Logic, Mathematics and the A Priori, Part II: Core logic as analytic, and as the basis for Natural Logicism

by

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Abstract

This is Part II of a two-part study of the foundations of mathematics through the lenses of (i) apriority and analyticity, and (ii) the resources supplied by Core Logic.

[In Part I we explained what is meant by apriority, as the notion applies to knowledge and possibly also to truths in general. We distinguished grounds for

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knowledge from grounds of truth, in light of our recent work on truthmakers. We then examined the role of apriority in the realism/anti-realism debate. We raised a hitherto unnoticed problem for any Orthodox Realist who attempts to explain the *a priori*.]

In Part II we now examine the sense in which logic is *a priori*, and explain how mathematical theories, while *a priori*, can be dichotomized non-trivially into analytic and synthetic portions. We go on to argue that Core Logic (the system of logic that the anti-realist should espouse) contains exactly the *a-priori-because-analytically*-valid deductive principles. We introduce the reader to the system of Core Logic by explaining its relationship to other logical systems, and setting out its rules of inference. Important metatheorems about Core Logic are reported, and its important features are noted, in setting out the case for Core Logic. Core Logic can serve as the basis for a foundational program that could be called *natural logicism*, an exposition of which will build on the (meta)logical ideas explained here.

1 Logic as *a priori*

In Part I we addressed, and disposed of (at least, from the vantage point of the Moderate Anti-Realist), the complaint that a category-mistake must be involved in seeking to speak of *truths* being *a priori*, rather than of *knowledge* being so.

An even more serious category-mistake, it may now be suggested, would be involved if we were to to divert our attention from *propositions* that are known, or that are true, and direct it instead to *inferences* involving such propositions as premises and conclusions. Inferences are not truth-*bearers*; rather, they are truth-*transmitters*. So, the suggestion might go, one should shy away from talking of *a priori* ***inferences***, and limit oneself instead to ***propositions*** known, or knowable, or perhaps even unknowably true, when considering the extent of the *a priori*.

This would be a methodological mistake. It derives from what we could call the Sententialist, or Representationalist, Fallacy. This is the fallacy of thinking that one's system of beliefs, in so far as objectivity is concerned, consists only of *propositions* firmly believed or claimed to be known. And it is the same methodological orientation on the part of certain logicians, such as Quine, that makes them focus on *theoremhood* at the expense of *deducibility*. The Quinean tries to register facts about inferrability as sentential *nodes* in the thinker's web of belief, rather than as *transitions*, or *formal patterns of inference* from nodes to nodes. We would argue that this is a serious methodological shortfall.[1]

[1]A simple example suffices to make the general point here. In a propositional language based on $\neg$ and $\wedge$, Intuitionistic Logic and Classical Logic have the same theorems. But

The correct view of the matter is that a thinker's web of belief, or knowledge, consists, rather, of richly structured *justifications*, which, to be sure, *involve* propositions, both as starting points and as termini.[2] But in between or among them there are *inferential transitions* to be made, which can be made in accordance with various *truth-preserving rules*. Such rules are typically multi-premise and single-conclusion rules. Foremost among these are the basic rules governing the logical operators of our language. And when it comes to *revising* one's system of beliefs, it is not only *propositions* that can be re-assessed (as to truth-value); it is also the *rules* in use that can (in principle) be reassessed (as to validity, or guaranteed truth-transmission). Any expressivism or non-factualism about logical rules of inference is bound to broaden its scope to include those propositions claimed to be known on the basis of applications of those very rules.[3] So, if we take the propositional nodes of a belief-scheme to enjoy their truth-values objectively, then so too should we take the validity of the transitional steps involving them as being an objective matter. It is on this assumption that our discussion will proceed.

Moreover, having made this assumption we immediately face the problem of either justifying our chosen inference rules as valid, or explaining why they are valid. Actually, to be precise without being captious, we do not have to show that, or explain why, these rules are *valid*—rather, we have to show that, or explain why, they are *validity-preserving*. For rules of inference build more complex proofs (of arguments) from simpler proofs (of immediate sub-arguments); so it is really validity-of-argument that they preserve, not truth.

Now, since either of the aforementioned activities (justifying or explaining) will make use of the inference rules themselves, there is the serious risk of what has been called 'rule-circularity'.[4] The task will be to show that

they differ on deducibilities: the classically derivable sequent $\neg\neg\varphi : \varphi$ is not intuitionistically derivable.

More to the point, for the present discussion, is the fact that while both Core Logic (for which, see below) and Intuitionistic Logic contain the theorem $(\varphi \wedge \neg\varphi) \rightarrow \psi$, Core Logic does *not* contain the intuitionistically derivable sequent $\varphi, \neg\varphi : \psi$.

[2] Taking this view of the structure of a rational agent's belief-scheme entails a marked departure from the (now conventional) account of belief-revision known as *AGM*-theory. For a detailed account of a radically different, and computationally implementable, account of belief-revision, see Tennant [2012a].

[3] *Cf.* Boghossian [2012], at second page within the text: '... if she has no entitlement to her most basic rules, then she has no entitlement to anything that is based upon them ...'.

[4] See Tennant [1978], at p. 74, where in discussing a detailed proof of soundness of the system of natural deduction for classical first-order logic, we gave a clear indication of

rule-circularity *per se*, in the context of such an activity, is not vicious.[5]

We shall locate a set of logical rules that we shall argue are

1. objectively valid,
2. indefeasibly so,[6] and
3. *a priori.*

These are the rules of Core Logic. They form the minimal logical canon for the Moderate Anti-Realist. As the title suggests, these rules form the *inviolable core* of logic. They are *a priori* because they are *analytic*. Those

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rule-circularity in the soundness proof:

> ... in each case we *use* the rule concerned in the very metalinguistic reasoning carried out to establish its soundness. For example, in the case for $\forall$-I we used $\forall$-I in the metalanguage ('Let β be an arbitrary individual ... But β was arbitrary ...'). The reader will have no difficulty in locating metalinguistic uses of the other rules in the reasoning concerning each of them. So it appears that we establish the soundness of our rules of inference by using those very rules in the metalanguage, and doing so, moreover, in a conspicuously 'one-to-one' fashion.

[5] Peacocke [1993] is an early statement of optimism in this regard. He says (at p. 188) that

> ... there can be such a thing as a justification of the primitive rules of inference and axioms employed by a given speaker.

Writing of his so-called 'metasemantic' account, he goes on to say that it

> ... will offer a justification for primitive axioms and inference rules which ... appeals only to what can be inferred from the possession conditions and determination theories for the constituent concepts.

Possession conditions and determination theories for concepts are explained at greater length in Peacocke [1992]. Peacocke (still on p. 188) 'agrees with Carnap' that 'holding certain principles is constitutive of understanding certain of the expressions they contain'.

[6] Compare Kitcher [1980], at p. 9: 'a priori warrants are ultra-reliable; they never lead us astray.' It should be noted, however, that Peacocke [1993], at p. 192, is less sanguine about the indefeasibility of the a priori:

> Any optimist who tries to sharpen the standard characterization in a way that requires justifications for a priori propositions to be epistemically invulnerable is heading for a definition of the a priori under which nothing falls.

Peacocke, however, adduces as support for this view the rather inconclusive consideration that mathematicians are sometimes uncertain whether their 'attempted proofs are sound'. But such uncertainty arises only because mathematicians typically provide *informal* proofs, which are far from being fully regimented as proofs in a formal system of logic.

orthodox rules of Classical Logic that are not in Core Logic might well be *a priori*, if ever they are valid; but they will be *synthetic*, not analytic.

1.1 Might there be unknowable validities?

At this point the reader might wonder whether there might be any problem for *inferences* in general, analogous to the one that we raised for the Orthdox Realist, concerning the possible apriority of her allegedly unknowable truths.[7] Recall that an *a priori* proposition is one whose *truth* can in principle be known without any recourse to sensory experience. The analogue here is an *a priori inference*: one whose *validity* can be known without any recourse to sensory experience. Recall that in Part I we argued for an 'abrupt divide' confronting the Orthodox Realist, regarding the *a priori* status of sentences in a discourse. The analogous problem here, for inference, is this: might there be valid inferences whose validity is unknowable? If so, then how are we to think of *all* deductive inferences as being *a priori*, if valid?

The answer to this question, from the inferentialist standpoint of the present author, is that there can be no problem of allegedly knowledge-transcendent validity of inference(s). This is because validity is always *constituted by* the existence of a proof in accordance with primitive rules of inference, the latter wearing their validity on their sleeves. Proofs are finite, hence knowable. So validity of inferences in general is always epistemically accessible. Our answer is the straightforward inferential analogue of our solution to the original problem of unknowable-but-somehow-*a priori* truths: there aren't any. This is because there aren't any unknowable truths. Just as, for the anti-realist, truth consists in the existence of a suitable proof from suitable first principles, so too does validity of inference consist (merely) in the existence of a suitable proof.

Matters might be different for, say, the second-order logician who wishes to work with a potentially knowlege-transcendent *semantic* notion of logical consequence, defined by reference to 'standard' Henkin models (ones using the *full power set* of the domain as constituting the range of the second-order quantifiers). Here, the prospect would arise of some second-order inference $\Delta : \varphi$ being *valid* but, because lacking any proof, *unknowably so*. The 'abrupt divide' would then present itself again: how could second-order validity-of-inference be taken, intuitively, to obtain only *a priori* if it obtains at all, if one cannot, in principle, come to *know* that it obtains? That, we

[7]The author is indebted to an anonymous referee for raising this problem concerning inference.

should stress, is yet another problem for *the realist*, *qua* model-theoretic semanticist; it does not arise at all for the anti-realist inferentialist.

2 Syntheticity and analyticity in mathematics

Our new classification of the rules of logic provides reason for renewed interest in Kant's two major distinctions, and deploys them to unorthodox effect.

Our first departure from orthodoxy is our assertion that rules or laws of Classical Logic that are not part of Core Logic are, if valid at all, not *analytically* so, but rather *synthetically*. The thinker who uses them, and who professes adherence to them (in the sense that she allows them to govern her thinking), is *manifesting a realist attitude* towards the litmus proposition involved. She is claiming that *reality is determinate in the regard in question*. Suppose, for example, that she is willing to argue by Dilemma that ψ holds because she can prove ψ from φ (the 'first horn' of the dilemma), and then prove ψ from $\neg\varphi$ (the 'second horn' of the dilemma). Here, the litmus proposition is φ. It is being taken by her to 'behave classically'. She is thereby committing herself to the realist metaphysical view that *reality is determinate in the φ-regard.*[8]

Our second departure from orthodoxy is that various branches of mathematics, too, may be dichotomized into their analytic portions and their synthetic portions. Peano–Dedekind arithmetic, for example, is arguably analytic, especially given its neo-Fregean derivability (in Core Logic) from much deeper, 'logical', first principles.[9] But we know that any such axiomatic system for arithmetic is incomplete. So, any sentence independent of PA of whose truth we eventually become convinced might involve grounds$_K$ that make use of resources going beyond the merely analytic.

[8]This position was stated and argued for in Tennant [1996]. In Tennant [20XX], it is pointed out, with technical detail too lengthy to enter into here, that with applications of Dilemma the 'classical focus' can be taken either as the sentence φ or as the sentence ψ (because of the ways the applications of Dilemma can be re-arranged within a proof). The upshot is that when one has proofs

$$\begin{array}{c} \Delta, \varphi \\ \Pi_1 \\ \psi \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma, \neg\varphi \\ \Pi_2 \\ \psi \end{array} ,$$

one can use Dilemma to construct a proof of ψ from Δ, Γ in such a way as to ensure that, whenever φ and ψ are of different complexities, the classical focus is on the *less complex* of the two.

[9]See Tennant [1984], Tennant [1987] and Tennant [2008].

Another example of a non-trivial dichotomy between analytic and synthetic is provided by set theory. One can formulate beautiful introduction and elimination rules for the set-abstraction operator (within a system of free logic), which afford non-trivial deductions of such principles as the Axiom of Extensionality for sets, and the so-called Church conversion schemata. Indeed, the portion of set theory generated by these rules corresponds to what Quine once called 'virtual set theory'.[10] It makes no existential claims; it is (arguably) *analytic* (though Quine would turn in his grave on hearing this). The remaining portion of set theory, generated by the important outright and conditional set-existence claims (with the possible exception of the existence of the empty set), is (arguably) synthetic.

Note that this holds true even of an *anti-realist*, i.e. intuitionistic, development of set theory. In *mathematics*, the synthetic portion has to be acknowledged as such, even if one uses only Core Logic to prove one's mathematical theorems. Both the analytic and the synthetic portions of arithmetic and of set theory remain, of course, *a priori.* And here it remains a substantive challenge to explain why these *synthetic* portions are indeed *a priori.* Compare Kitcher [1980], at p. 4:

> Somebody might protest that current practice is to define the notion of an a priori proposition outright, by taking the class of a priori propositions to consist of the truths of logic and mathematics (for example). But when philosophers allege that truths of logic and mathematics are a priori, they do not intend merely to recapitulate the definition of a priori propositions. Their aim is to advance a thesis about the epistemological status of logic and mathematics.

We are contemplating here various extensions of the dichotomization of the 'logical' *a priori* into an analytic and a synthetic portion (roughly, Core v. strictly Classical) in order to deal with logico-*mathematical* expressions (such as the number-abstraction operator $\#x \ldots x \ldots$ and the set-abstraction operator $\{x| \ldots x \ldots\}$). Such extensions might reveal good grounds for regarding the pertinent *portions* of mathematics as analytic, while regarding the residual portions as synthetic. *All* of mathematics, however, remains *a priori.* And it remains a challenge to the philosopher of logic and mathematics—*particularly* the realist-minded one—to explain why this is so.

[10]See Tennant [2004].

3 Core Logic in relation to familiar systems of logic

We have yet to tell the reader what sort of system Core Logic is. By way of orientation at the outset, this is how Core Logic and its classicized counterpart, Classical Core, sit in relation to the well-known systems of Classical, Intuitionistic and Minimal Logic, which have well-behaved natural-deduction formulations; and in relation to the relevance logic R of Anderson and Belnap:[11] Note that Core Logic, indicated with dashed lines, is the intersection of Classical Core with Intuitionistic Logic.

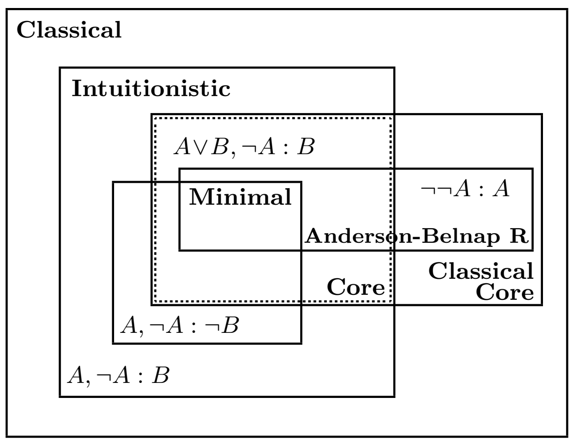


Figure 1: System Containments

Note that Core Logic contains Disjunctive Syllogism ($A \vee B, \neg A : B$), but neither one of the two closely related Lewis paradoxes $A, \neg A : B$ and $A, \neg A : \neg B$.

Core Logic (which we shall call $\mathbb{C}$) can be otained from classical logic **C** by following two lines of reform: constructivization and relevantization. If one constructivizes classical logic, the result is intuitionistic logic **I**. The remaining question is then how best to relevantize intuitionistic logic. Historically, the system **M** of *minimal* logic has been regarded as the natural contender for the title of 'relevantized intuitionistic logic'.[12] **M**'s claim to that title might seem to be underwritten by what appears to be a very nat-

[11] See Anderson and Nuel D. Belnap, Jr. [1975].

[12] The system of minimal logic is due to Johansson [1936].

ural nesting of sets of natural-deduction rules in, for example, the canonical presentation of the three systems **C**, **I** and **M** in Prawitz [1965]. There, the system **M** consisted of just the introduction and elimination rules for the logical operators, *in their original formulation due to* Gentzen [1934, 1935]. By adding the Absurdity Rule to **M**, one obtains the system **I**. By then adding one of the usual strictly classical rules for negation (Law of Excluded Middle; Rule of Dilemma; Classical *Reductio ad absurdum*; or Double Negation Elimination—see §4.1), one obtains the system **C**.

Taking the sequence in reverse: if one starts with **C**, and gives up whatever strictly classical rule(s) of negation one has chosen, one obtains **I**. If one then gives up the Absurdity Rule, one obtains **M**. It seems all very natural and 'layered'.

Our contention is that the second step, of relevantization, has not been properly carried out. The relevantization of intuitionistic logic should not be taken to be **M**. This is because matters have been skewed by an unsatisfactory choice of formulation of the *Ur*-rules of introduction and elimination. Gentzen, ironically, did not get them quite right. He should have addressed the problem of how best to formulate the introduction and elimination rules under the constraint that the intrinsic behavior of the logical operators was *all* one wished thereby to capture. The project of rule-formulation needs to be carried out not only under the constraint that one is not to commit oneself to any non-constructive inferential moves, but also under the constraint that one is not to commit oneself to any fallacies of relevance.

This is the path that we follow instead. On traversing it, we find that the high road to constructivization and relevantization has a terminus other than **M**. The correct terminus is the system $\mathbb{C}$ of Core Logic.

In order to acquaint the reader with the internal details of Core Logic, we shall first state the rules that are *not* to be found in Core Logic, and only thereafter state the rules of Core Logic itself.

4 Familiar Rules of Classical Logic that are not part of Core Logic

4.1 Strictly Classical Negation Rules

We follow here the notational conventions of Prawitz [1965]: parenthetically enclosed numerals are used to label both the application of a discharge-rule, and the assumption-occurrences above it that are discharged by that application. Also, an axiom is written as a 'zero-premise' rule: it has an inference stroke above it, but there are no premises above the inference stroke itself. The reader can therefore think of an axiom as not itself in need of any justification.

Law of Excluded Middle (LEM)

$$\frac{}{\varphi \vee \neg\varphi}$$

Dilemma (Dil)

$$\dfrac{\begin{array}{cc} {}^{(i)}\overline{\varphi} & \overline{\neg\varphi}^{(i)} \\ \vdots & \vdots \\ \psi & \psi \end{array}}{\psi}(i)$$

Double Negation Elimination (DNE)

$$\frac{\neg\neg\varphi}{\varphi}$$

Classical Reductio (CR)

$$\dfrac{\begin{array}{c} \overline{\neg\varphi}^{(i)} \\ \vdots \\ \bot \end{array}}{\varphi}(i)$$

4.2 ***Ex Falso Quodlibet (EFQ)*, *a.k.a.* the Absurdity Rule**

$$\frac{\bot}{\varphi}$$

5 The Rules of Core Logic

We pass now to the primitive rules of inference that constitute Core Logic. They might *look* familiar to the reader; but careful attention must be paid to the detailed conditions for their applicability. These are conditions to which the formulators and users of the formal systems **C**, **I** or **M** have paid insufficient attention. They have thereby allowed irrelevancies to creep into the fields of their formal deducibility relations. In order to state these conditions, we employ some extra notation.

5.1 Preliminaries

Boxes next to discharge strokes over 'assumptions for discharge' indicate that vacuous discharge is not allowed. That is to say, there really must be an assumption of the indicated form in the subordinate proof in question, available to be discharged upon application of the rule. With (∧-E) and (∀-E) we require only that at least one of the indicated assumptions should have been used, and be available for discharge.

The *diamond* next to the discharge stroke in the second half of (→-I) indicates that it is not required that the assumption in question should (have been used and) be available for discharge. But if it is available, then it is discharged.

5.2 The rules, in graphic form

(¬-I)
$$\dfrac{\begin{array}{c}\overline{\varphi}^{\,\Box\,(i)}\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i)$$

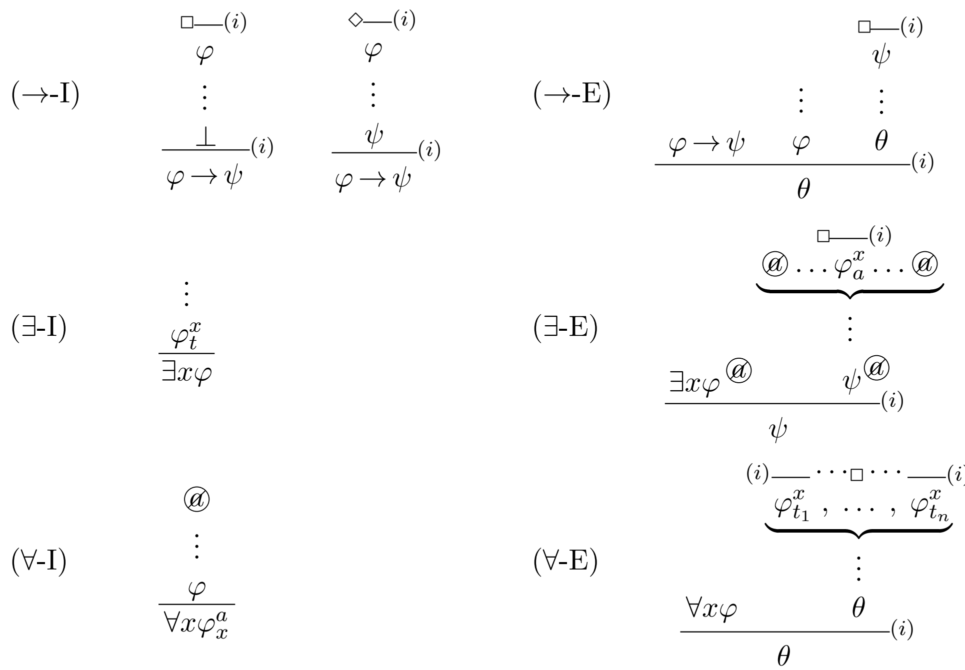
(¬-E)
$$\dfrac{\neg\varphi \qquad \begin{array}{c}\vdots\\ \varphi\end{array}}{\bot}$$

(∧-I)
$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\wedge\psi}$$

(∧-E)
$$\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}(i)\overline{\quad}\Box\overline{\quad}(i)\\ \underbrace{\varphi\,,\psi}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

(∨-I)
$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi\vee\psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\vee\psi}$$

(∨-E)
$$\dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{\,\Box\,(i)}\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\overline{\psi}^{\,\Box\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i) \qquad \dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{\,\Box\,(i)}\\ \vdots\\ \bot\end{array} \quad \begin{array}{c}\overline{\psi}^{\,\Box\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i) \qquad \dfrac{\varphi\vee\psi \quad \begin{array}{c}\overline{\varphi}^{\,\Box\,(i)}\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\overline{\psi}^{\,\Box\,(i)}\\ \vdots\\ \bot\end{array}}{\theta}\,(i)$$



5.3 Comments on the rules

Note that every elimination rule is in *parallelized* form.[13] The reader will probably be more familiar with the more frequently encountered *serial* forms of (∧-E), (→-E) and (∀-E). These were the ones employed by Gentzen and Prawitz:

$$\frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi} \; ; \qquad \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \; ; \qquad \frac{\forall x \varphi}{\varphi^x_t} \; .$$

The serial and parallelized forms of (∧-E), (→-E) and (∀-E) are respectively interderivable. There are considerable systemic advantages in having *all* one's elimination rules in parallelized form. The most important advantage is that one is thereby afforded a uniform way of ensuring *normality of proof*. One simply insists that no major premise for an elimination may have

[13]These rules were introduced in Tennant [1992].

any proof-work above it. That is to say: each major premise must *stand proud*. This has the further nice consequence that normal natural deductions (i.e. Core proofs) are structurally isomorphic to (cut-free, thinning-free) proofs in the sequent calculus for Core Logic.

Every rule of Core Logic can be derived in Intuitionistic Logic. So, if one is prepared to assert that the rules of Intuitionistic Logic are analytically valid, then one is immediately committed to the analytic validity of the rules of Core Logic. We do not, however, need to argue for the analyticity of the rules of Core Logic by hitching a ride in this way on the coat-tails of the intuitionist analyticity-theorist. There is a more direct and robust route to the analytic validity of the rules of Core Logic: simple reflection on the rules themselves, and on the obvious reduction procedures (see below) that establish, for each logical operator, the harmonious balance between its Introduction and Elimination rules.

5.4 Reduction Procedures

A reduction procedure is used in order to get rid of those unnecessary detours in reasoning that arise when the conclusion of a terminal introduction in a proof Π, say, is used as a major premise in an application of the corresponding elimination rule in a proof Σ, say. For technical reasons that are unnecessary to set out in full here,[14] we can confine ourselves to considering just those cases where the elimination in question is the terminal step of Σ. We shall denote by $[\Pi\ \Sigma]$ the result of 'reducing' Π and Σ, so that the dominant operator of the conclusion of Π is not introduced and then immediately eliminated in the combined train of deductive reasoning that begins with Π and proceeds through Σ. The following reduction procedures form the heart of the normalization or cut-elimination process described in Tennant [2012b]. They have the added advantage that they sometimes afford us reducts establishing stronger results than might have been anticipated.

$$\left[\begin{array}{cc} {}^{(i)}\overline{A}\,,\ \Xi & \Delta \\ \Theta & \Pi \\ \dfrac{\bot}{\neg A}{\scriptstyle (i)} & \dfrac{\neg A \quad A}{\bot}{\scriptstyle (1)} \end{array}\right] = \left[\begin{array}{cc} \Delta & A\,,\ \Xi \\ \Pi & \Theta \\ A & \bot \end{array}\right]$$

[14] They can be found in Tennant [2012b].

$$
\left[\begin{array}{cc}
\dfrac{\begin{array}{cc}\Xi_1 & \Xi_2\\ \Theta_1 & \Theta_2\\ A & B\end{array}}{A\wedge B}
&
\dfrac{\begin{array}{cc} & \Gamma,\ \overline{A}^{(1)},\ \overline{B}^{(1)}\\ & \Sigma\\ A\wedge B & \theta\end{array}}{\theta}{\scriptstyle(1)}
\end{array}\right]
=
\left[\begin{array}{cc}
\Xi_1 & \left[\begin{array}{cc}\Xi_2 & \Gamma,\ A,\ B\\ \Theta_2 & \Sigma\\ B & \theta\end{array}\right]\\
\Theta_1 & \\
A &
\end{array}\right]
$$

$$
\left[\begin{array}{cc}
\dfrac{\begin{array}{c}\Xi_i\\ \Theta_i\\ A_i\end{array}}{A_1\vee A_2}
&
\dfrac{\begin{array}{ccc} & {}^{(j)}\overline{A_1},\Gamma_1 & {}^{(j)}\overline{A_2},\Gamma_2\\ & \Sigma_1 & \Sigma_2\\ A_1\vee A_2 & \theta & \theta\end{array}}{\theta}{\scriptstyle(j)}
\end{array}\right]
=
\left[\begin{array}{cc}
\Xi_i & A_i,\Gamma_i\\
\Theta_i & \Sigma_i\\
A_i & \theta
\end{array}\right]
\qquad (i=1,2)
$$

$$
\left[\begin{array}{cc}
\dfrac{\begin{array}{c}\Xi_1\\ \Theta_1\\ A_1\end{array}}{A_1\vee A_2}
&
\dfrac{\begin{array}{ccc} & {}^{(j)}\overline{A_1},\Gamma_1 & {}^{(j)}\overline{A_2},\Gamma_2\\ & \Sigma_1 & \Sigma_2\\ A_1\vee A_2 & \theta & \bot\end{array}}{\theta}{\scriptstyle(j)}
\end{array}\right]
=
\left[\begin{array}{cc}
\Xi_1 & A_1,\Gamma_1\\
\Theta_1 & \Sigma_1\\
A_1 & \theta
\end{array}\right]
$$

$$
\left[\begin{array}{cc}
\dfrac{\begin{array}{c}\Xi_2\\ \Theta_2\\ A_2\end{array}}{A_1\vee A_2}
&
\dfrac{\begin{array}{ccc} & {}^{(j)}\overline{A_1},\Gamma_1 & {}^{(j)}\overline{A_2},\Gamma_2\\ & \Sigma_1 & \Sigma_2\\ A_1\vee A_2 & \theta & \bot\end{array}}{\theta}{\scriptstyle(j)}
\end{array}\right]
=
\left[\begin{array}{cc}
\Xi_2 & A_2,\Gamma_2\\
\Theta_2 & \Sigma_2\\
A_2 & \bot
\end{array}\right]
$$

$$
\left[\begin{array}{cc}
\dfrac{\begin{array}{c}\Xi_1\\ \Theta_1\\ A_1\end{array}}{A_1\vee A_2}
&
\dfrac{\begin{array}{ccc} & {}^{(j)}\overline{A_1},\Gamma_1 & {}^{(j)}\overline{A_2},\Gamma_2\\ & \Sigma_1 & \Sigma_2\\ A_1\vee A_2 & \bot & \theta\end{array}}{\theta}{\scriptstyle(j)}
\end{array}\right]
=
\left[\begin{array}{cc}
\Xi_1 & A_1,\Gamma_1\\
\Theta_1 & \Sigma_1\\
A_1 & \bot
\end{array}\right]
$$

$$\left[\begin{array}{cc}\begin{array}{c}\Xi_2\\\Theta_2\\\dfrac{A_2}{A_1\vee A_2}\end{array} & \dfrac{A_1\vee A_2\quad\begin{array}{c}\overset{(j)}{\overline{A_1,\Gamma_1}}\\\Sigma_1\\\bot\end{array}\quad\begin{array}{c}\overset{(j)}{\overline{A_2,\Gamma_2}}\\\Sigma_2\\\theta\end{array}}{\theta}\,(j)\end{array}\right] = \left[\begin{array}{cc}\Xi_2 & A_2,\Gamma_2\\\Theta_2&\Sigma_2\\A_2&\theta\end{array}\right]$$

$$\left[\begin{array}{cc}\begin{array}{c}\overset{(i)}{\overline{A}}\,,\ \Xi\\\Theta\\\dfrac{\bot}{A\to B}\,(i)\end{array} & \dfrac{A\to B\quad\begin{array}{c}\Delta\\\Pi\\A\end{array}\quad\begin{array}{c}\overset{(j)}{\overline{B}}\,,\ \Gamma\\\Sigma\\\theta\end{array}}{\theta}\,(j)\end{array}\right] = \left[\begin{array}{cc}\Delta & A\,,\ \Xi\\\Pi&\Theta\\A&\bot\end{array}\right]$$

$$\left[\begin{array}{cc}\begin{array}{c}\overset{(i)}{\overline{A}}\,,\ \Xi\\\Theta\\\dfrac{B}{A\to B}\,(i)\end{array} & \dfrac{A\to B\quad\begin{array}{c}\Delta\\\Pi\\A\end{array}\quad\begin{array}{c}\overset{(j)}{\overline{B}}\,,\ \Gamma\\\Sigma\\\theta\end{array}}{\theta}\,(j)\end{array}\right] = \left[\begin{array}{c}\Delta\\\Pi\\A\end{array}\left[\begin{array}{cc}A\,,\ \Xi & B\,,\ \Gamma\\\Theta&\Sigma\\B&\theta\end{array}\right]\right]$$

$$\left[\begin{array}{cc}\begin{array}{c}\Xi\\\Theta\\\dfrac{A^x_t}{\exists xA}\end{array} & \dfrac{\exists xA\quad\begin{array}{c}\overset{(i)}{\overline{A^x_a}}\,,\ \Gamma\\\Sigma\\\theta\end{array}}{\theta}\,(i)\end{array}\right] = \left[\begin{array}{cc}\Xi & A^x_t\,,\ \Gamma\\\Theta&\Sigma^a_t\\A^x_t&\theta\end{array}\right]$$

$$\left[\begin{array}{cc}\begin{array}{c}\Xi\\\Theta\\\dfrac{A}{\forall xA^a_x}\end{array} & \dfrac{\forall xA^a_x\quad\begin{array}{c}\underbrace{\overset{(i)}{\overline{A^a_{t_1}}}\ ,\ldots,\ \overset{(i)}{\overline{A^a_{t_n}}}\ ,\ \Gamma}\\\Sigma\\\theta\end{array}}{\theta}\,(i)\end{array}\right] = \left[\begin{array}{c}\Xi\\\Theta^a_{t_1}\\A^a_{t_1}\end{array}\ldots\left[\begin{array}{cc}\Xi & \underbrace{A^a_{t_1},\ldots,A^a_{t_n}\ ,\ \Gamma}\\\Theta^a_{t_n}&\Sigma\\A^a_{t_n}&\theta\end{array}\right]\ldots\right]$$

These reduction principles are a formal way of saying the following.

1. *The listener is* not *entitled, by employing the elimination rule for its dominant operator, to extract from an assertion* more *than its assertor would have been obliged to put into it by using that operator's introduction rule.*

2. *The listener* is *entitled, when employing the elimination rule for its dominant operator, to extract from an assertion* all *that its assertor would have been obliged to put into it by using that operator's introduction rule.*

3. *The assertor is* not *obliged, when employing the introduction rule for its dominant operator, to put* more *into an assertion (to back it up) than a listener would be entitled to extract from it by means of that operator's elimination rule.*

4. *The assertor* is *obliged, when employing the introduction rule for its dominant operator, to put into an assertion (to back it up)* all *that a listener would be entitled to extract from it by means of that operator's elimination rule.*

Speaker's obligations and listener's entitlements are in harmonious balance.

5.5 A-priori-because-analytically-valid inferences

Core Logic, then, contains only *a-priori-because-analytically*-valid deductive principles. But is it the case that Core Logic contains *exactly* the *a-priori-because-analytically*-valid deductive principles? An affirmative answer requires that we make a case for the view that any classically valid argument not provable in Core Logic is not analytically valid. The view in question can be backed up in two complementary ways.

First, wearing his hat as a *relevantist*, the Core Logician challenges the validity, let alone *analytic* validity, of any and all of the classical logician's *irrelevant* but classically provable arguments. (Paradigm example: the first Lewis paradox $A, \neg A : B$.) But *even if* some kind of validity were to be conceded to such arguments, their very defect of irrelevance (i.e., the lack of appropriate *meaning connections* between premises and conclusion) would render them *not analytically* valid.

Secondly, wearing his hat as an *anti-realist inferentialist*, the Core Logician challenges the validity, let alone *analytic* validity, of any and all of the classical logician's *non-constructive* but classically provable arguments. (Paradigm example: $\neg\neg A : A$.) He urges that the classicist's commitment to regarding such an argument as valid is to be construed as *manifesting a* *metaphysically realist stance*, and accordingly revealing the argument as, at best, *synthetically* valid. (We explained this plank of the platform in §1, as our 'first departure from orthodoxy'.)

It has already been remarked that Core Logic contains Disjunctive Syllogism ($A \vee B, \neg A : B$), but neither one of the two closely related Lewis paradoxes $A, \neg A : B$ and $A, \neg A : \neg B$. Here are some further important points to (re-)emphasize about Core Logic:

- Major premises for eliminations (MPEs) *stand proud*
- So, all Core proofs are in *normal form*
- *Vacuous discharge* of assumptions is prohibited where need be
- The rules of $\rightarrow$-I and $\vee$-E are *liberalized*
- The rule *Ex Falso Quodlibet* is banned
- So, all Core proofs are *relevant*

Every argument provable in Core Logic is provable in Intuitionistic Logic, but not *vice versa*. So Core Logic is a proper subsystem of Intuitionistic Logic.[15] One can, however, *classicize* Core Logic while still preserving its virtues as a canon of *relevance* (of premises to conclusions). The classicized system is a proper subsystem of Classical Logic. It does not, however, contain Intuitionistic Logic as a subsystem, since it is free of the irrelevancies that afflict the latter.

6 Metatheorems about Core Logic

Definition 1 *Unless otherwise indicated, we shall assume of the deducibility relation $\vdash$ that it satisfies no more than the rules of the system $\mathbb{C}$ of Core Logic. (To be precise, $\Delta \vdash \varphi$ means that φ is core-deducible from premises that are drawn from Δ. The set of premises in question may be a proper subset of Δ.)*

Definition 2 *We write Δ_1, Δ_2 for $\Delta_1 \cup \Delta_2$, and we write Δ, φ for $\Delta \cup \{\varphi\}$.*

[15] As we indicated above, Core Logic is what Johansson *should* have arrived at when he sought to relevantize intuitionistic logic. But, because he was dealing with suboptimal formulations of proof systems, he ended up avoiding $\varphi, \neg\varphi : \psi$ while unfortunately allowing derivation of $\varphi, \neg\varphi : \neg\psi$. So his Minimal Logic is not minimal enough.

Definition 3 *$\forall\neg\neg[\theta]$ is the result of inserting an occurrence of $\neg\neg$ immediately after each occurrence in θ of a universal quantifier prefix. $\neg\neg\Delta$ is $\{\neg\neg\theta \mid \theta \in \Delta\}$. $\forall\neg\neg[\Delta]$ is $\{\forall\neg\neg[\theta] \mid \theta \in \Delta\}$.*

Theorem 1 (Cut Elimination for Core Proof)
There is an effective method [,] that transforms any two core proofs

$$\begin{array}{cc} \Delta & \varphi, \Gamma \\ \Pi & \Sigma \\ \varphi & \theta \end{array} \qquad \text{(where } \varphi \notin \Gamma \text{ and } \Gamma \text{ may be empty)}$$

into a core proof $[\Pi, \Sigma]$ of θ or of $\bot$ *from (some subset of)* $\Delta \cup \Gamma$.

Proof. See Tennant [2012b].

Corollary 1 (CUT WITH POTENTIAL EPISTEMIC GAIN)
If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then either $\Delta, \Gamma \vdash \bot$ or $\Delta, \Gamma \vdash \psi$.

Corollary 2 (CUT FOR ABSURDITY)
If $\Delta \vdash \varphi$ and $\Gamma, \varphi \vdash \bot$, then $\Delta, \Gamma \vdash \bot$.

Corollary 3 (CUT ON CONSISTENT PREMISES)
If $\Delta \nvdash \bot$ and $\Delta \vdash \gamma$ for each $\gamma \in \Gamma$ and $\Gamma \vdash \psi$, then $\Delta \vdash \psi$.

Theorem 2 (A Gödel–Glivenko–Gentzen Theorem for Core Logic)
If $\Delta \vdash_C \varphi$, then $\forall\neg\neg[\neg\neg\Delta] \vdash \forall\neg\neg[\neg\neg\varphi]$.

Theorem 3
If $\Delta \vdash_I \varphi$, then for some $\Gamma \subseteq \Delta$, either $\Gamma \vdash \varphi$ or $\Gamma \vdash \bot$.

Theorem 4
If $\Delta \vdash_C \varphi$, then for some $\Gamma \subseteq \Delta$, either $\Gamma \vdash_{\mathbb{C}+} \varphi$ or $\Gamma \vdash_{\mathbb{C}+} \bot$.

Theorem 1 and Corollary 1 reassure us that any alleged 'loss' of unrestricted transitivity of deduction in Core Logic brings with it a fully compensatory measure of *epistemic gain*. (The obvious analogues of these results can be proved also for *Classical* Core Logic.)

Theorem 3 tells us that Core Logic suffices for Intuitionistic Mathematics. This is because the axioms Δ of intuitionistic mathematics are consistent; whence it is impossible that for some $\Gamma \subseteq \Delta$, $\Gamma \vdash \bot$.

Likewise, Theorem 4 tells us that Classical Core Logic suffices for Classical Mathematics.

The implications of these results are clear for a choice of logic for the foundations of mathematics. Mathematical reasoning can now be formalized using only core proofs (classical ones in the case of classical mathematics). We *need* to formalize *only* those stretches of reasoning involving no cuts. If Δ and Γ are collections of axioms of our mathematical theory, and we deduce a lemma φ from Δ, and thereafter deduce a theorem θ from Γ, φ, we shall have two proofs Π and Σ like this:

$$\begin{array}{cc} \Delta & \varphi, \Gamma \\ \Pi & \Sigma \\ \varphi & \theta \end{array}$$

The lemma φ is a 'cut sentence'. Our formalization, however, need not incorporate any application of CUT as a rule of formal proof-formation. All we need, for certainty about θ, is the core proof Π and the core proof Σ. Theorem 1 tells us that we can effectively determine from Π and Σ a core proof of *some subsequent of* $\Delta, \Gamma : \theta$. Typically we would not even bother to find it and write it down, given our confidence in the consistency of our mathematical axioms. In that regard we would be just like the constructivists who rely on indirect proofs rather than canonical proofs, and the assurance that *if* they were to apply the effective procedures encoded in the indirect proofs to hand, they would eventually find a canonical proof for a theorem they are already minded to assert on the basis only of their indirect proofs.

Theorem 4 assures us also that Classical Core Logic suffices for the hypothetico-deductive testing of scientific theories, if classical reasoning happens to be employed. But by Theorem 2, such reasoning can always be constructivized, by resorting to a language with $\exists$ as its only quantifier. Hence Core Logic itself suffices for the hypothetico-deductive testing of scientific theories.

There is an automated deducer for (propositional) Core Logic, whose decision-problem is PSPACE-complete (like that of Intuitionistic Logic).

7 The Case for Core Logic

Core Logic enjoys philosophical, mathematical, methodological, computational, proof-theoretic and revision-theoretic credentials.

The *philosophical case* for Core Logic was set out in Tennant [1987] and Tennant [1997]. Those works argue for Core Logic as the correct logic, on the basis of Dummettian considerations of manifestability of grasp of meaning. Core Logic lies at the intersection of two orthogonal lines of logical reform: *constructivization* and *relevantization*. Conceived this way, Core Logic $\mathbb{C}$ might look like a mere *residue* of sacrificial reforms, beginning with classical logic **C**:

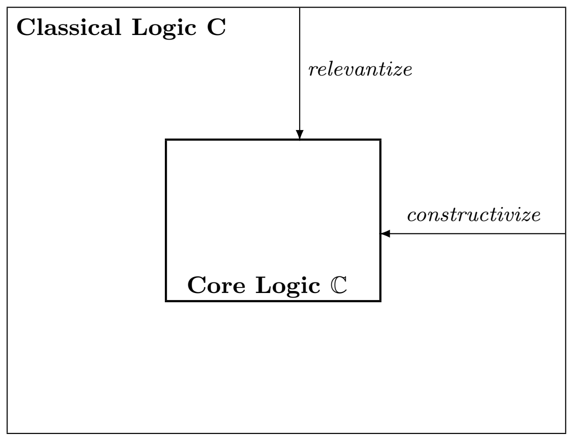


Figure 2: Two lines of logical reform

Such a picture gives the impression that Core Logic is to be characterized only as what is left over when one eschews certain principles of classical reasoning (those that are not constructive, and those that embody irrelevancies). But this impression, unsupplemented by any other perspective on how Core Logic might be the canon of choice, would be mistaken.

The philosophical claims about Core Logic that were made in the two works mentioned above can be supplemented by three further considerations of a philosophical kind.

First, Core Logic is the result of a natural generalization, to finite sets of *complex* sentences as premises, of the inferential methods that we employ in order to determine the truth-values of complex sentences from the 'atomic information' in a model (coded in the form of *literals*). Basically, the proto-logic involved in handling truth-table computations is but a small step of smooth extrapolation away from Core Logic. This proto-logic could be called the Logic of Evaluation, and will accordingly be denoted as $\mathbb{E}$. The picture above may now be amended so as to show that Core Logic is *also* a natural terminus of a process of generalizing from the Logic of Evaluation $\mathbb{E}$:

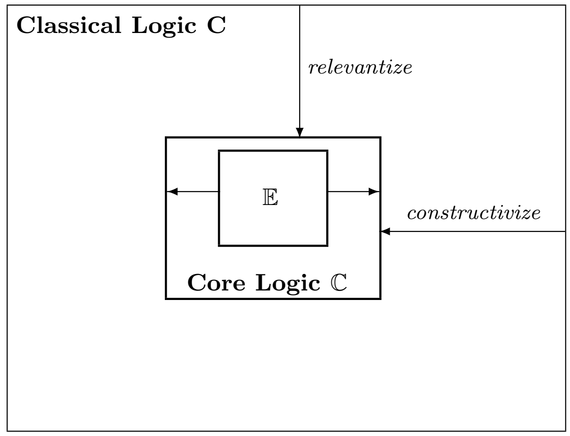


Figure 3: Pressure from without meets pressure from within

Using $\mathbb{E}$ to explicate and explain the (perhaps unconscious) inferential processing accomplished by subjects in the various imagined situations described in Casalegno [2004] provides a crucial theoretical ingredient for a mix that can dispose of all of Casalegno's objections to the Gentzen–Prawitz thesis that 'a subject knows the meaning of a logical constant if and only if she accepts a certain set of logical rules of inference' (p. 396).[16] The other ingredient that is needed in order to draw the counterexemplifying sting of Casalegno's imagined situations is the intuitionistic proof-theorist's notion of a *basis* of atomic rules of inference, among them possibly rules with $\perp$ as their conclusion. (See Prawitz [1974].) Such rules can express contrarieties, such as those between simultaneous but different color-attributions to one and the same object. Indeed, once one has $\perp$ playing a role as the conclusion in primitive atomic rules of inference for an interpreted language, the way is clear to an account of negation as having its meaning constituted by the rule of $\neg$-Introduction. (See Tennant [1999].)

The assurance of relevance, within $\mathbb{E}$, between (the evaluation of) a compound and (the evaluations of) its constituents can be transferred to the wider and more general deductive setting in which conclusions may be drawn from premises of any complexity (and not just from literals). That inferential drawing is thereby guaranteed to take place along lines of genuinely relevant connection between premises and conclusions. This characterization

[16]Casalegno attributes the thesis to Peacocke and Boghossian, apparently unaware of how the technical work of these major proof-theorists (Gentzen and Prawitz) had been both motivated by, and subsequently able to enrich and inform, an inferentialist theory of meaning.

of Core Logic was provided in Tennant [2010] and Tennant [forthcoming].

Secondly, the present author would argue that Core Logic is the correct *logic of conceptual constitution*. That is, constitutive logical interrelationships among concepts are to be exhibited strictly within the confines of Core Logic. Conceptual interconnections should not trade on any logical irrelevancies such as are supplied by intuitionistic and by classical logic. Moreover, along the lines of Tennant [1996], we stress once again: any strictly classical logical moves governing negation should be understood not as turning on meanings—for Core Logic exhausts all logical connections forged by meanings—but rather as *expressing a metaphysically realist attitude* as to the determinacy of truth-values of the 'litmus-sentences' to which those rules are applied.

Thirdly: consonant with Core Logic's being the correct logic of conceptual constitution (the canon for 'unpacking concepts') is a closely related thesis, advanced in Tennant [Unpublished typescript], concerning the logic needed in order to uncover logico-semantic paradoxes. The claim is that these paradoxes are never strictly classical. The kind of conceptual trouble that such a paradox reveals will afflict the intuitionist (and relevantist) just as seriously as it does the classicist. Therefore, attempted solutions to these paradoxes, if they are to be genuine solutions, must be available to the Core logician. Nothing about an attempted solution to a logico-semantic paradox should imply that the trouble it reveals has its origin in moves of classical (or even intuitionistic) reasoning that lie beyond the confines of Core Logic.

So much for the philosophical credentials of Core Logic. What about its mathematical credentials? These are not widely known or appreciated. The completely formal derivations, in Tennant [1987], of the Peano-Dedekind axioms for arithmetic from deeper logicist principles of inference governing the primitive notions 'the number of Φs' and 'n is a natural number' were given entirely within Core Logic. (This is in keeping with Core Logic being the 'logic of conceptual constitution'.) But this is not a provincial result. The adequacy of Core Logic for (all of) *intuitionistic mathematics* (already remarked on in §6) was established as a metatheorem in Tennant [1994]. The adequacy of the classicized version of Core Logic for (all of) *classical* mathematics is established in a similar fashion.

The *methodological* adequacy of Core Logic—its sufficiency for the hypothetico-deductive method of scientific theory-testing, already remarked on in §6—was established in Tennant [1997], building on earlier results concerning Minimal Logic in Tennant [1985].

The advantages for *computational logic* (or automated theorem-proving) of using systems such as Core Logic, in which all proofs are in a very exigent kind of normal form (without loss of completeness), were set out in Tennant [1992].

Core Logic enjoys the *proof-theoretic* distinction that its proofs have exactly the same structure whether they are presented as natural deductions or as sequent proofs. This is a consequence of the fact that the natural deductions of Core Logic are defined in such a way that major premises of eliminations (MPEs) always *stand proud*, with no proof-work above them. This of course ensures, as noted above, that all proofs are in normal form. Isomorphism between natural deductions and cut-free, thinning-free sequent proofs is then immediate. This isomorphism property would be enjoyed by any logical system that resembled Core Logic in these key respects (MPEs standing proud in natural deductions, and sequent proofs always being available in cut-free and thinning-free forms). But, to the best of the author's knowledge, there is no extant rival to Core Logic that matches it in these respects.

Finally, Core Logic has lately emerged in a new role, in a way that both explains and justifies the author's preference for its new label. An argument is presented in Tennant [2012a] for the following revision-theoretic thesis:

> *Core Logic is the minimal inviolable core of logic without any part of which one would not be able to establish the rationality of belief-revision.*

This provides yet another 'endogenous' argument for Core Logic, showing that $\mathbb{C}$ is a body of principles exactly responding to *pressures from within*:

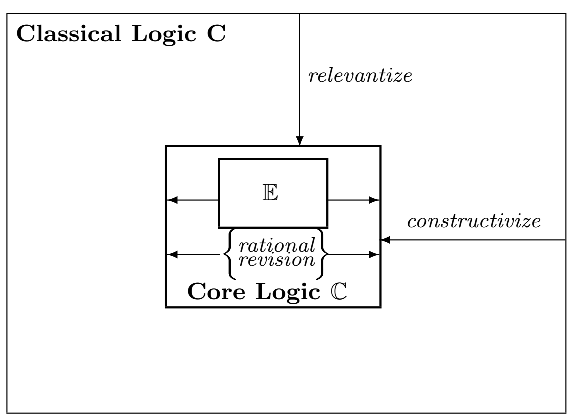


Figure 4: Pressure from without meets more pressure from within

The fact that such disparate perspectives on the problem of optimal choice of a logical system all converge on this single system of Core Logic is an indication that it captures a very stable, central and robust notion of logical deducibility.

Indeed, the present author would claim: Core Logic is *the* system of *a priori because analytically valid* logical inferences. Other inferences that are *classically* valid without exploiting irrelevance, but are not part of Core Logic, are *synthetic a priori*, if they are valid at all.

8 Natural Logicism

In what sense might logicism still be, in this post-Gödelian era, a viable and illuminating doctrine, even if only under suitable, self-imposed, limitations? The *natural logicist* answers:

> *Logicism remains viable for that part of mathematics whose mathematical operators admit of introduction and elimination rules that allow one to establish the appropriate analogue of Theorem 1 for the extended proof system containing them.*

The 'natural-deduction' methods and systems that Gentzen pioneered in logic afford the possibility of an important re-orientation of the logicist's agenda. According to the standard version of logicism, logic furnishes definitions of the primitive concepts of mathematics, allowing one to derive the mathematician's 'first principles' (such as the Peano–Dedekind postulates for arithmetic) as results within logic itself. Gödel's celebrated incompleteness theorems, however, put paid to this conception of what the logicist might in principle be able to achieve within some one formal system of deductive logic. *All* of mathematics?—surely not.

But what about *some interesting and significant core* of mathematics? In response to *this* question, the jury is still out. In so far as logic still aspires to provide the formal proofs that, in principle, can serve to 'regiment' *all* the informal proofs of ordinary mathematics, one might hope to recover, or develop afresh, a certain set of 'logicist' insights. They may not amount, collectively, to the sweeping doctrine of logicism as originally conceived; but they might go a long way towards illuminating, in a philosophically interesting way, the conceptual basis of modern mathematics.

Logicism would thereby cease to be the old, and now largely discredited, monolithic doctrine about the nature of mathematics as a whole. It would instead take a 'proof-theoretic turn', seeking to reveal, perhaps, those central *parts* of the whole body of mathematics that might justifiably lay claim to a logicist interpretation or re-formulation—even if the whole branch of mathematics for which such parts form its conceptual skeleton needs, in addition, some sinews of a more synthetic nature.

It is interesting to inquire about the basic concepts and pre-set-theoretic, 'native' intuitions of the mathematicians who were able to formulate and prove important results that have only subsequently acquired their predominantly set-theoretical trappings. We have in mind here not the kind of not-altogether-trustworthy geometric intuitions against which real analysis[17] (so it was thought) had to be guarded via the arithmetization undertaken by Cauchy and Weierstraß. Rather, we have in mind the *analytic*[18] intuitions[19] of the competent mathematician, which, when clear and distinct, betoken a thorough grasp of the mathematical concept(s) involved. A case in point would be the intuition that the natural numbers obey the Principle of Mathematical Induction. *Pace* Poincaré, we consider this intuition to be *analytic*, not synthetic. Its analyticity can be exhibited by furnishing it with a constructive logicist derivation using only rules that are analytic of the notions 'number of Fs', 'successor' and 'zero'. The Principle of Mathematical Induction turns out to be the logical *elimination rule* for the predicate '... is a natural number' (which is defined in terms of the aforementioned primitive notions).

In the course of laying a natural-logicist foundation for each mathematical discipline, it is imperative to achieve clarity about the norms of logical inference within a free logic with abstraction operators. With the appropriate logic formulated, one can turn one's attention to the concepts and operations specific to each mathematical discipline in turn.

Defined concepts should be manageable, fruitful, and of wide application. They should help to *atomicize* the reasoning. Introduction and elimination rules for conceptual primitives, and for concepts defined in terms of them, pin down the concepts in question. This is what justifies use of the label 'logicism'. Judicious choice of definitions, in which the definienda are furnished with introduction and elimination rules, enable one to minimize the logical complexity of sentences appearing in the formal proofs provided as

[17]Here, 'analysis' is meant in the mathematical sense—the study of real numbers and functions of reals.

[18]Here, 'analytic' is meant not in the mathematical sense mentioned in footnote 17, but in the Kantian sense, as arising from the meanings of the words involved.

[19]Here, 'intuitions' is used in the sense of ordinary mathematical parlance, and not in the special Kantian sense of 'telling us something informative about the world', which Kantians regard as the contradictory of the Kantian sense of 'analytic'!

regimentations of passages of informal mathematical reasoning. These formal proofs, we stress again, will be core proofs (perhaps classical ones) in the extended system.

The natural logicist maintains that formal proofs should be homologues of informal ones; formalization should merely 'supply missing details'. The early forms of logicism tended to obscure the virtues of logical rigor (in the regimentation of mathematical proofs) because they were tied to a quite orthogonal project. This was the project of trying to furnish an all-embracing, over-arching theory of classes or theory of types. The universe of discourse of the sought unifying theory, it was hoped, would accommodate (through appropriate surrogates) all the various kinds of mathematical objects that different mathematical theories are 'about'.

This Fregean and Russellian bent had the consequence that Logicism, as a philosophy and foundations for mathematics, appeared to be overambitious. Yet Logicism can and should be prosecuted without any concern for the unification of mathematics via class theory or set theory or category theory or the theory of types (to name the most important 'unifying theories' on offer). A logicism worthy of the name could confine itself to simply making existing proofs in the main corpora of rigorous[20] but informal mathematics, *perfectly* rigorous because completely formal and symbolic.

Formalization should reveal points of non-constructivity, impredicativity, 'purity' etc. This is one of the less appreciated benefits of full formalization of mathematical proofs. It enables the maturing mathematician to become aware of which steps of reasoning might be especially controversial or methodologically significant.

The natural logicist seeks to treat the objects of the theory as *sui generis*, rather than as surrogate objects within a 'more foundational' theory such as set theory. As remarked by Harrington, Morley, Ščedrov and Simpson in Harrington et al. [1985] at p. vii:

> . . . ZFC . . . is not appropriate . . . for a more delicate study of the nature of mathematical proof. Standard mathematics is not inherently or peculiarly set-theoretic.

This remark was intended to set the stage, however, for their subsequent explanation of how arresting it was that Friedman had been able to demonstrate *necessary* uses of abstract set theory in order to prove results in 'rel-

[20]Here we mean 'rigorous' to be understood in the usual way that a well-trained mathematician understands it. All main steps are explicitly indicated. Appeals to intuition are made only when the writer and the reader can be expected to know how to eliminate them in favor of more rigorous symbolic reasoning.

atively concrete mathematical situations' (*ibid.*, p. viii). What that means, however, is that the concrete result in question (φ, say) is provable in ZFC plus some large-cardinal axiom, and in turn implies (*modulo* some weak base theory, such as EFA)[21] the consistency of ZFC plus all smaller large-cardinal axioms. *If one's main concern is to calibrate the consistency strength* of a particular concrete-looking conjecture φ in this way, then of course it behooves one to translate both φ and the 'native' axioms of the theory T (to which φ might or might not belong) into the language of set theory, so that the calibration can proceed. *If, however, one's main concern is to clarify the logical structure of the reasoning by which all the known results of the 'native' theory have been established*, then it is better to *eschew* the set-theoretical trappings that help only with the calibration question, and deal with T directly, natively, *sui generis*.

Mathematical theories are learned, developed and communicated 'natively'. Each theory has its own special stock of concepts; and is 'about' its own special kinds of mathematical object. The early proofs by great expositors of these theories treat these objects as *sui generis*, without presenting them as complicated sets drawn from the cumulative hierarchy of pure sets. The 'Bourbakization' of mathematics—the re-definition of all the concepts of different branches of mathematics in terms of sets alone—makes it harder for a beginner to understand what any particular mathematical theory is *about*. It makes mathematics, which is already abstract enough, seem *utterly* abstract, to the point of enjoying no enlivening or illuminating connection whatsoever with any other area of human thought—be it physics, computer science or economics.

The neo-logicist should seek to explain how a given branch of mathematics is *applicable* (if it is). The more searingly abstract a presentation one provides for a mathematical theory, the more difficult it becomes to explain how it is in the very nature of the mathematical objects concerned that one's theory about them can be applied in reasoning about real-world phenomena and the regularities that underly them. Ironically, it was Frege who made the most of the requirement that such applicability be explained—and who then did the most damage to that very prospect.

The foregoing aims have important consequences. One attends more carefully to what is really 'built in' to a (defined) concept, as opposed to what is assumed in the hypotheses for one's reasoning. One 'carves informal proofs at their joints'. *Regimentation is anatomization!* One can more easily motivate the study of formal proofs for practicing mathematicians. One can

[21] EFA is exponential function arithmetic.

devise proof-search strategies in automated or interactive theorem-proving that are tailored to the branch of mathematics in question. One can address the issue of analytic v. synthetic truth in mathematics with sharper tools at one's disposal. Occasionally one detects a deeply hidden fallacy in even the best extant texts.

The question naturally arises, for any mathematical theory T: how far might this natural-logicist approach be extended to T? Could T be laid out in its own 'native' terms, shorn of the specifically set-theoretic notions that are employed in contemporary treatments in textbooks? Could one avoid the 'ontological riches' of a set-theoretic foundation, by helping oneself only to what is specifically needed, both conceptually and ontologically, in order to attain the results one is after?

Various branches of pure mathematics, such as arithmetic, different geometries, set theory etc. have been axiomatized by the pure mathematicians who practice in those fields. These mathematicians are interested first and foremost in the abstract *structures* formed by the mathematical objects under investigation, even when the intention is to try to characterize the structure in question up to isomorphism. Questions of applicability are usually set to one side, as are questions concerning the ultimate logical foundations of that branch of mathematics within rational thought as a whole. One of the (perhaps unintended) consequences of this 'pure isolationist' approach to particular mathematical theories is that their axioms are chosen with a pragmatic eye on how quickly they can yield desired consequences, and how readily they will be accepted (without proof) by the intended audience. Both consistency and certainty are desiderata, to be sure; but pragmatic compromises are also struck in pursuit of both brevity of proof and power of single axioms.

This means, in the case of some of the traditional axiomatizations of different branches of geometry, that there is a trade-off between the length of axioms and their number—usually increasing the former and decreasing the latter. The axioms eventually chosen serve mainly as convenient starting points for deductions, provided only that they will be accepted as true of the intended subject matter. There is no uncompromising concern, on the part of practicing mathematicians, to ensure that all the axioms laid down are conceptually basic, or—even better—*analytic* of the concepts involved. Nor is there any concern to keep the axioms within some tightly constrained syntactic class, involving, say, a minimal number of quantifier alternations.

One fruitful departure from this established precedent in mathematical axiomatization is a natural-logicist treatment of synthetic projective geometry. Rather than stating *axioms*—which are (usually complex) sentences of a formal language—one can state *transitional atomic rules of inference*. These are rules of inference, in natural-deduction format, in which only atomic sentences feature. Some of them may contain parameters, thereby enabling one to express existential import—but still the only sentences in view are atomic. One can state a great many rules, arranged, as far as possible, in thematically coherent groups. The basic methodological principle is: state more simply and more frequently, rather than less simply and less frequently. Fundamental principles of geometry should be like so many little ants, making for a supple organic whole, rather than like heavy foundation stones that are difficult to put in place.

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