{\langle I(z,\underset{\sim}{\rho}_1) \rangle \langle I(z,\underset{\sim}{\rho}_2) \rangle}$$

$$= \exp\ [4P_\chi(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)] - 1, \qquad (5\text{-}127)$$

where we have put $k_1 = k_2$.

(b) Joint-normal Distribution

Let

$$u(z,s) = \langle u(z,s) \rangle + u_1(z,s) \qquad (5\text{-}128a)$$

and

$$u_1(z,s) = u_{1c}(z,s) + u_{1s}(z,s) \quad , \tag{5-128b}$$

where $u_{1c}(z,s)$ is the component of u_1 "in phase" with $\langle u(z,s)\rangle$ and $u_{1s}(z,s)$ is the component of u_1 "out of phase" with $\langle u(z,s)\rangle$.

Under the joint-normal assumption, u is a Gaussian random variable, or u_{1c} and u_{1s} are joint-normal random variables with zero mean. When u is a normal random variable, all the higher moments can be expressed in terms of the three lower moments $\Gamma_{1,0}$, $\Gamma_{1,1}$ and $\Gamma_{2,0}$ given by Eqs. (5-125a,b,c). It is easy to show that the normalized intensity correlation function is given by

$$P_I(z,s_1,s_2) = \frac{\langle I(z,s_1)\; I(z,s_2)\rangle - \langle I(z,s_1)\rangle\langle I(z,s_2)\rangle}{\langle I(z,s_1)\rangle\langle I(z,s_2)\rangle}$$

$$= \frac{|\Gamma_{1,1}(z,s_1,s_2)|^2 + |\Gamma_{2,0}(z,s_1,s_2)|^2 - 2\,\langle u(z,s_1)\rangle^2\langle u(z,s_2)\rangle^2}{\Gamma_{1,1}(z,s_1,s_1)\;\Gamma_{1,1}(z,s_2,s_2)} \quad . \tag{5-129}$$

Consider two special cases of Eq. (5-129):

(a) When $k_1=k_2$, we have the normalized <u>spatial</u> intensity correlation function

$$P_I(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2) = \frac{|\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)|^2 + |\Gamma_{2,0}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)|^2 - 2\,\langle u(z,\underset{\sim}{\rho}_1)\rangle^2\langle u(z,\underset{\sim}{\rho}_2)\rangle^2}{\Gamma_{1,1}(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_1)\;\Gamma_{1,1}(z,\underset{\sim}{\rho}_2,\underset{\sim}{\rho}_2)} \tag{5-129a}$$

(b) When $\underset{\sim}{\rho}_1=\underset{\sim}{\rho}_2$, we have the normalized intensity correlation function with <u>different frequencies</u>

$$P_I(z,k_1,k_2) = \frac{|\Gamma_{1,1}(z,k_1,k_2)|^2+|\Gamma_{2,0}(z,k_1,k_2)|^2 - 2\langle u(z,k_1)\rangle^2\langle u(z,k_2)\rangle^2}{\Gamma_{1,1}(z,k_1,k_1)\;\Gamma_{1,1}(z,k_2,k_2)} \tag{5-129b}$$

where each quantity is measured at the same point.

§(3) Results for the Log-normal Assumption

Throughout this thesis we consider the case that the incoming wave is a plane wave and the medium is statistically homogeneous. Then $\langle u(z,\rho)\rangle$ is independent of the transverse coordinate ρ and $\Gamma_{1,1}(z,\rho_1,\rho_2)$ and $\Gamma_{2,0}(z,\rho_1,\rho_2)$ depend only on z and $|\rho| = |\rho_1-\rho_2|$. We also normalize u such that $u(z=0,\rho) = 1$. Then Eqs. (4-45), (4-46) and (4-47) reduce to

$$\frac{\partial\Gamma_{1,0}(z)}{\partial z} = -\frac{A_\beta(0)}{4k^2}\,\Gamma_{1,0}\ , \tag{5-130}$$

$$\frac{\partial\Gamma_{1,1}(z,\rho)}{\partial z} = -\frac{1}{2k^2}\,[A_\beta(0)-A_\beta(\rho)]\,\Gamma_{1,1}(z,\rho) \tag{5-131}$$

and

$$\frac{\partial\Gamma_{2,0}(z,\rho)}{\partial z} = \frac{i}{k}\nabla_\rho^2\,\Gamma_{2,0}(z,\rho) - \frac{1}{2k^2}[A_\beta(0)+A_\beta(\rho)]\,\Gamma_{2,0}(z,\rho)\ . \tag{5-132}$$

From Eqs. (5-130) and (5-131), we obtain immediately

$$\Gamma_{1,0}(z) = \langle u(z)\rangle = e^{-\frac{A_\beta(0)z}{4k^2}} \tag{5-133}$$

and

$$\Gamma_{1,1}(z,\rho) = \Gamma_{1,1}(z=0,\rho)\,e^{-\frac{z}{2k^2}[A_\beta(0)-A_\beta(\rho)]} = e^{-\frac{z}{2k^2}[A_\beta(0)-A_\beta(0)]}\ . \tag{5-134}$$

The second moment $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ is also called the mutual coherence function (MCF), and is the quantity that describes the loss of coherence of an initial coherent wave propagating in the random medium.

Eqs. (5-127) and (5-126c) can then be written respectively as

$$P_I(z,\underset{\sim}{\rho}) = \exp\left[4P_\chi(z,\underset{\sim}{\rho})\right] - 1 \qquad (5\text{-}135)$$

and

$$P_\chi(z,\underset{\sim}{\rho}) = \frac{1}{2}\,\ell n \left| \frac{\Gamma_{1,1}(z,\underset{\sim}{\rho})\ \Gamma_{2,0}(z,\underset{\sim}{\rho})}{\overline{u}^4} \right| \quad . \qquad (5\text{-}136)$$

Eq. (5-132) cannot be solved analytically. Writing $\Gamma_{2,0}(z,\underset{\sim}{\rho})$ as

$$\Gamma_{2,0}(z,\underset{\sim}{\rho}) \equiv e^{-\frac{A_\beta(0)z}{2k^2}}\ \Gamma_c(z,\underset{\sim}{\rho}) \quad , \qquad (5\text{-}137)$$

we obtain from Eq. (5-132) an equation for $\Gamma_c(z,\underset{\sim}{\rho})$,

$$\frac{\partial}{\partial z}\Gamma_c(z,\underset{\sim}{\rho}) = \frac{i}{k}\,\nabla_{\underset{\sim}{\rho}}^{\,2}\Gamma_c(z,\underset{\sim}{\rho}) - \left[\frac{A_\beta(0)}{2k^2}\,H(\underset{\sim}{\rho})\right]\Gamma_c(z,\underset{\sim}{\rho}) \qquad (5\text{-}138)$$

where $H(\underset{\sim}{\rho}) = A_\beta(\underset{\sim}{\rho})/A_\beta(0)$. The initial condition for $\Gamma_c(z,\underset{\sim}{\rho})$ is

$$\Gamma_c(z,\underset{\sim}{\rho}) = 1, \qquad (5\text{-}139)$$

and the boundary condition is

$$\text{at } |\underset{\sim}{\rho}| = \infty,\ \Gamma_c(z,\underset{\sim}{\rho}) = 1. \qquad (5\text{-}140)$$

For Gaussian spectrum, we have from Eq. (1-26)

$$A_\beta(0) = \frac{Bk^4}{8\pi}\,q_o^{\,2} \qquad (5\text{-}141)$$

and

$$H(\underset{\sim}{\rho}) = e^{-(q_o^{\,2}\rho^2/4)} \quad . \qquad (5\text{-}142)$$

Define the following new variables (c.f. Eq. (5-95)),

$$\rho_c = \frac{2}{q_o}$$

$$z_c = k\,\rho_c^{\,2} = 4k/q_o^{\,2}$$

$$\underset{\sim}{\xi} = \underset{\sim}{\rho}/\rho_c$$

$$\eta = z/z_c \tag{5-143}$$

$$\beta_A = \frac{A_\beta(0)}{2k^2}$$

and

$$\gamma = \beta_A z_c = \frac{2A_\beta(0)}{kq_o^{\,2}} \quad .$$

Then Eq. (5-138) can be written as

$$\frac{\partial}{\partial\eta}\Gamma_c(\eta,\underset{\sim}{\xi}) = \frac{i}{\gamma}\nabla_{\underset{\sim}{\xi}}^{\,2}\Gamma_c(\eta,\underset{\sim}{\xi}) - H(\underset{\sim}{\xi})\Gamma_c(z,\underset{\sim}{\xi}) \tag{5-144}$$

where $H(\underset{\sim}{\xi}) = e^{-\xi^2}$. γ is the diffraction parameter, which determines the importance of the diffraction term $(\frac{i}{\gamma}\nabla_{\underset{\sim}{\xi}}^{\,2}\Gamma_c)$. From Eqs. (5-133)-(5-137), we have

$$P_\chi(\eta,\underset{\sim}{\xi}) = \frac{1}{2}\,\ell n\,|\Gamma_c(\eta,\underset{\sim}{\xi})| + \frac{1}{2}\,\eta\,H(\underset{\sim}{\xi}) \tag{5-145a}$$

and

$$P_I(\eta,\underset{\sim}{\xi}) = \exp\,[4P_\chi(\eta,\underset{\sim}{\xi})] - 1 = \exp\,[2\eta H(\underset{\sim}{\xi})] \times |\Gamma_c|^2 - 1. \tag{5-145b}$$

Techniques for numerical solution of Eq. (5-144) are given in Appendix (D). The numerical results for P_I and P_χ are presented in Figures (5-8a,b,c). In the following we will also give some approximate analytic solutions for Γ_c.

Figure Captions

Figure (5-8a). This figure shows the log-amplitude variance σ_x^2 v.s. the normalized propagating distance $\eta = \beta_A z$ under the log-normal assumption. Curve (1) is the theoretical curve for the diffraction parameter $\gamma = 0$. Curves (2-5) are the numerical results for $\gamma = 0.1$, 1.0, 5.0 and 10.0 respectively. Curve (6) is the MSP variance for $\gamma = 10$. In the region $\eta < 1$, curves (5) and (6) are about the same. Note that σ_x^2 grows without a saturation value for all curves.

Figure (5-8b). This figure shows the scintillation index m_z^2 as a function of η for $\gamma = 0, 0.1$, 1 and 10. Curve (1) is the theoretical curve for $\gamma = 0$ and curves (3), (4), and (5) are the numerical results.

Figure (5-8c). For $\gamma = 100$, the computed log-amplitude variance σ_x^2 as a function of η is shown in curve (3). Curve (1) is the MSP variance $\sigma_T^2 = \frac{8}{3}\left[1 - \frac{\gamma}{4\eta} \tan^{-1}\left(\frac{4t}{\eta}\right)\right]$ and curve (2) is the theoretical variance $\sigma_x^2 = \frac{8}{3}\frac{\eta^3}{\gamma^2}\left[1 - \frac{\eta}{4}\right]$ under large γ approximation. The peak of σ_x^2 is at $\eta = 3$ and σ_x^2 goes to negative value for $\eta > 4$.

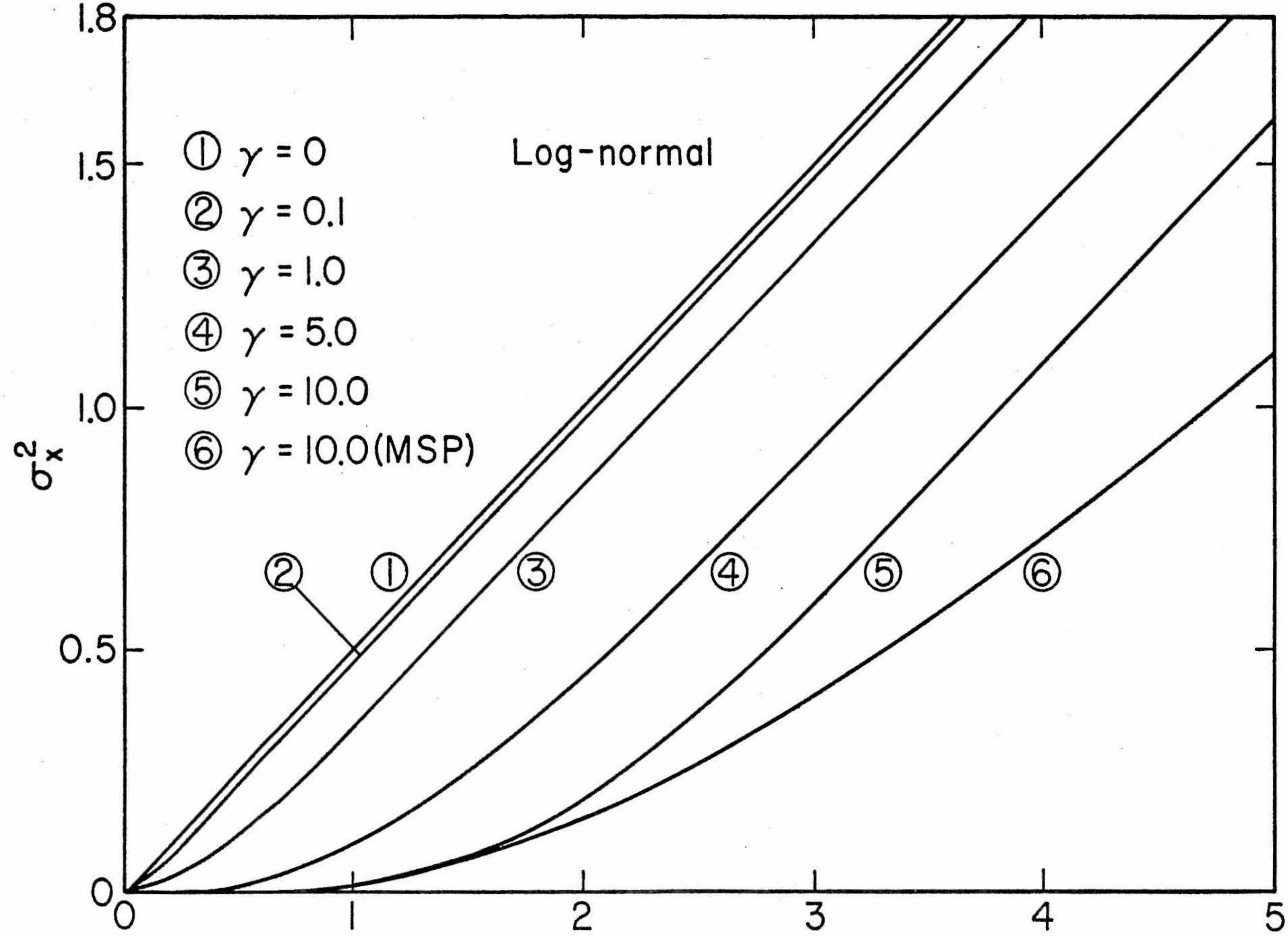

Log-normal
① γ = 0
② γ = 0.1
③ γ = 1.0
④ γ = 5.0
⑤ γ = 10.0
⑥ γ = 10.0(MSP)
1.8
1.5
1.0
0.5
0
σ_x^2
0
1
2
3
4
5
η

Figure (5-8a)

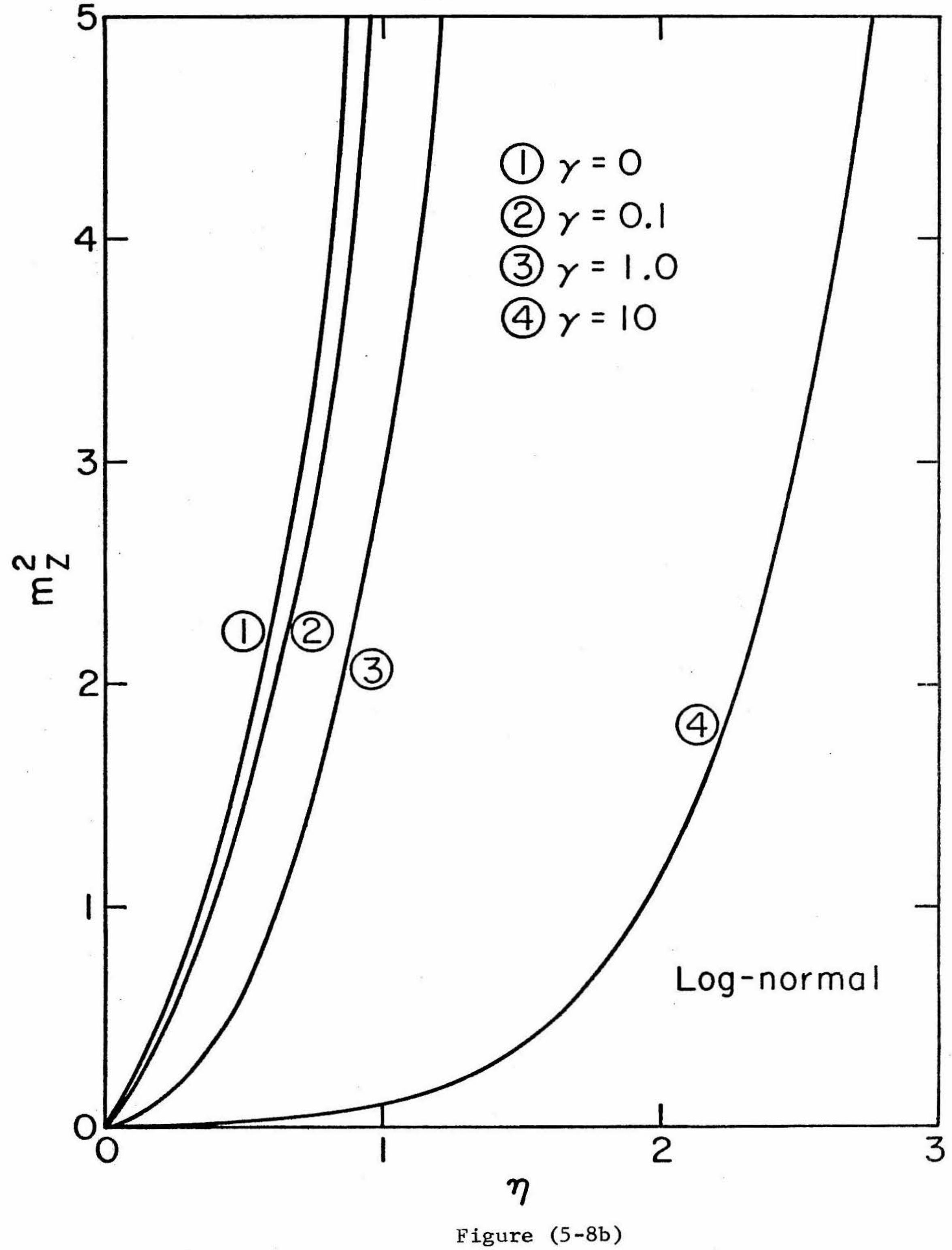
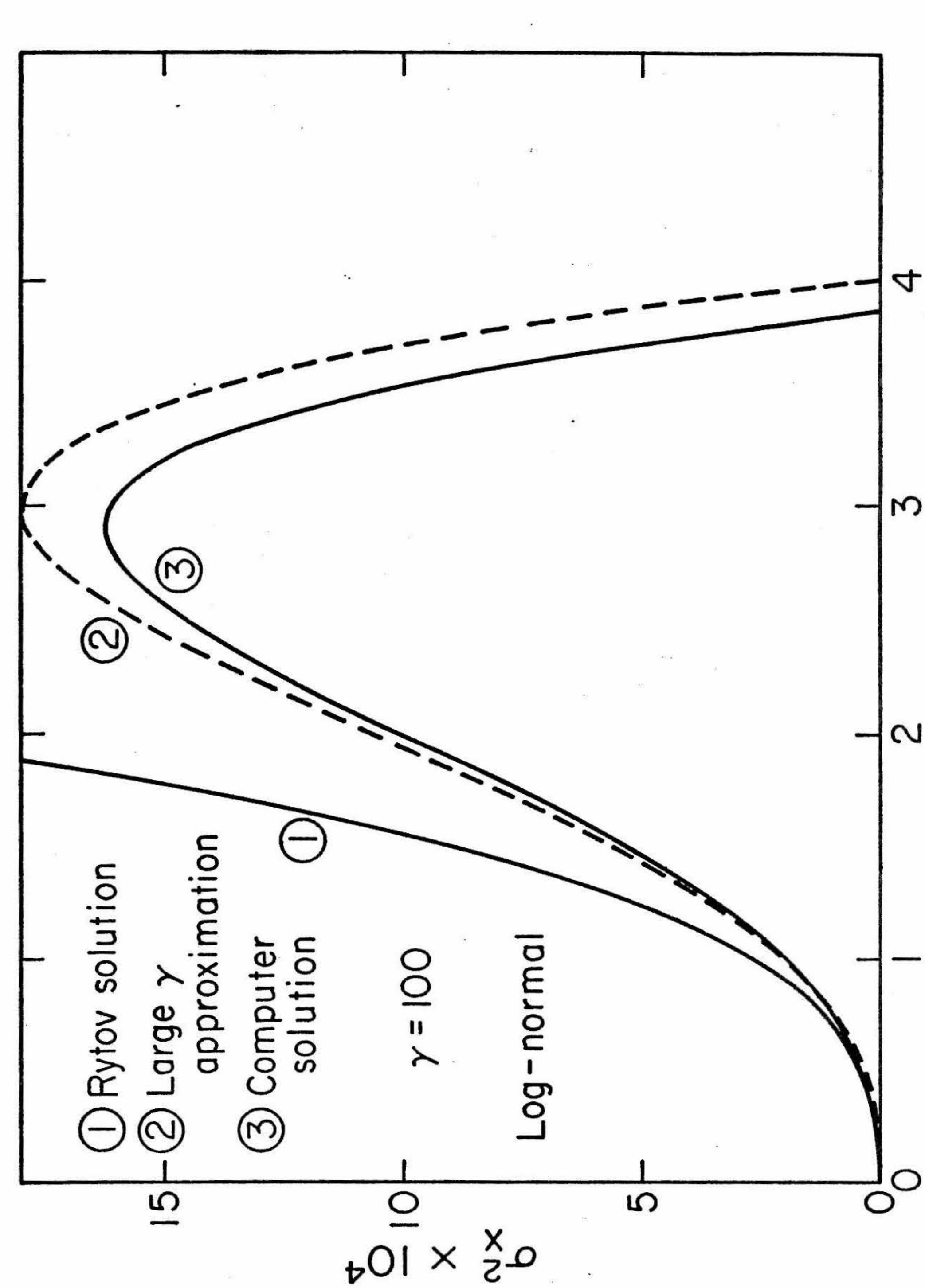


Figure (5-8b)



① Rytov solution
② Large γ approximation
③ Computer solution
γ = 100
Log-normal
$\sigma_\chi^2 \times 10^4$
η

Figure (5-8c)

(a) Method of Smooth Perturbation (MSP)

We let

$$\Gamma_c(\eta, \underset{\sim}{\xi}) = e^{\Psi_c(\eta, \underset{\sim}{\xi})} \tag{5-146}$$

and obtain an equation for $\Psi_c(\eta, \underset{\sim}{\xi})$ from Eq. (5-144),

$$\frac{\partial \Psi_c}{\partial \eta} = \frac{i}{\gamma} [\nabla_{\underset{\sim}{\xi}}^2 \Psi_c(\eta, \underset{\sim}{\xi}) + (\nabla \Psi_c(\eta, \underset{\sim}{\xi}))^2] - H(\underset{\sim}{\xi}) \ . \tag{5-147}$$

Under the method of smooth perturbation, the non-linear term $\frac{i}{\gamma}(\nabla \Psi_c)^2$ is neglected and one has

$$\frac{\partial \Psi_c(\eta, \underset{\sim}{\xi})}{\partial \eta} = \frac{i}{\gamma} \nabla_{\underset{\sim}{\xi}}^2 \Psi_c(\eta, \xi) - H(\underset{\sim}{\xi}) \quad . \tag{5-148}$$

The initial condition for $\Psi_c(\eta, \underset{\sim}{\xi})$ is

$$\Psi_c(\eta = 0, \underset{\sim}{\xi}) = 0 \quad . \tag{5-149}$$

We then have from Eqs. (5-148) and (5-149)

$$\Psi_c(\eta, \underset{\sim}{\xi}) = \frac{i\gamma}{2\pi} \iint_{-\infty}^{\infty} \frac{\widetilde{H}(\underset{\sim}{q})}{q^2} (1 - e^{-\frac{iq^2 t}{\gamma}}) e^{i\underset{\sim}{q} \cdot \underset{\sim}{\xi}} d^2 \underset{\sim}{q} \tag{5-150}$$

where

$$\widetilde{H}(\underset{\sim}{q}) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} H(\underset{\sim}{\xi}) e^{-i\underset{\sim}{\xi} \cdot \underset{\sim}{q}} d^2 \underset{\sim}{\xi} \quad . \tag{5-151}$$

For Gaussian spectrum, we have

$$H(\underset{\sim}{\xi}) = e^{-\xi^2} \text{ and } \widetilde{H}(\underset{\sim}{q}) = \frac{1}{2} e^{-\frac{q^2}{4}} \tag{5-152}$$

from which we obtain

$$\Psi_c(\eta,\xi=0) = -\frac{\gamma}{4}\tan^{-1}\left(\frac{4\eta}{\gamma}\right) + i\left[\frac{1}{4}\eta\tan^{-1}\left(\frac{4\eta}{\gamma}\right) - \frac{\gamma}{32}\ln\left(1+\frac{16\eta^2}{\gamma^2}\right)\right]. \tag{5-153}$$

Eqs. (5-136) and (5-153) give us

$$\sigma_\chi^{\;2}(\eta) = P_\chi(\eta,\xi=0) = \frac{\eta}{2}\left[1-\frac{\gamma}{4\pi}\tan^{-1}\left(\frac{4\eta}{\gamma}\right)\right] \equiv \sigma_T^{\;2}(\eta). \tag{5-154}$$

$\sigma_x^{\;2}(\eta)$ is the variance of the log-amplitude and $\sigma_T^{\;2}(\eta)$ is called the MSP variance (or Rytov variance).

$$\text{As } \left(\frac{4\eta}{\gamma}\right) \ll 1, \sigma_x^{\;2}(\eta) \approx \frac{8}{3}\frac{\eta^3}{\gamma^2}\quad . \tag{5-155a}$$

$$\text{As } \left(\frac{4\eta}{\gamma}\right) \gg 1, \sigma_x^{\;2}(\eta) \approx \frac{\eta}{2} - \frac{\pi\gamma}{16}\quad . \tag{5-155b}$$

From Eq. (5-150) we also find that

$$\text{as } \eta \geq 1, \left|\frac{\partial}{\partial\xi}\Psi(\eta,\xi)\right|^2 \gtrsim \left|\nabla_{\underset{\sim}{\xi}}^{\;2}\Psi(\eta,\xi)\right| \tag{5-156}$$

and method of smooth perturbation is no longer valid. As shown in Figure (5-8), Eq. (5-154) for $\sigma_x^{\;2}(\eta)$ is consistent with the numerical results only when $\eta \lesssim 1$. $\sigma_x^{\;2}(\eta)$ given in Eq. (5-154) is the same as obtained by MSP without using the log-normal assumption (Tatarskii 1961.)

(b) Approximation for large γ

When $\gamma \gg 1$, the diffraction term is not important. For a Gaussian spectrum, we have from Eq. (5-144)

$$\frac{\partial \Gamma_c(\eta,\xi)}{\partial \eta} = i\delta\,(\nabla_{\underset{\sim}{\xi}}{}^2 \Gamma_c) - e^{-\xi^2}\,\Gamma_c \tag{5-157}$$

where $\delta = \frac{1}{\gamma}$. We then expand Γ_c in term of the small parameter δ as

$$\Gamma_c(\eta,\xi) = \Gamma_o(\eta,\xi) + \delta\Gamma_1(\eta,\xi) + \delta^2\Gamma_c(\eta,\xi) + \ldots, \tag{5-158}$$

and set

$$\Gamma_c(\eta{=}0,\xi) = 1 \text{ and } \Gamma_1(\eta{=}0,\xi) = \Gamma_2(\eta{=}0,\xi) = \ldots = 0.$$

Inserting Eq. (5-158) into Eq. (5-157), we have

$$\frac{\partial \Gamma_o(\eta,\xi)}{\partial \eta} = -\,e^{-\xi^2}\,\Gamma_o \tag{5-159a}$$

$$\frac{\partial \Gamma_1(\eta,\xi)}{\partial \eta} = i\,\nabla_{\underset{\sim}{\xi}}{}^2\,\Gamma_o - e^{-\xi^2}\,\Gamma_1 \tag{5-159b}$$

$$\frac{\partial \Gamma_2(\eta,\xi)}{\partial \eta} = i\,\nabla_{\underset{\sim}{\xi}}{}^2\,\Gamma_1 - e^{-\xi^2}\,\Gamma_2 \tag{5-159c}$$

and etc... We solve Eq. (5-159) and obtain to second order in δ,

$$\sigma_\chi{}^2(\eta) = \frac{8}{3}\,\delta^2(\eta^3 - \frac{\eta^4}{4}) = \frac{8\eta^3}{3\gamma^2}\,(1 - \frac{\eta}{4}). \tag{5-160}$$

For $\eta \underset{\sim}{<} 1$, $\sigma_\chi{}^2(\eta) \simeq \frac{8}{3}\frac{\eta^3}{\gamma^2}$ and $\sigma_\chi{}^2(\eta)$ is the same as given by the method of smooth perturbation. But as $\eta \geq 1$, the non-linear term $\frac{i}{\gamma}(\nabla\Psi)^2$ in Eq. (5-147) cannot be neglected and Eq. (5-160) gives the correct $\sigma_\chi{}^2(\eta)$. The peak of $\sigma_\chi{}^2(\eta)$ is

$$\sigma_{\chi(\max)}^2 = \frac{2}{3}\,\frac{\eta^3}{\gamma^2} = \frac{18}{\gamma^2}\,,\ \text{at } \eta = 3 \tag{5-161a}$$

We also note that

$$\sigma_\chi^2(\eta) = 0, \text{ at } \eta = 4, \tag{5-161b}$$

and $\sigma_\chi^2(\eta)$ goes to negative when $\eta > 4$. Figure (5-8c) shows that at $\gamma = 100$, the approximate analytic solution is consistent with the numerical results. $\sigma_\chi^2(\eta)$ has its peak near $\eta = 3$ and goes to negative near $\eta = 4$. Eq. (5-160) for $\sigma_\chi^2(\eta)$ also fits very well the numerical solution at $\gamma \simeq 1.429\text{x}10^5$ in Figure (7) of Wang & Strohbehn (1974a). The series expansion for Γ_c in Eq. (5-158) is valid only when

$$|\Gamma_o(\eta,\xi)| \underset{\sim}{\gg} |\delta\Gamma_1(\eta,\xi)| \underset{\sim}{\gg} |\delta^2\Gamma_2(\eta,\xi)| \ldots$$

from which we get

$$\left(\frac{2\eta^2}{\gamma}\right) \underset{\sim}{\ll} 1 \quad . \tag{5-162}$$

If we require the solution for $\sigma_\chi^2(\eta)$ in Eq. (5-160) to be valid for $\eta \lesssim 4$, then we have from Eq. (5-162)

$$\gamma \underset{\sim}{\gg} 32. \tag{5-163}$$

The results in Figure (5-8) are consistent with our prediction in Eq. (5-163). For $\gamma = 100 > 32$, $\sigma_\chi^2(\eta)$ is about the same as given by Eq. (5-160), while for $\gamma = 10 < 32$, the numerical result is completely different from that given by Eq. (5-160).

In the extreme case that $\gamma = \infty$, the diffraction term can be neglected and

$$\Gamma_c(\eta,\xi) = \Gamma_o(\eta,\xi) \simeq e^{-H(\xi)\eta}$$

and we have $P_\chi = P_I \simeq 0$.

(c) Approximation for Small γ

When $\gamma \ll 1$, we have

$$\frac{\partial \Gamma_c(t,\underset{\sim}{\xi})}{\partial t} = i\, \nabla_{\underset{\sim}{\xi}}^2 \Gamma_c - \gamma\, e^{-\underset{\sim}{\xi}^2} \Gamma_c(t,\underset{\sim}{\xi}) \qquad (5\text{-}164)$$

where $t \equiv \eta/\gamma$ and $\Gamma_c(t,\underset{\sim}{\xi}) = \Gamma_c(\eta,\underset{\sim}{\xi})$. Expand Γ_c in terms of γ as

$$\Gamma_c(t,\underset{\sim}{\xi}) = \Gamma_o(t,\underset{\sim}{\xi}) + \gamma\, \Gamma_1(t,\underset{\sim}{\xi}) + \gamma^2 \Gamma_2(t,\underset{\sim}{\xi}) + \ldots , \qquad (5\text{-}165)$$

and set $\Gamma_o(t=0,\underset{\sim}{\xi}) = 1$ and $\Gamma_1(t=0,\underset{\sim}{\xi}) = \Gamma_2(t=0,\underset{\sim}{\xi}) = \ldots = 0$. Inserting Eq. (5-165) into Eq. (5-164), we have

$$\frac{\partial \Gamma_o(t,\underset{\sim}{\xi})}{\partial t} = i\, \nabla_{\underset{\sim}{\xi}}^2\, \Gamma_o(t,\underset{\sim}{\xi}) \qquad (5\text{-}166a)$$

$$\frac{\partial \Gamma_1(t,\underset{\sim}{\xi})}{\partial t} = i\, \nabla_{\underset{\sim}{\xi}}^2\, \Gamma_1(t,\underset{\sim}{\xi}) - e^{-\xi^2} \Gamma_o \qquad (5\text{-}166b)$$

and etc... We solve Eq. (5-166) and obtain to first order in γ,

$$\sigma_\chi^2(\eta) = \frac{\eta}{2}\left[1 - \frac{\gamma}{4\eta} \tan^{-1}\left(\frac{4\eta}{\gamma}\right)\right] \quad . \qquad (5\text{-}167)$$

It is also easy to show that the expansion in Eq. (5-165) for Γ_c is valid only when

$$\gamma t = \eta \lesssim 1. \qquad (5\text{-}168)$$

We note that the result for $\sigma_\chi^2(\eta)$ here is the same as that given by method of smooth perturbation, which is also valid for $\eta \lesssim 1$.

In the extreme case when $\gamma \to 0$, the diffraction term in Eq. (5-164) dominates and $\Gamma_c(\eta,\xi) = \Gamma_o = 1$, from which we have $\sigma_\chi^2(\eta) = \frac{\eta}{2}$ and $M_z^2 = (e^{2\eta}-1)$, which is also shown in Figure (5-8).

From the numerical results shown in Figure (5-8) and the above approximate analytical solutions we find that

(i) For $\eta = \beta_A z \lesssim 1$, MSP is valid and the calculated log-amplitude variance σ_χ^2 is consistent with the MSP variance σ_T^2. When MSP is valid, it can be shown that the probability distribution of intensity (or amplitude) is log-normal (Tatarskii 1971, Young 1971). We conclude that for $\eta = \beta_A z \lesssim 1$, the probability of the random wave is log-normal.

(ii) For $\eta = \beta_A z \gtrsim 1$, MSP is no longer valid. Under the log-normal assumption numerical calculations show that for large γ, σ_χ^2 becomes negative when $\eta > 4$ and for small γ, m_z^2 (or σ_χ^2) becomes much larger than unity when $\eta \ggg 1$. A negative variance is physically impossible and the fact that m_z^2 goes to infinity without a saturation value is inconsistent with experiment results (Tatarskii 1971) and the theoretical prediction in last section. We conclude that for $\eta = \beta_A z \geq 1$, the probability distribution of u can <u>not</u> be log-normal.

(iii) The criterion here for the validity of MSP is different from that given by Tatarskii (1971). Our condition for MSP is $\eta = \beta_A z \leq 1$ while Tatarskii's condition is the log-amplitude $\sigma_\chi^2 \lesssim 1$. As shown in Figure (5-8c), for $\gamma = 100$ at $\beta_A z = 3$, MSP is no longer valid but the log-amplitude σ_χ^2 is still much less than 1. ($\sigma_\chi^2 = 1.8\text{x}10^{-3} \ll 1$). The numerical results of Wang & Strohbehn (1974a) for Kolmogorov spectrum also show that MSP is not valid even when $\sigma_\chi^2 \ll 1$. Figure (3) of Wang & Strohbehn (1974a) shows that the calculated σ_χ^2 deviates from the MSP variance σ_T^2 for $\sigma_\chi^2 \geq 0.05$. Our condition that $\beta_A z \lll 1$ is equivalent to the condition that the mean square phase fluctuation is much less than unity. When $\gamma \leq 1$, the condition that $\beta_A z \lll 1$ is equivalent to $\sigma_\chi^2 \lll 1$. But for $\gamma > 1$, our condition is more stringent than that of Tatarskii.

§(4). Results of the Joint-normal Distribution

In this subsection we consider the spatial intensity correlation function $P_I(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2)$ under the joint-normal assumption. Using the notation of the last subsection (§(3)), we have

$$P_I(z,\underset{\sim}{\rho}) \equiv P_I(z,\underset{\sim}{\rho}_1,\underset{\sim}{\rho}_2) = |\Gamma_{1,1}(z,\underset{\sim}{\rho})|^2 + |\Gamma_{2,0}(z,\underset{\sim}{\rho})|^2 - 2\langle u\rangle^4 \tag{5-169}$$

where $\underset{\sim}{\rho} = \underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_2$. And using the normalized coordinates as given Eq. (5-143), we have

$$P_I(\eta,\underset{\sim}{\xi}) = e^{-2\eta[1-\hat{H}(\xi)]} + e^{-2\eta}|\Gamma_c(\eta,\underset{\sim}{\xi})|^2 - 2e^{-2\eta} \ . \tag{5-170}$$

The numerical calculation for Γ_c in Appendix (D) gives $P_I(\eta,\underset{\sim}{\xi})$. The results are shown in Figures (5-9a,b,c,d,e,f). The scintillation index m_z^2 is given by

$$m_z^2 = P_I(\eta,\xi=0) = 1-2\, e^{-2\eta} + e^{-2\eta}|\Gamma_c(\eta,\xi=0)|^2 \quad . \tag{5-171}$$

The calculated m_z^2 vs the normalized propagating path η is shown in Figure (9a) for various values of the diffraction parameter γ. Also shown in the figure are two theoretical curves for m_z^2 in the extreme cases that $\gamma = 0$ and $\gamma = \infty$. In the case that $\gamma = 0$, $\Gamma_c = 1$ and we have

$$P_I(\eta,\xi) = e^{-2\eta[1-H(\xi)]} - e^{-2\eta} \quad (5\text{-}172a)$$

and

$$m_z^2 = 1 - e^{-2\eta} \quad . \quad (5\text{-}172b)$$

Similarly for $\gamma = \infty$, $\Gamma_c = e^{-\eta\hat{H}(\xi)}$, and we have

$$P_I(\eta,\xi) = [e^{-\eta(1-\hat{H}(\xi))} - e^{-\eta(1+\hat{H}(\xi))}] \quad (5\text{-}173a)$$

and

$$m_z^2 = (1 - e^{-2\eta})^2 \quad . \quad (5\text{-}173b)$$

For all other values of γ, numerical curves for m_z^2 lie between these two theoretical curves. Eqs. (5-172a) and (5-173a) are the same as those derived by Uscinskii (1968a, 1968b), who employed a complicated physical model under the joint-normal assumption. The numerical result in Figure (5-9) shows that for $\eta \gtrsim 1$, $m_z^2 = 1$. We note that since $|\Gamma_c| \le 1$ for all γ, Eq. (5-170) gives us for $\eta \gtrsim 1$,

$$P_I(\eta,\xi) \approx e^{-2\eta[1-\hat{H}(\xi)]} = |\Gamma_{1,1}(\eta,\xi)|^2 \text{ and } m_z^2 = 1, \quad (5\text{-}174)$$

which check with the numerical results. For $\eta >> 1$, Eqs. (5-172) and (5-173) also reduce to Eq. (5-174).

Figure (9b) gives us the normalized spatial intensity correlation function $P_I^{(N)}(z,\underset{\sim}{\rho}) \equiv P_I(z,\underset{\sim}{\rho})/P_I(z,0)$ at various normalized distance η for $\gamma = 1$. The normalized intensity functions are presented in Figures (5-9c), (5-9d), (5-9e) and (5-9f) for other values of the parameter γ. As shown in the figures, $P_I(z,\underset{\sim}{\rho})$ is essentially equal to $|\Gamma_{1,1}(z,\underset{\sim}{\rho})|^2$ for $\beta_A z \gtrsim 2.0$, which checks with Eq. (5-174). The correlation scale $\rho_{c.s.}$ shown in Figure (5-9) is

Figure Captions

Figure (5-9a). This figure shows the scintillation index m_z^2 vs the propagating distance $\eta = \beta_A z$ under joint-normal assumption for various values of the diffraction parameter γ. Curve (1) is the theoretical curve for $\gamma = \infty$, where $m_z^2 = (1- e^{-2\eta})^2$. For $\gamma = 5$ and 10, the computed index m_z^2 is essentially on curve (1). Curves (2), (3), and (4) are the computed curves for $\gamma = 1$, 0.5 and 0.1, respectively. Curve (5) is the theoretical curve for $\gamma = 0$, where $m_z^2 = (1- e^{-2\eta})$. Note that for $\eta \geq 2$, $m_z^2 \simeq 1$.

Figure (5-9b). This figure shows the computed normalized intensity correlation function $P_I^{(N)}(\rho)$ vs the normalized transverse coordinate $\xi = \rho/\rho_c$ under joint-normal assumption for various values of distance $\eta = \beta_A z$. The diffraction parameter γ is set equal to 1. The correlation scale of $P_I^{(N)}(\rho)$ is nearly constant for $\beta_A z \leq 1$ and decreases as $(\beta_A z)$ increases for $\beta_A z > 1$.

Figures (5-9c,d,e, & f). The computed $P_I^{(N)}(\rho)$'s are shown in these figures under the joint-normal assumption for various values of $(\beta_A z)$ and γ. We note that for $\eta = \beta_A z \geq 2$, the computed curves for $P_I^{(N)}(\rho)$ are essentially the same as the theoretical curve $P_J(\rho) = \exp\left\{ -2\eta[1-H(\rho)] \right\}$.

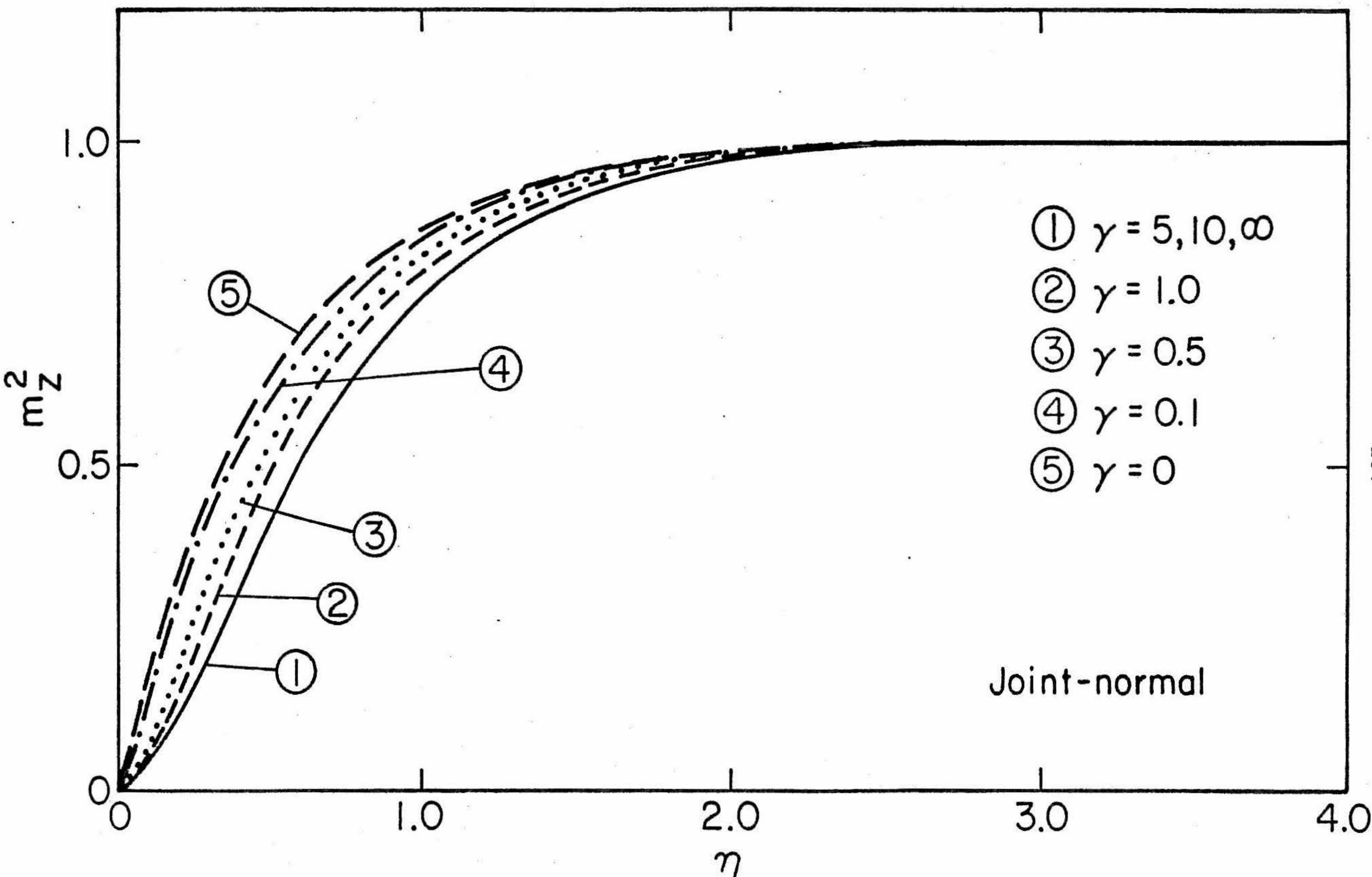
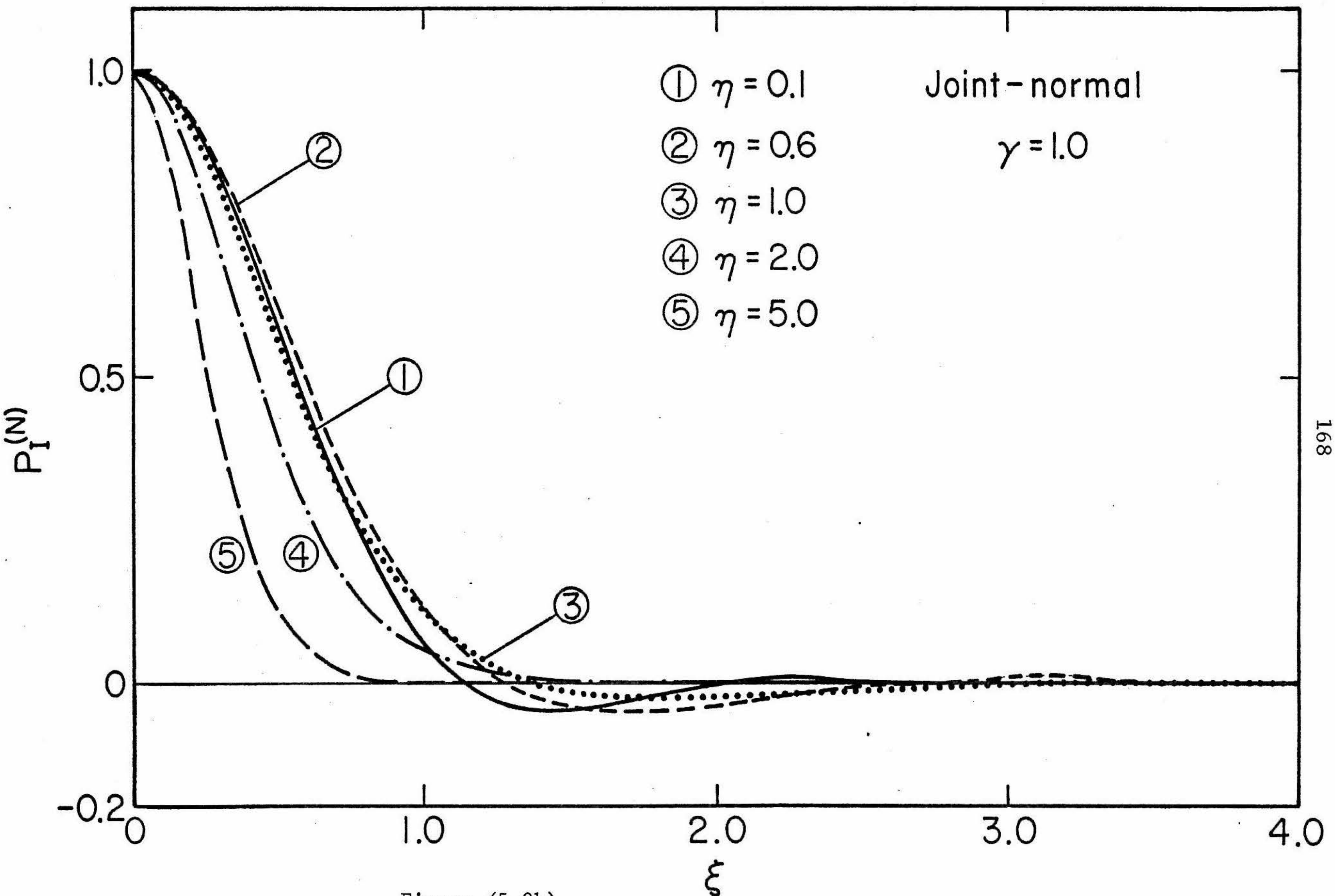
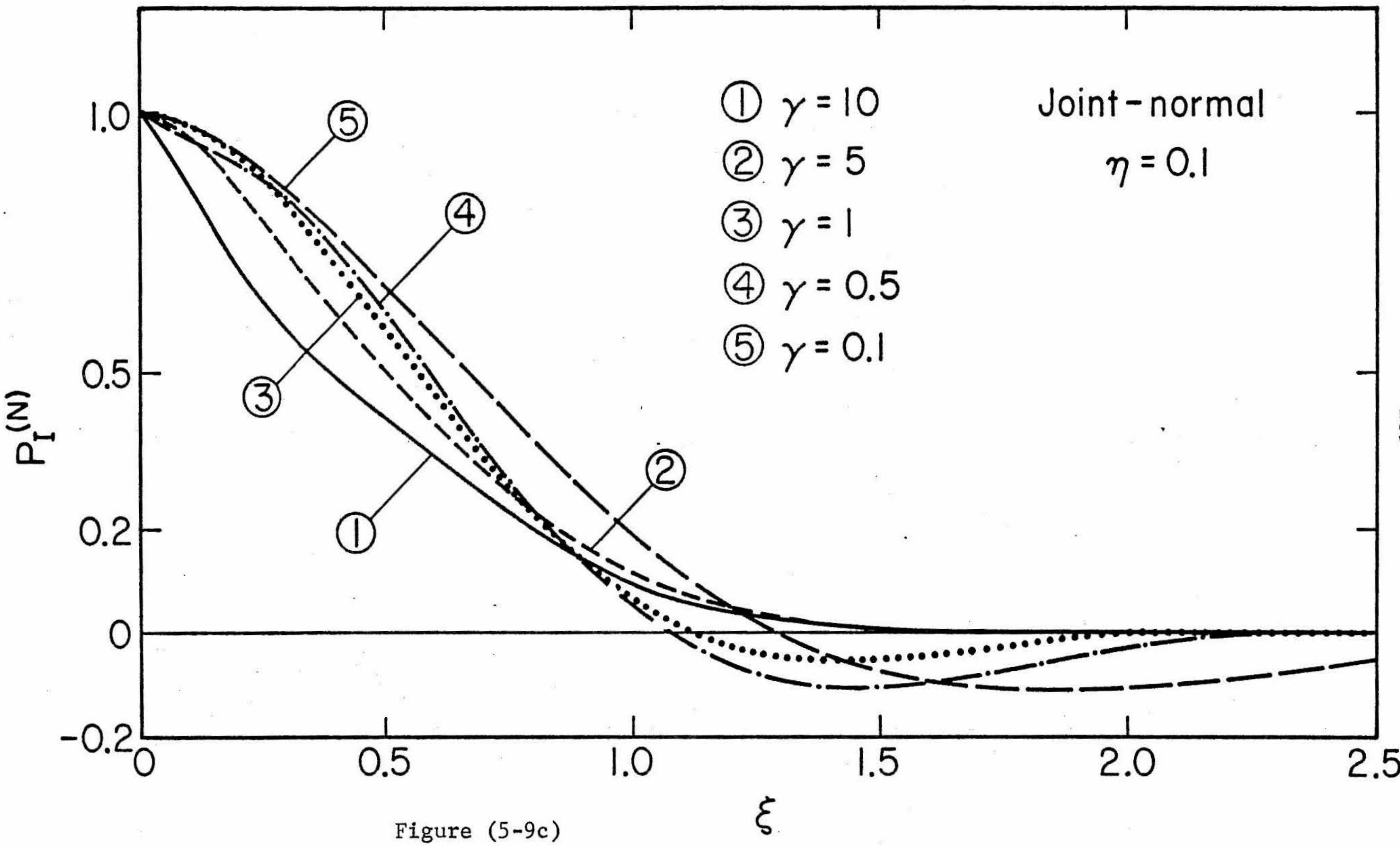

1.0
0.5
0
m_z^2
0
1.0
2.0
3.0
4.0
η
① γ = 5,10,∞
② γ = 1.0
③ γ = 0.5
④ γ = 0.1
⑤ γ = 0
Joint-normal

Figure (5-9a)



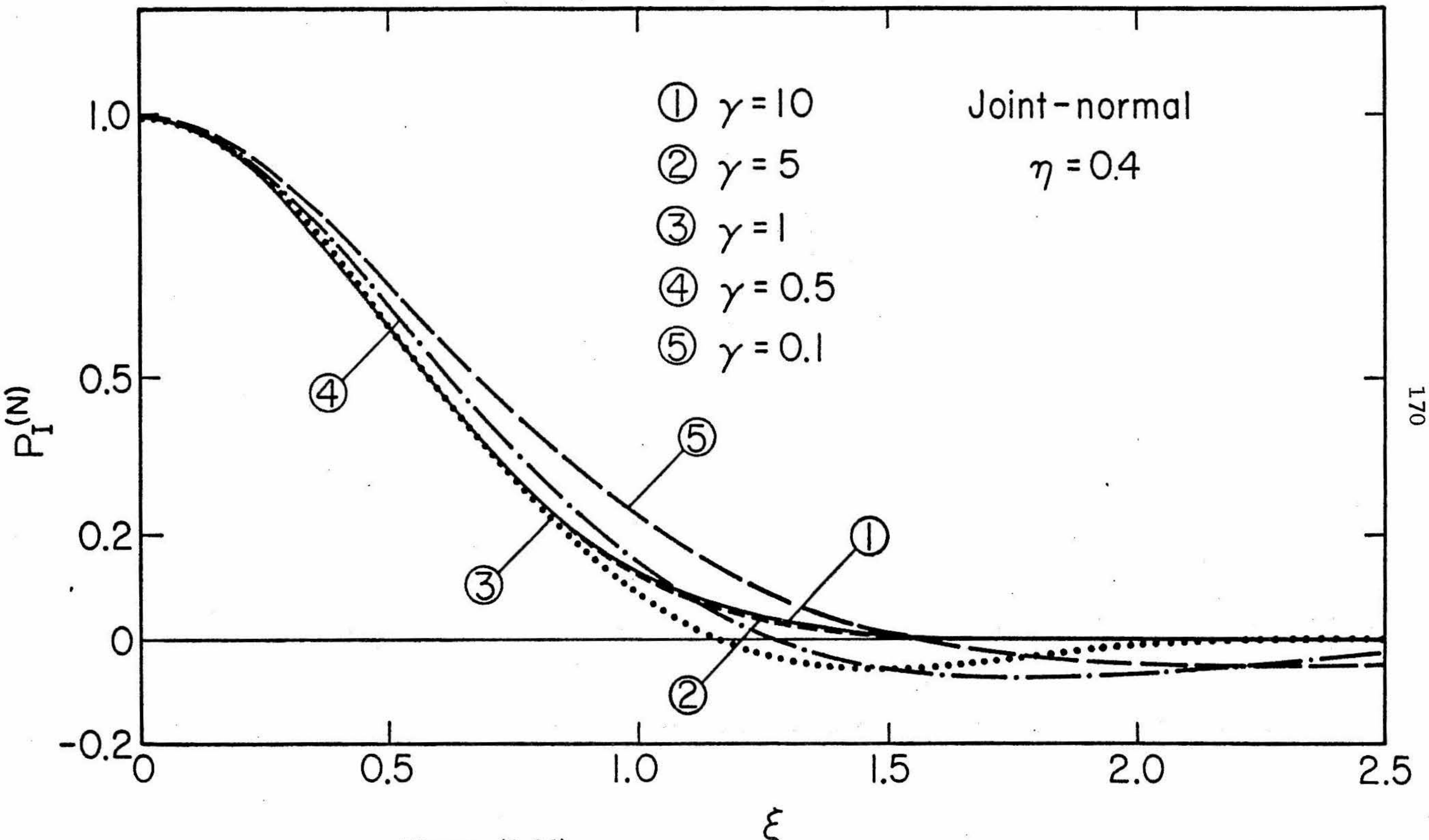
1.0
0.5
0
-0.2
$P_I^{(N)}$
0
1.0
2.0
3.0
4.0
ξ
① $\eta = 0.1$
② $\eta = 0.6$
③ $\eta = 1.0$
④ $\eta = 2.0$
⑤ $\eta = 5.0$
Joint-normal
$\gamma = 1.0$

Figure (5-9b)



① γ = 10
② γ = 5
③ γ = 1
④ γ = 0.5
⑤ γ = 0.1
Joint-normal
η = 0.1
$P_I^{(N)}$
1.0
0.5
0.2
0
-0.2
0
0.5
1.0
1.5
2.0
2.5
ξ

Figure (5-9c)



① γ = 10
② γ = 5
③ γ = 1
④ γ = 0.5
⑤ γ = 0.1
Joint-normal
η = 0.4
1.0
0.5
0.2
0
-0.2
0
0.5
1.0
1.5
2.0
2.5
ξ

Figure (5-9d)

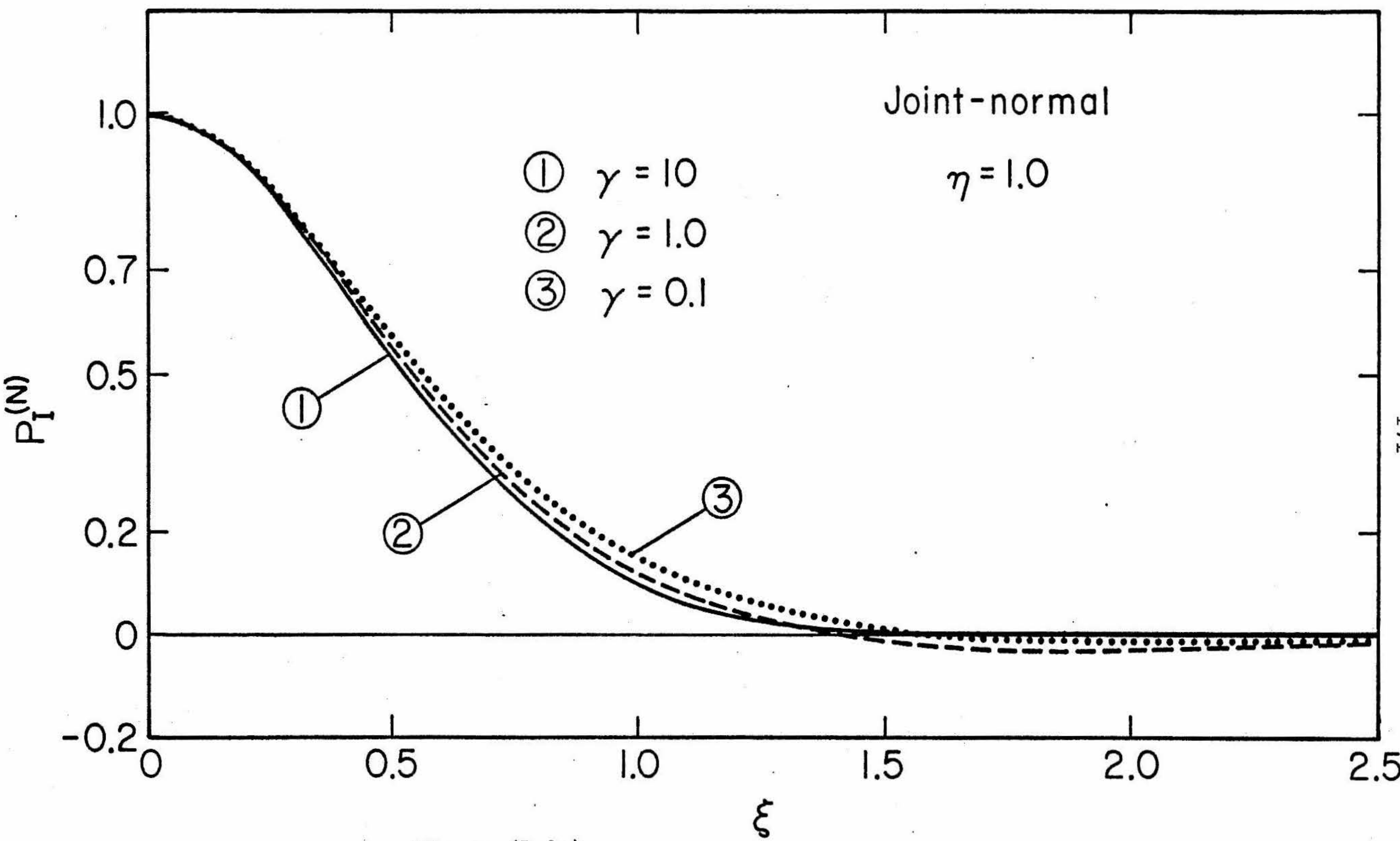

Joint-normal
η = 1.0
① γ = 10
② γ = 1.0
③ γ = 0.1
1.0
0.7
0.5
0.2
0
-0.2
0
0.5
1.0
1.5
2.0
2.5
$P_I^{(N)}$
ξ

Figure (5-9e)

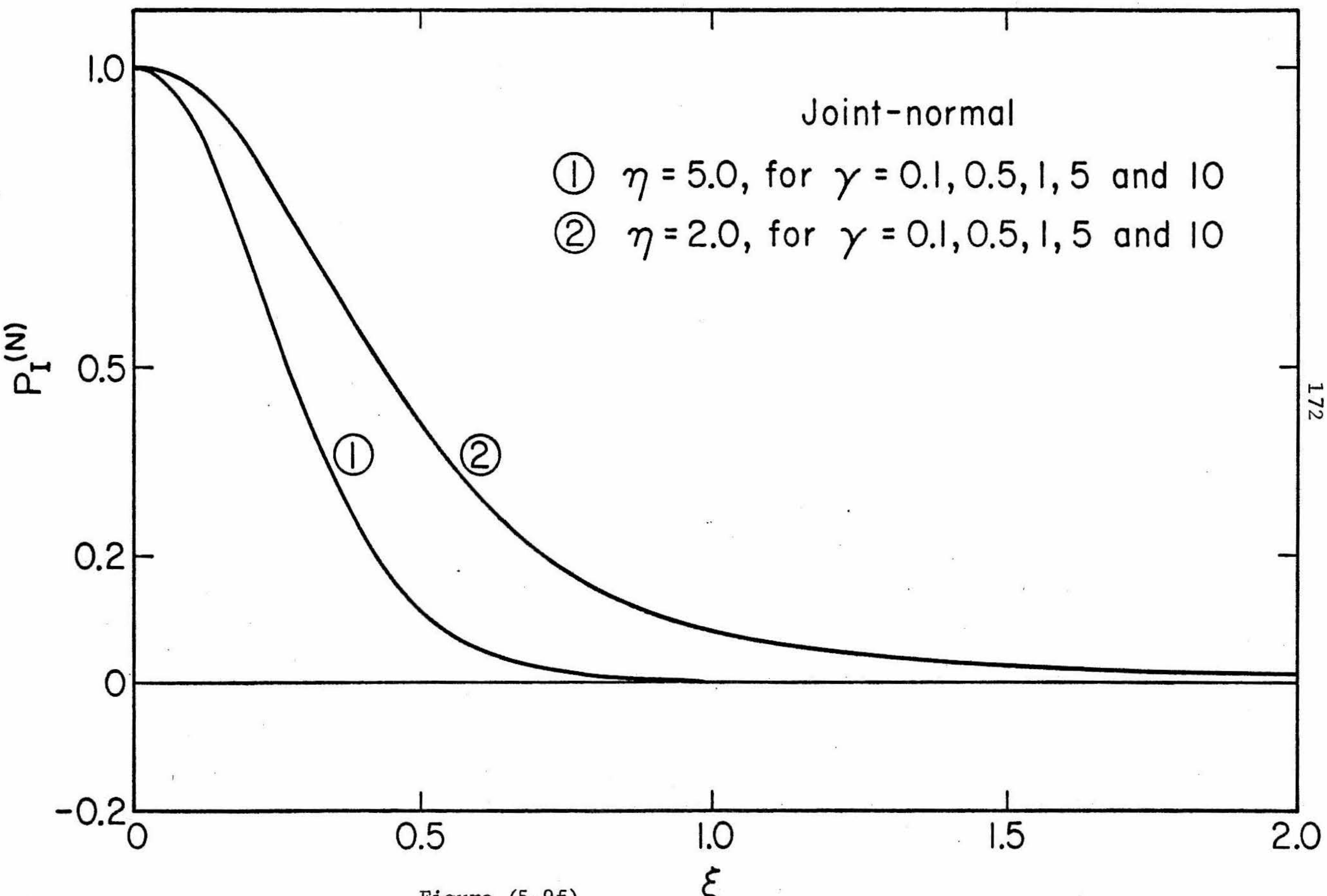

Joint-normal
① η = 5.0, for γ = 0.1, 0.5, 1, 5 and 10
② η = 2.0, for γ = 0.1, 0.5, 1, 5 and 10
①
②
1.0
0.5
0.2
0
-0.2
0
0.5
1.0
1.5
2.0
P_I^(N)
ξ

Figure (5-9f)

$$\rho_{c.s.} = \frac{2}{q_o} \quad , \text{ for } \beta_A z \le 1 \qquad (5\text{-}175a)$$

and

$$\rho_{c.s.} = \frac{2}{q_o} \times \frac{1}{\sqrt{2\beta_A z}} \quad , \text{ for } \beta_A z \ge 1 \ . \qquad (5\text{-}175b)$$

Under the joint-normal assumption, the scintillations are strong ($m_z^{\ 2} \simeq 1$) for $\beta_A z > 1$, which is different from the result of last section for $\gamma > 1$. Clearly the criterion here for strong scintillation under the joint-normal assumption is not correct. In particular, in the limit $\gamma = \infty$, the diffraction term is zero and there is no amplitude fluctuation. Thus for $\gamma > 1$, and $1 < \beta_A z < \gamma^{2/3}$ for Gaussian spectrum (or $1 < \beta_A z < \gamma^{5/11}$ for Kolmogorov spectrum), the probability distribution of the random wave cannot be joint-normal. In subsection §(3), we have showed that for $\beta_A z \le 1$, the probability distribution is log-normal, so that for $\beta_A z \le 1$, the distribution is not joint-normal either. In next subsection we will show that the log-normal assumption is valid in the strong scintillation region in predicting the spatial intensity correlation function.

§(5). Validity of Joint-normal Assumption

In order to prove the validity of a certain kind of probability distribution, one must check the result of the distribution with <u>every</u> moment of the random wave, which is an impractical work. Instead, we will check the joint-normal assumption only with the 4-th moment Γ_4.

From Eq. (5-96), we have for the 4-th moment Γ_4,

$$[\frac{\partial}{\partial\eta} + L(\underset{\sim}{\alpha},\underset{\sim}{\beta})]\ \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0 \tag{5-176}$$

and

$$L(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \frac{-i}{\gamma}\ \nabla_{\underset{\sim}{\alpha}}\cdot\nabla_{\underset{\sim}{\beta}} - f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \tag{5-177}$$

where

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \langle u(\eta,\underset{\sim}{\alpha}_o)u^*(\eta,\underset{\sim}{\alpha}_o+\underset{\sim}{\alpha})u(\eta,\underset{\sim}{\alpha}_o+\underset{\sim}{\alpha}+\underset{\sim}{\beta})u^*(\eta,\underset{\sim}{\alpha}_o+\underset{\sim}{B})\rangle \tag{5-178}$$

and

$$f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 2 - 2\hat{H}(\underset{\sim}{\alpha})-2\hat{H}(\underset{\sim}{\beta})+\hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{B})+\hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta}) \quad . \tag{5-179}$$

Under the joint-normal assumption, it is easy to show that 4-th moment Γ_4 is given by

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}{}^2(\eta,\underset{\sim}{\alpha})+\Gamma_{1,1}{}^2(\eta,\underset{\sim}{\beta})+\Gamma_{2,0}(\underset{\sim}{\alpha}+\underset{\sim}{\beta})\Gamma_{2,0}{}^*(\underset{\sim}{\alpha}-\underset{\sim}{\beta})-2\bar{u}^4$$

$$\equiv \Gamma_4{}^{(J)}(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \tag{5-180}$$

where $\Gamma_{1,1}$, $\Gamma_{2,0}$ and $\bar{u}$ are given by Eqs. (5-130), (5-131), and (5-180). We call $\Gamma_4{}^{(J)}(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ the 4-th moment under joint-normal assumption. It is easy to show that

$$[\frac{\partial}{\partial\eta} + L(\underset{\sim}{\alpha},\underset{\sim}{\beta})]\ \Gamma_4{}^{(J)}(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = E(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \tag{5-181}$$

where

$$E(\eta,\alpha,\beta) = E_A+E_B+E_C+E_D\ , \tag{5-182a}$$

$$E_A = -\ [2\hat{H}(\underset{\sim}{\beta})-\hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{\beta})-\hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})]\cdot\exp\left\{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]\right\}, \tag{5-182b}$$

$$E_B = -\ [2\hat{H}(\underset{\sim}{\alpha})-\hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{B})-\hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{B})]\cdot\exp\left\{-2\eta[1-\hat{H}(\underset{\sim}{\beta})]\right\}, \tag{5-182c}$$

$$E_C = -\ 2[H(\underset{\sim}{\alpha})+H(\underset{\sim}{\beta})]\ \Gamma_{2,0}(\underset{\sim}{\alpha}+\underset{\sim}{\beta})\ \Gamma_{2,0}{}^*(\underset{\sim}{\alpha}-\underset{\sim}{\beta}), \tag{5-182d}$$

and
$$E_D = 2[2H(\underset{\sim}{\alpha})+2H(\underset{\sim}{\beta}) - H(\underset{\sim}{\alpha}+\underset{\sim}{\beta}) - H(\underset{\sim}{\alpha}-\underset{\sim}{\beta})]\ \bar{u}^4 \quad . \tag{5-182e}$$

Since $\bar{u}^4 = e^{-2\eta}$ and $|\Gamma_{2,0}| \le e^{-2\eta}$, the last two terms in the right-hand side of Eq. (5-182a), E_C and E_D, can be neglected for large η. Consider the E_A in Eq. (5-182b). We see that E_A is of order of $e^{-2\eta}$, which is small for large η, except for $|\underset{\sim}{\alpha}| \ll 1$, since $[1-\hat{H}(\underset{\sim}{\alpha})]$ in the exponential of Eq. (5-182b) is of order of 1 except near $|\underset{\sim}{\alpha}| = 0$. For $|\underset{\sim}{\alpha}| \ll 1$, it is easy to demonstrate that $|E_A(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})|$ has its maximum value at $\underset{\sim}{\beta} = 0$. Thus

$$\text{for } |\underset{\sim}{\alpha}| \ll 1, \quad |E_A(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})| \le |E_A(\eta,\underset{\sim}{\alpha},0)|$$
$$= 2[1-\hat{H}(\underset{\sim}{\alpha})]\ e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} \le (\frac{e^{-1}}{\eta}) \quad . \tag{5-183}$$

Similarly, E_B can be shown to be of the order of, or less than $(\frac{e^{-1}}{\eta})$. Thus

$$|E(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})| \le 0\ (\frac{1}{\eta}) \tag{5-184}$$

and for $\eta \gg 1$, $\Gamma_4^{(J)}$ satisfies the differential equation for Γ_4 in Eq. (5-176). Furthermore, we have from Eq. (5-105) for strong scintillation

$$\Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\alpha}) + \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\beta}) - \bar{u}^4$$
$$\simeq \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\alpha}) + \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\beta}) \quad . \tag{5-185}$$

The criterion for strong scintillation is given by Eq. (5-113) and Eq. (5-114). For $\eta \gg 1$, we have from Eq. (5-180)

$$\Gamma_4^{(J)}(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \simeq \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\alpha}) + \Gamma_{1,1}^{\ 2}(\eta,\underset{\sim}{\beta}) \tag{5-186}$$

by noting that $|\Gamma_{2,0}| \le e^{-\eta}$. Comparing Eqs. (5-185) and (5-186), we find that $\Gamma_4^{(J)}$, the 4-th moment under the joint-normal assumption, is the same as the true 4-th moment Γ_4 for strong scintillation. Thus the joint-normal assumption is valid in strong scintillation region in predicting spatial intensity correlation function.

Finally we draw the following conclusions from the above discussion

(i) For $\beta_A z \le 1$, the probability distribution is log-normal.

(ii) For strong scintillation, the distribution is joint-normal as far as the intensity correlation function is concerned. The criterion for strong scintillation is given by Eqs. (5-113) and (5-114).

(iii) For $\gamma > 1$ and $1 < \beta_A z < \gamma^{5/11}$ for Kolmogorov spectrum (or $1 < \beta_A z < \gamma^{2/3}$ for Gaussian spectrum), the distribution is not log-normal, nor is joint-normal.

(iv) It can be shown that for large η, the joint-normal distribution goes to Rice-distribution. Since $\langle u \rangle$ is real, let

$$u = \langle u \rangle + u_c + i\, u_s \tag{5-187}$$

where u_c and u_s are real. Under joint-normal assumption, u_c and u_s are joint-normal with zero mean. From Eq. (5-187) we have

$$\Gamma_{1,1}(z, \underset{\sim}{\rho}=0) = \langle u\, u^* \rangle = \langle u \rangle^2 + \langle u_c^{\,2} \rangle + \langle u_s^{\,2} \rangle \tag{5-188a}$$

and

$$\Gamma_{2,0}(z, \underset{\sim}{\rho}=0) = \langle u \rangle^2 + \langle u_c^{\,2} \rangle - \langle u_s^{\,2} \rangle + 2i\, \langle u_1 u_2 \rangle \quad . \tag{5-188b}$$

we note $\langle u \rangle = e^{-\frac{\eta}{2}}$, $|\Gamma_{2,0}| \le e^{-\eta}$ and $\Gamma_{1,1}(z,\underset{\sim}{\rho}=0) = 1$. For $\eta \gg 1$, we then have from Eq. (5-188) that

$$\langle u_c^{\,2} \rangle \simeq \langle u_s^{\,2} \rangle \simeq \frac{1}{2}\,, \text{and } \langle u_c u_s \rangle \simeq 0 \ . \qquad (5\text{-}189)$$

Thus for strong scintillation, u_c and u_s are two independent Gaussian variable with same variance. Therefore the random wave u has a Rice-distribution (Rice 1954).

Furthermore, since for strong scintillation $\langle u \rangle = e^{-\frac{\eta}{2}} \ll 1$, and from Eq. (5-189), $\langle u_c^{\,2} \rangle = \langle u_s^{\,2} \rangle = \frac{1}{2}$, Rice distribution reduces to Rayleigh distribution where we have for the probability distribution P(I),

$$P(I) \quad = e^{\,I/\langle I \rangle} \ / \ \langle I \rangle \qquad (5\text{-}190)$$

where $\langle I \rangle$ is the mean intensity and $\langle I \rangle = 1$ for normalized incoming wave.

The above discussions are for waves propagating in a <u>thick</u> medium. Similarly, for the <u>thin screen</u> case (see Chapter 3), one can also show that the joint-normal assumption is valid in predicting the spatial intensity correlation function and the probability distribution for intensity I is given by Eq. (5-190) in the strong scintillation region.

VII. Intensity Correlation Function with Different Frequencies

§(1). Introduction

In interstellar scintillations, the intensity correlation function with different frequencies has been measured (Scott & Collins 1968, Komesaroff et al. 1971, Rickett 1969, Lang 1971, Sutton 1971.) In particular, the decorrelation frequency $f_I = (\frac{\omega_I}{2\pi})$ of the intensity correlation function can be determined. The decorrelation frequency f_I is defined as the frequency difference of two observing frequency-channels, beyond which the intensities measured at these two channels are nearly uncorrelated. Many authors (Salpeter 1969, Lang 1971, Cronyn 1970a ,Sutton 1970) have tried to relate the decorrelation frequency f_I to the characteristic time t_c of pulse broadening and they found $2\pi f_I t_c=1$ under geometrical optics. But as pointed out in Chapter 2, the region of validity of the geometrical optics is limited. In Section IV of this chapter, we found that it is the decorrelation frequency $\frac{\omega_c}{2\pi}$ $(= \frac{ck_c}{2\pi})$ of the second moment $\Gamma_{1,1}$ that relates directly to the characteristic pulse broadening time t_c and that $\omega_c t_c \simeq 1$. It is the purpose of this section to find the intensity correlation function and the decorrelation frequency f_I.

§(2). Intensity Correlation Function in the Strong Scintillation Region

The intensity correlation function can be obtained by solving Eq. (4-44) of the general 4-th moment $\Gamma_{2,2}$. However, it is a formidable job to solve the complicated equation of $\Gamma_{2,2}$. In Section VI of this chapter, we pointed out that the joint-normal distribution of the random wave is valid for strong scintillation in predicting the spatial intensity correlation function, which leads us to believe the probability distribution of the random wave is joint-normal. Therefore we will find in this section the intensity correlation function with different frequencies under the joint-normal assumption.

For simplicity, we will assume in the initial plane $u(z=0,\underset{\sim}{\rho},k) = 1$ for all k and $\underset{\sim}{\rho}$. It follows from Eqs. (5-129b) and (5-133) that the intensity correlation function measured at $(z,\underset{\sim}{\rho}_1)$ between two wavenumbers k_1 and k_2

$$P_I(z,k_1,k_2) = |\Gamma_{1,1}(z,k_1,\underset{\sim}{\rho}_1,k_2,\underset{\sim}{\rho}_2=\underset{\sim}{\rho}_1)|^2+|\Gamma_{2,0}(z,k_1,\underset{\sim}{\rho}_1,k_2,\underset{\sim}{\rho}_2=\underset{\sim}{\rho}_1)|^2$$

$$- 2\, e^{-\frac{A_\beta(0)z}{2}\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)} \quad . \tag{5-191}$$

The function $\Gamma_{1,1}$ in Eq. (5-191) has been solved and discussed in Section IV. The last term in the right-hand side of Eq. (5-191) is small and can be neglected for strong scintillation. We now consider $\Gamma_{2,0}$. From the complete set of moment equations in Eq. (37), we have in particular for $\Gamma_{2,0}$,

$$\frac{\partial}{\partial z}\Gamma_{2,0}(z;k_1,\underset{\sim}{\rho}_1;k_2,\underset{\sim}{\rho}_2) = \frac{i}{2}\left[\frac{\nabla_1^2}{k_1}+\frac{\nabla_2^2}{k_2}\right]\Gamma_{2,0}$$

$$- \frac{1}{4}\left[\frac{A_\beta(0)}{k_1^2}+\frac{A_\beta(0)}{k_2^2}+2\,\frac{A_\beta(\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2)}{k_1k_2}\right]\Gamma_{2,0} \quad . \tag{5-192}$$

Since the initial condition is homogeneous and the random medium is also statistically homogeneous in the transverse plane, $\Gamma_{2,0}$ depends on $|\underset{\sim}{\rho}| = |\underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2|$ only. Therefore $\nabla_1^2 = \nabla_2^2 = \nabla_{\underset{\sim}{\rho}}^2$ where $\underset{\sim}{\rho} = \underset{\sim}{\rho}_1-\underset{\sim}{\rho}_2$ and Eq. (5-192) can be written as

$$\frac{\partial}{\partial z}\Gamma_{2,0}(z;k_1,k_2;\underset{\sim}{\rho}) = \frac{i}{2}\left[\frac{1}{k_1}+\frac{1}{k_2}\right]\nabla_{\underset{\sim}{\rho}}^{2}\Gamma_{2,0}$$

$$-\frac{1}{4}\left[\frac{A_\beta(0)}{k_1^2}+\frac{A_\beta(0)}{k_2^2}+2\,\frac{A_\beta(\underset{\sim}{\rho})}{k_1k_2}\right]\Gamma_{2,0} \quad . \qquad (5\text{-}192')$$

Again, for small difference between k_1 and k_2, we set as in Eq. (5-50) of Section IV,

$$k_1 = k + \frac{\Delta k}{2} \; , \; k_2 = k - \frac{\Delta k}{2} \quad , \qquad (5\text{-}193)$$

from which we obtain to the order of $(\frac{\Delta k}{2})^2$,

$$\frac{\partial}{\partial z}\Gamma_{2,0}(z;k_1,k_2;\underset{\sim}{\rho}) = \frac{i}{k}\left[1+\frac{\Delta k^2}{4k^2}\right]\Delta_{\underset{\sim}{\rho}}^{2}\Gamma_{2,0} - \frac{\Gamma_{2,0}}{2}\left\{\left[1+\frac{3\Delta k^2}{4k^2}\right]\frac{A_\beta(0)}{k^2}\right.$$

$$\left. + \frac{A_\beta(\rho)}{k^2}\left[1+\frac{\Delta k^2}{4k^2}\right]\right\} \quad . \qquad (5\text{-}194)$$

Suppose that $|\frac{\Delta k}{k}| \ll 1$, which is always true in interstellar scintillation. Then we have for $\Gamma_{2,0}$

$$\frac{\partial}{\partial z}\Gamma_{2,0}(z;k_1,k_2;\underset{\sim}{\rho}) \simeq \frac{i}{k}\nabla_{\underset{\sim}{\rho}}^{2}\Gamma_{2,0} - \frac{1}{2k^2}\left[A_\beta(0)+A_\beta(\underset{\sim}{\rho})\right]\Gamma_{2,0} \; , \qquad (5\text{-}195)$$

which shows that $\Gamma_{2,0}(z;k_1,k_2;\underset{\sim}{\rho})$ is about the same as the function $\Gamma_{2,0}(z,k,\underset{\sim}{\rho})$ in Eq. (5-132), where the wave numbers are equal. From the discussions in Section VI, we know $|\Gamma_{2,0}(z,\underset{\sim}{\rho})| \le e^{-\frac{A_\beta(0)z}{2k^2}} \ll 1$

for strong scintillations. Therefore we have for strong scintillation

$$P_I(z,k_1,k_2) \simeq |\Gamma_{1,1}(z,\underset{\sim}{\rho} = 0, k_1,k_2)|^2 \qquad (5\text{-}196)$$

where $\Gamma_{1,1}(z,\underset{\sim}{\rho},k_1,k_2)$ is given by Eq. (5-49').

It follows from Eqs. (5-56) and (5-53) that

$$P_I(z,k,\Delta k) = P_I(z,k_1,k_2) = P_{IR}(z,k,\Delta k) \times P_{ID}(z,k,\Delta k) \quad (5\text{-}197)$$

where
$$P_{IR}(z,k,\Delta k) = |\Gamma_R(z,\underset{\sim}{\rho} = 0,\Delta k)|^2 = \exp\left\{-\frac{\Delta k^2}{2k^4} A_\beta(0)z\right\}, \qquad (5\text{-}198a)$$

and
$$P_{ID}(z,k,\Delta k) = |\Gamma_D(z,\underset{\sim}{\rho}=0,\Delta k)|^2 . \qquad (5\text{-}198b)$$

P_{IR} is the intensity correlation function where only "pure refraction" exists and P_{ID} is caused by the diffraction of the wave by the random medium. The total intensity correlation P_I is simply the multiplication of these two functions P_{IR} and P_{ID}. P_{IR} has a decorrelation frequency ω_{IR}

$$\omega_{IR} = \sqrt{2}k^2 c \quad (A_\beta(0)z)^{\frac{1}{2}} . \qquad (5\text{-}199)$$

The numerical solution of Γ_D has been given in Section IV. From the results of Γ_D, we can easily compute P_{ID}. Figure (5-10) shows the numerical result of $P_{ID}(z,k,\Delta k)$ for the medium with Kolmogorov spectrum. We note that the decorrelation frequency $f_{ID} = \omega_{ID}/2\pi$ of P_{ID} is the same as that of Γ_D and is given from Eq. (5-66) by

$$\omega_{ID} = \Delta k_c \cdot c = c\beta_o^{-2/\nu}\, k^{2(\nu+2)/\nu}(z/2)^{-(\nu+2)/\nu} \qquad (5\text{-}200)$$

Figure (5-10)

The frequency decorrelation function $P_{ID}(\Delta\omega)$ is plotted as a function of frequency difference $\Delta\omega$ for a Kolmogorov spectrum. ω_{ID} is the decorrelation frequency given by Eq. (5-200). Note that the abscissa is $(\Delta\omega/\omega_{ID})^{\frac{1}{2}}$.

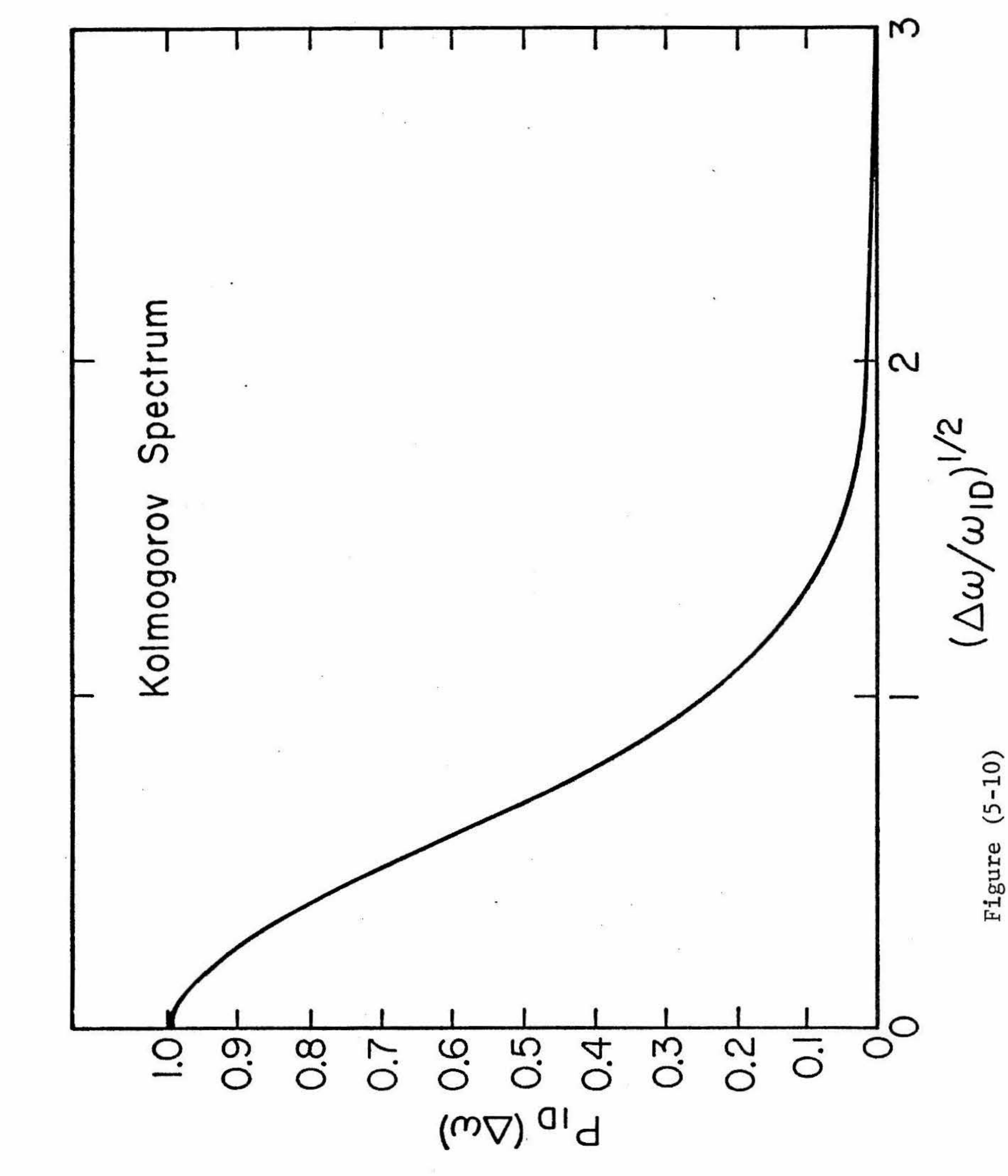

Kolmogorov Spectrum
$P_{1D}(\Delta\omega)$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
$(\Delta\omega/\omega_{1D})^{1/2}$

Figure (5-10)

where β_o is given by Eqs. (B-3') and (B-18), and $\gamma = 2$ and 5/3 for the Gaussian and the Kolmogorov spectra respectively. We also note that $\omega_I \approx \omega_{ID}$ for $\omega_{ID} \ll \omega_{IR}$ and $\omega_I \approx \omega_{IR}$ for $\omega_{ID} \gg \omega_{IR}$.

Thus for strong scintillation, the decorrelation frequency $(\omega_I/2\pi)$ of intensity correlation function is the same as the decorrelation frequency of the second moment $\Gamma_{1,1'}$, which is directly related to the characteristic pulse broadening time t_c. The relation between ω_I(or f_I) and t_c can be written as

$$\omega_I \, t_c = 1 \qquad (5\text{-}201)$$

or

$$2\pi \, f_I \, t_c = 1 \quad . \qquad (5\text{-}202)$$

The above discussions are for waves propagating in a thick medium. Similarly one can show that Eq. (5-202) also holds for the decorrelation frequency f_I of intensity and the width t_c of pulse-broadening in the thin screen case (see Chapter 3).

Chapter 6

Selected Applications to the Problem of Interplanetary Scintillations

1. Introduction

Interplanetary scintillations of radio waves provide an inexpensive and sensitive probe which can determine many of the properties of the solar wind. Since the discovery of these effects a decade or so ago there have been a large number of papers concerning observations and interpretation in terms of solar-wind fluctuations. (Clarke 1964, Hewish et al. 1964, Cohen et al. 1967, Dennison & Hewish 1967, Salpeter 1967, Hollweg 1968, Cronyn 1970b, Hollweg 1970, Jokipii 1970, Jokipii & Hollweg 1970, Ekers & Little 1971, Little & Ekers 1971, Lovelace et al. 1970, Young 1971, Armstrong & Coles 1972, Jokipii & Lee 1972, 1973.) The scintillations are the manifestation of the diffraction and refraction of radio waves by electron-density fluctuations in the interplanetary plasma.

Observations of interplanetary scintillations are one of two kinds, they either refer to the correlation of the fluctuations in intensity at one point as a function of time, or they refer to observations at more than one point and measure the correlation as a function of position and time. As the plasma turbulence is convected outward from the sun, a stationary observer on the earth will see temporal fluctuations of the radio waves and will be able to calculate the correlation of the fluctuations in intensity as function of time. The frequency power spectrum of intensity fluctuations can then be obtained from the Fourier transform of the correlation function in time. For two-station observations, one defines the "pattern velocity" of the solar wind as the distance between the two stations divided by the correlation time at which the cross-correlation function between the two stations gains its maximum value. (Jokipii & Lee, 1973.) (See Figures (6-1) and (6-2).) The measured "pattern velocity" is, in general, different from the true mean wind velocity. Thus in probing the solar-wind velocity by two-station (or multi-station) observations, one has to determine the relation between the "pattern velocity" and the true wind velocity.

The fluctuations in the solar-wind velocity affect the shape of the frequency power spectrum of intensity fluctuations and the ratio between "pattern velocity" and the wind velocity because in the existence of the velocity fluctuations, different parts of the plasma turbulence is convected past at different velocity and the fluctuations of velocity can destroy the fluctuation pattern of the plasma turbulence. It is the purpose of this chapter to study the effects of velocity fluctuations on the frequency power spectrum and on the ratio between pattern velocity and true wind velocity in three-dimensional model of interplanetary scintillations (Young 1971; Rytov 1971; Jokipii & Lee, 1972, 1973).

The effects of velocity fluctuations on the frequency power spectrum and on the ratio between pattern velocity and mean wind velocity has been estimated under "thin screen" model, in which the scattering medium is replaced by a "phase-changing screen" at some "mean" distance $\bar{z}$ from the observer (Little & Ekers 1971, Ekers & Little 1971, Armstrong & Coles 1972). Although the thin screen model provides a first-order estimate of the effects of velocity fluctuations, it is not realistic in interplane-

Figure Captions

Figure (6-1). Schematic representation of the parameters used in the calculations. The y-axis is chosen normal to the ray path in the plane formed by the source, sun and observer. L is the distance along the ray path from the point of closest approach to the sun to the observer. A and B are two observing points that determine the cross-correlation $C_k(x=0,y,z=L,\tau)$. θ is the elongation angle.

Figure (6-2). Schematic representation of observations at two points A and B and the cross correlation. The parameter y/τ^* is related to the wind speed, as discussed in the text.

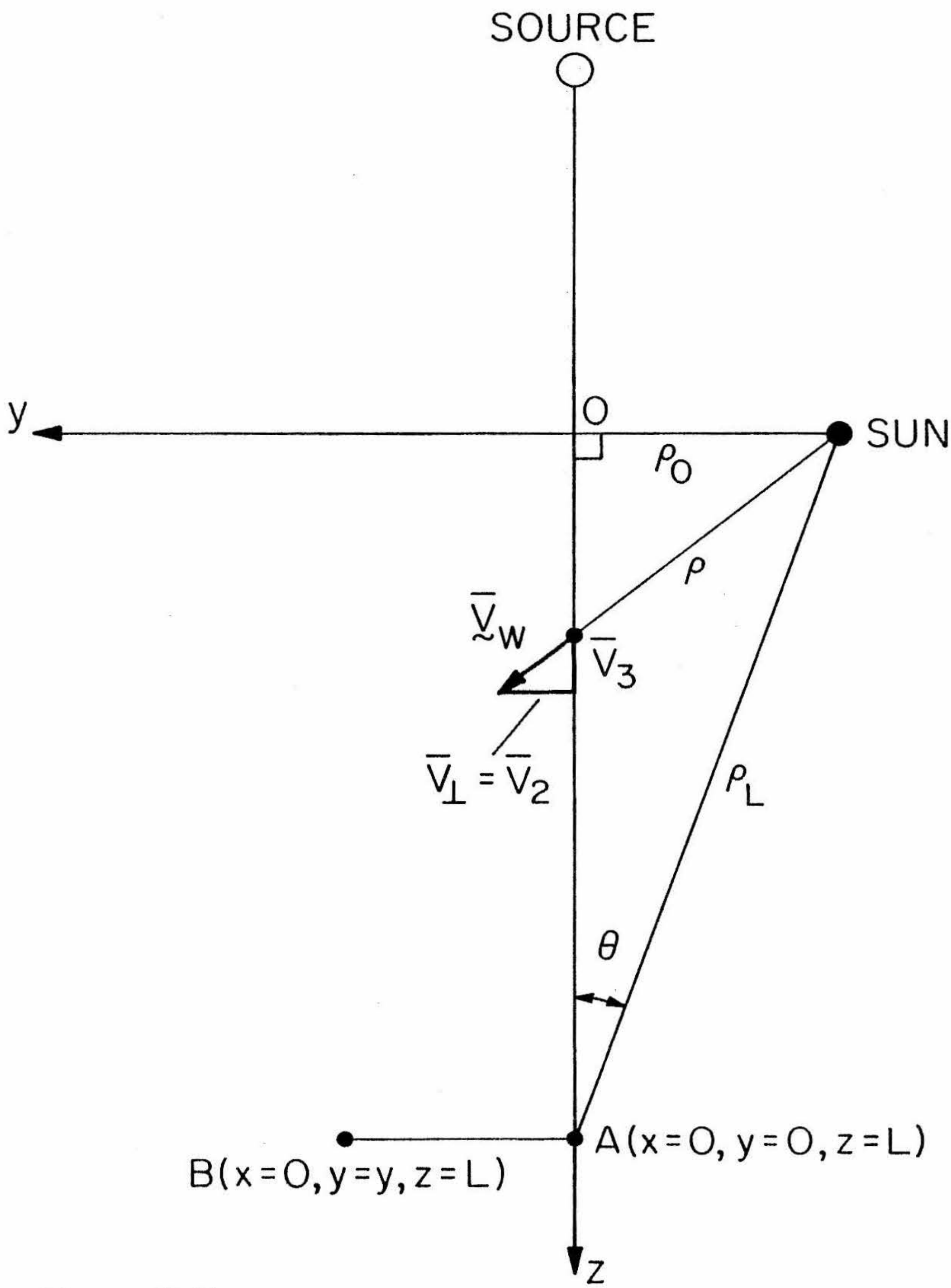

SOURCE
y
O
SUN
ρ_O
ρ
$\bar{V}_W$
$\bar{V}_3$
$\bar{V}_\perp = \bar{V}_2$
ρ_L
θ
B(x=O, y=y, z=L)
A(x=O, y=O, z=L)
z

Figure (6-1)

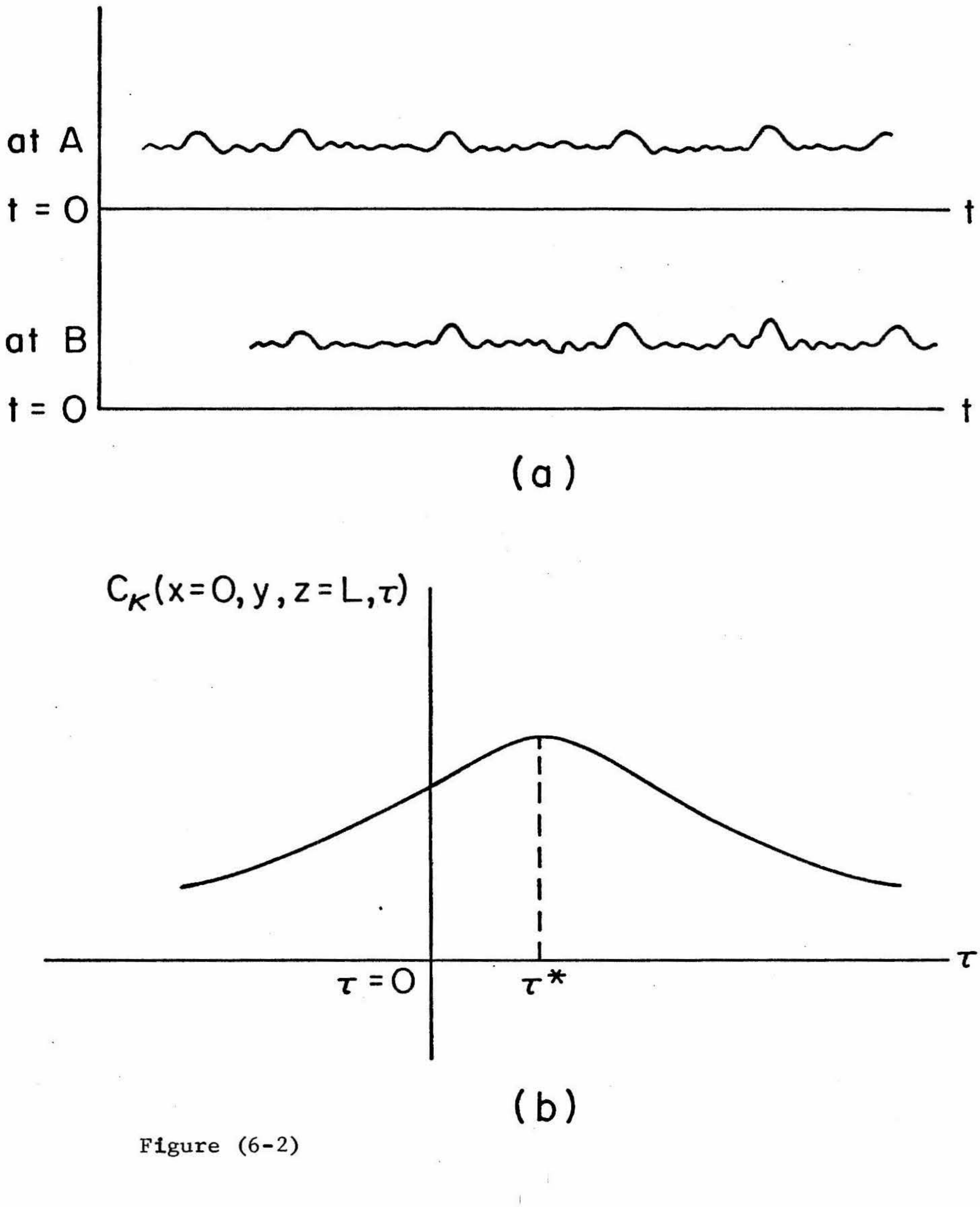


Figure (6-2)

tary scintillations for the following reasons.

(a) In the case of weak scattering, let $M_z^2(\underset{\sim}{q})$ be the power spectrum of intensity fluctuation contributed by a thin layer of plasma turbulence at a distance z from the observer, and let $\Phi_o^2 P_{\Phi}(z;\underset{\sim}{q})$ be the two-dimensional Fourier transform of the phase fluctuation caused by the thin layer of plasma turbulence. Then one can show that

$$M_z^2(\underset{\sim}{q}) = 4 \sin^2(\frac{q^2 z}{2k}) \, \Phi_o^2 \, P_{\Phi}(z;\underset{\sim}{q}) \qquad (6\text{-}1)$$

(c.f., Eq. (3-14)) and that the total power spectrum $M_T^2(\underset{\sim}{q})$ of the intensity fluctuation is the sum of the power spectrum contributed by each layer (Salpeter 1967, Young 1971, Jokipii & Lee 1972, 1973). Since there is only one thin screen in "thin screen" model, the total power spectrum would be the same as given by Eq. (6-1) with z replaced by the mean distance $\bar{z}$. But in interplanetary scintillation, each layer of the turbulence has a different z and, therefore, a different modulation factor $\sin^2(\frac{q^2 z}{2k})$ for the power spectrum $M_z^2(\underset{\sim}{q})$. The spread of z gives us the total power spectrum $M_T^2(\underset{\sim}{q})$ different from that predicted by the thin screen model. Furthermore, the determination of the mean distance $\bar{z}$ is complicated by the modulation factor $\sin^2(\frac{q^2 z}{2k})$.

(b) Since the solar wind flows radially outward from the sun, each layer of turbulence perpendicular to the line of sight has different drift velocity over the observer. This structure of the solar wind will reduce the ratio between the pattern velocity and the wind velocity. Jokipii & Lee (1972, 1973) have calculated the ratio in the absence of velocity fluctuations when the radio source is at a position with small solar elongation angle (see Fig. (6-1)). In this chapter, we will also calculate the ratio at large elongation angle. Thin screen model is not able to calculate this effect.

(c) Little & Ekers (1971) found that the velocity fluctuations have the effect of smearing the frequency power spectrum of intensity fluctuations. In three-dimensional model, different drift velocity of the turbulence layer has the same effect of smearing the frequency power spectrum as do the velocity fluctuations. Thus if one uses "thin screen" model to predict the velocity fluctuations, then one tends to overestimate the magnitude of velocity fluctuations.

It is the purpose of this chapter to study the effects of wind-velocity fluctuations on the shape of intensity power spectrum and on the ratio between the "pattern" and the true wind-velocity in the three-dimensional model of interplanetary scintillation.

II. Formulation

In this section, we will generalize the three-dimensional model of interplanetary scintillation to take the velocity fluctuations into account. The three-dimensional model has been presented by Young (1971), Rytov (1971), and Jokipii & Lee (1972, 1973). The model is based on the method of smooth perburbations (MSP) (see Chapter 2, Section III) presented by Tatarskii (1961). For completeness, we will repeat the derivations of Tatarskii (1961) and Jokipii & Lee (1972, 1973), and then generalize the formulation to include the effects of velocity fluctuations.

From Eqs. (3-38), (3-39a) and (3-39b) of Chapter 2, we have

$$u \equiv e^{\varphi} \equiv A\, e^{iS} \tag{6-2}$$

$$2ik \frac{\partial \varphi(\underset{\sim}{r})}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi(\underset{\sim}{r}) + k^2 \epsilon_k(\underset{\sim}{r}) = 0 \tag{6-3}$$

and

$$\begin{cases} \chi = \log A = \mathrm{Re}\, \varphi & \text{(6-4a)} \\ S = \mathrm{Im}\, \varphi \quad . & \text{(6-4b)} \end{cases}$$

Following Tatarskii (1961), we define $d\Psi(z,\underset{\sim}{q})$, $d\nu(z,\underset{\sim}{q})$, $da(z,\underset{\sim}{q})$ and $ds(z,\underset{\sim}{q})$ as the two-dimensional Fourier components of $\varphi(\underset{\sim}{r})$, $\epsilon_k(\underset{\sim}{r})$, $\chi(\underset{\sim}{r})$ and $S(\underset{\sim}{r})$, respectively. We have

$$\varphi(\underset{\sim}{r}) = \iint e^{i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}\, d\Psi(z,\underset{\sim}{q}) \tag{6-5a}$$

$$\epsilon_k(\underset{\sim}{r}) = 2\iint e^{i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}\, d\nu_k(z,\underset{\sim}{q}) \tag{6-5b}$$

$$\chi(\underset{\sim}{r}) = \iint e^{i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}\, da_k(z\ \underset{\sim}{q}) \tag{6-5c}$$

and

$$S(\underset{\sim}{r}) = \iint e^{i\underset{\sim}{q}\cdot\underset{\sim}{\rho}}\, ds_k(z,\underset{\sim}{q}) \tag{6-5d}$$

where $\underset{\sim}{\rho} \equiv (x,y)$ and $\underset{\sim}{q} = (q_1,q_2)$. Take the Fourier transform (two-dimensional) of Eq. (6-3) and get

$$\frac{\partial}{\partial z} d\Psi(z,\underset{\sim}{q}) + \frac{iq^2}{2k} d\Psi(z,\underset{\sim}{q}) - ikd\nu_k(z,\underset{\sim}{q}) = 0 \quad . \tag{6-6}$$

If the perturbations are zero at $z = z_o$, then Eq. (6-6) gives us

$$d\Psi(z,\underset{\sim}{q}) = ik \int_{z_o}^{z} e^{-\frac{iq^2(z-z')}{2k}} d\nu_k(z',\underset{\sim}{q})\, dz' \quad . \tag{6-7}$$

From Eqs. (6-4a), (6-4b) and (6-7), we can easily get

$$da_k(z,\underset{\sim}{q}) = k \int_{z_o}^{z} dz' \sin\left[\frac{q^2(z-z')}{2k}\right] d\nu(z',\underset{\sim}{q}) \tag{6-8}$$

and

$$ds_k(z,\underset{\sim}{q}) = k \int_{z_o}^{z} dz' \cos\left[\frac{q^2(z-z')}{2k}\right] d\nu(z',\underset{\sim}{q}) \tag{6-9}$$

where $q^2 = q_1^2 + q_2^2$.

Typically, one is interested in $F_k(z,\underset{\sim}{q})$, the two-dimensional power spectrum of the fluctuations in χ in the plane z = constant. By the Wiener-Khinchine theorem, this may be obtained by considering Eq. (6-8) and the product $\langle da_k da_k^* \rangle$. We generalize this result to the case where the statistical properties of the medium vary with position, so that we may apply the results to the solar wind. Such a generalization is particularly simple if, as in the solar wind, the scale of variation of the means is large compared with the correlation scale. This can be shown to be true in the solar wind. If this condition of gradual variation is satisfied, one may define a local spectrum $d\nu_k(z,\underset{\sim}{q})$ precisely as above since the random function is approximately stationary or homogeneous. In this case, we may define a power spectrum as a function of position $\underset{\sim}{r}$, $P_k(\underset{\sim}{r},q_1,q_2,q_3)$ where

$$P_k(\underset{\sim}{r},q_1,q_2,q_3) \equiv \frac{1}{4(2\pi)^3}\iiint \langle \epsilon_k(\underset{\sim}{r})\,\epsilon_k(\underset{\sim}{r}+\underset{\sim}{R})\rangle\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{R}}\, d\underset{\sim}{R}^3 \quad . \tag{6-10}$$

Then one can readily obtain the following result from Eq. (6-8) assuming $|z-z_o|$ is much greater than the correlation scale of $\epsilon_k(\underset{\sim}{r})$,

$$F_k(z,q_1,q_2) = 2\pi\, k^2 \int_{z_o}^{z} dz' \,\sin^2\left[\frac{q^2(z-z')}{2k}\right] P_k(z', q_1,q_2,q_3=0) \tag{6-11}$$

where P_k is to be evaluated along the line of sight, which is in the z direction.

A similar result may be obtained for the spectrum of fluctuations in phase,

$$G_k(z,q_1,q_2) = 2\pi k^2 \int_{z_o}^{z} dz' \cos^2\left[\frac{q^2(z-z')}{2k}\right] P_k(z',q_1,q_2,q_3=0) \quad . \quad (6\text{-}12)$$

One can measure the spatial power spectrum given by Eq. (6-11) by simultaneously recording the intensity of radio wave at a large number of stations. But this kind of observation is not practical. As mentioned above, observations of interplanetary scintillations are either to measure the correlation of fluctuations in intensity at one point as a function of time, or to measure at two (or more) stations the correlation as a function of position and time. For this reason, it is useful to compute the two-point, two-time correlation of the intensity. This may be done as follows. Let $\underset{\sim}{V}_w$ be the plasma velocity and assume the wind velocity is much larger than any wave velocities, so we may treat the density fluctuations as being frozen into the solar wind. Then we have the frozen-in condition as

$$\epsilon_k(\underset{\sim}{r},t) = \epsilon_k(\underset{\sim}{r}+\underset{\sim}{V}_w\tau, t+\tau) \quad (6\text{-}13)$$

which is true over distances small compared with the mean density variations. There is one more requirement for Eq. (6-13) to be true, namely, within the region with radius $|\underset{\sim}{V}_w\tau|$, the velocity $\underset{\sim}{V}_w$ must be nearly constant. This condition cannot be satisfied if the fluctuations of velocity exist within the scale size we are interested in. However, one can divide the plasma turbulence into many <u>classes</u>. In each class, the plasma has the same wind velocity. Then Eq. (6-13) is still correct for each class of plasma turbulence with same velocity. Since MSP is a linear approximation, the total power spectrum can be written as the sum of the power spectrum contributed by each class of plasma turbulence with same velocity provided there is no correlation between different classes. In the following approach, we will assume the density fluctuations and turbulent velocity fluctuations are uncorrelated.[1]

For each class, we have

$$\epsilon_{k,\underset{\sim}{V}_w}(\underset{\sim}{r},t) = \epsilon_{k,\underset{\sim}{V}_w}(\underset{\sim}{r}+\underset{\sim}{V}_w\tau, t+\tau) \tag{6-14}$$

where the subscript $\underset{\sim}{V}_w$ denotes the class of turbulence convected at velocity $\underset{\sim}{V}_w$. This can be shown to imply for the instantaneous spatial Fourier transform of $\epsilon_{k,\underset{\sim}{V}_w}(\underset{\sim}{r},t)$

$$d\nu_{k,\underset{\sim}{V}_w}(\underset{\sim}{q},t+\tau) = \exp\,(-i\underset{\sim}{q}\cdot\underset{\sim}{V}_w\tau)\; d\nu_{k,\underset{\sim}{V}_w}(\underset{\sim}{q},t) \quad . \tag{6-15}$$

Of interest is $d\nu_{k,\underset{\sim}{V}_w}(z,q_1,q_2,t)$, which is the two-dimensional transform of $\epsilon_{k,\underset{\sim}{V}_w}$, as a function of z and t. Manipulation of Eq. (6-15) yields

$$d\nu_{k,\underset{\sim}{V}_w}(z,q_1,q_2,t+\tau) = \exp\,(-iq_2V_2\tau-iq_1V_2\tau)d\nu_{k,\underset{\sim}{V}_w}(z-V_3\tau,q_1,q_2,t) \tag{6-16}$$

where $\underset{\sim}{V}_w = (V_1,V_2,V_3)$.

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[1]This is a reasonable assumption since in the solar wind the density scales that are important in producing scintillations ($\ell \sim$ 100-200 km) are small compared with the scales that contribute to fluctuations in $\underset{\sim}{V}_w$.

From Eq. (6-11), we may write the time-dependent amplitude contributed from the class of velocity $\underset{\sim}{V}_w$

$$da_{k,\underset{\sim}{V}_w}(z,q_1,q_2,t) = k \int_{z_o}^{z} dz' \sin\left[\frac{q^2(z-z')}{2k}\right] d\nu_{k,\underset{\sim}{V}_w} \quad . \qquad (6\text{-}17)$$

Using Eqs. (6-16) and (6-17), one may write

$$\langle da_{k,\underset{\sim}{V}_w}(z,q_1,q_2,t+\tau)\, da^*_{k,\underset{\sim}{V}_w}(z,q_1',q_2',t)\rangle$$

$$= k^2 \int_{z_o}^{z} dz' \int_{z_o}^{z} dz'' \sin\left[\frac{q^2(z-z')}{2k}\right] \sin\left[\frac{q^2(z-z'')}{2k}\right] \cdot \exp(-iq_1V_1\tau - iq_2V_2\tau)$$

$$\times \langle d\nu_{k,\underset{\sim}{V}_w}(z'-V_3\tau,q_1,q_2,t)\, d\nu^*_{k,\underset{\sim}{V}_w}(z'',q_1',q_2',t)\rangle \qquad (6\text{-}18)$$

Next make the following arguments:

(i) Following the argument of Tatarskii (1961, p.135), we see that since $\langle d\nu_{k,\underset{\sim}{V}_w}(z',q_1,q_2) d\nu^*_{k,\underset{\sim}{V}_w}(z'',q_1,q_2)\rangle$ is important only for $q|z'-z''| \lesssim 1$, we may replace z" by $z'-V_3z$ in the sine function provided $q_c/k \ll 1$ which is true in interplanetary trubulence. q_c is the characteristic scale of $P_k(\underset{\sim}{r},\underset{\sim}{q})$.

(ii) $$\int \langle d\nu_{k,\underset{\sim}{V}_w}(z',q_1,q_2,t)\, d\nu^*_{k,\underset{\sim}{V}_w}(z'',q_1',q_2',t)\rangle\, dz''$$

$$= 2\pi\, P_{k,\underset{\sim}{V}_w}(z',q_1,q_2,q_3=0)\, \delta(q_1-q_1')\delta(q_2-q_2')dq_1dq_2dq_1'dq_2' \quad .$$

(iii) $\langle da_{k,\underset{\sim}{V}_w}(z,q_1,q_2,t+\tau)\ da^*_{k,\underset{\sim}{V}_w}(z,q_1',q_2',t)\rangle$

$$= C_{k,\underset{\sim}{V}_w}(q_1,q_2,z,\tau)\ \delta(q_1-q_1')\delta(q_2-q_2')dq_1\ dq_2\ \ dq_1'\ dq_2' \quad .$$

These allow us to write immediately

$$C_{k,\underset{\sim}{V}_w}(q_1,q_2,z,\tau) = 2\pi k^2 \int_{z_o}^{z} dz'\ \sin\left[\frac{q^2(z-z')}{2k}\right] \sin\left[\frac{q^2(z-z'-V_3\tau)}{2k}\right]$$

$$\exp\ (-iq_1V_1\tau-iq_2V_2\tau)\quad P_{k,\underset{\sim}{V}_w}(z',q_1,q_2,q_3=0) \tag{6-19}$$

Since we are interested in cases where $(q_c^2V_3\tau/k) \ll 1$ we can drop $(V_3\tau)$ in the sine factor of the integral in Eq. (6-19). Thus we have

$$C_{k,\underset{\sim}{V}_w}(q_1,q_2,z,\tau) = 2\pi k^2 \int_{z_o}^{z} dz'\ \sin^2\left[\frac{q^2(z-z')}{2k}\right] \exp\ (-iq_1V_1\tau-iq_2V_2\tau)$$

$$\times\ P_{k,\underset{\sim}{V}_w}(z',q_1,q_2,q_3=0) \tag{6-20}$$

The total power $C_k(q_1,q_2,z,\tau)$ is the integration of $C_{k,\underset{\sim}{V}_w}(q_1,q_2,z,\tau)$ over all velocity space if we treat $P_{k,\underset{\sim}{V}_w}$ as the power density in velocity space (see footnote 1). Since velocity fluctuations and density fluctuations are independent, we may write

$$P_{k,\underset{\sim}{V}_w}(z',q_1,q_2,q_3=0) = f_{z'}(\underset{\sim}{V}_w)\ P_k(z',q_1,q_2,q_3=0) \quad . \tag{6-21}$$

And the total power spectrum $C_k(q_1,q_2,z,\tau)$

$$C_k(q_1,q_2,z,\tau) = 2\pi k^2 \int_{z_o}^{z} dz' \sin^2\left[\frac{q^2(z-z')}{2k}\right] \langle \exp(-iq_1V_1\tau - iq_2V_2\tau)\rangle_{z',\underset{\sim}{V}_w}$$

$$\times P_k(z',q_1,q_2,q_3=0) \tag{6-22}$$

where

$$\langle \exp(-iq_1V_1\tau - iq_2V_2\tau)\rangle_{z',\underset{\sim}{V}_w} = \int f_{z'}(\underset{\sim}{V}_w) \exp(-iq_1V_1\tau - iq_2V_2\tau)\, d\underset{\sim}{V}_w \; . \tag{6-23}$$

Since $\exp(-iq_1V_1\tau - iq_2V_2\tau)$ is independent of V_3, we can carry out the interpretation over V_3, and write

$$\langle \exp(-iq_1V_1\tau - iq_2V_2\tau)\rangle_{z,\underset{\sim}{V}_w} = \langle \exp(-iq_1V_1\tau - iq_2V_2\tau)\rangle_{z,V_1,V_2}$$

$$= \iint F_z(V_1,V_2) \exp(-iq_1V_1\tau - iq_2V_2\tau)\, dV_1 dV_2 \tag{6-24}$$

where $\quad F_z(V_1,V_2) \equiv \int f_z(\underset{\sim}{V}_w)\, dV_1 \quad .$

Thus

$$C_k(q_1,q_2,z,\tau) = 2\pi k^2 \int_{z_o}^{z} dz' \sin^2\left[\frac{q^2(z-z')}{2k}\right] \langle \exp(-iq_1V_1\tau - iq_2V_2\tau)\rangle_{z',V_1,V_2}$$

$$\times P_k(z',q_1,q_2,q_3=0) \quad . \tag{6-25}$$

Here $C_k(q_1,q_2,z,\tau)$ may be interpretated as the contribution to the cross-correlation of fluctuations in intensity at z from wavenumbers q_1 and q_2 at time-lag τ. Thus, the actual cross correlation for two points in the (x,y) plane separated by x and y at lag τ is given by

$$C_k(x,y,z,\tau) = \frac{1}{4\pi^2}\int_{\infty}^{\infty} dq_1 \int_{\infty}^{\infty} dq_2 \exp[i(q_1x+q_2y)] \; C_k(z,q_1,q_2,\tau) \; . \tag{6-26}$$

This is the desired result.

Consider some limiting cases. In the absence of the velocity fluctuations, Eqs. (6-25) and (6-26) reduce to that obtained by Jokipii & Lee (1972, 1973). In the thin screen case, we replace the integral over z' by some function $G(q_1,q_2)$ and Eq. (6-25) becomes

$$C_k(q_1,q_2,z,\tau) = G(q_1,q_2)\langle\exp(-iq_1V_1\tau-iq_2V_2\tau)\rangle_{V_1,V_2} \, . \tag{6-27}$$

Eq. (6-27) is just the equation used by Little & Ekers (1971).

Regarding to the validity of the MSP approximation, we will quote from Tatarskii (1961,1971) that MSP is valid if $|\nabla S| \ll k$, $\epsilon_k \ll 1$, $z-z_o \ll \ell^4/\lambda^3$ and $\langle\chi^2\rangle \lesssim 1$, where ℓ is the smallest scale of the fluctuations. For weak scattering where the scintillation index $m_z^2 \approx \langle\chi^2\rangle \lesssim 1$, these conditions are well satisfied in the solar wind for $k \sim 10^{-2}$ cm^{-1} and $z-z_o \ll 10^{22}$ cm.

III. Effects on the Ratio between Pattern Velocity and Mean Wind Velocity

From Eqs. (6-25) and (6-26) obtained in the last section, we can calculate the ratio between pattern velocity and mean wind velocity provided the velocity distribution function $F(V_1,V_2)$ and the electron density power spectrum $P_k(z',q_1,q_2,q_3)$ are known. Eqs. (6-25) and (6-26) are quite general. Consider now a spherically symmetric solar wind and the specific geometry illustrated in Figure (6-1). Let the mean velocity of the solar wind $\bar{\underset{\sim}{V}}_w$ be radial and constant and assume that the density spectrum is separable in the form

$$P_k(z',q_1,q_2,q_3=0) = g(z')\ h\ (q_1,q_2,q_3=0) \quad . \tag{6-28}$$

We note that in Figure (6-1), the mean drift velocity perpendicular to the line of sight is in the y-direction and the mean velocity in the x-direction $\bar{V}_1$ is zero.

For demonstration, we assume that the velocity distribution $F_z(V_1,V_2)$ is Gaussian with mean velocity $\bar{V}_1,\bar{V}_2$ and variances σ_1^2 and σ_2^2. Referring to Figure (6-1), we express z' in terms of ρ, the distance from the Sun. Clearly, $\bar{V}_1=0$ and $\bar{V}_2(z') = \bar{V}_w\rho_o/\rho$, where ρ_o is the distance of closest approach. Note that σ_2^2 and σ_3^2 are in general functions of the position z'. Then we have

$$\langle e^{-i(q_2V_2+q_1V_1)\tau}\rangle_{z,V_1,V_2} = e^{-iq_2\bar{V}_2(z')\tau}\ e^{-\frac{\tau^2}{2}\left[q_2{}^2\sigma_2{}^2(z')+q_1{}^2\sigma_1{}^2(z')\right]} \tag{6-29}$$

Let
$$g(z') = g(\rho) = A_o(\frac{\rho_o}{\rho})^n \quad . \qquad (6\text{-}30)$$

From Eqs. (6-25), (6-26), (6-28), (6-29) and (6-30), one obtains

$$C_k(0,y,L,\tau) = (\frac{k^2}{2\pi}) \; A_o \int_{-\infty}^{L} dz' \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 \; h(q_1,q_2,q_3=0)$$

$$\sin\left[\frac{q^2(L-z')}{2k}\right] (\frac{\rho_o}{\rho})^n \cdot e^{iq_2[y-\bar{V}_2(z')\tau]} \; e^{-\frac{\tau^2}{2}[q_2^2\sigma_2^2(z')+q_1^2\sigma_1^2(z')]} \; .$$

(6-31)

The observation of $C_k(0,y,L,\tau)$ is illustrated schematically in Figures (6-2). We are interested in the time-lag τ^* for which the cross correlation is a maximum, for a given separation of stations y. One defines the pattern velocity $V_p = y/\tau_*$. The computation of V_p from Eq. (6-31) using a computer is quite straightforward. One calculates C_k for different τ and finds τ_* for which C_k gets maximum value. Then $V_p = y/\tau_*$. For simplicity we put $\sigma_1=0$. The ratio $\gamma \equiv \bar{V}_w/V_p$ depends in general on $\sigma_2(z)$, the separation y, the solar elongation angle θ, the index n, and the power spectrum h $(\underset{\sim}{q})$.. Assume Kolmogorov spectrum for h(q),

$$h(q) = B_o \, q^{-11/3} \qquad (6\text{-}32)$$

and assume

$$\sigma_2(z') = \sigma_{2o}(\frac{\rho_o}{\rho})^m \quad . \qquad (6\text{-}33)$$

Define also
$$\eta \equiv \sigma_{2o}/\bar{V}_w \qquad (6\text{-}34)$$

The computer results are shown in Figures (6-3a,b,c,d,e,f). Computer calculations also show that γ is insensitive to the separation y. In Figure (6-3) we put y = 75 km and m = 1. For m = 1, $(\sigma_2/\bar{V}_2)$ is constant for all ρ(or z). The figures show the ratio γ as a function of elongation angle θ, the index n and the parameter $\eta \equiv \frac{\sigma_{2o}}{\bar{V}_w}$.

From the results, we can see the following features.

(1) For small elongation angle θ, and $\eta = 0$, the ratio γ is about the same as given by Jokipii & Lee (1973) under analytic approximation.

(2) For elongation angle θ near 90^o or greater than 90^o, the ratio γ increases rapidly as the angle θ increases. In this region, thin screen model works very poorly.

(3) The larger the random velocity, the smaller the ratio γ.

Figure Captions

Figure (6-3a). The ratio γ between the mean wind velocity $\bar{V}_w$ and the pattern velocity V_p is plotted as a function of the elongation angle θ for various value of the index n (N is used in the figure). In this figure, the velocity fluctuation is assumed to be zero (η=0).

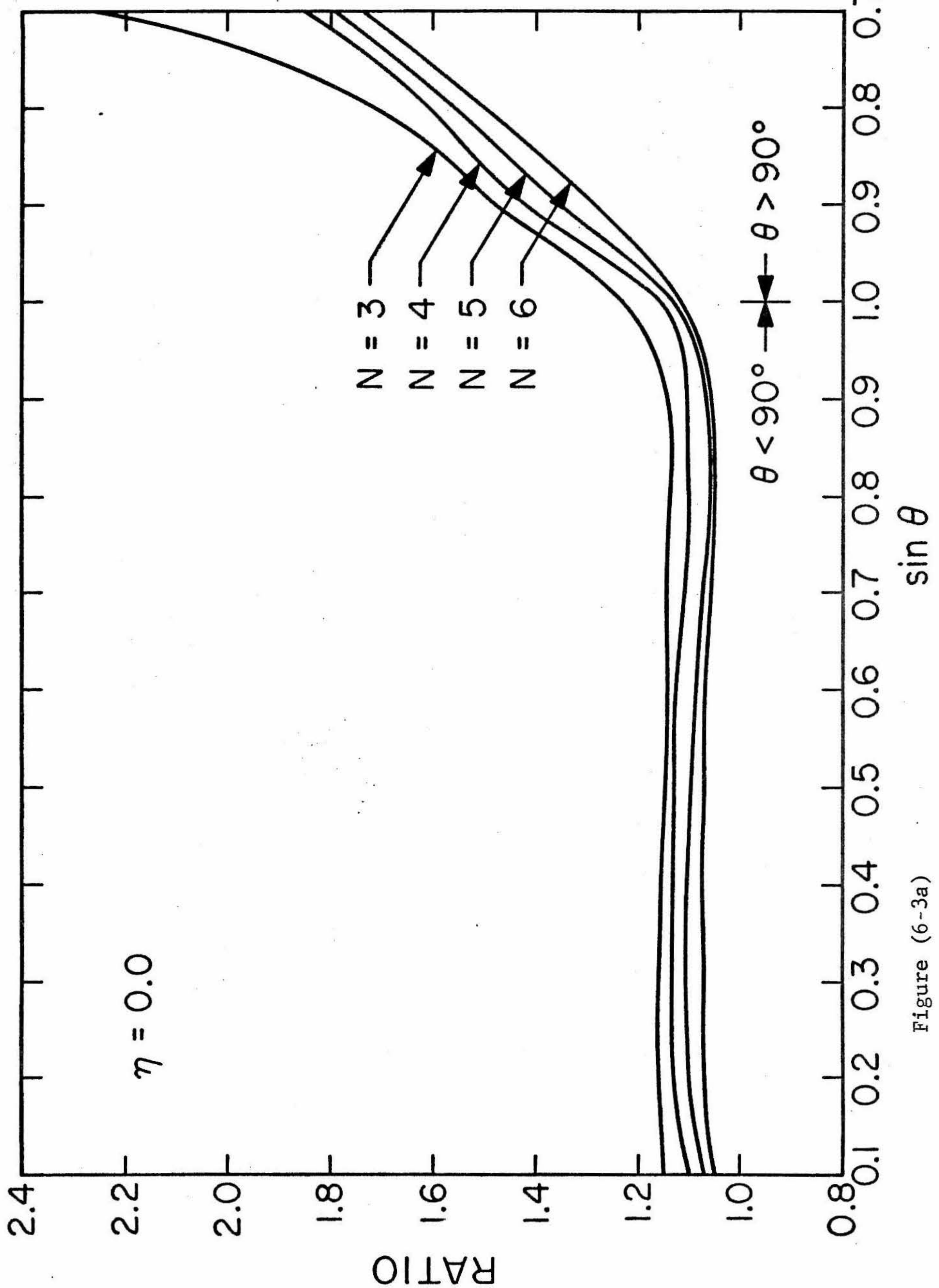
Figure (6-3b). The figure shows the ratio γ vs sin θ (θ is the elongation angle) for various value of the velocity-fluctuation parameter η when the index n = 3. Note that the parameter $\eta = \sigma_{20}/\bar{V}_w$.

Figure (6-3c). As in Figure (6-3b) when n = 4.

Figure (6-3d). As in Figure (6-3b) when n = 5.

Figure (6-3e). As in Figure (6-3b) when n = 6.

Figure (6-3f). The ratio γ is plotted as a function of the parameter η at sin θ = 0.5 for various values of the index n.



RATIO
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.9
0.8
0.7
sin θ
η = 0.0
N = 3
N = 4
N = 5
N = 6
θ < 90°
θ > 90°

Figure (6-3a)

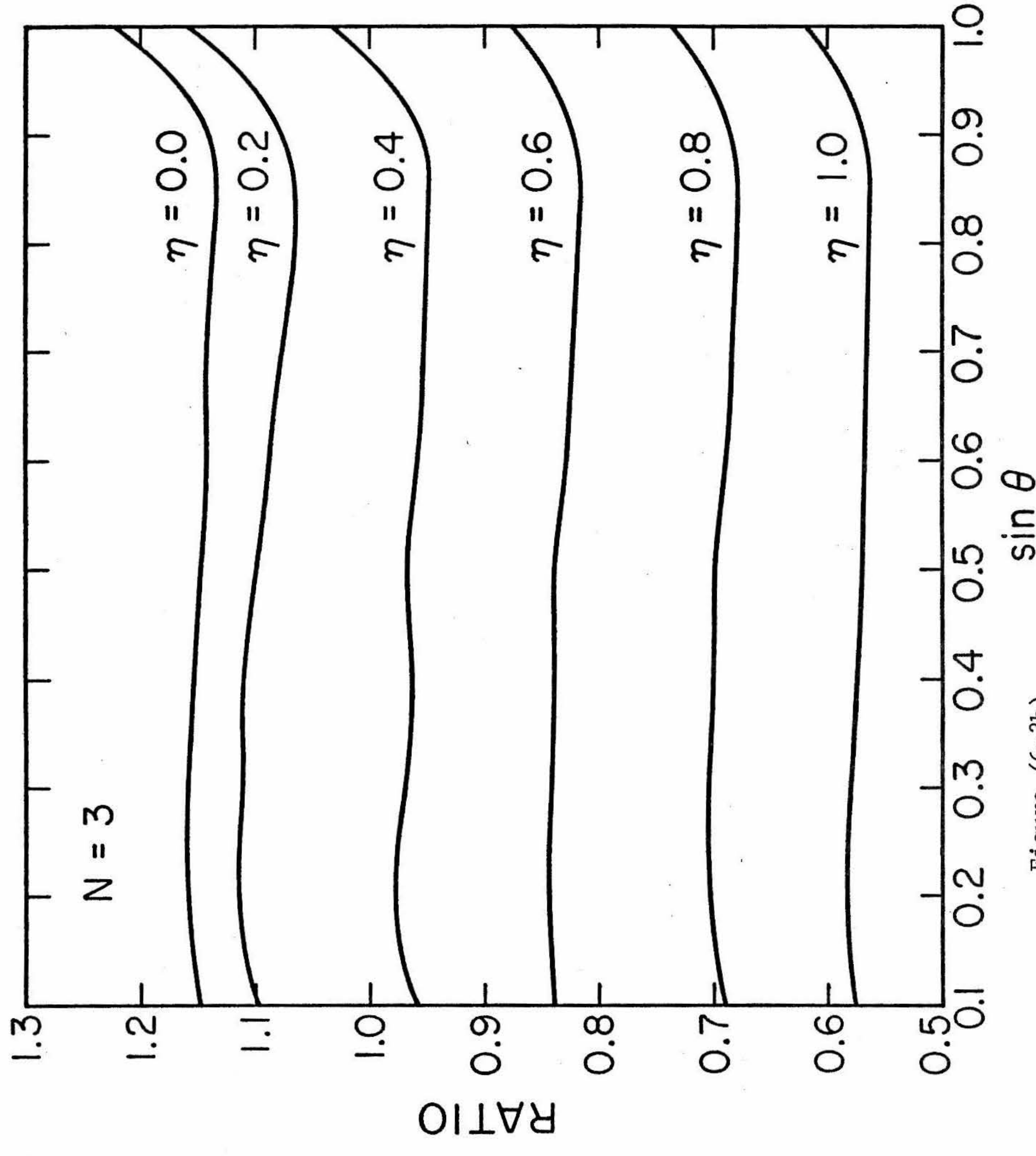

N = 3
η = 0.0
η = 0.2
η = 0.4
η = 0.6
η = 0.8
η = 1.0
RATIO
sin θ
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure (6-3b)

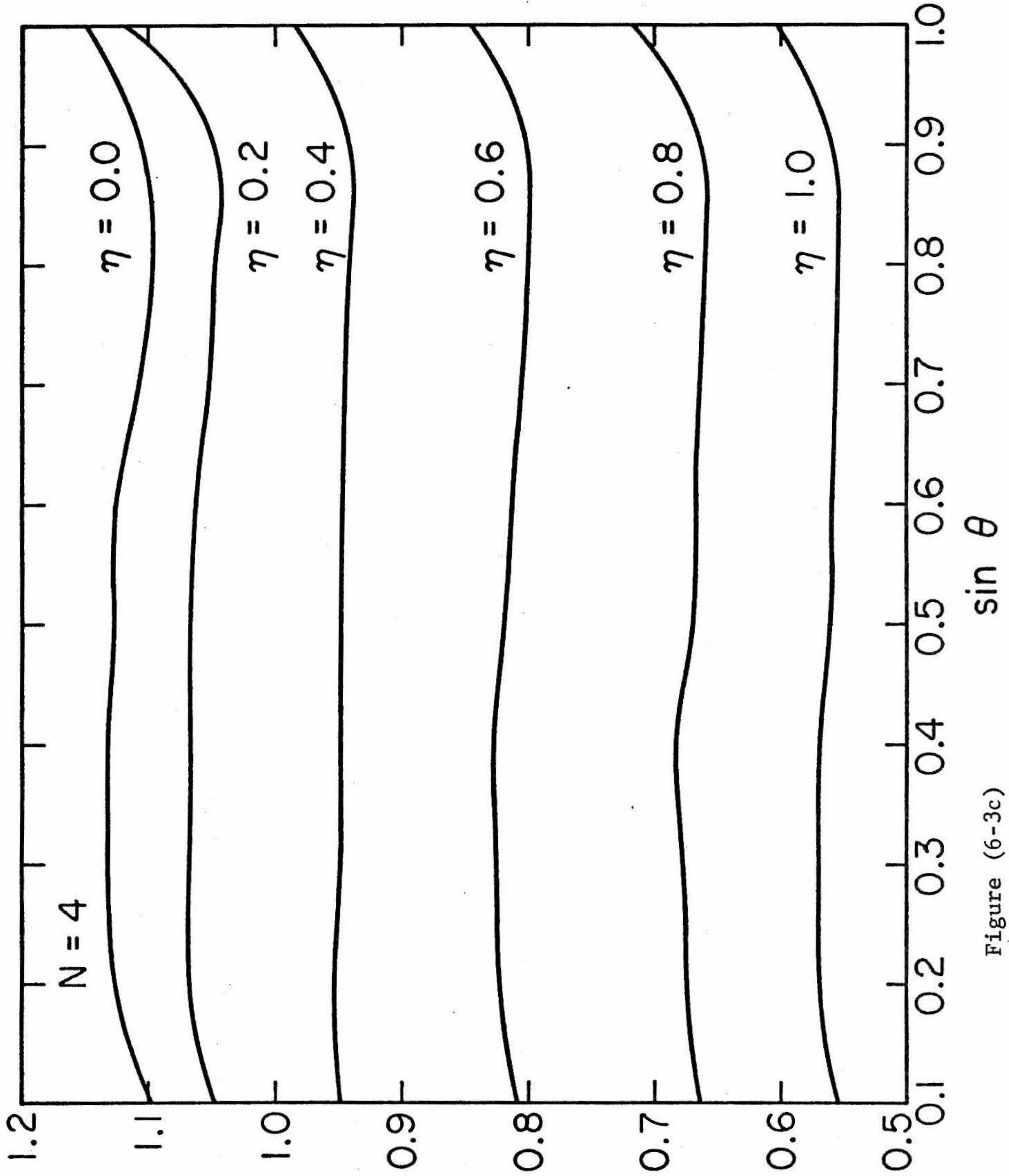

N = 4
η = 0.0
η = 0.2
η = 0.4
η = 0.6
η = 0.8
η = 1.0
RATIO
sin θ
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure (6-3c)

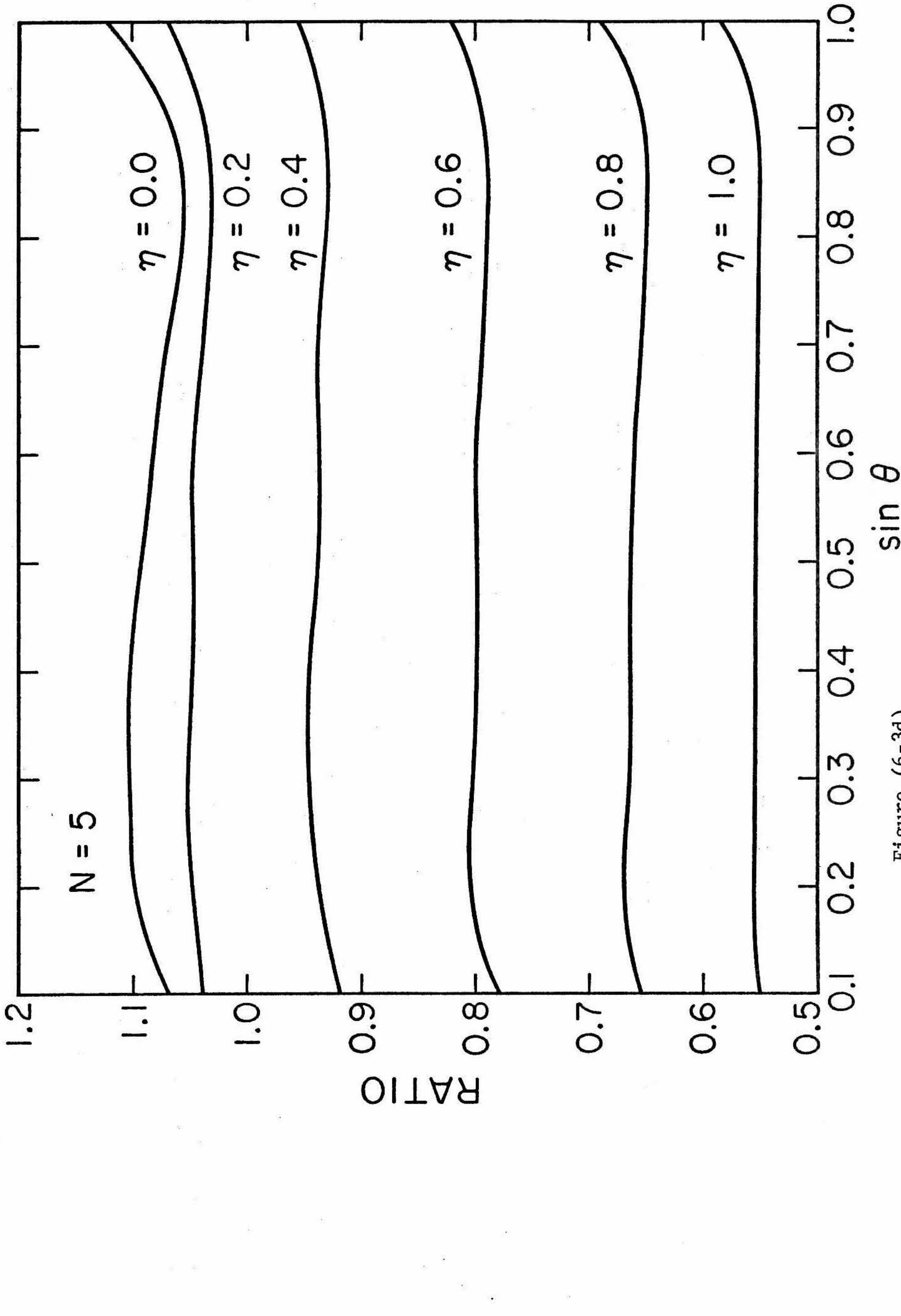
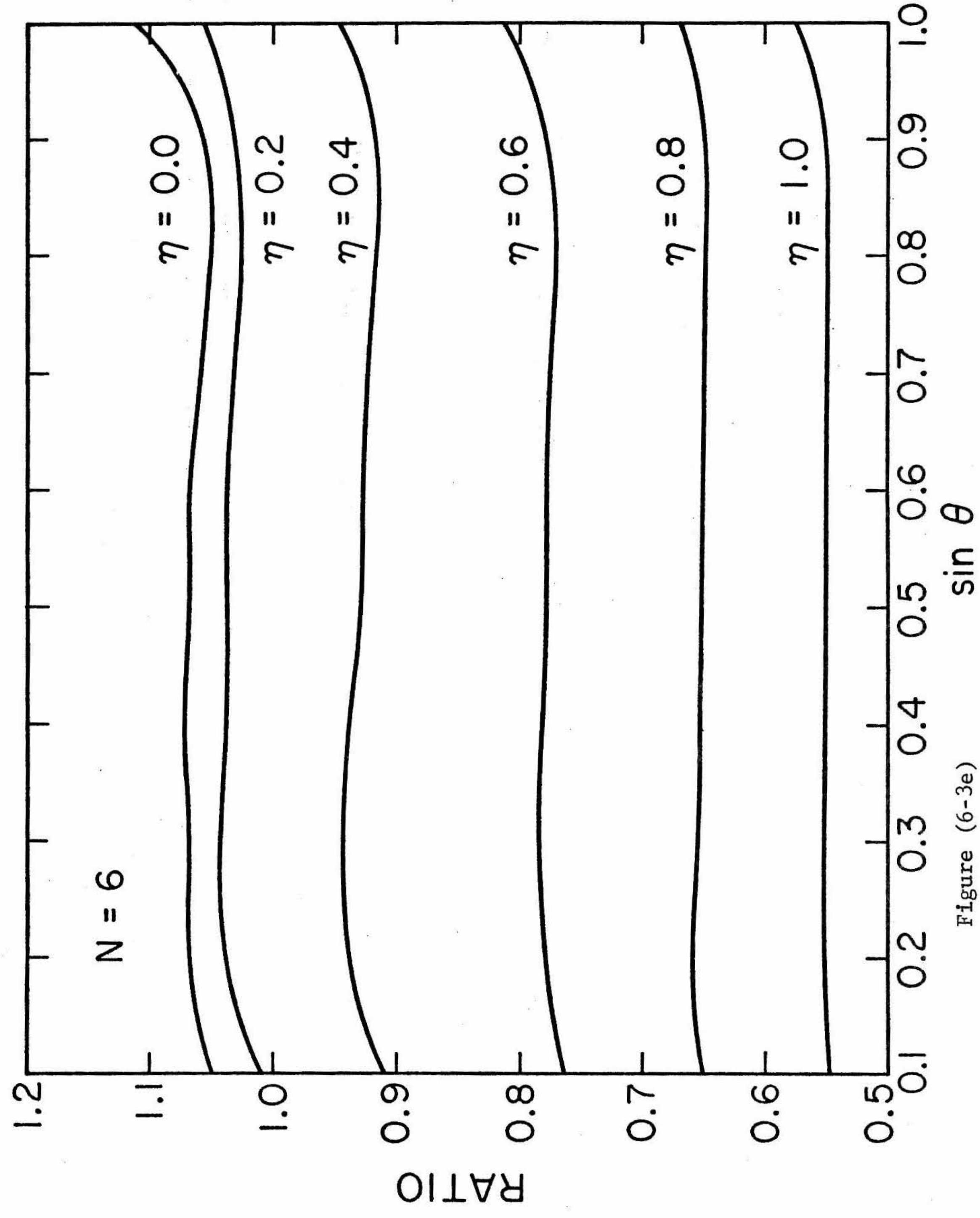

N = 5
η = 0.0
η = 0.2
η = 0.4
η = 0.6
η = 0.8
η = 1.0
RATIO
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
sin θ

Figure (6-3d)



N = 6
η = 0.0
η = 0.2
η = 0.4
η = 0.6
η = 0.8
η = 1.0
RATIO
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
sin θ

Figure (6-3e)

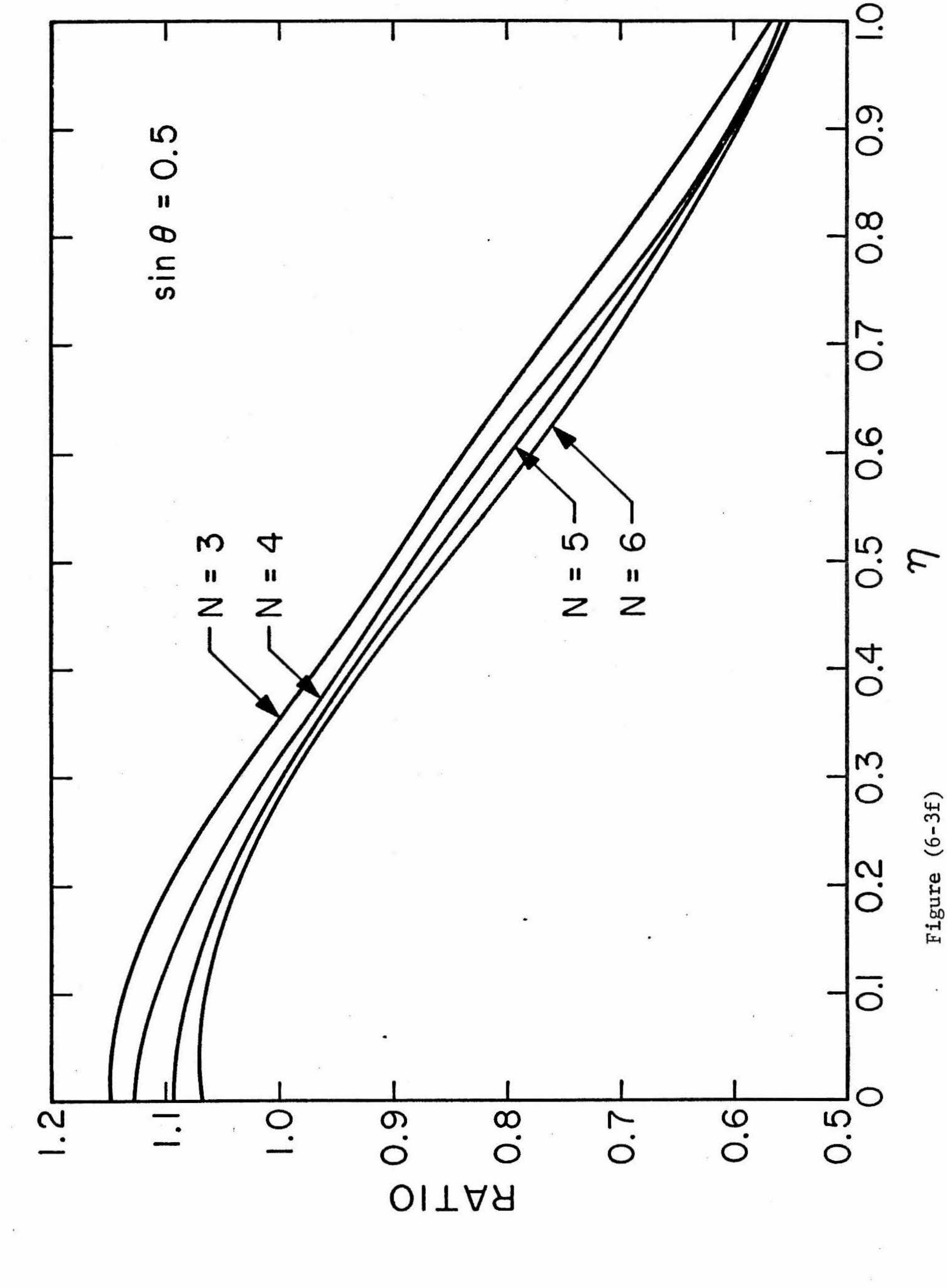

sin θ = 0.5
N = 3
N = 4
N = 5
N = 6
RATIO
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
η

Figure (6-3f)

IV. Effects on the Frequency Power Spectrum

The frequency power spectrum observed at one station can be obtained from Eq. (6-31) by taking the Fourier transform in time,

$$C_k(0,0,L,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} C_k(x=0,Y=0,L,\tau)\, d\tau$$

$$= \frac{k^2}{(2\pi)^2} \int_{-\infty}^{\infty} e^{i\omega\tau} d\tau \int_{-\infty}^{L} dz' \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2\, g(z')h(q_1,q_2,q_3=0)\sin^2\left[\frac{q^2(L-z')}{2k}\right]$$

$$\times\, e^{iq_2[y-\bar{V}_2(z')\tau]}\, e^{-\frac{\tau^2}{2}[q_2^2\sigma_2^2(z')+q_1^2\sigma_1^2(z')]} \quad . \qquad (6\text{-}35)$$

Carrying out the integration in τ, one obtains

$$C_k(0,0,L,\omega) = \frac{k^2}{(2\pi)^2} \int_{-\infty}^{L} dz' \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2\, g(z')h(q_1,q_2,q_3=0)\sin^2\left[\frac{q^2(L-z')}{2k}\right]$$

$$\times \frac{1}{\sqrt{2\pi}\sqrt{q_2^2\sigma_2^2+q_1^2\sigma_1^2}}\, e^{-\left[\frac{(\omega-q_2\bar{V}_2(z'))^2}{2(q_2^2\sigma_2^2+q_1^2\sigma_1^2)}\right]} \quad . \qquad (6\text{-}36)$$

Eq. (6-36) gives the frequency power spectrum observed at one station with the effect of velocity fluctuations considered. Although the integration is complicated, the calculation is very straightforward. We will not do the integration here. Instead we will consider an example to demonstrate the effect of velocity turbulence on the frequency power spectrum of wave intensity.

Young (1971) has carried out the numerical computation of the frequency power spectrum $C_k(0,0,L,\omega)$ when the velocity fluctuations do not exist. In the case $\sigma_1,\sigma_2 \to 0$, Eq. (6-36) becomes

$$C_k^{(0)}(0,0,L,\omega) = \frac{k^2}{(2\pi)^2} \int_{-\infty}^{L} dz' \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 g(z') h(q_1,q_2,q_3=0) \sin^2\left[\frac{q^2(L-z')}{2k}\right] \delta(\omega - q_2 V_2(z')) \quad . \qquad (6\text{-}37)$$

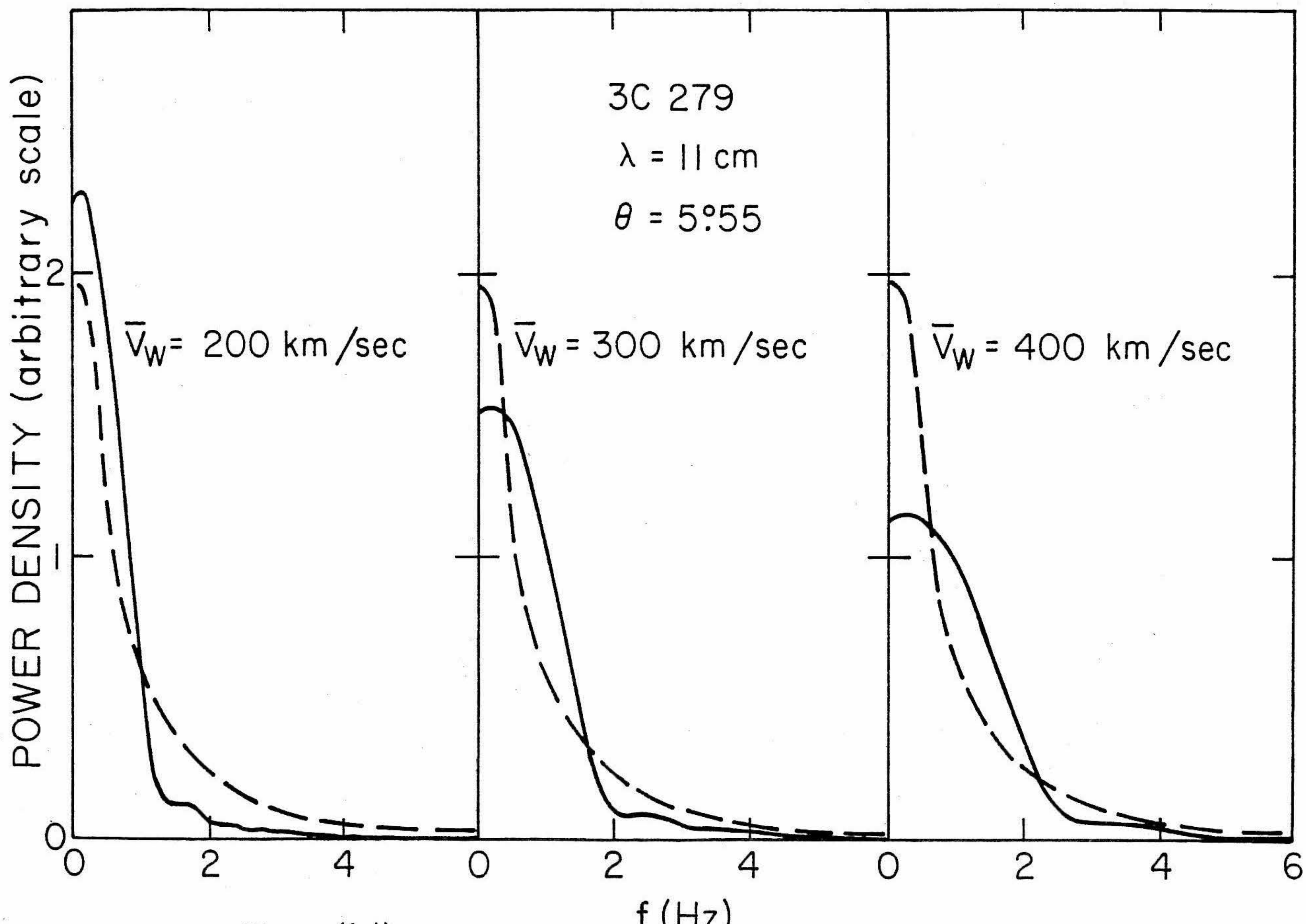
For large solar elongation angle, the $C_k^{(0)}(0,0,L,\omega)$ in Eq. (6-37) fits the observation data quite well for radial solar velocity of 300-350 km/sec. However, for small elongation angles ($\theta \approx 5^o$), Young found that there exists a commonly observed discrepancy between computed and observed spectra: the observed spectra are stronger than the computed at high and low frequencies, and weaker in the region of the knee as shown in Figure (6-4). In Figure (6-4), there is no way to fit the data by varying the wind velocity. The observed data in Figure (6-4) is published by Cohen & Gundermann (1969) for 3c 279 at λ = 11 cm.

The discrepancy between computed and observed spectra can be explained by the existence of velocity fluctuation. Let $\sigma_1 = \sigma_2 = \sigma$ in Eq. (6-36). We try to find the best fit by varying σ for a given mean wind velocity $\overline{V}_w$ using an approximate method, which is presented in Appendix (E). The

Figure Captions

Figure (6-4). Computed (solid) and observed (dashed) power spectra at $\theta = 5.^{o}55$, $\lambda = 11$ cm, and m_z (scintillation index) = 0.20, for three different values of $\overline{V}_w$. The computed curves are based on the assumption that there is no velocity fluctuation. (From Young 1971.)

Figure (6-5). Theoretical (dashed and dotted) and observed (solid) power spectrum at $\theta = 5.^{o}55$, $\lambda = 11$ cm. The observed spectrum is the same as that in Figure (6-4). Curve (2) is the theoretical curve for $V_{20} = \overline{V}_w =$ 200 km/sec and $\eta = 1.13$, and curve (3) is for $V_{20} = \overline{V}_w = 300$ km/sec and $\eta = 0.7$



POWER DENSITY (arbitrary scale)
0
1
2
$\overline{V}_W$ = 200 km/sec
$\overline{V}_W$ = 300 km/sec
$\overline{V}_W$ = 400 km/sec
3C 279
λ = 11 cm
θ = 5°55
0
2
4
6
f (Hz)

Figure (6-4)

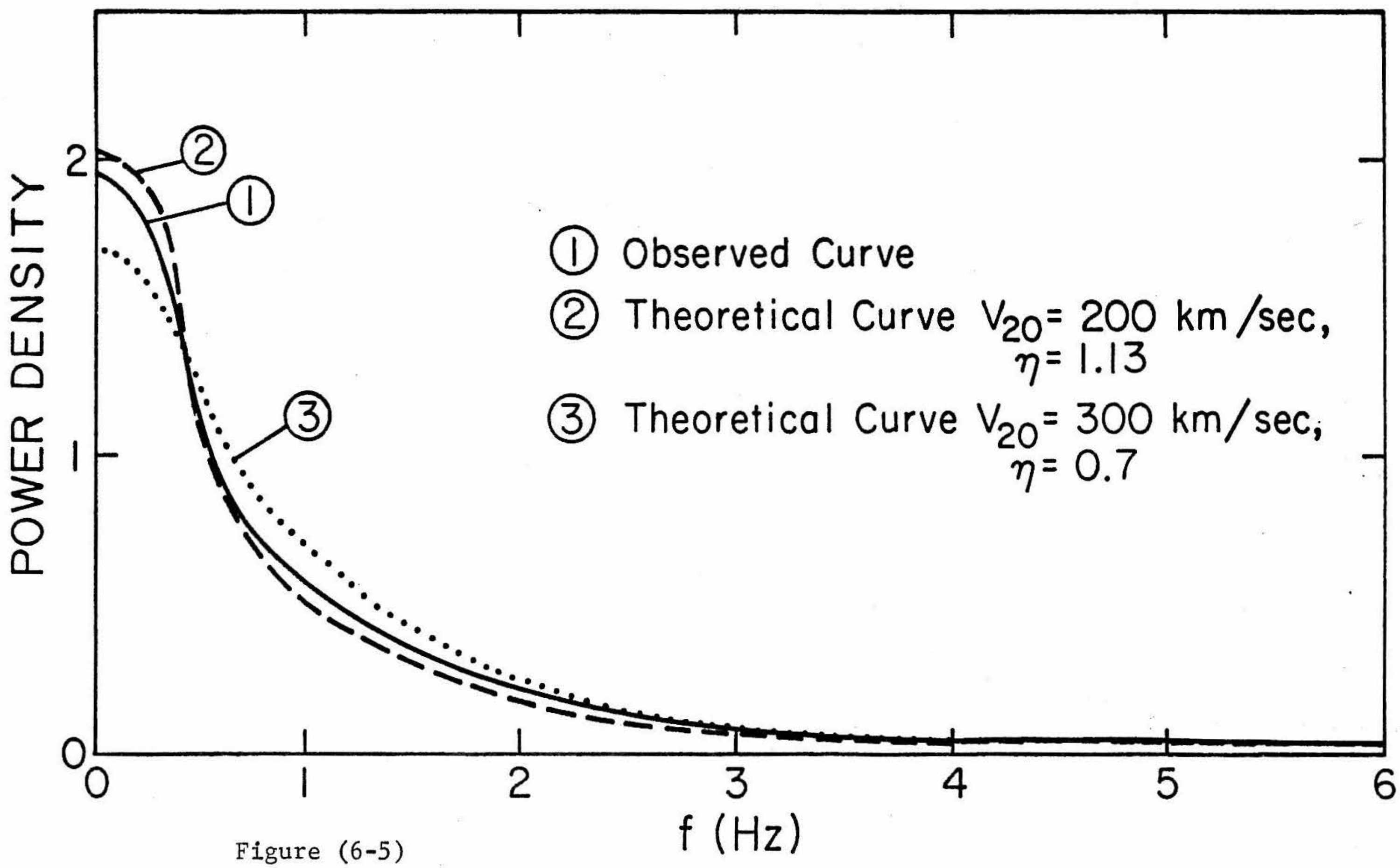

1 Observed Curve
2 Theoretical Curve V20= 200 km/sec, η= 1.13
3 Theoretical Curve V20= 300 km/sec, η= 0.7
POWER DENSITY
2
1
0
0
1
2
3
4
5
6
f (Hz)

Figure (6-5)

results for 3c 279 data (Cohen & Gundermann 1969) are shown in Figure (6-5) with $\overline{V}_w$ = 200 and 300 km/sec. For $\overline{V}_w$ = 200 km/sec, the best fit is at $\sigma_{10}=\sigma_{20}=$ 210 km/sec and for $\overline{V}_w$ = 300 km/sec, $\sigma_{10}=\sigma_{20}=$ 226 km/sec. The curves in Figure (6-5) show that the discrepancy as pointed out by Young (1971) no longer exists. From Figure (6-5), we expect that at $\overline{V}_w=$ 230 km/sec and $\sigma_{10}=\sigma_{20}=$ 220 km/sec, the computed power spectrum would fit the observed one more or less exactly.

Thus our result predicts that the velocity turbulence exists even in the three-dimensional model of interplanetary scintillation. And we find that the data presented by Cohen & Gundermann (1969) show that near the Sun the wind velocity is about 230 km/sec and the velocity fluctuation is near 220 km/sec.

Chapter 7

Interstellar Scintillations

I. Introduction

Interstellar scintillations of the radio waves from pulsars provide a probe of the characteristics of the turbulent plasmas in the interstellar medium (Hewish et al. 1968, Pilkington et al. 1968, Lyne & Rickett 1968, Scheuer 1968, Salpeter 1969, Rickett 1969, 1970, Lang 1969, 1971, Huguenin & Taylor 1969, Cronyn 1970, Ables et al. 1970, Rankin et al. 1970, Ewing et al. 1970, Sutton 1971, Counselman & Rankin 1971, Downs & Reichley 1971, Higgins et al. 1971, Komesaroff et al. 1971, 1972, Komesaroff 1971, Williamson 1972, Shitov 1972, Rankin & Counselman 1973, Little & Matheson 1973, Backer 1974, Mutel et al. 1974, and Cohen & Cronyn 1974.) The observed phenomena include intensity fluctuations, angular broadening, temporal pulse smearing and decorrelation frequency. The scintillations of the radio waves in the interstellar medium are strong ($m_z \simeq 1$) and the correct theory must take the multiple-scattering effect into account. The geometrical optics and the single-scattering theory are not valid for interstellar scintillations.

In this chapter, we try to interpret the observed interstellar scintillation data using the theories developed in Chapters 3,4 and 5, which are valid for strong scintillation. In Section II, we summarize the results of the strong scintillation theory and compare them with the observed data of interstellar scintillations. We also plot the values of the correlation scale of intensity fluctuation, the characteristic time of pulse broadening and the condition for strong scintillation for various values of the parameters in interstellar scintillations. In Section III, we will analyze the data of three pulsars, namely, CP 0328, PSR 0833-45 and NP 0532. We find that the Kolmogorov spectrum of the turbulent medium fits the observed data.

II. Strong Scintillation in the Interstellar Medium

In this section, we will summarize the results of the strong scintillation theory developed in the previous chapters and compare them with the observed data in interstellar scintillations when possible. First we rewrite the parameters B and $A_\beta(0)$ in terms of the properties of the propagating wave and the plasma medium. For a medium with a Gaussian spectrum in Eq. (1-20), we have

$$B = 128\pi^{7/2} r_e^2 k^{-4} q_o^{-3} \langle \delta N_e^2 \rangle \tag{7-1a}$$

and

$$A_\beta(0) = \frac{Bk^4 q_o^2}{8\pi} \tag{7-1b}$$

where $q_o^{-1} = L$ is the correlation scale of the random medium. For the power-law spectrum in Eq. (1-21),

$$B = 128\pi^{7/2} r_e^2 k^{-4} q_o^{-3} \langle \delta N_e^2 \rangle \Gamma(\tfrac{\alpha}{2}) / \Gamma(\tfrac{\alpha}{2} - \tfrac{3}{2}) \tag{7-2a}$$

and

$$A_\beta(0) = \frac{Bk^4 q_o^2}{4\pi(\alpha - 2)} \quad . \tag{7-2b}$$

(For a Kolmogorov spectrum, $\alpha = \frac{11}{3}$.)

(A) Scintillation Index

For strong scintillations, the scintillation index is

$$m_z \simeq 1. \tag{7-3}$$

(See Eq. (3-67) and Chapter 5, Section V.) In interstellar scintillations, it has been observed that the scintillation index is unity for most pulsars (Rickett 1969, 1970, Downes & Reichley 1971).

(B) Probability Distribution of Intensity

For strong scintillations, the probability distribution of the random wave is a Rayleigh-distribution (special case of a Rice-distribution), and the probability distribution of intensity is exponential,

$$P(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}. \tag{7-4}$$

(See Eq. (5-190).) Eq. (7-4) fits the data in interstellar scintillations for most pulsars (Rickett 1969, 1970, Downes & Reichley 1971). The exponential distribution in Eq. (7-4) is different from the log-normal distribution for the <u>weak</u> scintillation (Tatarskii 1971, Young 1971).

(C) Phase Fluctuations

The mean square phase fluctuation Φ_o^2 for waves scattered by the random medium is approximately

$$\Phi_o^2 = \frac{z}{2k^2} A_\beta(0) \tag{7-5}$$

where $A_\beta(0)$ is given by Eqs. (7-1) and (7-2) for the Gaussian and the Kolmogorov spectra respectively. Eq. (7-5) can be used to estimate the phase fluctuations in interstellar scintillations. Note that $\Phi_o^2 > 1$ is one of the conditions for strong scintillations.

(D) Spatial Correlation Scale and Decorrelation Time of Intensity Fluctuations

The spatial intensity correlation function, $P_I(z,\underset{\sim}{\rho})$, for strong scintillations is given by

$$P_I(z,\underset{\sim}{\rho}) = \exp\left\{ -\frac{z}{k^2}\left[A_\beta(0)-A_\beta(\underset{\sim}{\rho})\right]\right\} . \qquad (7\text{-}6)$$

(See Eq. (5-110) or Eq. (5-115).) The correlation scale $\rho_{c.s}$ is simply that value of ρ for which

$$\frac{z}{k^2}\left[A_\beta(0) - A_\beta(\rho)\right] = 1 \quad . \qquad (7\text{-}7)$$

For our Gaussian spectrum in Eq. (1-20) this becomes

$$\frac{Bzk^2}{32\pi}\left(\rho^2_{c.s}q_o^{\ 4}\right) = 1 \qquad (7\text{-}8a)$$

for $\rho_{c.s.} \ll \frac{1}{q_o} = L$ (equivalent to $\Phi_o^{\ 2} \gg 1$), which is always true for the strong scintillations. Eq. (7-8a) gives immediately

$$\rho_{c.s.} = \left(\frac{Bzk^2q_o^{\ 4}}{32\pi}\right)^{-\frac{1}{2}} \propto f\, L^{\frac{1}{2}}\, z^{-\frac{1}{2}} \langle \delta N_e^{\ 2}\rangle^{-\frac{1}{2}} \quad . \qquad (7\text{-}8b)$$

The value of $\rho_{c.s.}$ in Eq. (7-8b) for various values of the parameters in interstellar scintillations is plotted in Figure (7-1a).

Proceeding similarly for the power law spectrum in Eq. (1-21), we expand $A_\beta(\rho)$ as in Eq. (1-28) for $\ell < \rho < L$, and obtain

Figure Captions

Figure (7-1a). The correlation scale $\rho_{c.s.}$, of intensity fluctuation plotted versus the coherent scale L for a Gaussian refractive index spectrum. In all cases $f = 3 \times 10^8$Hz and $z = 10^3$pc. In curve (1) $\langle \delta N_e^{\ 2} \rangle = 10^{-8} cm^{-6}$, in curve (2) $\langle \delta N_e^{\ 2} \rangle = 10^{-6} cm^{-6}$, and in curve (3) $\langle \delta N_e^{\ 2} \rangle = 10^{-4} cm^{-6}$. The value of $\rho_{c.s.}$ scales $\propto f\, L^{0.5} z^{-0.5} \langle \delta N_e^{\ 2} \rangle^{-0.5}$.

Figure (7-1b). As in Figure (7-1a) for a Kolmogorov power-law refractive index spectrum with $\ell < \rho^{(0)}_{c.s.} < L$. In curve (1) $\langle \delta N_e^{\ 2} \rangle = 10^{-8} cm^{-6}$, in curve (2) $\langle \delta N_e^{\ 2} \rangle = 10^{-6}$, and in curve (3) $\langle \delta N_e^{\ 2} \rangle = 10^{-4}$. In all cases, $f = 3 \times 10^8$Hz and $z = 10^3$pc. The value of $\rho_{c.s.}$ scales $\propto f^{\ 1.2} L^{\ 0.4} z^{\ -0.6} \langle \delta N_e^{\ 2} \rangle^{-0.6}$. Note that $\Delta N \equiv \langle \delta N_e^{\ 2} \rangle^{\frac{1}{2}}$.

Figure (7-1c). The correlation scale $\rho_{c.s.}$ of intensity fluctuation plotted versus the mean square electron density fluctuation $\langle \delta N_e^{\ 2} \rangle$ for the Kolmogorov spectrum with $\ell > \rho^{(0)}_{c.s.}$. In curve (1) $\ell = 10^{12}$cm and $L = 10^{19}$cm, in curve (2) $\ell = 10^{12}$cm and $L = 10^{18}$cm, in curve (3) $\ell = 10^{10}$cm and $L = 10^{18}$cm, in curve (4) $\ell = 10^8$cm and $L = 10^{18}$cm, in curve (5) $\ell = 10^{12}$cm, $L = 10^{15}$cm, and finally in curve (5) $\ell = 10^{10}$cm and $L = 10^{15}$cm. In all cases $f = 3 \times 10^8$Hz and $z = 10^3$pc. The value of $\rho_{c.s.}$ scales $\propto f\, z^{-0.5} \langle \delta N_e^{\ 2} \rangle^{-0.5} L^{\frac{1}{3}}\, \ell^{\frac{1}{6}}$.

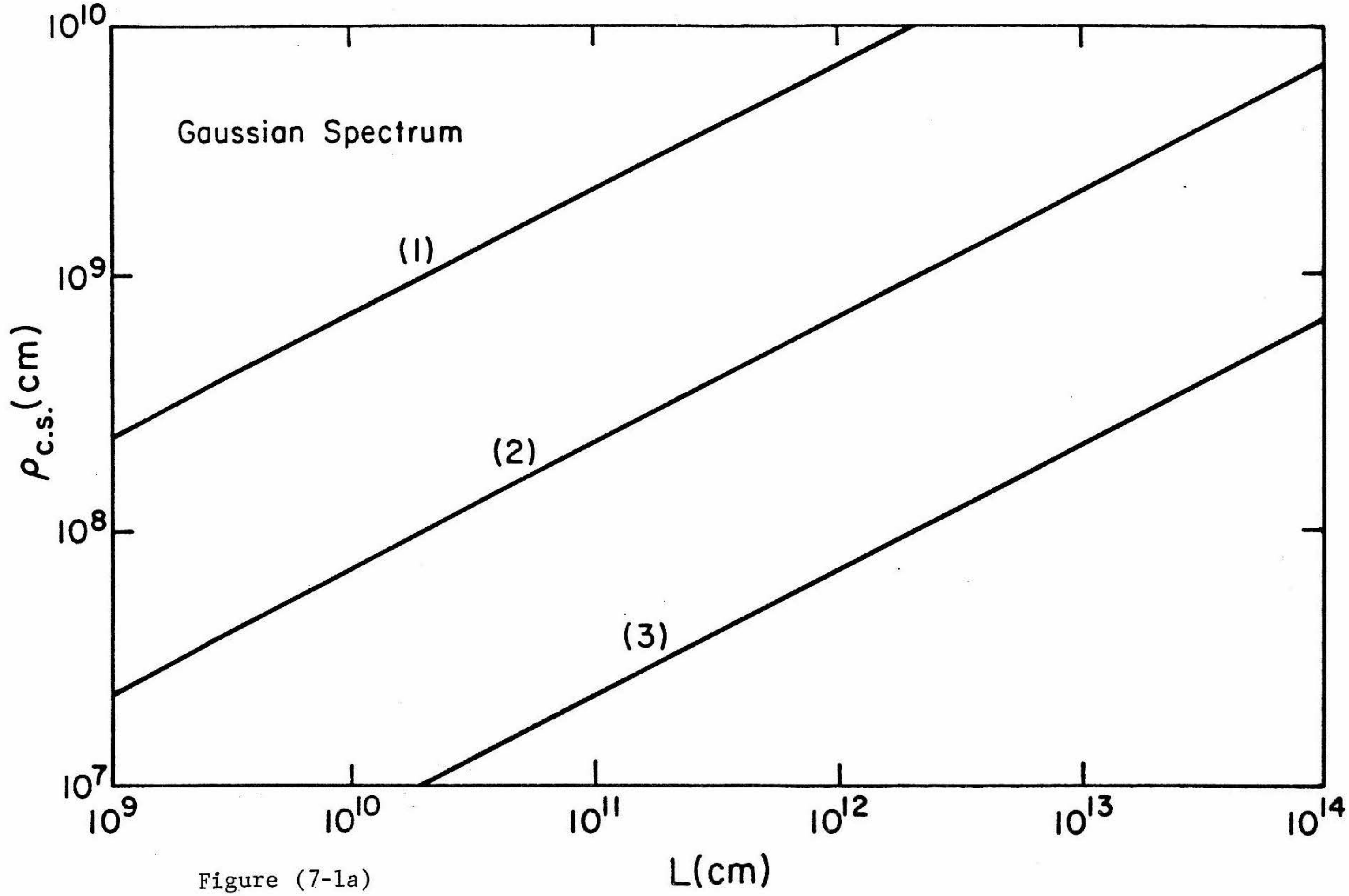

Gaussian Spectrum
(1)
(2)
(3)
ρc.s.(cm)
10^10
10^9
10^8
10^7
10^9
10^10
10^11
10^12
10^13
10^14
L(cm)

Figure (7-1a)

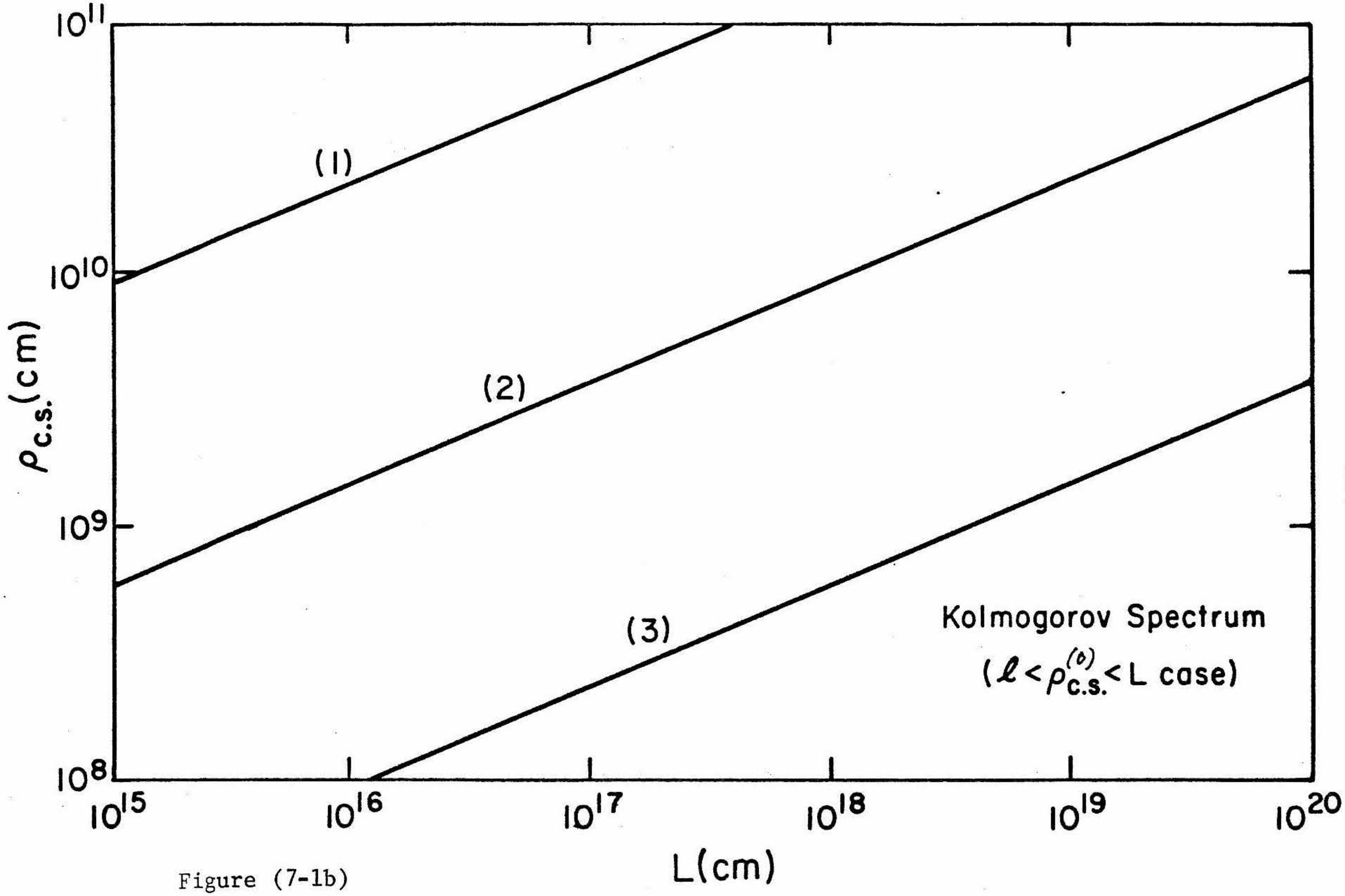

10^11
10^10
10^9
10^8
ρ_c.s.(cm)
10^15
10^16
10^17
10^18
10^19
10^20
L(cm)
(1)
(2)
(3)
Kolmogorov Spectrum
(ℓ < ρ^(0)_c.s. < L case)

Figure (7-1b)

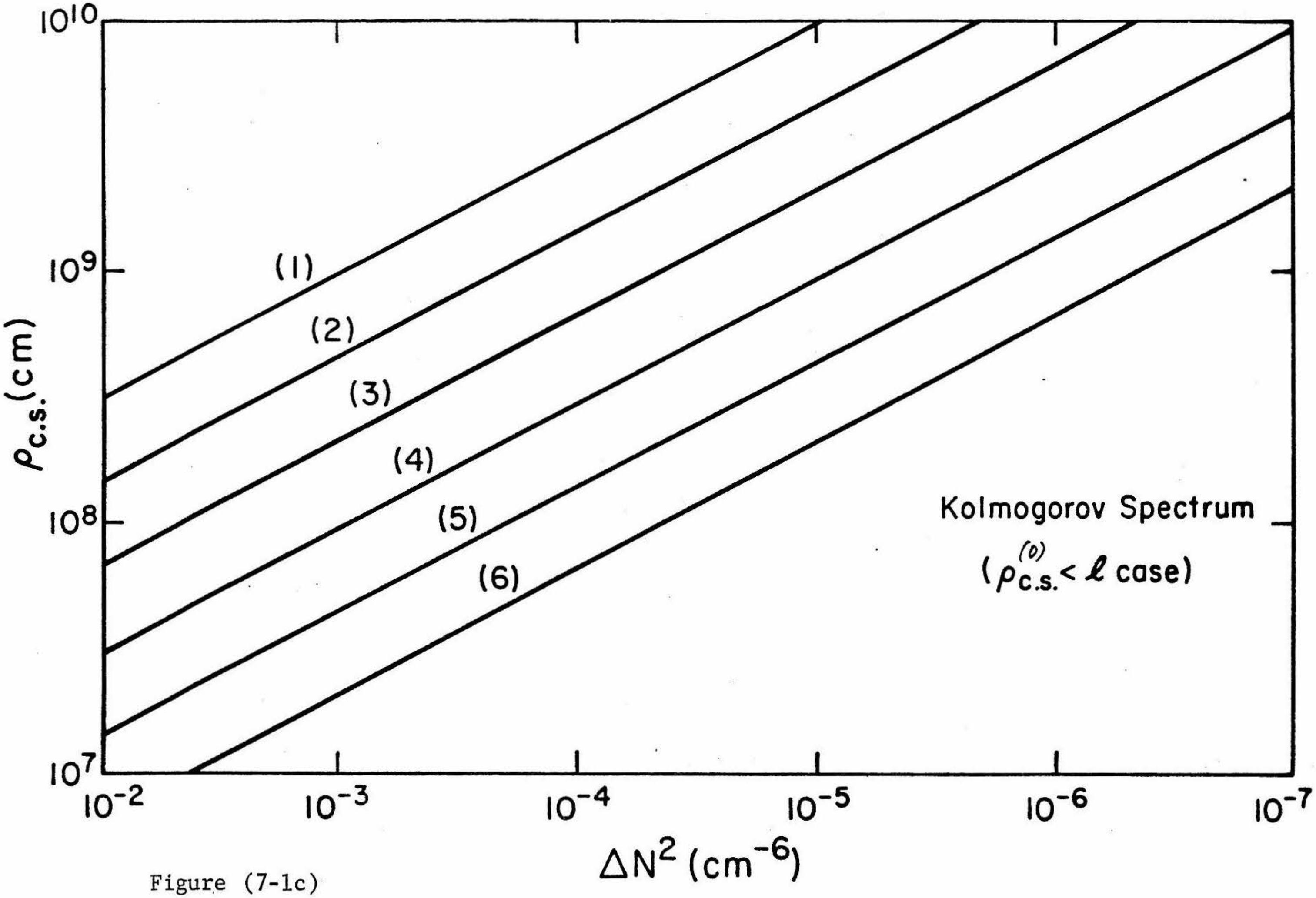

10^10
10^9
10^8
10^7
ρc.s.(cm)
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
ΔN^2 (cm^-6)
(1)
(2)
(3)
(4)
(5)
(6)
Kolmogorov Spectrum
(ρ(0)c.s. < ℓ case)

Figure (7-1c)

$$\rho_{c.s.} = \frac{2}{q_o}\left[\frac{zBk^2 q_o^{\,2}\Gamma(2-\frac{\alpha}{2})}{4\pi(\alpha-2)\Gamma(\frac{\alpha}{2})}\right]^{\frac{-1}{(\alpha-2)}} \equiv \rho^{(0)}_{c.s.} \qquad (7\text{-}9a)$$

for $2 < \alpha < 4$. However Eq. (7-9a) is correct only for the case where $\ell < \rho^{(0)}_{c.s} < L$. The condition $\rho^{(0)}_{c.s} < L$ (corresponding to $\Phi_o^{\,2} > 1$) is always satisfied for the strong scintillation. However when $\rho^{(0)}_{c.s.} < \ell$, we must use the expansion in Eq. (1-29) and we obtain

$$\rho_{c.s} = \left[(\frac{1}{32\pi})\; zBk^2 q_o^{\,\alpha} q_1^{\,(4-\alpha)}\Gamma(\frac{4-\alpha}{2})\right]^{-\frac{1}{2}} \equiv \rho^{(1)}_{c.s} \;. \qquad (7\text{-}9b)$$

Again, the value of $\rho_{c.s}$ in Eq. (7-9a) and Eq. (7-9b) for the Kolmogorov spectrum ($\alpha=\frac{11}{3}$) is illustrated in Figures (7-1b,c) for various values of the parameters. We note that for the Kolmogorov spectrum,

$$\rho^{(0)}_{c.s} \propto f^{1.2} L^{0.4} z^{-0.6} \langle \delta N_e^{\,2}\rangle^{-0.6} \qquad (7\text{-}10a)$$

and

$$\rho^{(1)}_{c.s} \propto f\, L^{\frac{1}{3}}\, \ell^{\frac{1}{6}} \langle \delta N_e^{\,2}\rangle^{-0.5} z^{-0.5} \;. \qquad (7\text{-}10b)$$

The condition that $\rho^{(0)}_{c.s.} = \ell$ for the Kolmogorov spectrum ($\alpha=\frac{11}{3}$) is plotted in Figure (7-2), in which $\rho^{(0)}_{c.s.} < \ell$ is on the upper side of each curve and $\rho^{(0)}_{c.s} > \ell$ is on the lower side.

The correlation scale $\rho_{c.s.}$ is related to the decorrelation time $\tau_{c.s.}$ by

$$\rho_{c.s.} = \tau_{c.s.} \times v \qquad (7\text{-}10c)$$

Figure (7-2)

The relation $\rho_{c.s.} = \ell$ is plotted on the ℓ - z plane for the Kolmogorov power-law index spectrum. In all curves $f = 3 \times 10^8$ z and $L = 10^{18}$cm. From curve (1) to curve (6), $\langle \delta N_e^2 \rangle = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ and 10^{-7}cm^{-6} respectively. Note that the value of z scales $\propto f^2 L^{2/3} \ell^{-\frac{5}{3}} \langle \delta N_e^2 \rangle^{-1}$.

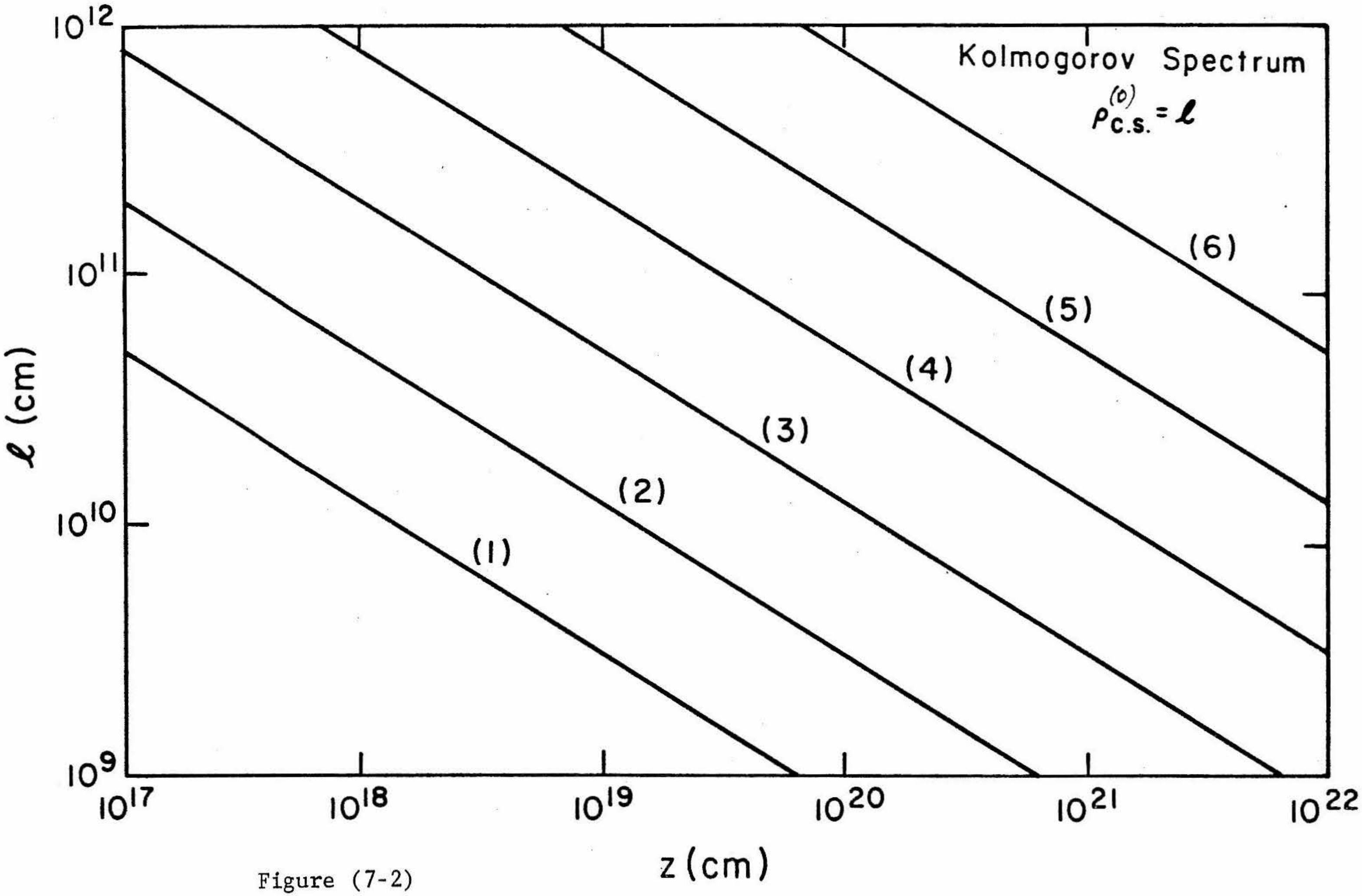

Kolmogorov Spectrum
$\rho^{(0)}_{c.s.} = \ell$
(1)
(2)
(3)
(4)
(5)
(6)
10^{12}
10^{11}
10^{10}
10^{9}
ℓ (cm)
10^{17}
10^{18}
10^{19}
10^{20}
10^{21}
10^{22}
z (cm)

Figure (7-2)

for the turbulent plasmas move at a velocity v, transverse to the line of sight to the pulsar (frozen-in condition being assumed). Lang (1969) has shown that the decorrelation times $\tau_{c.s.}$ for the three pulsars considered varies as $f^{0.7-1.3}$. Rickett (1970) has also presented data which indicate that $\tau_{c.s.} \propto f^n$, where n lies in the range $0.3 \le n \le 1.5$ for all observations be made. The data for PSR 0833-45 (Backer 1974) show that $\tau_{c.s.} \propto f^{1.0-1.3}$. The observation data are in general consistent with that predicted by Eq. (7-8b) or Eqs. (7-10a,b).

(E) Characteristic Scattering Angle

The angular power spectrum of the random waves, $\Psi(\underset{\sim}{\theta})$, is given by Eq. (5-5). From the angular power spectrum, one can calculate the characteristic scattering angle, θ_c, due to the random medium and has

(i) for the Gaussian spectrum,

$$\theta_c = q_o^2 \left(\frac{Bz}{64\pi}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}\, k\rho_{c.s.}} \propto f^{-2} L^{-0.5} z^{0.5} \langle \delta N_e^2 \rangle^{0.5} \qquad \text{(7-11a)}$$

(ii) for the power-law spectrum with $2 < \alpha < 4$,

$$\theta_c = \frac{q_o}{2k} \left[\frac{Bzk^2 q_o^2 \Gamma(2-\frac{\alpha}{2})}{8\pi(\alpha-2)\Gamma(\frac{\alpha}{2})}\right]^{\frac{1}{\alpha-2}} = \frac{1}{2^{(\frac{1}{\alpha-2})} k\rho_{c.s.}} \qquad \text{(7-11b)}$$

(if $\ell < \rho^{(0)}_{c.s.} < L$),

$$\theta_c = \frac{1}{k} \left[\frac{zBk^2 q_o^{\alpha} q_1^{(4-\alpha)} \Gamma(\frac{4-\alpha}{2})}{64\pi}\right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}k\rho_{c.s.}} \qquad \text{(7-11c)}$$

(if $\ell > \rho^{(0)}_{c.s.}$). For the Kolmogorov spectrum ($\alpha=\frac{11}{3}$), one has

$$\theta_c \propto f^{-2.2} L^{-0.4} z^{0.6} \langle \delta N_e^2 \rangle^{0.6} \qquad (7\text{-}12a)$$

(if $\ell < \rho^{(0)}_{c.s.} < L$),

$$\theta_c \propto f^{-2} L^{-\frac{1}{3}} \ell^{-\frac{1}{6}} z^{0.5} \langle \delta N_e^2 \rangle^{0.5} \qquad (7\text{-}12b)$$

(if $\ell > \rho^{(0)}_{c.s.}$).

In the interplanetary and the interstellar scintillations, it has been observed $\theta_c \propto f^{-n}$, where $n = 2.05 \pm 0.25$, which checks in general with that given by Eq. (7-11a) or (7-12a,b) (Erickson 1964), Readhead & Hewish 1972, Mutel et al. 1974, Cohen & Cronyn 1974).

(F) Pulse Broadening and Decorrelation Frequency

From the results of Chapter 5, Section V, we learned that the pulse emitted from a pulsar is broadened by the turbulent interstellar medium. There are three physical effects causing the broadening of the pulse: a dispersion effect, a pure refraction effect and a diffraction effect. The characteristic broadening times for these three effects are respectively t_1, t_R and t_D where

$$t_1 = \sqrt{1+R^2\Delta^4}/\Delta \simeq R\Delta = \frac{zN_e r_e c\Delta}{2\pi^2 f^3} \qquad (7\text{-}13a)$$

$$t_R = \frac{1}{k^2 c}\left(\frac{A_\beta(0)z}{2}\right)^{\frac{1}{2}} \qquad (7\text{-}13b)$$

and

$$t_D = \frac{z\theta_c^{\,2}}{2c} \, , \qquad (7\text{-}13c)$$

where $(\frac{\Delta}{2\pi})$ is the bandwidth of measurement instrument, and where θ_c is given by Eqs. (7-11a,b,c) for various cases.

Let t_c be the characteristic time of the temporal pulse broadening due to the combination of pure refraction and the diffraction effects. Then the decorrelation frequency f_I for strong scintillation to t_c by

$$2\pi \, f_I \, t_c = 1 \qquad (7\text{-}14)$$

(See Eq. (5-202).)

Since the observation time for a pulse profile is short (usually less than 1 day), the pure refraction effect (t_R) can not be detected and we will neglect this effect. Thus $t_c \simeq t_D$. We then have for the Gaussian spectrum,

$$t_D = \frac{1}{2\pi f_I} \propto f^{-4} \, z^2 \, L^{-1} \, \langle \delta N_e^{\,2} \rangle \, . \qquad (7\text{-}15a)$$

For a Kolmogorov spectrum, we have

(i) when $\ell < \rho_{c.s.} < L$

$$t_D = \frac{1}{2\pi f_I} \propto f^{-4.4} \, z^{2.2} \, L^{-0.8} \, \langle \delta N_e^{\,2} \rangle^{1.2} \qquad (7\text{-}15b)$$

and

(ii) when $\rho_{c.s.} < \ell$,

$$t_D = \frac{1}{2\pi f_I} \propto f^{-4} \, L^{-\frac{2}{3}} \, \ell^{-\frac{1}{3}} \, z^2 \, \langle \delta N_e^{\,2} \rangle . \qquad (7\text{-}15c)$$

The value of t_D(and f_I) for various values of the parameters in interstellar scintillations is illustrated in Figures (7-3a,b,c).

The "dispersion measure", DM, of the plasma medium from the radio star to the observer is defined as

$$DM = \int_{star}^{observer} N_e(z')dz' = \langle N_e \rangle z. \tag{7-16}$$

Suppose that for <u>all</u> interstellar plasma media,

$$\langle \delta N_e^2 \rangle^{\frac{1}{2}} \propto \langle N_e \rangle \quad . \tag{7-17}$$

We then have for the Gaussian spectrum and the Kolmogorov spectrum with $\rho_{c.s.}^{(0)} < \ell$,

$$t_D = \frac{1}{2\pi f_I} \propto (DM)^2 \tag{7-18a}$$

and for the Kolmogorov spectrum with $\ell < \rho_{c.s.}^{(0)} < L$,

$$t_D = \frac{1}{2\pi f_I} \propto (DM)^{2.2-2.4} \quad . \tag{7-18b}$$

The measurements of the decorrelation frequency f_I and the pulse broadening t_D show that $f_I \propto f^{4\pm1}$ and $f_I^{-1} \propto (DM)^n$ where $n = 1.6-3.6$ (Lang 1971, Sutton 1971 and Backer 1974), which are also in general consistent with those given above.

Figure Captions

Figure (7-3a). The characteristic time of pulse broadening, t_D, versus the correlation scale L for a Gaussian refractive index spectrum with f = 300 MHz and z = 1000 pc. From curve (1) to curve (4), $\langle \delta N_e^2 \rangle = 10^{-4}$, 10^{-5}, 10^{-6} and $10^{-7} cm^{-6}$ respectively. The value of t_D scales $\propto f^{-4} z^2 L^{-1} \langle \delta N_e^2 \rangle$. Also shown in the figure is the decorrelation frequency f_I.

Figure (7-3b). As in Figure (7-3a) for the Kolmogorov spectrum with $\ell < \rho_{c.s.}^{(0)} < L$. In all cases, f = 300 MHz and z = 1000 pc. From curve (1) to curve (5), $\langle \delta N_e^2 \rangle = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5} and $10^{-6} cm^{-6}$ respectively. The value of t_D scales $\propto f^{-4.4} z^{2.2} \langle \delta N_e^2 \rangle L^{-0.8}$.

Figure (7-3c). As in Figure (7-3a) for the Kolmogorov spectrum with $\ell > \rho_{c.s.}^{(0)}$. From curve (1) to curve (4), $\langle \delta N_e^2 \rangle = 10^{-2}$, 10^{-3}, 10^{-4} and $10^{-5} cm^{-6}$ respectively. In all cases, f = 300 MHz, z = 1000 pc and $L = 10^{18}$ cm. The value of t_D scales $\propto f^{-4} z^2 \langle \delta N_e^2 \rangle L^{-\frac{2}{3}} \ell^{-\frac{1}{3}}$.

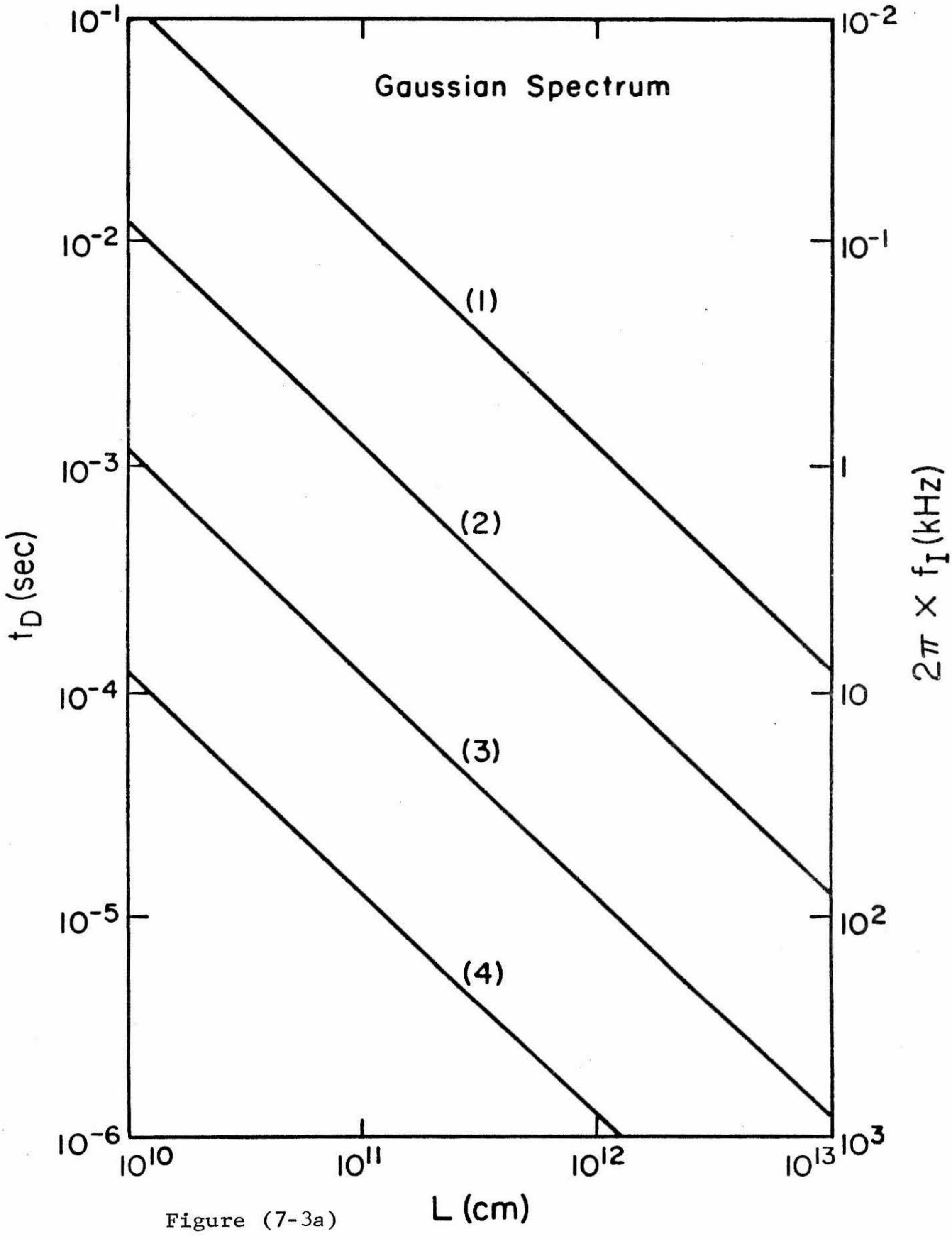
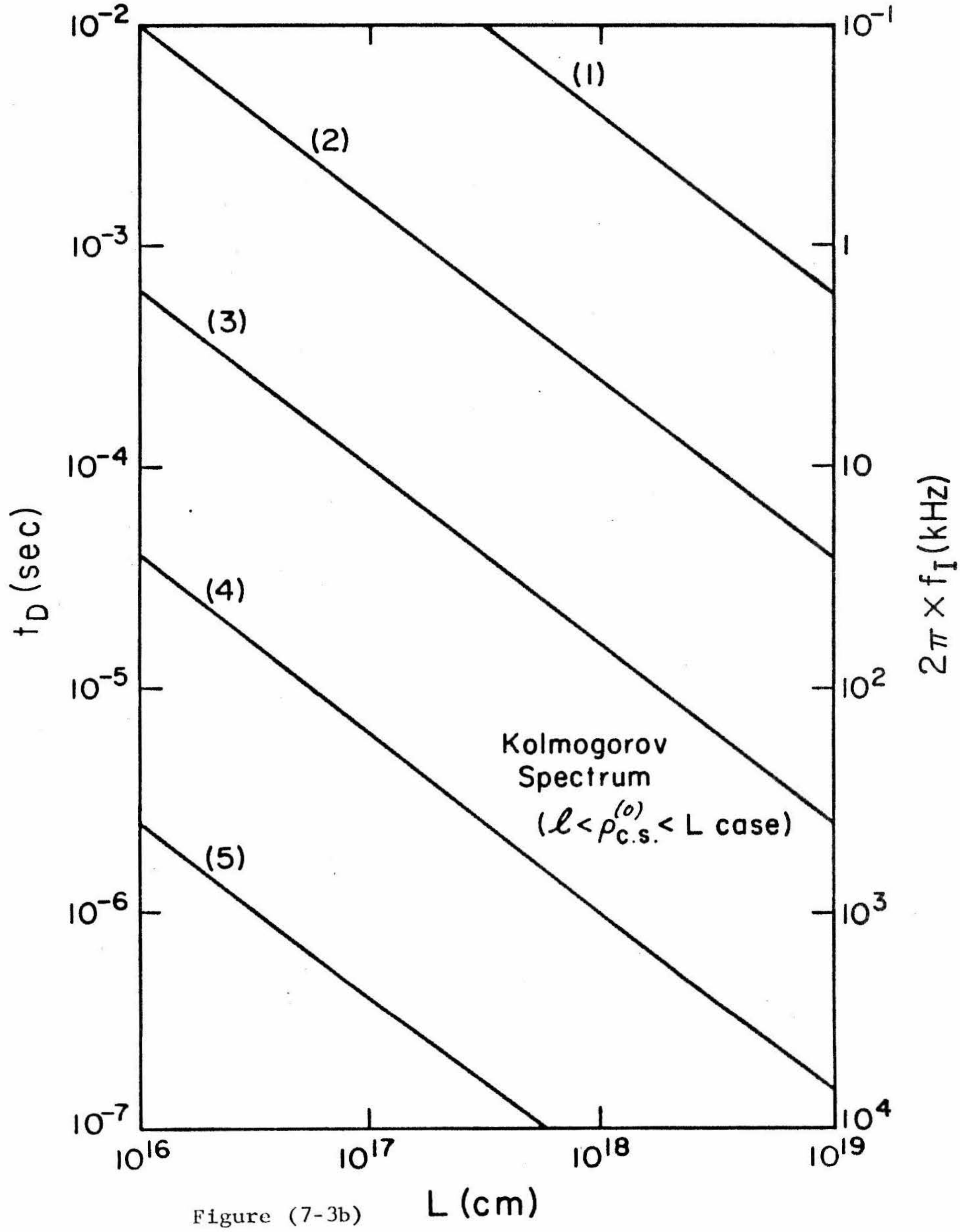

Gaussian Spectrum
(1)
(2)
(3)
(4)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
t_D (sec)
10^{-2}
10^{-1}
1
10
10^{2}
10^{3}
$2\pi \times f_I$(kHz)
10^{10}
10^{11}
10^{12}
10^{13}
L (cm)

Figure (7-3a)



10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
t_D (sec)
10^{-1}
1
10
10^2
10^3
10^4
$2\pi \times f_I$ (kHz)
(1)
(2)
(3)
(4)
(5)
Kolmogorov
Spectrum
($\ell < \rho^{(0)}_{c.s.} < L$ case)
10^{16}
10^{17}
10^{18}
10^{19}
L (cm)

Figure (7-3b)

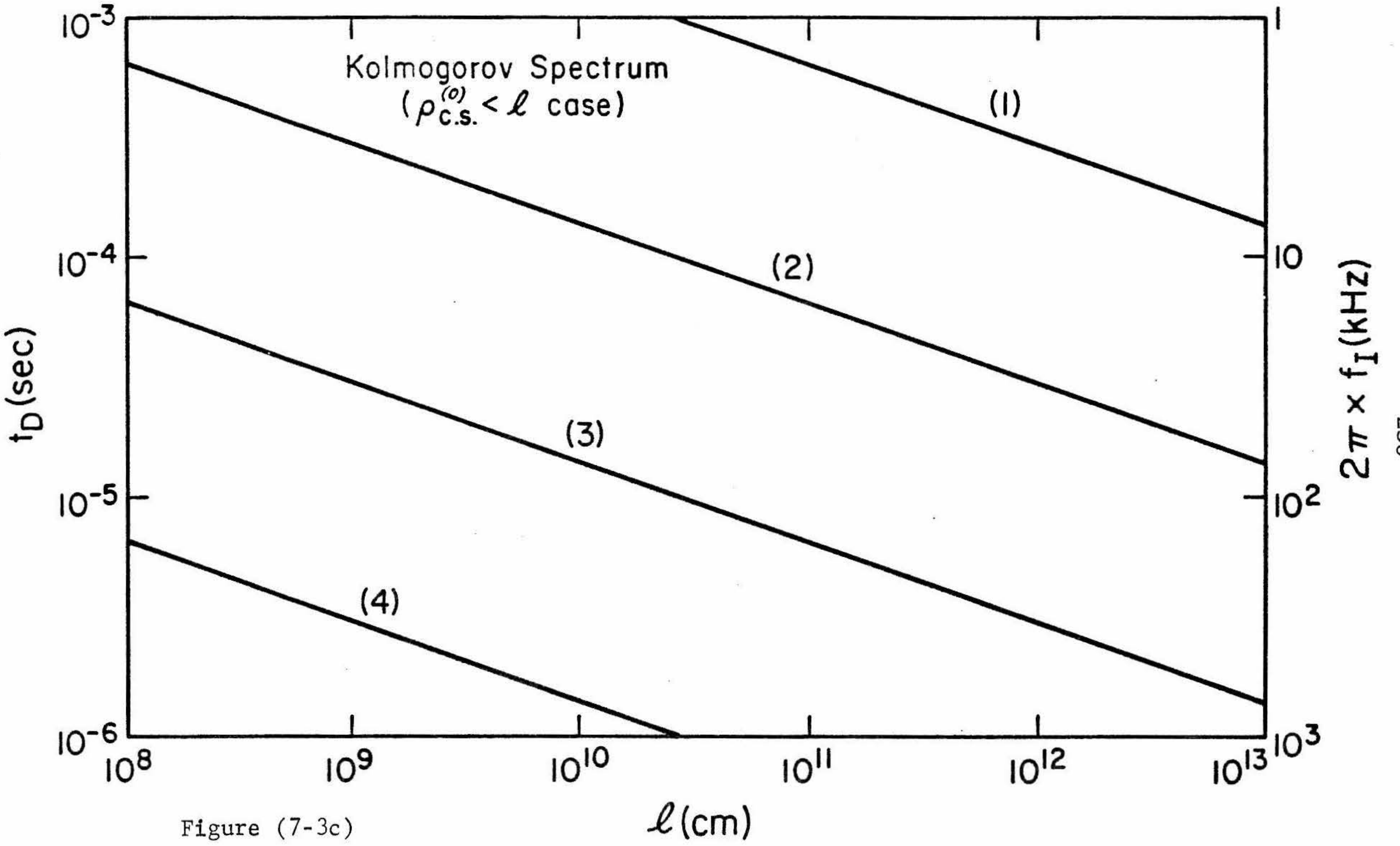

Kolmogorov Spectrum
($\rho^{(0)}_{c.s.} < \ell$ case)
(1)
(2)
(3)
(4)
10^{-3}
10^{-4}
10^{-5}
10^{-6}
t_D(sec)
1
10
10^2
10^3
$2\pi \times f_I$(kHz)
10^8
10^9
10^{10}
10^{11}
10^{12}
10^{13}
ℓ(cm)

Figure (7-3c)

For the "thin screen" case θ_c and $\rho_{c.s.}$ are the same as given in this section with z replaced by the thickness of the thin layer D. The characteristic broadening time t_D and the decorrelation frequency f_I are related to the scattering angle θ_c by the same relation as Eq. (7-13c)

$$t_D = \frac{z\theta_c^2}{2c} \qquad (7\text{-}19)$$

where z is the distance between the thin screen and observers.

(G) Conditions for Strong Scintillation

Let

$$\gamma = A_\beta(0)/2kq_o^2 \qquad (7\text{-}20a)$$

and

$$\eta = z\ A_\beta(0)/2k^2\ (=\Phi_o^2). \qquad (7\text{-}20b)$$

From Eqs. (5-111a,b) and (5-112a,b) we find that the conditions for strong scintillation are

(i) for the Gaussian spectrum,

$$\eta = \Phi_o^2 > 1 \qquad (7\text{-}21a)$$

and

$$\eta > \gamma^{2/3}\ , \qquad (7\text{-}21b)$$

(ii) and for the Kolmogorov spectrum with $\ell < \rho_{c.s.}^{(0)} < L$,

$$\eta = \Phi_o^2 > 1 \qquad (7\text{-}22a)$$

and
$$\eta > \gamma^{5/11} \quad . \tag{7-22b}$$

Using the same arguments as in obtaining Eq. (5-111), we find that

(iii) for the Kolmogorov spectrum with

$$\rho^{(0)}_{c.s.} < \ell, \tag{7-23}$$

the conditions for strong scintillations are

$$\eta = \Phi_o^{\,2} > 1 \tag{7-24a}$$

and
$$z > k \left(\frac{48k}{A_\beta^{(4)}(0)}\right)^{\frac{1}{3}} \tag{7-24b}$$

where
$$A_\beta^{(4)}(0) = \left.\frac{d^4 A_\beta(0)}{d\rho^4}\right|_{\rho=0} = \frac{3}{64\pi} k^4 q_o^{\,11/3} q_1^{\,7/3} \Gamma(\tfrac{7}{6}) . \tag{7-25}$$

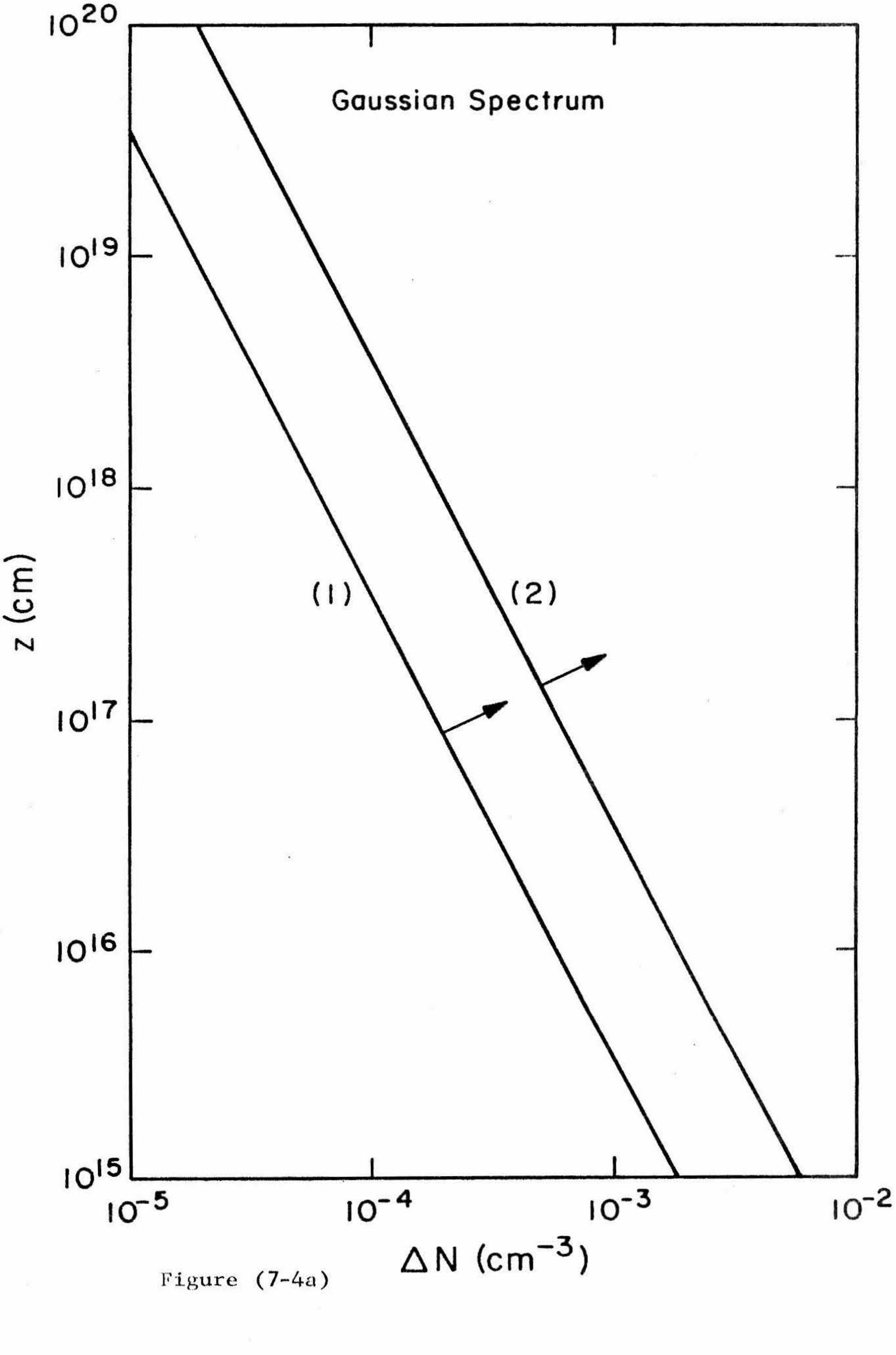
The condition $\Phi_o^{\,2} > 1$ in Eqs. (7-21a), (7-22a) and (7-24a), for both the Gaussian and the Kolmogorov spectra is illustrated in Figures (7-4a,b) for various values of the parameters in interstellar scintillations. We note that for the Kolmogorov spectrum, the condition $\Phi_o^{\,2} > 1$ is always satisfied for reasonable values of the parameters in interstellar scintillations.

The conditions in Eqs. (7-21b), (7-22b) and (7-24b) are respectively plotted in Figures (7-5a,b and c). We note that our results are valid only for the point radio source.

Figure Captions

Figure (7-4a). The value of z, beyond which the condition $\Phi_o^2 > 1$ is satisfied, is plotted versus the root mean square electron density fluctuation $\Delta N = \langle \delta N_e^2 \rangle^{\frac{1}{2}}$, for the Gaussian refractive index spectrum. In curve (1) $L = 10^{11}$cm and in curve (2) $L = 10^{10}$cm. In both curves f = 300 MHz. The value of z scales $\propto f^2 L^{-1} \langle \delta N_e^2 \rangle^{-1}$.

Figure (7-4b). The value of z, beyond which the condition $\Phi_o^2 > 1$ is satisfied, is plotted versus the correlation scale L, for the Kolmogorov refractive index spectrum. In curve (1) $\Delta N = \langle \delta N_e^2 \rangle^{\frac{1}{2}} = 10^{-6}\text{cm}^{-3}$, in curve (2) $\Delta N = 10^{-4}\text{cm}^{-3}$ and in curve (3) $\Delta N = 10^{-2}\text{cm}^{-3}$. In all cases f = 300 MHz. The value of z scales $\propto f^2 L^{-1} \langle \delta N_e^2 \rangle^{-1}$.



Gaussian Spectrum
(1)
(2)
z (cm)
10^{20}
10^{19}
10^{18}
10^{17}
10^{16}
10^{15}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
ΔN (cm^{-3})

Figure (7-4a)

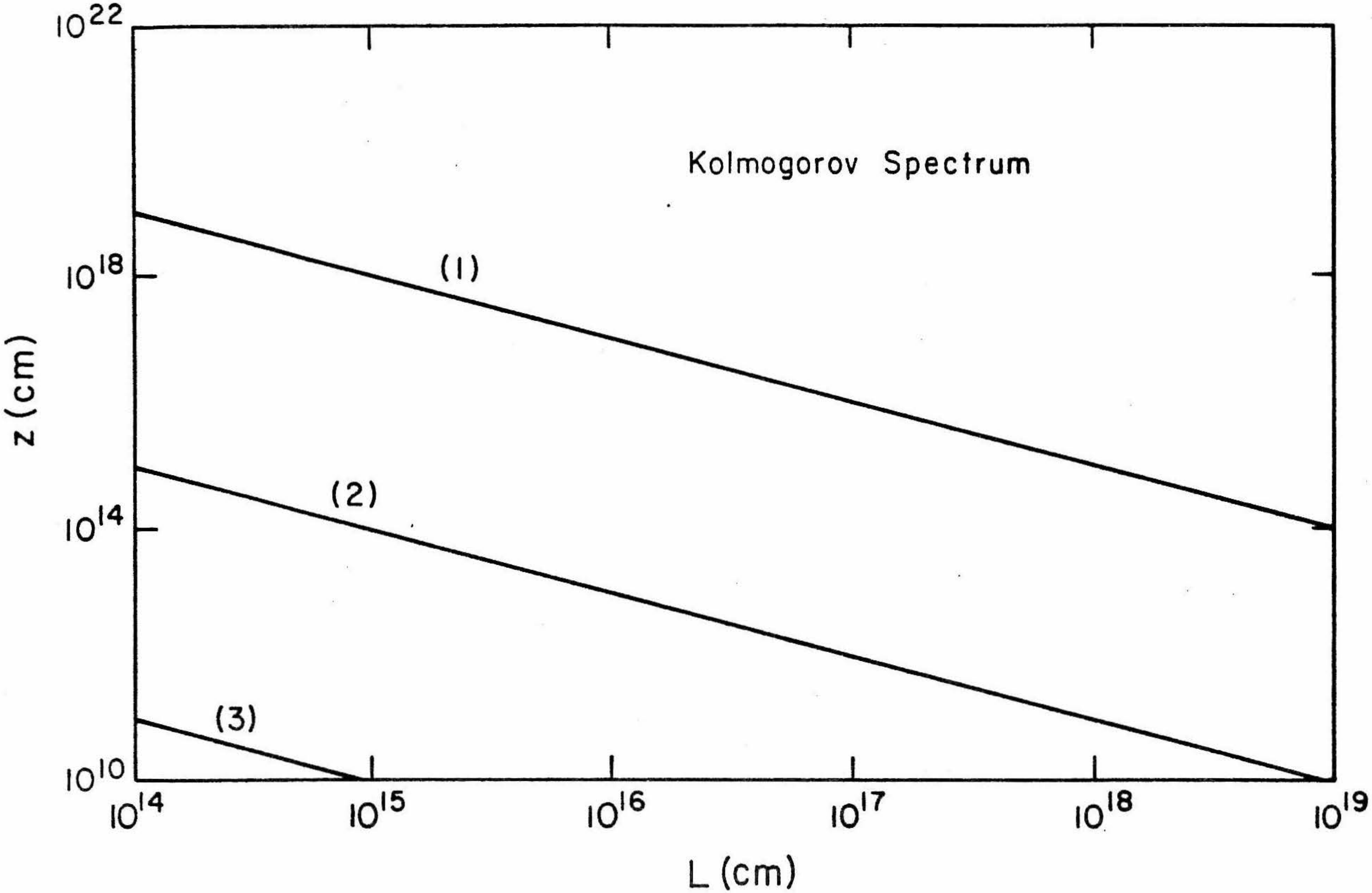

Kolmogorov Spectrum
(1)
(2)
(3)
z (cm)
L (cm)
10^22
10^18
10^14
10^10
10^14
10^15
10^16
10^17
10^18
10^19

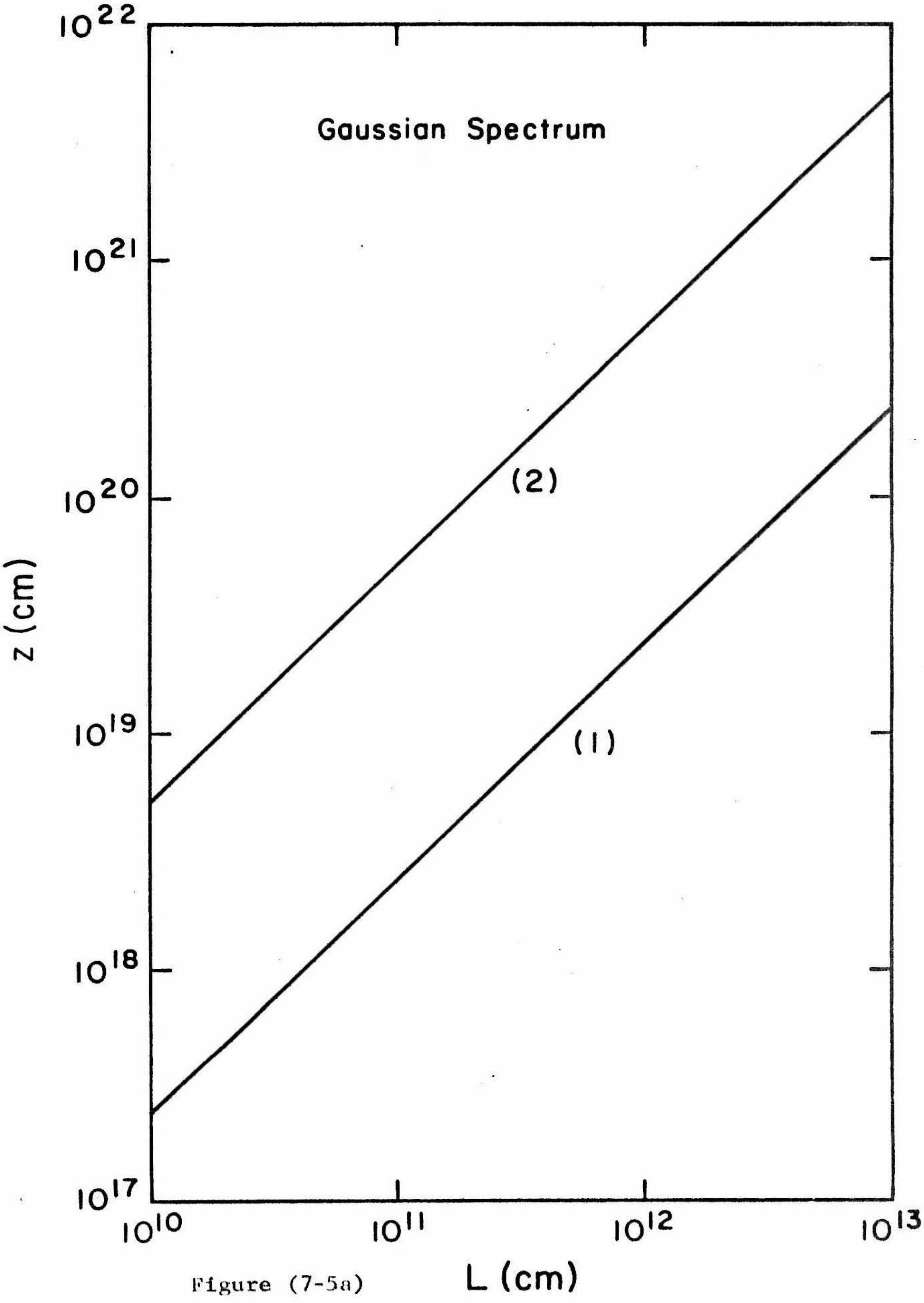
Figure (7-4b)

Figure Captions

Figure (7-5a). The value of z, beyond which the strong scintillation condition $\eta > \gamma^{2/3}$ is satisfied, versus the correlation scale L, for the Gaussian refractive index spectrum. In curve (1) $\langle \delta N_e^{\,2} \rangle = 10^{-4} \mathrm{cm}^{-6}$, and in curve (2) $\langle \delta N_e^{\,2} \rangle = 10^{-8} \mathrm{cm}^{-6}$. In both cases f = 300 MHz. The value of z scales $\propto f^{\frac{4}{3}} L \langle \delta N_e^{\,2} \rangle^{-\frac{1}{3}}$.

Figure (7-5b). The value of z, beyond which the strong scintillation condition $\eta > \gamma^{5/11}$ is satisfied, versus the correlation scale L, for the Kolmogorov spectrum ($\ell < \rho^{(0)}_{c.s.} < L$ case). From curve (1) to curve (3), $\langle \delta N_e^{\,2} \rangle = 10^{-4}$, 10^{-6} and $10^{-8} \mathrm{cm}^{-6}$ respectively. In all cases f = 300 MHz. The value of z scales $\propto f^{17/11} L^{4/11} \langle \delta N_e^{\,2} \rangle^{-\frac{6}{11}}$.

Figure (7-5c). The value of z, beyond which the strong scintillation condition in Eq. (7-24b) is satisfied, versus the inner scale ℓ, for the Kolmogorov spectrum ($\ell > \rho^{(0)}_{c.s.}$ case). From curve (1) to curve (5), $\langle \delta N_e^{\,2} \rangle = 10^{-8}$, 10^{-6}, 10^{-4}, 10^{-2} and $1.0 \mathrm{~cm}^{-6}$ respectively. In all cases $L = 10^{18}$ cm and f = 300 MHz. The value of z scales $\propto f^{\frac{4}{3}} \langle \delta N_e^{\,2} \rangle^{-\frac{1}{3}} L^{\frac{2}{9}} \ell^{\frac{7}{9}}$.



Gaussian Spectrum
(2)
(1)
z (cm)
10^{22}
10^{21}
10^{20}
10^{19}
10^{18}
10^{17}
10^{10}
10^{11}
10^{12}
10^{13}
L (cm)

Figure (7-5a)

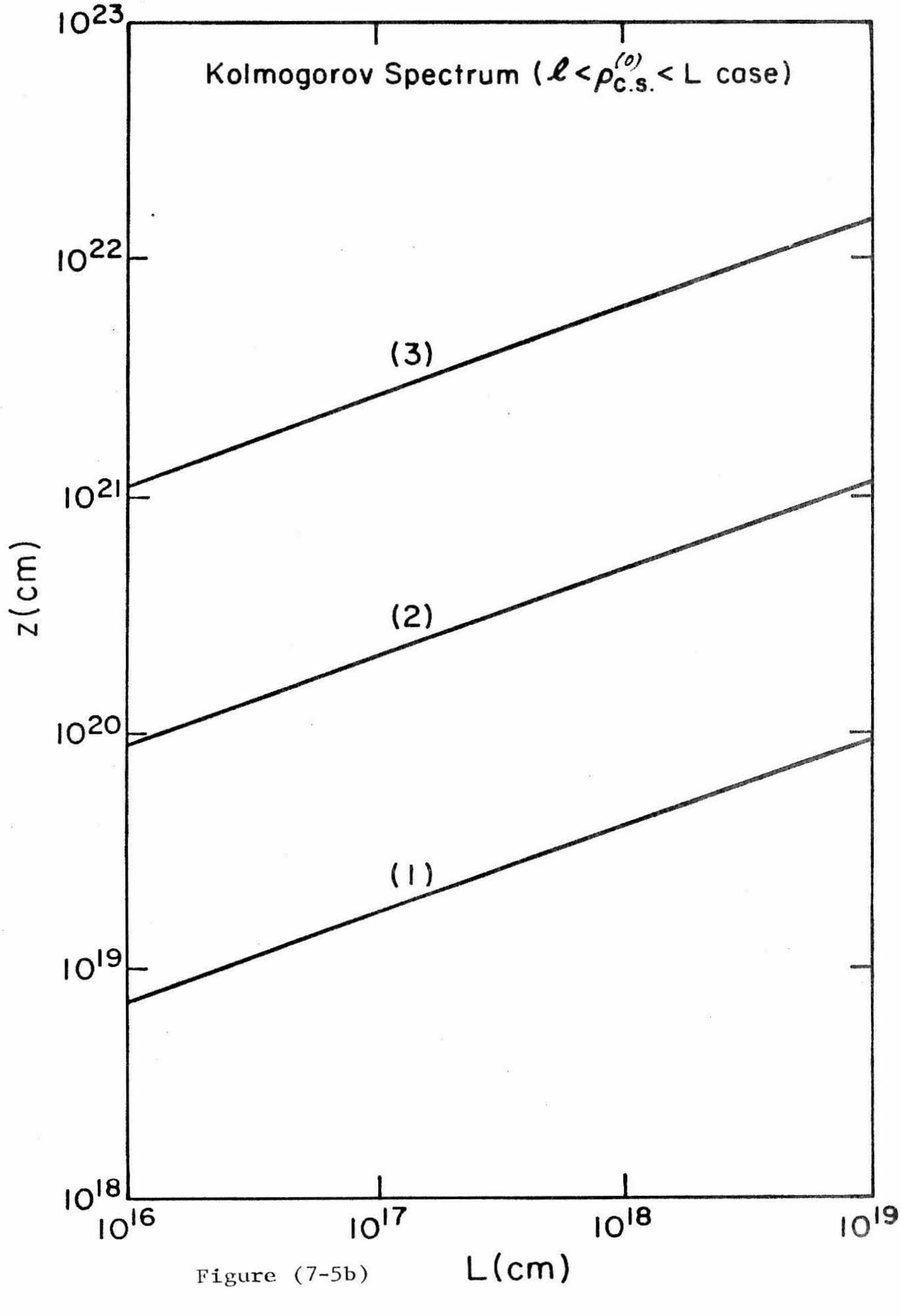
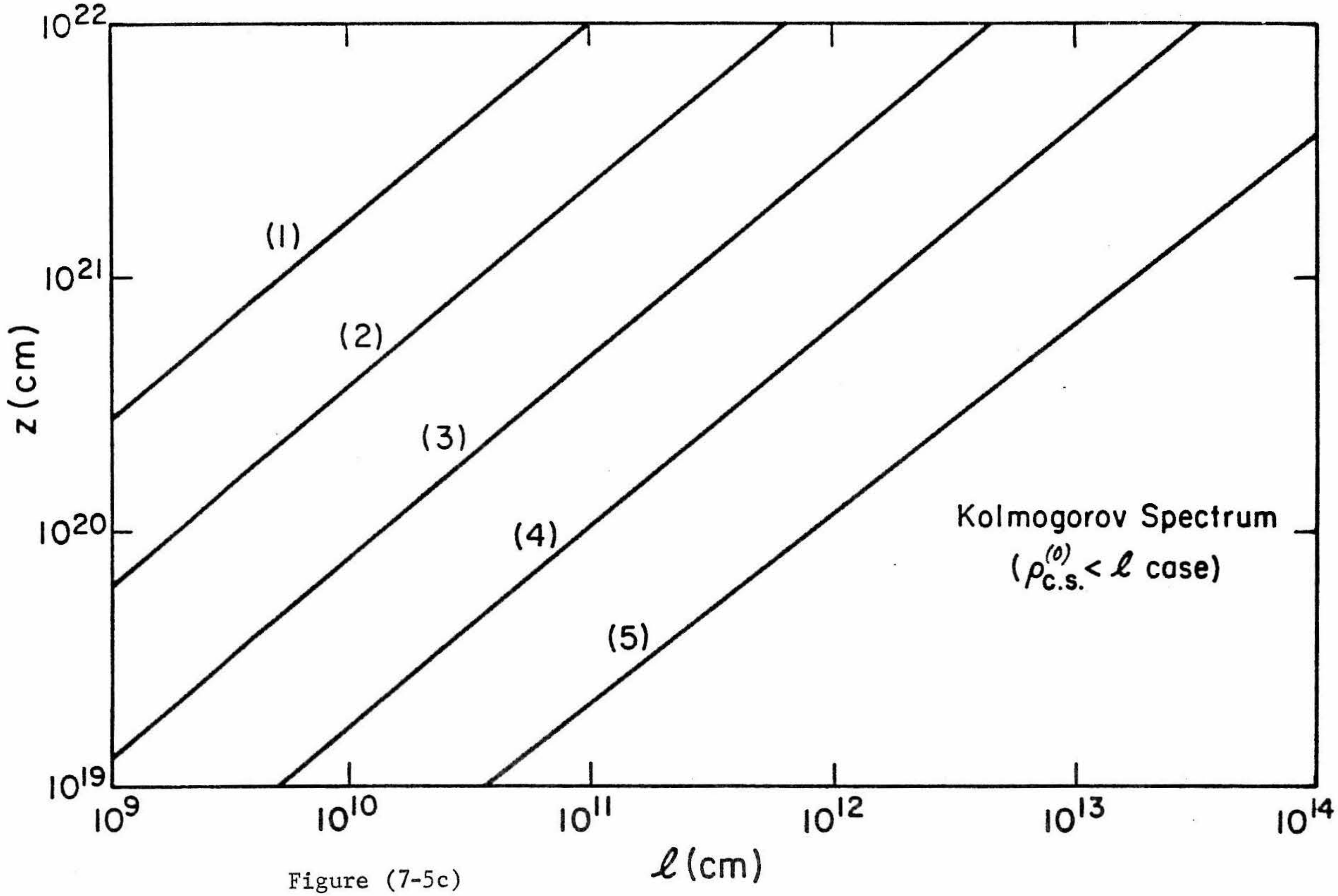

Kolmogorov Spectrum ($\ell < \rho^{(0)}_{c.s.} < L$ case)
(3)
(2)
(1)
z(cm)
10^{23}
10^{22}
10^{21}
10^{20}
10^{19}
10^{18}
10^{16}
10^{17}
10^{18}
10^{19}
L(cm)

Figure (7-5b)



10^{22}
10^{21}
10^{20}
10^{19}
z (cm)
(1)
(2)
(3)
(4)
(5)
Kolmogorov Spectrum
($\rho^{(0)}_{c.s.} < \ell$ case)
10^{9}
10^{10}
10^{11}
10^{12}
10^{13}
10^{14}
ℓ (cm)

Figure (7-5c)

III. Analysis of the Data for CP 0328, PSR 0833-45 and NP 0532

In this section, we will analyze in detail the data of three pulsars, namely, CP 0328, PSR 0833-45 and NP 0532.

(A) CP 0328

The CP 0328 pulsar has been observed by Rickett (1970). The dispersion measure is

$$DM = 26.7 \text{ pc cm}^{-3} = 8.01 \times 10^{19} \text{cm}^{-2}. \qquad (7\text{-}25)$$

The decorrelation frequency f_I and the characteristic broadening time t_c are related to the measured half-visibility bandwidth B_h by

$$2\pi f_I = t_c^{-1} = B_h \qquad (7\text{-}26)$$

(Lovelace 1970). From the result of Rickett (1970), one has $B_h(408 \text{ MHz}) = 8 \times 10^5 \text{Hz}$. Therefore

$$f_I(408 \text{ MHz}) = 1.27 \times 10^5 \text{Hz} \qquad (7\text{-}27a)$$

and

$$t_c(408 \text{ MHz}) = 1.25 \times 10^{-6} \text{sec}. \qquad (7\text{-}27b)$$

The decorrelation time $\tau_{c.s.}$ at 408 MHz can be obtained from Figure (11) of Rickett (1970) and one has

$$\tau_{c.s.}(408 \text{ MHz}) \simeq 720 \text{ sec}. \qquad (7\text{-}28)$$

Setting the propagating distance

$$z = 1000 \text{ pc} = 3 \times 10^{21} \text{cm} \qquad (7\text{-}29)$$

(c.f. Rickett 1970, Lang 1971), one has from Eq. (7-25)

$$\langle N_e \rangle = \frac{DM}{z} = 0.0267 \text{ cm}^{-3} \quad . \tag{7-30}$$

Using the relations between $\rho_{c.s.}$, θ_c and t_D in Eqs. (7-11) and (7-13c), one obtains from Eqs. (7-28) and (7-29) the correlation scale

$$\rho_{c.s.} = 1.7 \times 10^9 \text{cm} \quad . \tag{7-31}$$

$\tau_{c.s.}$ and $\rho_{c.s.}$ in Eqs. (7-28) and (7-31) give us the relative transverse velocity of the plasma medium

$$v \simeq 24 \text{ km/sec} \tag{7-32}$$

which is of the correct order for interstellar space.

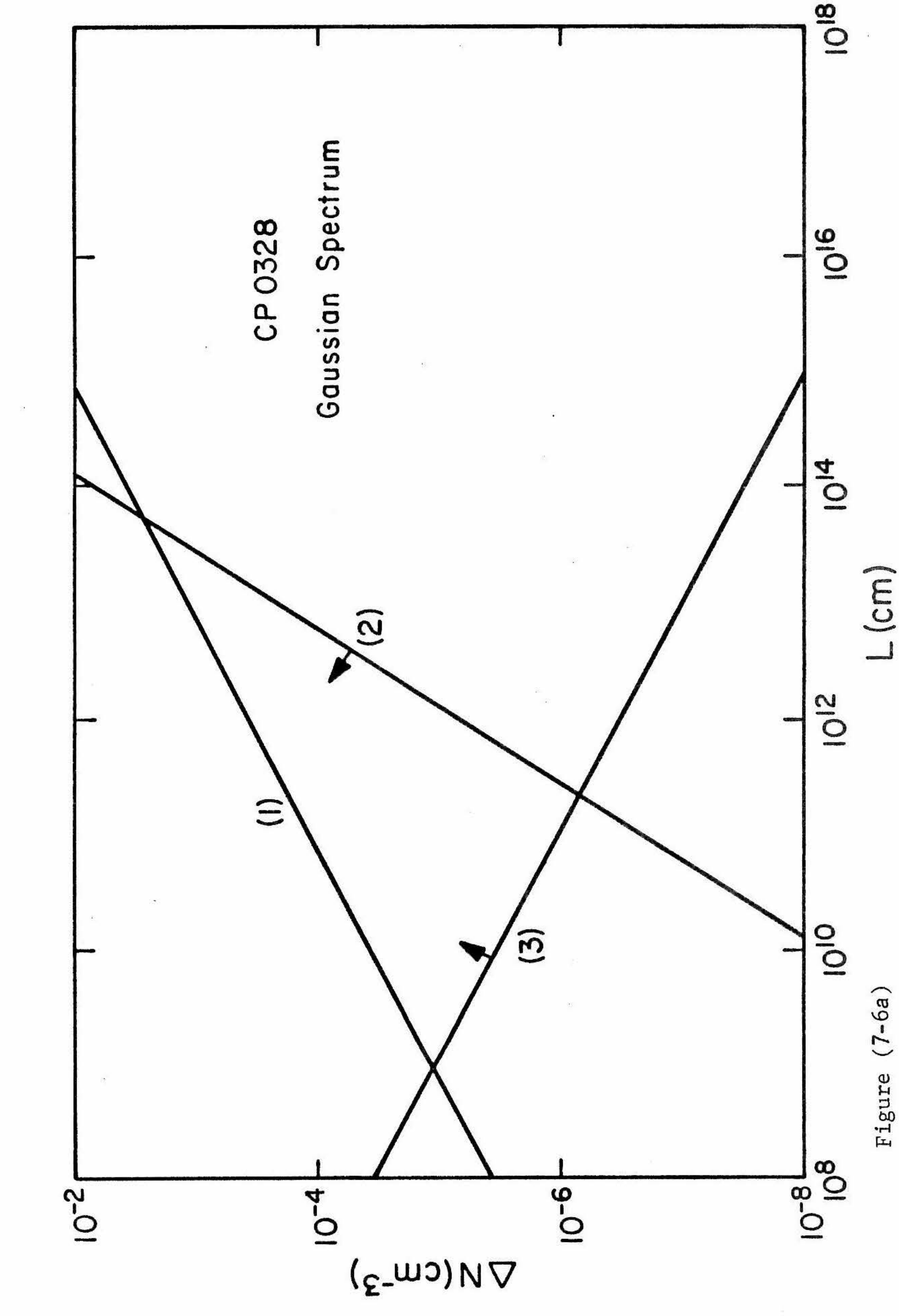
Assume that the turbulent plasmas have a Gaussian power spectrum and are statistically evenly distributed between the pulsar and the earth. Figure (7-6a) is a logarithmic plot of the ΔN, L plane as plotted by Scheuer (1968) and Rickett (1970), where $\Delta N = \langle \delta N_e^2 \rangle^{1/2}$ is the root mean square electron density fluctuation and L is the coherent scale of electron density fluctuation in the interstellar medium between the CP 0328 pulsar and the earth. Curve (1) of Figure (7-6a) is the constraint for ΔN and L from the observed value of t_c in Eq. (7-27b) at f = 408 MHz and $z = 3 \times 10^{21}$cm. The values of ΔN and L must lie on the line of curve (1). Since the scintillation is strong for CP 0328 pulsar at 408 MHz (Rickett 1970), the conditions for strong scintillation in Eqs. (7-21a,b) for f = 408 MHz, $z = 3 \times 10^{21}$cm, are also plotted in the Figure. Curve (2) is $\eta = \gamma^{2/3}$ and curve (3) is $\eta = \Phi_o^2 = 1$. The fact that at f = 610 MHz the scintillation is still strong will have a stronger restriction on the range of ΔN and L. From Figure (7-6a), we see that

Figure Captions

Figure (7-6a). Conditions for the interstellar scintillation of CP 0328. Curve (1) is the pulse broadening (or decorrelation frequency) relation in Eq. (7-27). Curve (2) is the condition $\eta > \gamma^{2/3}$ and curve (3) is the condition $\Phi_o^{\ 2} > 1$ for f = 408 MHz. This figure is for the Gaussian refractive index spectrum.

Figure (7-6b). As in Figure (7-6a) for the Kolmogorov spectrum with $\ell < \rho_{c.s.}^{(0)} < L$. Curve (1) is the pulse broadening relation. Curve (2) is the condition for strong scintillation, $\eta > \gamma^{5/11}$. The values of ΔN and L must lie on curve (1).

Figure (7-6c). As in Figure (7-6a) for the Kolmogorov spectrum with $\ell > \rho_{c.s.}^{(0)}$. Curve (1a) is the pulse broadening relation for $L = 10^{20}$cm. Curve (1b) is the condition of strong scintillation in Eq. (7-24b) and curve (1c) is the condition $\rho_{c.s.}^{(0)} < \ell$ for $L = 10^{20}$cm. For $L = 10^{20}$cm, the values of ΔN and ℓ must be on the section of curve (1a) between curve (1b) and (1c). For $L = 10^{19}$ and 10^{18}cm, the constraints are plotted respectively in curves (2a,b,c) and (3a,b,c).



CP 0328
Gaussian Spectrum
(1)
(2)
(3)
10^{-2}
10^{-4}
10^{-6}
10^{-8}
$\Delta N(cm^{-3})$
10^{8}
10^{10}
10^{12}
10^{14}
10^{16}
10^{18}
L (cm)

Figure (7-6a)

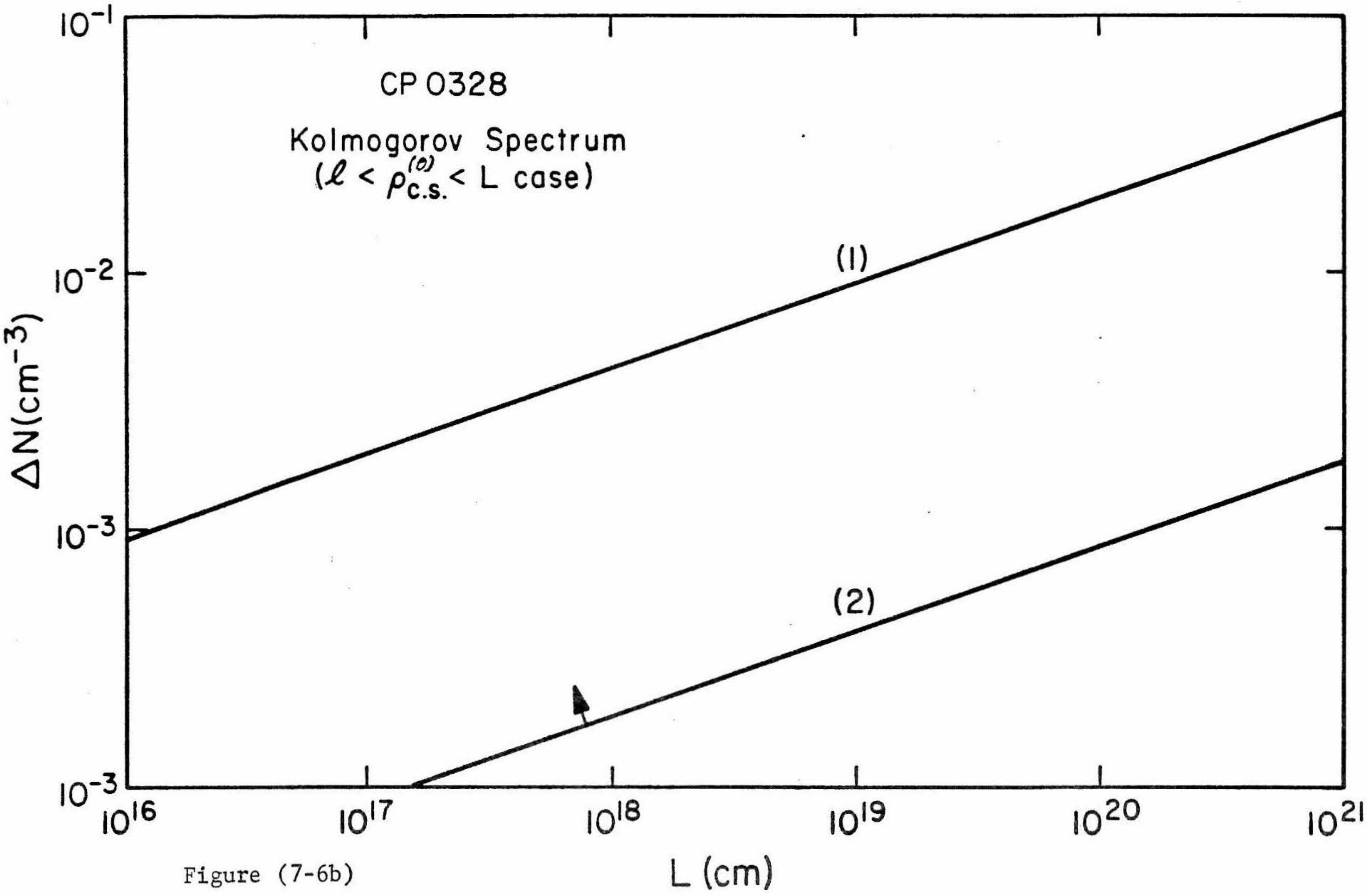
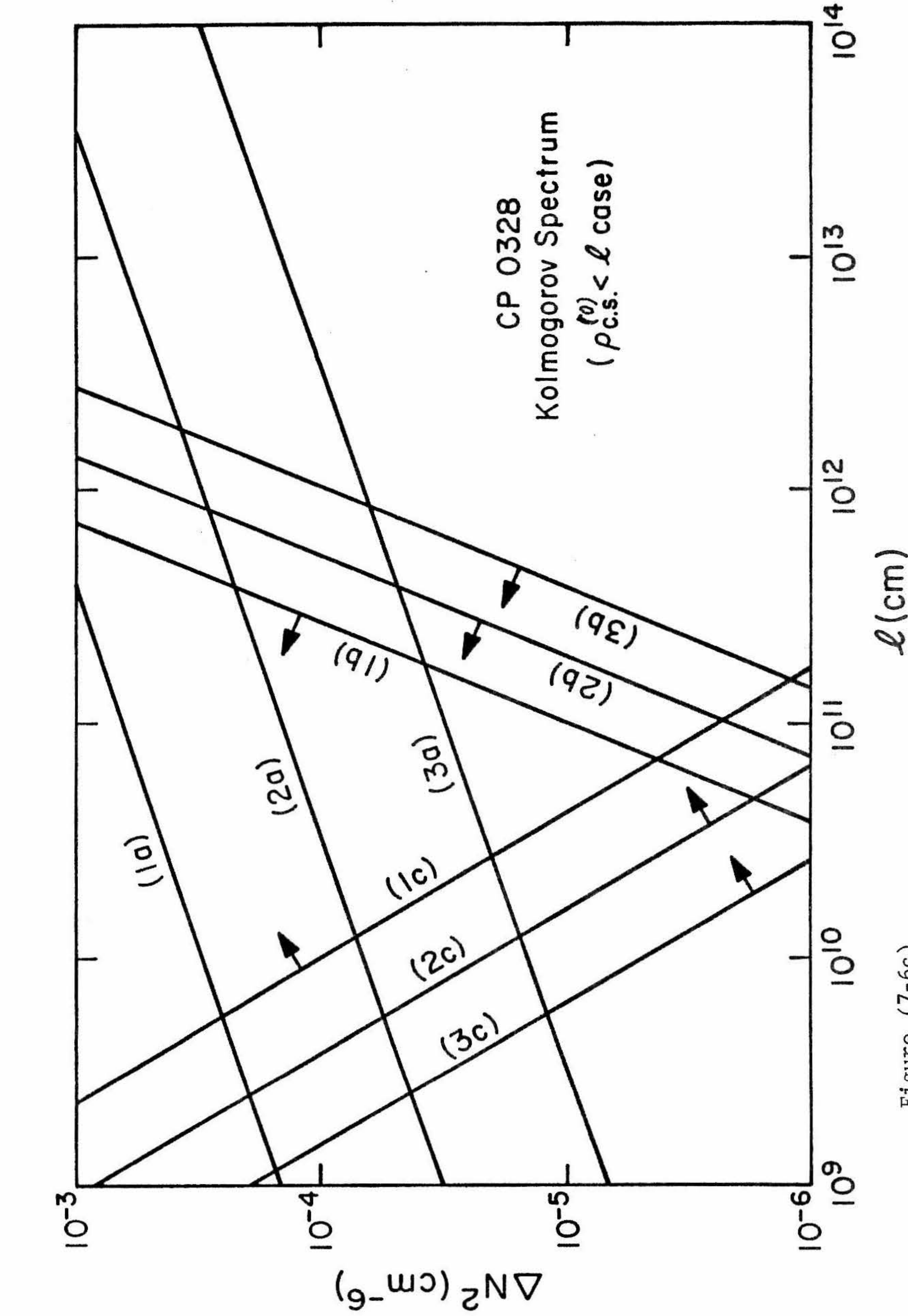

CP 0328
Kolmogorov Spectrum
($\ell < \rho^{(0)}_{c.s.} < L$ case)
(1)
(2)
10^{-1}
10^{-2}
10^{-3}
10^{-3}
$\Delta N(cm^{-3})$
10^{16}
10^{17}
10^{18}
10^{19}
10^{20}
10^{21}
L (cm)

Figure (7-6b)



CP 0328
Kolmogorov Spectrum
($\rho^{(0)}_{c.s.} < \ell$ case)
(1a)
(2a)
(3a)
(1b)
(2b)
(3b)
(1c)
(2c)
(3c)
ΔN^2 (cm^{-6})
10^{-3}
10^{-4}
10^{-5}
10^{-6}
ℓ(cm)
10^{9}
10^{10}
10^{11}
10^{12}
10^{13}
10^{14}

Figure (7-6c)

$$10^{10}\text{cm} \lesssim L \leq 10^{12}\text{cm} \qquad (7\text{-}33a)$$

and

$$10^{-5}\text{cm}^{-3} \leq \Delta N \lesssim 10^{-3}\text{cm}^{-3} \qquad (7\text{-}33b)$$

for the turbulent medium between CP 0328 pulsar and the earth.

The results in Eqs. (7-33a,b) are about the same as obtained by Rickett (1970). However as mentioned in Chapter 1, the Gaussian spectrum is rather artificial in interpreting the scintillation data and the power-law spectrum is a more realistic form. Observed turbulent plasmas tend to have power spectra (e.g. Jokipii 1973). Figure (7-6b) is a similar plot as Figure (7-6a) for the turbulent medium having a Kolmogorov spectrum with $\ell < \rho^{(0)}_{c.s.} < L$. Again curve (1) is t_c (408 MHz) $= 1.25 \times 10^{-6}$ sec. Curve (2) shows one of the two conditions for strong scintillation, $\eta > \gamma^{5/11}$. The other condition for strong scintillation ($\Phi_o^2 = \eta > 1$) is well satisfied for reasonable values of the parameters, ΔN and L, and is not shown in the figure. The values of ΔN and L must lie on curve (1). We note that in Figure (7-6b), the condition $\eta > \gamma^{5/11}$ has no constraint on the values of ΔN and L since curve (1) and curve (2) are parallel. This is physically expectable for the following reason. We note that the turbulent inhomogeneities (eddies) with a scale of size much greater than the Fresnel scale $\rho_f = \sqrt{\frac{z}{k}}$ have very little effect on the scintillations of the radio wave. Therefore the scintillation pattern of the radio wave must remain the same if one adds inhomogeneities of larger size (equivalent to increase the value of the largest scale L) while keeping constant the magnitude of the turbulent inhomogeneities which are important for the scintillations of the radio wave by increasing the value of ΔN. Thus from the scintillation data, one can not set an upper limit on L and the strong scintillation condition, $\eta > \gamma^{5/11}$ must have no restriction on the values of ΔN and L.

In the interstellar medium, we expect that the largest scale size of the turbulent inhomogeneities will be of the order of 10 pc to 100 pc (Jokipii and Lerche, 1969). For demonstrations, we choose

$$L = 10^{20} \mathrm{cm} \ (\simeq 33 \ \mathrm{pc}). \qquad (7\text{-}34)$$

Then from Figure (7-6b), we find that

$$\langle \delta N_e^{\,2} \rangle^{\frac{1}{2}} = \Delta N = 2 \times 10^{-2} \mathrm{cm}^{-3} \simeq 0.75 \ \langle N_e \rangle \quad . \qquad (7\text{-}35)$$

We note that the Kolmogorov spectrum gives a higher value of ΔN, the root mean square of the electron density fluctuations, than the Gaussian spectrum. The value of ΔN in the artificial Gaussian spectrum gives only the density fluctuation of those inhomogeneities which are important in causing the scintillation pattern.

We consider the other case in which the interstellar medium has a Kolmogorov spectrum with $\rho^{(0)}_{c.s.} < \ell$. Figure (7-6c) is a plot of ΔN^2 vs the smallest scale ℓ (inner scale or cut-off scale) for various value of the outer scale L. For $L = 10^{20}$cm, curve (1a) is $t_c = 1.25 \times 10^{-6}$sec at $f = 408$ MHz and $z = 3 \times 10^{21}$cm, curve (1b) is the condition of strong scintillation in Eq. (7-24b) and curve (1c) is the condition $\rho^{(0)}_{c.s.} < \ell$. The condition $\Phi_o^{\,2} > 1$ is satisfied and is not shown in the figure. For

$L = 10^{20}$cm, the values of ΔN^2 and ℓ must lie on the section of curve (1a) between curves (1b) and (1c). Similarly, the constraints are shown respectively in curves (2a,b,c) and (3a,b,c) for $L = 10^{19}$cm and $L = 10^{18}$cm. For all values of L, we have from Figure (7-6c),

$$\ell \simeq 5 \times 10^{10} \text{cm} \tag{7-36}$$

within a factor of 4. It is of interest to note that the value of the cut-off scale ℓ in Eq. (7-36) for the Kolmogorov spectrum is about the same as the coherent scale L in Eq. (7-33a) for the Gaussian spectrum. For $L = 10^{20}$cm, we have from Figure (7-6c)

$$\Delta N \simeq 2 \times 10^{-2} \text{cm}^{-3} , \tag{7-37}$$

which is the same as in Eq. (7-35) for the case without a cut-off at $\ell \simeq 5 \times 10^{10}$cm, since for the Kolmogorov spectrum, the value of ΔN comes mostly from the large scale inhomogeneities. We also note that the conditions for strong scintillation, in the case of a Kolmogorov spectrum with $\rho^{(0)}_{c.s.} < \ell$, do not give L an upper limit either. The increase of the value of L corresponds to an increase of ΔN while keeping constant the strength of the inhomogeneities which are important for scintillation. Comparing the above two cases ($\rho^{(0)}_{c.s.} < \ell$ and $\ell < \rho^{(0)}_{c.s.} < L$) for the Kolmogorov spectrum, we see that it is not necessary for the power spectrum to have a cut-off at $\ell \simeq 5 \times 10^{10}$cm to fit the scintillation data.

Little and Matheson (1973) also considered the case $\rho^{(0)}_{c.s.} < \ell$, but their results are incorrect because they used an incorrect condition for the strong scintillation. They used the condition $z\theta_c > L$, for strong scintillation which is too strong since the largest scale inhomogeneities have nearly no effect on the scintillations of the radio wave.

In conclusion, we point out that both the Gaussian and the Kolmogorov spectra fit the data of CP 0328 pulsar and that the Gaussian spectrum is quite artificial and gives only the density of those inhomogeneities which are important for the scintillation.

(B) PSR 0833-45

The average pulse profile of the pulses from the PSR 0833-45 pulsar has been measured for several frequencies between 300 and 1410 MHz (Ables et al. 1970, Komesaroff et al. 1972). Backer (1974) measured the decorrelation frequency f_I and the decorrelation time $\tau_{c.s.}$ for the same pulsar with frequencies ranging from 837 to 8085 MHz. The above results are combined and plotted in Figure (7-7). f_I at 300 MHz is from the pulse-broadening measurement (Komesaroff et al. 1972) by using the relation $2\pi f_I t_c = 1$ in Eq. (7-14). At f = 300 MHz,

$$t_c = 9.4 \text{ msec}, \qquad (7\text{-}38)$$

and therefore

$$f_I = 17 \text{ Hz} \quad . \qquad (7\text{-}39)$$

All other points in Figure (7-7) are from the measurement of Backer (1974). We note that the decorrelation bandwidth B_s used by Backer corresponds to $f_I/2$.

For the frequency dependence of the decorrelation time in Figure (7-7), curve (1a) is the best fit for $\tau_{c.s.} \propto f$, corresponding

Figure (7-7)

Estimates of the decorrelation bandwidth f_I (I symbol and left axis) and of the decorrelation time $\tau_{c.s.}$ (+ symbol and right axis) of interstellar scintillation for the PSR 0833-45 pulsar. The length of the I-symbol indicates the spread of measurements. For the decorrelation time, curve (1a) is the best fit for $\tau_{c.s.} \propto f$ and curve (1b) is for $\tau_{c.s.} \propto f^{1.2}$. For the decorrelation frequency, curve (2a) is the fit for $f_I \propto f^{4.0}$ and curve (2b) is for $f_I \propto f^{4.4}$.

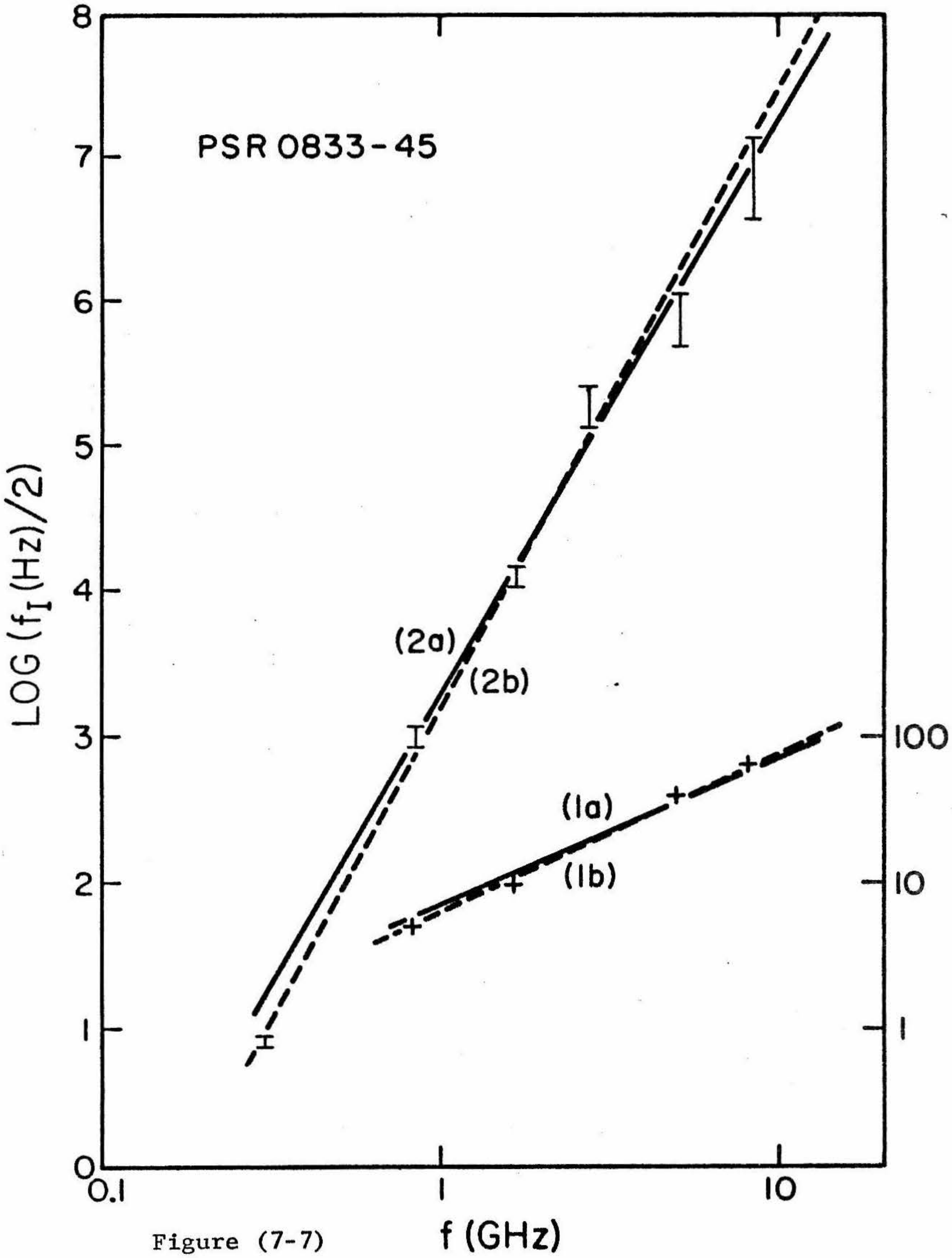

PSR 0833-45
LOG (f_I (Hz)/2)
$\tau_{c.s.}$ (sec)
f (GHz)
(2a)
(2b)
(1a)
(1b)
0
1
2
3
4
5
6
7
8
0.1
1
10
100
10
1

Figure (7-7)

to the Gaussian spectrum and curve (1b) is for $\tau_{c.s.} \propto f^{1.2}$, corresponding to the Kolmogorov spectrum ($\ell < \rho^{(0)}_{c.s.} < L$ case). Note that curve (1b) fits better than (1a). For the decorrelation frequency of f_I, curve (2a) is the best fit with a slope +4 ($f_I \propto f^4$) for the Gaussian spectrum, and curve (2b) is with a slope +4.4 ($f_I \propto f^{4.4}$) for the Kolmogorov spectrum. As shown in Figure (7-7), both the Gaussian and the Kolmogorov spectra fit the data and the relation $2\pi f_I t_c = 1$ is satisfied.

From Figure (7-7), we have the decorrelation time at f = 300 MHz (by extrapolation),

$$\tau_{c.s.}(300 \text{ MHz}) = 2.0 \text{ sec} \quad . \tag{7-40}$$

The dispersion measure, DM, for PSR 0833-45 is

$$\text{DM} = 69.2 \pm 0.1 \text{ pc cm}^{-3} , \tag{7-41}$$

(Ables et al. 1970). If we set the distance of the pulsar from the earth to be

$$z = 500 \text{ pc} = 1.5 \times 10^{21} \text{cm}, \tag{7-42}$$

(c.f. Komesaroff et al. 1972) then one has for the mean electron density along the propagation path,

$$\langle N_e \rangle = 0.138 \text{ cm}^{-3}. \tag{7-43}$$

Again using the relations between $\rho_{c.s.}$, θ_c and t_D in Eqs. (7-11) and (7-13c), one obtains from Eqs. (7-38) and (7-42) the correlation scale

$$\rho_{c.s.}(300 \text{ MHz}) = 1.7 \times 10^9 \text{cm}, \tag{7-44}$$

which combining with the $\tau_{c.s.}$ in Eq. (7-40) gives the transverse velocity of the interstellar medium relative to the line between the pulsar and the earth,

$$v = 30 \text{ km/sec}, \tag{7-45}$$

which shows that the distance z in Eq. (7-42) is of the correct order.

For the turbulent plasmas with a Gaussian power spectrum, the relation t_c(300 MHz) = 9.4 msec and the strong scintillation conditions are plotted in Figure (7-8), similar to Figure (7-6a). From Figure (7-8), we find that

$$L \simeq 10^{13} \text{cm} \tag{7-46a}$$

and

$$\Delta N \simeq 0.1 \text{ cm}^{-3} \simeq 0.7 \langle N_e \rangle \quad . \tag{7-46b}$$

For the Kolmogorov spectrum with $\ell < \rho^{(0)}_{c.s.} < L$, the relation t_c (300 MHz) = 9.4 msec and the strong scintillation condition, $\eta > \gamma^{5/11}$, are respectively plotted on curves (1a) and (1b) in Figure (7-9), similar to Figure (7-6b). From the plot in Figure (7-9), we find that if we set the outer scale of the turbulent plasmas to be

$$L = 10 \text{ pc} = 3 \times 10^{19} \text{cm}, \tag{7-47}$$

then we have

$$\Delta N \simeq 0.85 \text{ cm}^{-3} \simeq 6 \langle N_e \rangle. \tag{7-48}$$

(If we set $L > 10$ pc, then $\Delta N > 6 \langle N_e \rangle$.)

Since the electron density N_e is always positive, the high ratio between ΔN and $\langle N_e \rangle$ ($\frac{\Delta N}{\langle N_e \rangle} \simeq 6$) for the Kolmogorov spectrum in Eq. (7-48) is very unusual and we do not expect to have such a high ratio. The

Figure Captions

Figure (7-8). As in Figure (7-6a) for the PSR 0833-45 pulsar.

Figure (7-9). As in Figure (7-6b), conditions for the interstellar scintillation with the Kolmogorov refractive index spectrum ($\ell < \rho^{(0)}_{c.s.} < L$ case). Curve (1a) is the pulse broadening relation and curve (1b) is the condition $\eta > \gamma^{5/11}$ for PSR 0833-45. Curve (2) is the pulse broadening relation for NP 0532.

Figure (7-10). As in Figure (7-6a) for the NP 0532 pulsar.

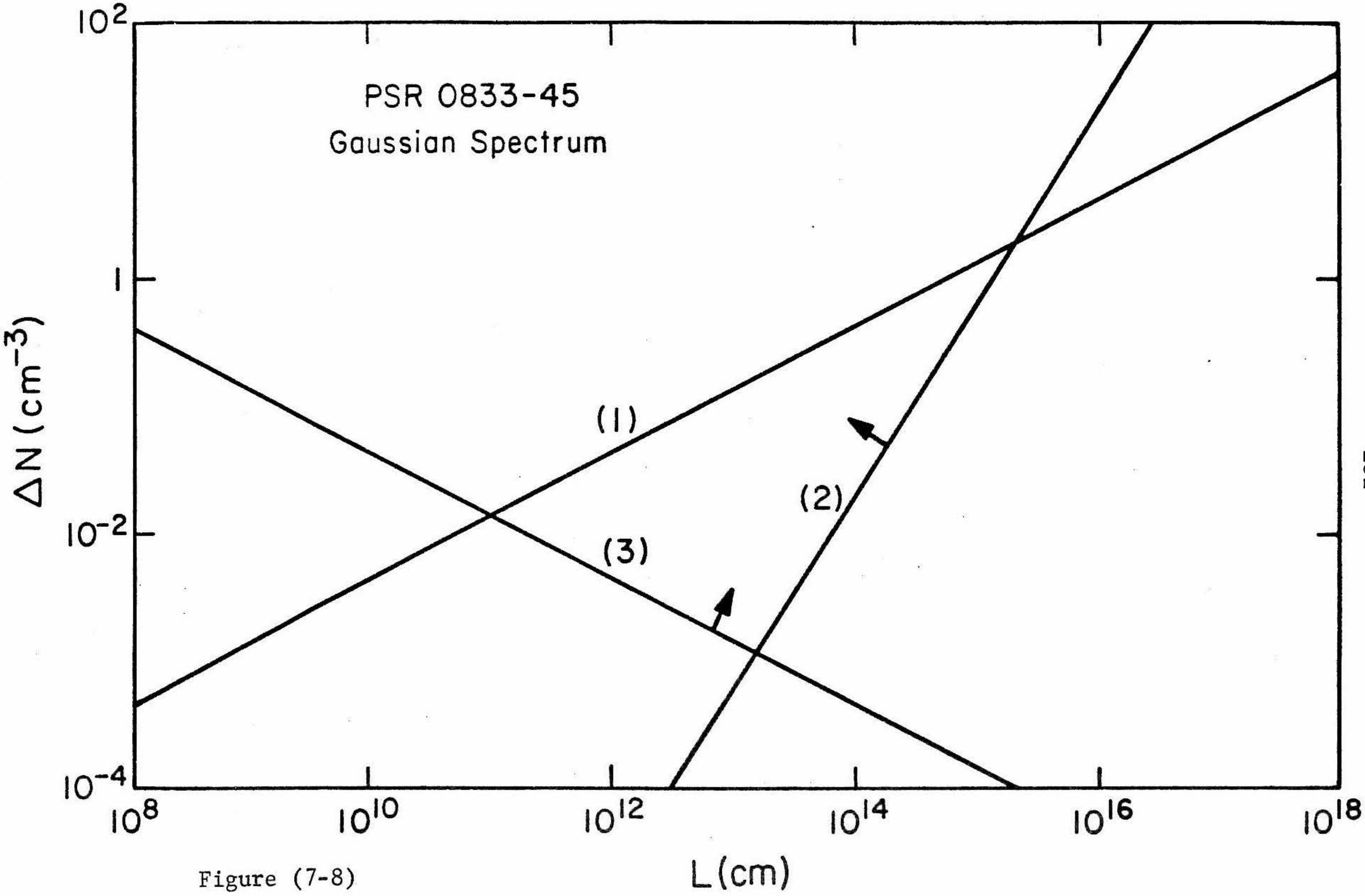

PSR 0833-45
Gaussian Spectrum
(1)
(2)
(3)
ΔN (cm^-3)
10^2
1
10^-2
10^-4
10^8
10^10
10^12
10^14
10^16
10^18
L (cm)

Figure (7-8)

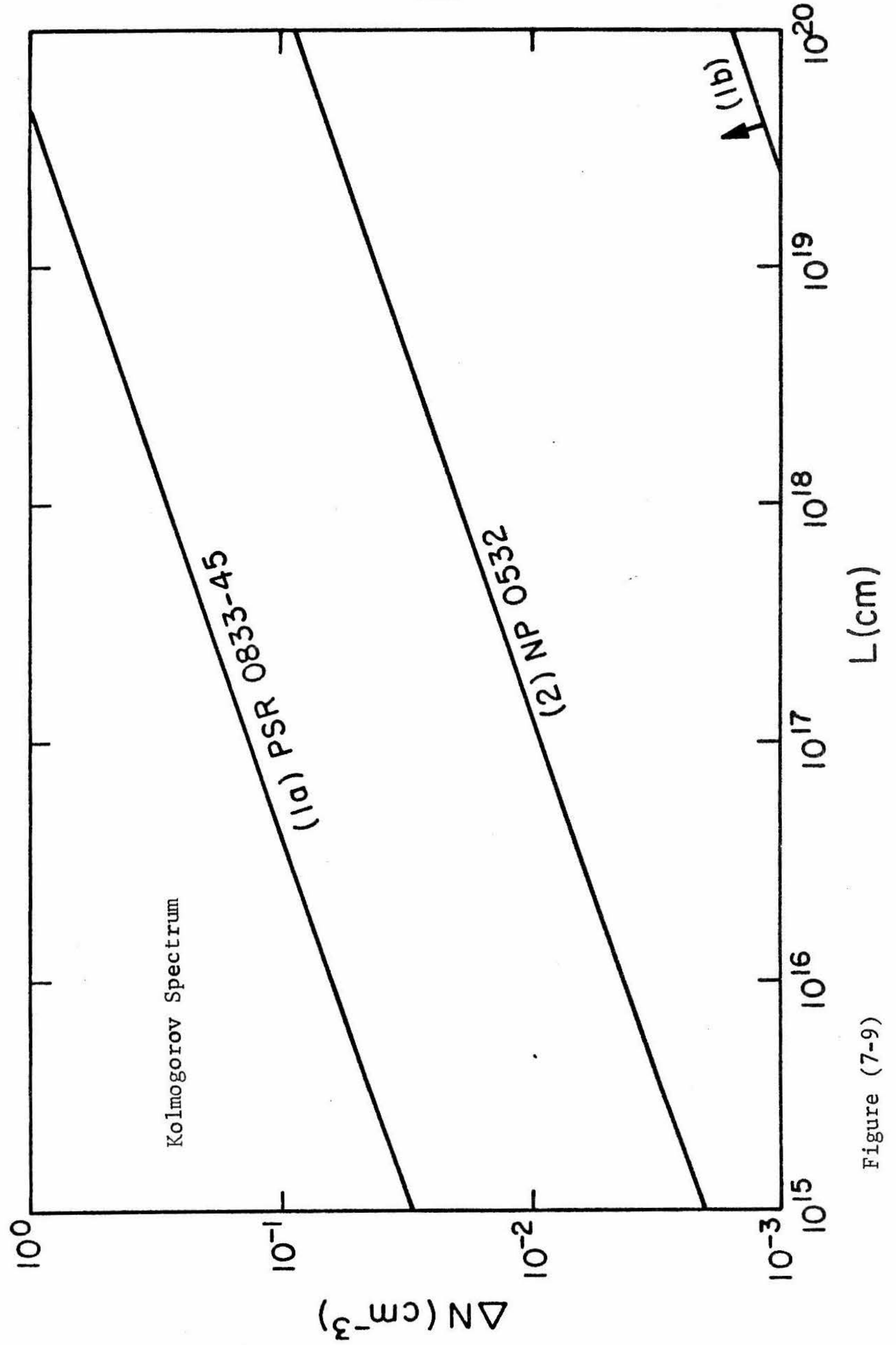

Kolmogorov Spectrum
(1a) PSR 0833-45
(2) NP 0532
(1b)
ΔN (cm⁻³)
L (cm)
10^0
10^-1
10^-2
10^-3
10^15
10^16
10^17
10^18
10^19
10^20

Figure (7-9)

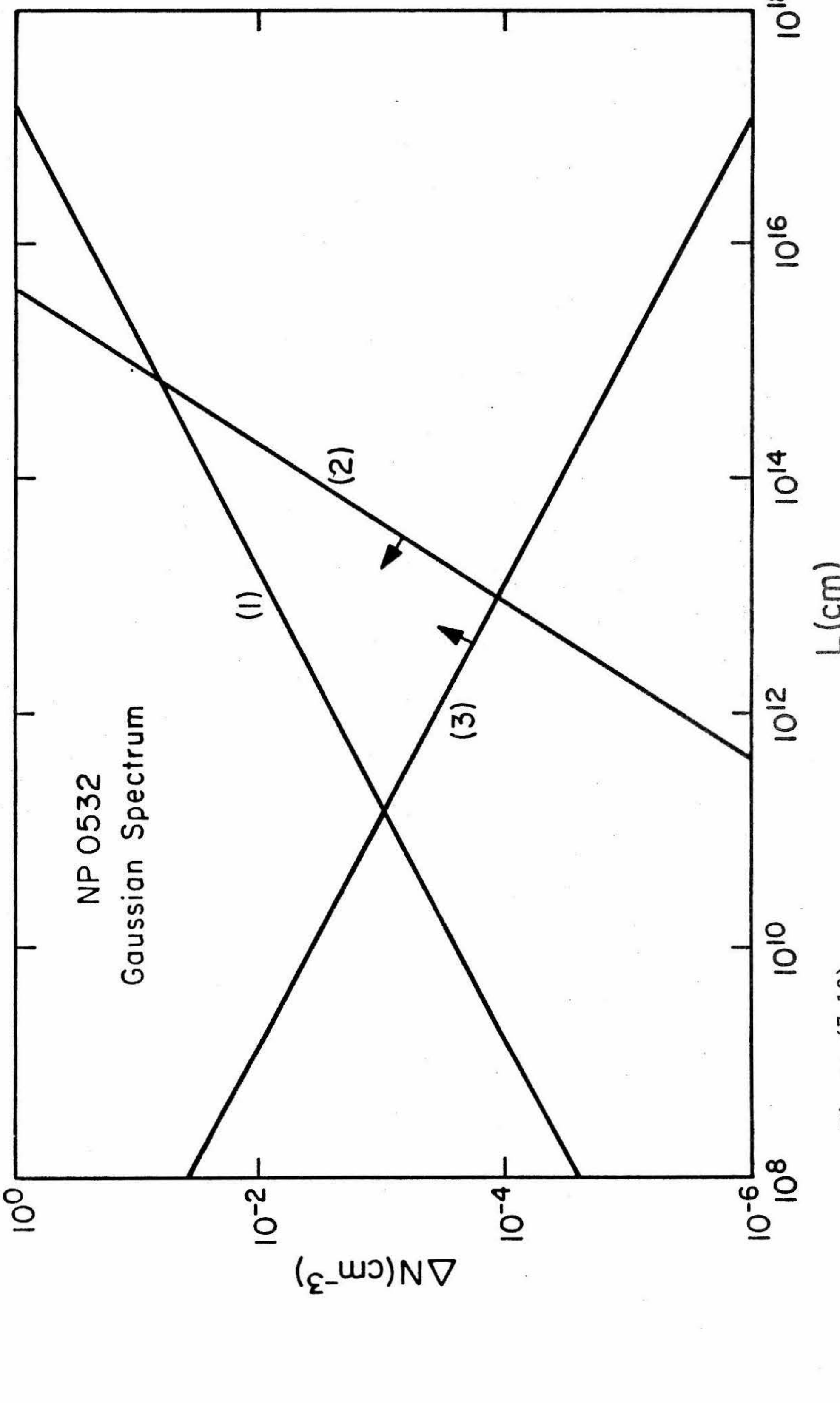

NP 0532
Gaussian Spectrum
(1)
(2)
(3)
L(cm)
ΔN(cm⁻³)
10^0
10^-2
10^-4
10^-6
10^8
10^10
10^12
10^14
10^16
10^18

Figure (7-10)

ratio $\frac{\Delta N}{\langle N_e \rangle} \simeq 0.7$ in Eq. (7-46b) for the Gaussian spectrum is also very unusual since ΔN in the Gaussian case measures only those inhomogeneities which are important for the scintillation. We note that we obtain such a high ratio of $\frac{\Delta N}{\langle N_e \rangle}$ because we have assumed the random medium is uniformly distributed between the pulsar and the observer. If we assume the random medium to be concentrated in a thin layer with thickness D, then we can use the thin-screen diffraction theory in Chapter 3 (or in Eq. (7-19)) to fit the data of PSR 0833-45. We find that for the thickness of the layer

$$D = 10 \text{ pc} \tag{7-49a}$$

and the outer scale of the Kolmogorov spectrum

$$L = 10 \text{ pc}, \tag{7-49b}$$

we have

$$\langle N_e \rangle_D \simeq 7 \text{ cm}^{-3} \tag{7-50a}$$

and

$$(\Delta N)_D \simeq 6 \text{ cm}^{-3} \simeq 0.86 \langle N_e \rangle_D \quad , \tag{7-50b}$$

where the subscript D in Eqs. (7-50a,b) denotes the average of the quantity over the plasma layer of thickness D. We note that if we require $(\Delta N)_D \leq \langle N_e \rangle_D$, $L \geq 10$ pc and $D \gtrsim L$, the values of D, L, $(\Delta N)_D$ and $\langle N_e \rangle_D$ in Eqs. (7-49a,b) and (7-50a,b) are unique. Similarly for the Gaussian spectrum with $D \simeq 10$ pc, we have

$$(\Delta N)_D \simeq 0.7 \text{ cm}^{-3} \simeq 0.1 \langle N_e \rangle_D . \tag{7-51}$$

Thus if we want to avoid a high ratio of $\Delta N/\langle N_e \rangle$, there must exist a region of strong electron density fluctuation with thickness $D \simeq 10$ pc and mean electron density $\langle N_e \rangle_D \simeq 7 \text{ cm}^{-3}$.

It is interesting to consider the mean shape of the pulse profile from the PSR 0833-45 pulsar. The mean pulse shape at the point just passing the plasma layer with thickness D is of the form shown in Figure (5-4b) or (5-5b) with a characteristic time

$$t_c^{(\text{layer})} = \frac{D\theta_c^{\;2}}{2c} \quad . \tag{7-52}$$

(c.f. Eq. (5-67').) However as the pulse propagating a distance z in the (relatively) free space, a mean pulse shape of truncated exponential form for the Gaussian spectrum, (or nearly exponential form for the Kolmogorov spectrum) with a characteristic time

$$t_c^{(\text{free space})} = \frac{z\theta_c^{\;2}}{2c} \tag{7-53}$$

will develop. (See Eqs. (5-84) and (5-86).) The resulting pulse shape is approximately the convolution of the two mean pulse shape with characteristic times $t_c^{(\text{layer})}$ and $t_c^{(\text{free space})}$. It is noted that the observed pulse shape can in general determine the ratio of D and z. In particular, in the case $\frac{z}{D} \gg 1$, the pulse shape is of truncated exponential form, and in the case D = z (i.e. the wave is scattered all the way), the pulse shape is of the form given by Figure (5-4b) or (5-5b) with a rise time of the order of $t_c = \frac{z\theta_c^{\;2}}{2c}$. For the PSR 0833-45 pulsar, $\frac{z}{D} \simeq 50$ and $t_c^{(\text{free space})}$ dominates. Thus the mean pulse of PSR 0833-45 due to the scattering of the interstellar medium will have an exponential (or nearly exponential) form. The above analysis is consistent with the observation of Komesaroff et al. (1972). They found that an exponential form for the broadened pulse fits best the observed data.

The above method of analysis can be used to probe the density and the structure of the interstellar medium between the pulsars and the earth. For the interstellar space between CP 0328 and the earth, the mean electron density is about $\langle N_e \rangle \simeq 0.0267 \text{ cm}^{-3}$ and there is probably no region of strong electron density fluctuation. However, for the interstellar medium between PSR 0833-45 and the earth, there exists a region of strong electron density fluctuation with thickness $D \simeq 10$ pc and mean electron density $\langle N_e \rangle_D \simeq 7 \text{ cm}^{-3}$.

(C) NP 0532

The dispersion, the temporal pulse smearing and the angular broadening of the NP 0532 pulsar have been measured. (Rankin et al. 1970, Counselman & Rankin 1971, Rankin & Counselman 1973, Mutel et al. 1974). The results are summarized as follows. The dispersion measure is

$$DM = 58.2 \text{ pc cm}^{-3}, \tag{7-54a}$$

the characteristic time of pulse smearing at $f = 300$ MHz is

$$t_c(300 \text{ MHz}) = 1.8 \times 10^{-4} \text{ sec}, \tag{7-54b}$$

and the characteristic scattering angle at 300 MHz is

$$\theta_c(300 \text{ MHz}) = 0.01 \text{ arc sec} = 4.83 \times 10^{-8} \text{ radians}. \tag{7-54c}$$

From the measured t_c and θ_c, and the relation $t_c = \dfrac{z\theta_c^{\,2}}{2c}$, we obtain the distance between NP 0532 and the earth,

$$z = 4.5 \times 10^{21} \text{cm} = 1500 \text{ pc}. \tag{7-55}$$

It follows from the dispersion measure DM in Eq. (7-54a) that the mean electron density between the pulsar and the earth is

$$\langle N_e \rangle \simeq 0.039 \text{ cm}^{-3}. \qquad (7\text{-}56)$$

Note that the values of z and $\langle N_e \rangle$ given above come directly from the observation without any further assumption.

For the turbulent plasmas with a Gaussian spectrum, the relation t_c (300 MHz) = 1.8 x 10^{-4} sec and the strong scintillation conditions are plotted in Figure (7-10), similar to Figures(7-6a) and (7-8), assuming the random medium is evenly distributed between the pulsar and the earth. We find that the coherent scale

$$L \simeq 10^{13} \text{ cm} \qquad (7\text{-}57a)$$

and the root mean square of electron density fluctuation

$$\Delta N \simeq 10^{-2} \text{ cm}^{-3} \simeq 0.25 \langle N_e \rangle. \qquad (7\text{-}57b)$$

For the Kolmogorov spectrum with $\ell > \rho^{(0)}_{c.s.} < L$, the relation t_c (300 MHz) = 1.8 x 10^{-4} sec is plotted in curve (2) of Figure (7-9). The conditions for the strong scintillation are satisfied and are not shown in the Figure. From Figure (7-9), we find, by setting the outer scale of the turbulent medium

$$L = 10^{20} \text{ cm} \approx 33 \text{ pc}, \qquad (7\text{-}58a)$$

the mean electron density

$$\Delta N = 0.09 \text{ cm}^{-3} \simeq 2.3 \langle N_e \rangle . \qquad (7\text{-}58b)$$

The relatively high value of ΔN in Eq. (7-57b) or (7-58b) indicates that there exists a layer of relatively strong electron density fluctuation. If we require $L = 10^{20}$ cm and $\frac{\Delta N}{\langle N_e \rangle} \simeq 1$ for the Kolmogorov spectrum, then we obtain the thickness of the plasma layer

$$D = 282 \text{ pc } (\simeq 0.2z), \tag{7-59a}$$

the mean electron density inside the layer

$$\langle N_e \rangle_D \simeq 0.212 \text{ cm}^{-3} \tag{7-59b}$$

and the electron density fluctuation

$$(\Delta N)_D \simeq 0.212 \text{ cm}^{-3}. \tag{7-59c}$$

For the Gaussian spectrum with D = 282 pc, we have

$$(\Delta N)_D = 0.023 \text{ cm}^{-3} = 0.11 \langle N_e \rangle_D \ . \tag{7-60}$$

Using the same argument as for PSR 0833-45, we find that for the observed mean pulse shape of NP 0532, the ratio of the two characteristic times due to respectively the propagation in the layer and the propagation in the (relatively) free space is

$$t_c^{(\text{layer})}/t_c^{(\text{free space})} \simeq \frac{D}{z} \simeq 0.2 \quad . \tag{7-61}$$

We note that $t_c^{(\text{free space})}$ is more important and the pulse shape is close to the exponential form. But since $t_c^{(\text{layer})}$ is not very small ($t_c^{(\text{layer})} \simeq 0.2\ t_c^{(\text{free space})}$), $t_c^{(\text{layer})}$ can not completely be neglected and the correct pulse shape can be obtained by the convolution of these two effects.

Note that in the above discussions, we have, for simplicity, assumed a single layer of strong electron density fluctuation between the pulsars and the earth. The existence of many regions of strong fluctuation is possible but it will complicate the analysis.

In conclusion, we summarize in Table (7-1) the properties of the interstellar media between the earth and the three pulsars discussed above in this section assuming a Kolmogorov spectrum. From Table (7-1), we believe that the mean electron density ($\langle N_e \rangle \simeq 0.0267$ cm^{-3}) between the CP 0328 pulsar and the earth is the ambient electron density in the interstellar medium. For PSR 0833-45, the mean electron density ($\langle N_e \rangle \simeq 0.138$ cm^{-3}) is much higher than the ambient density, which indicates the existence of a plasma layer of very high electron density (with thickness $D \simeq 10$ pc and $\langle N_e \rangle_D \simeq 7$ cm^{-3}) in the interstellar space between PSR 0833-45 and the earth. For NP 0532, $\langle N_e \rangle$ ($\simeq 0.039$ cm^{-3}) is slightly higher than the ambient density and there also exists a plasma layer of relatively high electron density ($D \simeq 280$ pc and $\langle N_e \rangle_D \simeq 0.21$ cm^{-3}).

Table (7-1)

The Characteristics of the Interstellar Media between the Earth and three Pulsars

Pulsar / Properties	CP 0328	PSR 0833-45	NP 0532
DM (pc cm^{-3})	26.7	69.2	58.2
z (pc)	1000	500	1500
$\langle N_e \rangle$ (cm^{-3})	0.0267	0.138	0.039
A layer of strong fluctuation	not needed	strongly needed	needed
D (thickness of layer) (pc)	-	10	282
$\langle N_e \rangle_D$ (cm^{-3})	-	7	0.2

Appendix A

Given a probability distribution P(A) which satisfies Eq. (5-29) in the text, we wish to find a lower bound on $\langle A \rangle = \int_0^\infty AP(A)dA$. Define

$$M_n = \int_0^\infty A^n P(A)dA, \tag{A-1}$$

so that our constraints are $M_o = M_2 = 1$, $M_4 = 1 + m_z^2$ and we look for a lower bound on M_1.

Now, M_o and M_2 can always be built up for any P(A) out of increments dm whose total must be 1 and which are added in such a way that $dM_o = dm$ and $dM_2 = dm$. To do this add α dm at $x_1 \le 1$ and $(1-\alpha)$ dm at $x_2 \ge 1$ where x_2 and x_1 are otherwise arbitrary and where

$$\alpha = \frac{x_2^2 - 1}{x_2^2 - x_1^2} \quad . \tag{A-2}$$

The contribution of each increment to M_1 is then

$$dM_1 = k_1 dm$$

where

$$k_1 = x_1 \alpha + x_2(1-\alpha) = \frac{1+x_1x_2}{x_2+x_1} \quad . \tag{A-3}$$

Similarly,

$$dM_4 = k_4 dm$$

where

$$k_4 = x_2^2 + x_1^2 - x_1^2 x_2^2 \tag{A-4}$$

It is readily demonstrated that $\partial k_1/\partial x_1 \geq 0$ and $\partial k_1/\partial x_2 \leq 0$, so that the minimum value of k_1 (and hence M_1) occurs when $x_1 = 0$ and x_2 is as large as possible. From Eq. (A-4) and the constraints $M_4 = 1+m_z^2$, we see that the average value of x_2^2 over all the increments dm must be $1+m_z^2$. But also, if $x_1=0$, $k_1=1/x_2$ and $k_4=x_2^2=1/k_1^2$. For this relationship between k_1 and k_4 it is readily seen that the minimum value of k_1 averaged over the increments dm occurs when $x_2=\sqrt{1+m_z^2}$ for <u>each</u> increment dm. To see this clearly, the reader should plot k_4 vs. k_1 and note that since the curve is concave and since the average k_4 over the distribution of dm is fixed, the minimum of the mean of k_1 occurs when all increments are at one value of k_4(or x_2).

Hence, the minimum value of M_1 is given by

$$(M_1)_{min} = \frac{1}{\sqrt{1+m_z^2}} \quad . \qquad \text{(A-5)}$$

Appendix B

In this appendix, we employ a numerical method to solve the following equation,

$$\frac{\partial \Gamma_D(z,\underset{\sim}{\rho},\Delta k)}{\partial z} + \frac{i\Delta k}{2k^2}\nabla_{\underset{\sim}{\rho}}^2 \Gamma_D + \frac{1}{4k^2}[A_\beta(0) - A_\beta(\rho)]\,\Gamma_D = 0 \quad . \tag{B-1}$$

For a Gaussian spectrum, we have from Eq. (5-52)

$$D_\beta(\rho) = A_\beta(0) - A_\beta(0) = \frac{Bk^4}{8\pi} q_o^2[1 - \exp(-\frac{\rho^2 q_o^2}{4})] \ . \tag{B-2}$$

Let the transverse scale of $\Gamma_D(z,\rho,\Delta k)$ be ρ_c. We expect ρ_c to be of the order of the characteristic scale of $\Gamma_{1,1}(z,\underset{\sim}{\rho})$ given by Eq. (4-60), which can be shown to be smaller than $L = q_o^{-1}$. Thus for $\rho_c \ll L$, we can expand $D_\beta(\rho)$ in Eq. (B-2) in powers of $(\frac{\rho}{L})^2$ and we have to first order

$$D(\rho) \simeq \frac{Bk^4}{32\pi} q_o^4 \rho^2 \equiv \beta_o\rho^2 \tag{B-3}$$

where

$$\beta_o = \frac{Bk^4 q_o^4}{32\pi} \quad . \tag{B-3'}$$

We introduce the dimensionless variables η and ξ

$$z = \frac{2k^2}{(\Delta k\beta_o)^{1/2}}\,\eta \tag{B-4}$$

and

$$\rho = (\frac{\Delta k}{\beta_o})^{\frac{1}{4}}\,\xi \quad , \tag{B-5}$$

and have

$$\frac{\partial\Gamma_D}{\partial\eta} + i\left[\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right]\Gamma_D + \xi^2\,\Gamma_D = 0 \quad . \qquad \text{(B-6)}$$

Note that in Eq. (B-6), the dependences of Γ_D on (Δk) and z have been collapsed into one variable η. The quantity we need is $\Gamma_D(z,\underset{\sim}{\rho}=0,\Delta k)$. For a given (Δk), we solve Eq. (B-6) and obtain Γ_D at z by setting $\eta = \frac{(\Delta k\beta_o)^{1/2}z}{2k^2}$. For another value $(\Delta k)'$, the equation to be solve is exactly Eq. (B-6) and we get $\Gamma_D(z,\underset{\sim}{\rho}=0,\Delta k')$ by setting $\eta = \frac{(\Delta k'\beta_o)^{1/2}z}{2k^2}$. Therefore it is enough to solve Eq. (B-6) only once to obtain the whole spectrum of $\Gamma_D(,z,\rho=0,\Delta k)$ at position z with all different Δk by varying η.

We employ the implicit schemes of numerical difference method to solve Eq. (B-6) (Richtmyer & Morton, 1967). The initial condition is that

$$\Gamma_D(\eta = 0,\ \xi) = 1 \quad . \qquad \text{(B-7)}$$

The boundary condition at $\xi \to \infty$ is $\Gamma_D = 0$. In practice, we cannot apply this boundary condition because this would require an infitite number of mesh points. The boundary condition that we use is

$$\text{at } \xi = 5,\ \frac{\partial^2\Gamma_D}{\partial\xi^2} = 0 \quad . \qquad \text{(B-8)}$$

The sensitivity of the solution of Eq. (B-6) to this assumption was tested by extending the truncation point to $\xi = 10$ and by setting

$$\text{at } \xi = 5,\ \frac{\partial\Gamma_D}{\partial\xi} = 0 \quad , \text{ or } \Gamma_D = 1. \qquad \text{(B-9)}$$

No significant difference in the solutions was detected for all these variations.

The boundary condition at $\xi = 0$ is

$$\frac{\partial \Gamma_D}{\partial \xi}\Big|_{\xi = 0} = 0 \qquad \text{(B-10)}$$

as can be seen from Eq. (B-6) by requiring $(\frac{1}{\xi}\frac{\partial}{\partial \xi}\Gamma_D)$ be finite at $\xi = 0$. In order to calculate the value of $(\frac{1}{\xi}\frac{\partial}{\partial \xi}\Gamma_D)$ at $\xi = 0$, we note that

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\Gamma_D = \frac{\partial^2}{\partial \xi^2}\Gamma_D, \text{ at } \xi = 0, \qquad \text{(B-11)}$$

Eq. (B-11) is obtained by assuming that $\frac{1}{\xi}\frac{\partial}{\partial \xi}\Gamma_D$ and $\frac{\partial^2}{\partial \xi^2}\Gamma_D$ are finite at $\xi = 0$.

The result is shown in Figure (5-4a) for $\xi = 0$. The characteristic wavenumber k_c is obtained by setting $\eta = 1$ in Eq. (B-4). We get

$$k_c = \frac{1}{\beta_o}\left(\frac{2k^2}{z}\right)^2 \quad . \qquad \text{(B-12)}$$

The characteristic correlation frequency $f_c = \frac{\omega_c}{2\pi} = \frac{k_c c}{2\pi}$.

From Eq. (B-1), it can be shown that

$$\Gamma_D(-\Delta k) = \Gamma_D^*(\Delta k). \qquad \text{(B-13)}$$

Next step is to calculate $P_D(z,t)$ as defined in Eq. (5-58).

Let

$$t^* \equiv [t - \frac{z}{v_g}]/t_c \tag{B-14}$$

where
$$t_c \equiv \frac{1}{\omega_c} = \frac{\beta_o z^2}{4ck^4} \quad . \tag{B-15}$$

Then we have

$$P_D(z,t) = \frac{1}{\pi t_c} \int_{-\infty}^{\infty} \left[\text{Re} \left\{ \Gamma_D(\eta, \xi=0) \right\} \cos\ (\eta^2 t_*) + \text{Im} \left\{ \Gamma_D(\eta, \xi=0) \right\} \sin(\eta^2 t_*) \right]$$

$$\times\ \eta\ d\eta \quad . \tag{B-16}$$

The numerical result is shown in Figure (5-4b).

For Kolmogorov spectrum with $L \gg \rho \gg \ell$, one has from Eq. (1-28)

$$D_\beta(\rho) = A_\beta(0) - A_\beta(\rho)$$

$$= \frac{Bk^4 q_o^{\ \alpha}\ \Gamma(2-\frac{\alpha}{2})}{4\pi(\alpha-2)\ 2^{(\alpha-2)}\Gamma(\frac{\alpha}{2})} \quad \rho^{\alpha-2} = \beta_o \rho^\nu \tag{B-17}$$

where $\alpha = \frac{11}{3}$, $\nu = \alpha - 2 = \frac{5}{3}$ and

$$\beta_o = \frac{Bk^4 q_o^{\frac{11}{3}}\ \Gamma(\frac{1}{6})}{4\pi\ \times (\frac{5}{3})\ \times\ 2^{(5/3)}\Gamma(\frac{11}{6})} \quad . \tag{B-18}$$

The numerical results are shown in Figure (5-5a,b). The characteristic time scale t_c and the characteristic wavenumber k_c are given by

$$t_c = \frac{1}{k_c c} = \frac{1}{c}\ \beta_o^{\ 1.2}\ k^{-4.4}\ (\frac{z}{2})^{2.2} \quad . \tag{B-19}$$

It can be shown from Eqs. (B-15) and (B-16) that for both Gaussian and Kolmogorov spectra,

$$t_c = z\theta_c^{\,2}/2c \tag{B-20}$$

where θ_c is given by Eq. (5-16). Eq. (B-20) can be interpreted as follows. For a ray travelling a distance z and having a scattered-angle θ_c, the additional ray path relative to the unscattered ray is $z\theta_c^{\,2}/2$, and therefore the delayed time for such a ray is $t_c = z\theta_c^{\,2}/2c$.

Appendix C

Define the operator L as

$$L(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \equiv -\frac{i}{\gamma}\nabla_{\underset{\sim}{\alpha}} \cdot \nabla_{\underset{\sim}{\beta}} + f(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \tag{C-1}$$

where $f(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is given by Eq. (5-97). Then Eq. (5-96) can be written as

$$(\frac{\partial}{\partial\eta} + L)\ \Gamma_4(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0 \quad . \tag{C-2}$$

Let

$$\Gamma_4^{(A)}(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \equiv e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} + e^{-2\eta[1-\hat{H}(\underset{\sim}{\beta})]} - e^{-2\eta} \tag{C-3}$$

be the asymptotic solution of Γ_4 at large η in Eq. (5-105').

We obtain immediately from Eqs. (C-1) and (C-3)

$$(\frac{\partial}{\partial\eta} + L)\Gamma_4^{(A)} = E_t(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \tag{C-4}$$

where

$$E_t(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = E_a(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) + E_b(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) + E_c(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}), \tag{C-5}$$

$$E_a = -[2\hat{H}(\underset{\sim}{\beta}) - \hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{\beta}) - \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})]\ e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} \quad , \tag{C-6a}$$

$$E_b = -[2\hat{H}(\underset{\sim}{\alpha}) - \hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{\beta}) - \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})]\ e^{-2\eta[1-\hat{H}(\underset{\sim}{\beta})]} \quad , \tag{C-6b}$$

and

$$E_c = [2\hat{H}(\underset{\sim}{\alpha}) + 2\hat{H}(\underset{\sim}{\beta}) - \hat{H}(\underset{\sim}{\alpha}+\underset{\sim}{\beta}) - \hat{H}(\underset{\sim}{\alpha}-\underset{\sim}{\beta})]\ e^{-2\eta} \quad . \tag{C-6c}$$

The term E_c is of order of $e^{-2\eta}$ and can be neglected for large η. Consider E_a in Eq. (C-6a). We see that $E_a(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of order of $e^{-2\eta}$, which is small for large η, except for $|\underset{\sim}{\alpha}| \ll 1$, since $[1-\hat{H}(\underset{\sim}{\alpha})]$ in the exponential of Eq. (C-6a) is of order 1 except near $|\underset{\sim}{\alpha}| = 0$. For $|\underset{\sim}{\alpha}| \ll 1$, it is easy to demonstrate that $|E_a(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})|$ has its maximum value at $\underset{\sim}{\beta} = 0$. Thus for $|\underset{\sim}{\alpha}| \ll 1$,

$$|E_a(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})| \le |E_a(\eta,\underset{\sim}{\alpha},0)|$$

$$= 2[1-\hat{H}(\underset{\sim}{\alpha})]e^{-2\eta[1-\hat{H}(\underset{\sim}{\alpha})]} \le (\frac{e^{-1}}{\eta}) . \qquad \text{(C-7)}$$

Similarly $|E_b|$ can be shown to be of the order of, or less than $(\frac{e^{-1}}{\eta})$. Therefore,

$$|E_t(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})| \le 0\ (\frac{1}{\eta}) \qquad \text{(C-8)}$$

and for $\eta \gg 1$, $\Gamma_4^{(A)}$ satisfies the differential equation for Γ_4 in Eq. (C-2) or (5-96) to terms of order $(\frac{1}{\eta})$.

Next we estimate the error of the asymptotic solution for Γ_4. Define the difference function

$$\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \Gamma_4 - \Gamma_4^{(A)} \quad . \qquad \text{(C-9)}$$

We then have from Eqs. (C-2) and (C-4),

$$(\frac{\partial}{\partial\eta} + L)\ \delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = -\ E_t(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}). \qquad \text{(C-10)}$$

The boundary condition for $\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is

$$\text{at } |\underset{\sim}{\alpha}| = \infty \text{ and/or } |\underset{\sim}{\beta}| = \infty,\ \delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0 \qquad \text{(C-11)}$$

since at the boundary $\Gamma_4 = \Gamma_4^{(A)}$.

From Eq. (C-8), we can write to order $(\frac{1}{\eta})$,

$$E_t(\eta,\alpha,\beta) = \frac{C(\underset{\sim}{\alpha},\underset{\sim}{\beta})}{\eta} + \text{(higher order terms)}. \qquad \text{(C-12)}$$

From the properties of the operator $(\frac{\partial}{\partial\eta} + L)$ discussed in Chapter 5, Section V, we find that Γ_4 does not grow as η increases, and we expect that Γ_4 will not oscillate as a function of η. Thus for $\Gamma_4^{(A)}$ in Eq. (C-3), the difference function $\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})(= \Gamma_4 - \Gamma_4^{(A)})$ will not grow and will not oscillate as η increases. Therefore we can estimate the difference function by expanding $\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ as[1]

$$\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = \sum_{n=0}^{\infty} \frac{\delta_n(\underset{\sim}{\alpha},\underset{\sim}{\beta})}{\eta^n}, \quad \text{for } \eta \gg 1. \qquad \text{(C-13)}$$

Substituting Eq. (C-12) and Eq. (C-13) into Eq. (C-10), we have

$$L\,\delta_o(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0 \qquad \text{(C-14a)}$$

$$L\,\delta_1(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = -c(\underset{\sim}{\alpha},\underset{\sim}{\beta}) \qquad \text{(C-14b)}$$

and etc.. The boundary condition for $\delta_n(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ are

$$\text{at } |\underset{\sim}{\alpha}| = \infty, \text{ and/or } |\underset{\sim}{\beta}| = \infty, \; \delta_n(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0, \text{ for } n = 0,1,2,\cdots. \qquad \text{(C-15)}$$

With the boundary condition in Eq. (C-15), we find that

$$\delta_o(\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 0 \qquad \text{(C-16)}$$

is a solution of Eq. (C-14a). From Eq. (C-14b), we see that $\delta_1(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is of order of unity relative to η since $\delta_1(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is independent of η. Thus

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[1] In writing $\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ as given by Eq. (C-13), we have neglected the error which decays exponentially as η increases.

to the order of $(\frac{1}{\eta})$,

$$\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta}) \simeq \frac{\delta_1(\underset{\sim}{\alpha},\underset{\sim}{\beta})}{\eta} \quad . \tag{C-17}$$

For large η, the error $\delta(\eta,\underset{\sim}{\alpha},\underset{\sim}{\beta})$ is small and can be neglected.

Appendix D

We write Eq. (5-144) for a Gaussian spectrum as

$$\frac{\partial}{\partial\eta}\Gamma_c(\eta,\xi) = \frac{i}{\gamma}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}\right)\Gamma_c - e^{-\xi^2}\Gamma_c \quad . \tag{D-1}$$

The initial conditions at $\eta = 0$ are

$$\Gamma_c(\eta=0,\xi) = 1 \quad . \tag{D-2}$$

The boundary condition at $\xi = \infty$ is

$$\Gamma_c(\eta,\xi=\infty) = 1 \quad . \tag{D-3}$$

The boundary condition at $\xi = 0$ is

$$\left.\frac{\partial\Gamma_c}{\partial\xi}\right|_{\xi=0} = 0 \tag{D-4}$$

as can be seen from Eq. (D-1) by requiring $(\frac{1}{\xi}\frac{\partial}{\partial\xi}\Gamma_c)$ to be finite at $\xi = 0$. In order to calculate the value of $(\frac{1}{\xi}\frac{\partial}{\partial\xi}\Gamma_c)$ at $\xi = 0$, we note that

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\Gamma_c = \frac{\partial^2}{\partial\xi^2}\Gamma_c \quad , \text{ at } \xi = 0 \tag{D-5}$$

by assuming that $\frac{1}{\xi}\frac{\partial}{\partial\xi}\Gamma_c$ and $\frac{\partial^2}{\partial\xi^2}\Gamma_c$ are finite near $\xi = 0$.

We employ the implicit schemes of numerical difference method to solve Eq. (D-1) (Richtmyer & Morton, 1967). Again the boundary condition at $\xi = \infty$ in Eq. (D-3) is truncated (c.f. Appendix B). The boundary condition that we use for large ξ is

$$\text{at } \xi = 5, \ \Gamma_c = 1 \quad . \tag{D-6}$$

The numerical results of Γ_c are then applied to calculate the functions P_χ and P_I in Eqs. (5-145a), (5-145b), and (5-170). The numerical values of P_χ and P_I are shown in Figures (5-8) and (5-9).

Appendix E

The power spectrum $C_k(0,0,L,\omega)$ in Eq. (6-36) can be obtained for various values of the parameters $\bar{V}_2$, σ_2 and σ_1 by direct integration. However, direct integration is very time-consuming. Instead we will use an approximate method, in which the power spectra for $\sigma_1=\sigma_2=0$ already computed by Young (1971) are employed to calculate the new $C_k(0,0,L,\omega)$.

First we note that if we put $\sigma_1=\sigma_2=\sigma=\eta\,\bar{V}_2$ in Eq. (6-36) for a constant η, we have at $\omega=0$,

$$\frac{C_k(0,0,L,\omega=0)}{C_k^{(0)}(0,0,L,\omega=0)} = \sqrt{\frac{\pi}{2}}\;\frac{1}{\eta}\; e^{\frac{1}{4\eta^2}}\; I_o\left(\frac{1}{4\eta^2}\right) \qquad \text{(E-1)}$$

where I_o is the modified Bessel function of the first kind and $C_k^{(0)}$ is the power spectrum in Eq. (6-37) without velocity fluctuation ($\sigma_1=\sigma_2=0$). For $\theta = 5.55^o$ and $\bar{V}_w$ = 200,300 or 400 km/sec, $C_k^{(0)}(0,0,L,\omega)$ is shown in Figure (6-4).

Next if we assume that $\sigma_1=0$ and $\frac{\sigma_2}{\bar{V}_2} = \eta$ = constant for all z, then the C_k in Eq. (6-36) can be written as

$$C_k(0,0,L,\omega) = \int_{-\infty}^{\infty} d\omega'\; G_c(\omega,\omega')\; C_k^{(0)}(0,0,L,\omega') \qquad \text{(E-2)}$$

where the weighting function $G_c(\omega,\omega')$ is given by

$$G_c(\omega,\omega') = \frac{1}{\sqrt{2\pi}}\;\frac{1}{\omega'\eta}\;\exp\left\{-\frac{(\omega-\omega')^2}{2\omega'^2\eta^2}\right\} \quad . \qquad \text{(E-3)}$$

Thus the desired power spectrum $C_k(0,0,L,\omega)$ is easily obtained by averaging the old one, $C_k^{(0)}$, with the weighting function $G_c(\omega,\omega')$.

Unfortunately the $C_k(0,0,L,\omega)$ in Eq. (E-2) is divergent at $\omega = 0$. Physically the divergence at $\omega = 0$ is caused by the fact that there exist some plasmas which are stationary ($V_2=\overline{V}_2+\delta V_2=0$) since the velocity probability distribution function is assumed Gaussian. Note that the divergence at $\omega = 0$ exists for both the thin screen model and the three-dimensional spherical model. For $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$, there is no divergence in $C_k(0,0,L,\omega)$ at $\omega=0$. In particular, if we put $\sigma_1=\sigma_2=\eta\,\overline{V}_2$, $C_k(0,0,L,\omega=0)$ is given by Eq. (E-1).

Our approximation scheme is to put $\sigma_1=\sigma_2=\sigma=\eta$ in calculating $C_k(\omega=0)$ and put $q_1^2\sigma_1^2 = q_2^2\sigma_2^2$ in Eq. (6-36) for $C_k(\omega\neq 0)$, which is equivalent to put $\sigma_1= 0$ and $\sigma_2= \sqrt{2}\sigma$ in calculating $C_k(\omega)$ for $\omega\neq 0$. Thus the power spectrum in Eq. (6-36) with $\sigma_1=\sigma_2=\sigma=\eta\,\overline{V}_2$ can be calculated from the old spectrum $C_k^{(0)}$ using Eqs. (E-1) and (E-2). For the 3c 279 data (Cohen & Gundermann, 1969) shown in Figure (6-4), we calculate the power spectrum $C_k(\omega)$ from the $C_k^{(0)}$ also shown in the figure for $\overline{V}_w= 200$ and 300 km/sec and for various values of η. For $\overline{V}_w= 200$ km/sec, the best fit is at $\eta = 1.13$ (or $\sigma_{10}=\sigma_{20}= 226$ km/sec and for $\overline{V}_w= 300$ km/sec, $\eta = 0.7$ (or $\sigma_{10}=\sigma_{20}= 210$ km/sec). The results are shown in Figure (6-5).

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