Logicism in Logical Empiricism

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1. Introduction

Logicism, as developed by Frege and Russell, is the thesis that pure mathematics is part of logic.[1] While the logicist thesis was a central doctrine in the philosophy of mathematics of the late nineteenth and early twentieth century, it did not present a uniform research project. Different scholars used the term "logicism" to describe different practices of reducing mathematical theories to higher-order logic or set theory.[2] This holds true, in particular, of work by philosophers related to modern empiricism. Logicism presents one of the cornerstones of logical empiricism.[3] At the same time, the views defended by Carnap, Hahn, and Hempel (among others) differ significantly from Frege's and Russell's original thesis.

The present chapter will focus on several accounts of logicism developed in the main phase of logical empiricism between 1920 and 1940. The aim here is twofold. The first aim is to survey how Frege's classical thesis was modified during the period in question. As we will show, this concerns not only a radically revised conception of the underlying logic, but also a new focus on non-arithmetical mathematical theories to be reduced to logic. More specifically, philosophers such as Carnap aimed to formulate a generalized logicism valid for all branches of pure mathematics, including different theories of geometry, topology, and algebra. As we will see in section 3, his and

[1] This article presents an extended version of Schiemer, G., "Nonstandard logicism", to appear in Uebel, T. (ed.), *Handbook of Logical Empiricism*, Routledge, (forthcoming).
[2] A more general study of the historical origins of logicism would have to consider also the "non-Fregean" line of early logicism, including the foundational work of Dedekind and Hilbert. See Reck's chapter in the present volume as well as Sieg & Schlimm (2005) on Dedekind's logicism. Compare Ferreirós (2009) on Hilbert's early logicism.
[3] Compare, e.g., Goldfarb (1996), Friedman (1999), and Awodey & Carus (2007).

related accounts are best described as a form of conditional logicism based on an *if-thenist* reconstruction of mathematics.

The second aim in this chapter is to clarify how the contributions to conditional logicism are related to other developments in the foundations of logic and mathematics at the time. One focus here will be Wittgenstein's account of the tautological status of logic and the general significance of this view for the logical empiricists' project. In particular, we will retrace how the shift from Frege's and Russell's "universalist" conception of logic to the view of logic as a system of tautologies led to a reformulation of the logicist thesis in work by Carnap. A second issue addressed here concerns Carnap's continued attempts to reconcile classical logicism with a structuralist account of mathematical theories related to the rise of modern axiomatics. A third focus in section 4 will be on the question how the logicist thesis was reformulated in his subsequent work on the foundations of mathematics from the late 1930s (and thus after Gödel's incompleteness results).

2. Classical logicism and the type-theoretic tradition

The history of classical logicism is well studied. The position is rooted in work on the foundations of mathematics in the nineteenth century, in particular, on the rigorization of number theory and analysis by Cantor, Weierstrass, and Dedekind (among others).[4] Frege's logicist project is often described as a direct continuation of this foundational work (see, e.g., Giaquinto (2002)). As is well known, Frege developed his program in several steps. He first introduced quantificational logic as the basis for the logicist reduction in his *Begriffsschrift* of 1879. Some years later, an informal characterization of the logicist thesis for arithmetic is outlined in *Grundlagen der Arithmetik* (1884). Based on a critical discussion of Mill's and Kant's respective views on the epistemological status of arithmetic, Frege presents here a new definition of the concept of natural numbers as well as the thesis that arithmetical notions are definable in pure higher-order logic. His main motivation for this reduction of arithmetic to logic is clearly an epistemological one: for Kant, all forms of mathematical knowledge, including arithmetic and geometry, consist of synthetic a priori truths

[4] Compare, in particular, Ferreirós (1999) and Grattan-Guiness (2000) for detailed historical studies of these developments.

and are thus grounded in *pure* intuition. Whereas Frege still upholds a version of this Kantian view in the case of geometry, his logicist reduction was to show, contra Kant, that the laws of arithmetic have an altogether different, namely purely analytic status.

The technical details of the logicist program are eventually presented in *Grundgesetze der Arithmetik* (1893/1903). In particular, Frege introduces here a higher-order logic together with a naive set theory describing concept extensions. Frege's central axiom on the logical behavior of such extensions of concepts is his notorious Basic Law V:

$$\hat{x}Px = \hat{x}Qx \leftrightarrow \forall x(Px \leftrightarrow Qx)$$

where 'P' and 'Q' are second-order variables ranging over concepts and '$\hat{x}Px$' and '$\hat{x}Qx$' are the extensions of concepts 'P' and 'Q' respectively. This principle is essentially an axiom for unrestricted set abstraction: it states that any two concepts have same extension if they are equivalent. Together with the basic logical laws stated in the *Begriffsschrift* of 1879 as well as a rule of substitution (equivalent to a modern principle of second-order property comprehension), these axioms form Frege's logical system.[5]

Frege's main objective in *Grundgesetze* is to present the technical details of the logicist reduction of arithmetic first outlined in *Grundlagen*. Specifically, based on his explicit definitions of the natural numbers and the successor relation between numbers, it is shown how one can derive each axiom of the Dedekind-Peano axiom system from his system of basic logical laws. Unfortunately, as Russell first pointed out in 1902, Frege's naive theory of classes based on Basic Law V turned out to be inconsistent. In particular, Russell's famous paradox follows from the following instance of naive comprehension:

$$\exists z \forall x \, (x \in z \leftrightarrow x \notin x)$$

which stipulates the existence of a set that contains as members all sets that do not contain themselves as members.

Contributions to logicism after the discovery of Russell's and related paradoxes were usually based

[5] See, in particular, Heck (2012) for a detailed presentation of Frege's logical system.

on the logical theory of types, a logical system first introduced in Appendix B of Russell's *Principles of Mathematics* (1903) and then developed systematically in Russell's & Whitehead's landmark *Principia Mathematica* (1910-1913). The type theory presented in the three volumes is a higher-order logic describing a rich universe that is stratified into distinct types of objects. Moreover, the logical system presents an intensional logic given that each type is further ramified into different orders, where the order of an object is determined by the kind of formula defining it.[6] Now, Russell's and Whitehead's system of ramified type theory was simplified significantly in subsequent work by Carnap, Tarski, Ramsey, and Gödel (among many others). The simplification meant primarily that the original predicativist approach of partitioning type domains into objects of different orders was eventually dropped. Consequently, Russell's "primacy of intensions" was given up in favor of purely extensional account of logic.

A second, equally important modification of Russell's original framework concerned the proper formalization of type-theoretic logic: for instance, the clear distinction between the syntax and the semantics of the logic, i.e. between the grammatical rules for a type-theoretic language on the one hand and its semantic interpretation on the other hand. While the specific interpretation of type theoretic languages varied from author to author, the general picture emerging in the late 1920s and early 1930s is that of type theory as a formal set theory, that is, a theory describing a rich universe of sets. Thus, far from being an ontologically neutral theory like first-order logic (as later suggested by Quine), type theory was conceived as a strong logic that describes a rich ontology of sets.[7]

Simple type theory was arguably the standard logical system in the main period of logical empiricism. As Ferreirós put it, before the consolidation of first-order logic, and "as late as 1930 type theory was still regarded by mathematical logicians as the most important and natural system of logic." (Ferreirós 1999, p. 445). In cases where the logical principles of this system (in addition to the standard laws of propositional logic) were explicitly discussed, these usually include an axiom scheme for typed comprehension:

[6] Compare, e.g., Giaquinto (2002), Ferreirós (1999), and Schiemer (forthcoming).
[7] There are other modifications of type theory in the 1930s not discussed here, for instance, the extension of (finite) type theory to theories of transfinite types. See again Ferreirós (1999) for further details.

$$\exists x^{i+1} \forall x^{i} (x^{i+1}(x^{i}) \leftrightarrow \varphi(x^{i}))$$

for formulae not containing x^{i+1} free, for all types $i \in \omega$.[8] Informally speaking, the axiom states that every well-formed formula with variable x^{i} determines a property or set of the objects the formula is true of. The second principle usually mentioned (by Tarski, Gödel, and others) is an axiom scheme of extensionality:

$$\forall x^{i} (x^{i+1}(x^{i}) \leftrightarrow y^{i+1}(x^{i})) \rightarrow x^{i+1} = y^{i+1}$$

This axiom scheme states that properties or sets of a given type $i + 1$ are identical if co-extensional. In Russell's & Whitehead's original presentation of ramified type theory in *Principia Mathematica*, three other axioms were taken to belong to the logical theory: a multiplicative axiom equivalent to the axiom of choice in set theory; an axiom of infinity; and the axiom of reducibility.

How can the logicist project of representing arithmetic in logic be developed in simple type theory? One way to specify the reduction relation is in terms of the formal notion of interpretability.[9] The interpretation of one theory into another is based on the notion of translation of one formal language into another formal language, usually defined as follows:

Definition 1 A *translation* τ of a language L_S into a language L_T consists of (i) an L_T-formula $\delta(x)$ and (ii) formulas $\varphi_{Ri}(x_1, \ldots x_n)$ (for each primitive n-ary predicate R_i in the language L_S) such that:

$$1.\ (R_i x_1 \ldots x_n)^{\tau} = \varphi_{Ri}(x_1, \ldots x_n)$$

$$2.\ (x = y)^{\tau} = (x = y)$$

$$3.\ (\neg\varphi)^{\tau} = \neg\varphi^{\tau}$$

$$4.\ (\varphi \wedge \psi)^{\tau} = \varphi^{\tau} \wedge \psi^{\tau}$$

$$5.\ (\forall x \varphi)^{\tau} = \forall x(\delta(x) \rightarrow \varphi^{\tau})$$

[8] See, e.g. Gödel (1931).

[9] See, e.g., Burgess (2005), Walsh (2014), and Schiemer (forthcoming). As we will see in the next two chapters, this approach to treat the logicist thesis as an interpretability result is closely connected to Carnap's account of logicism.

The formula $\delta(x)$ presents a 'domain formula' in language L_T for the variables occurring in L_S-formulas. Formulas $\varphi_{Ri}(x_1, \dots, x_n)$ provide 'interpretations', within the language L_T, of the non-logical terminology of the primitive terms of L_S. Given this notion of a translation, one can then define the notion of an interpretation as follows:

Definition 2 A translation τ is an *interpretation* of theory S in theory T if for every formula φ such that $S \vdash \varphi$, we have $T \vdash \varphi^\tau$.

Frege's project of reducing arithmetic to higher-order logic is often described in the literature as an interpretability result of this form. Roughly speaking, it expresses the fact that (second-order) Dedekind-Peano arithmetic is interpretable in logical type theory. In particular, one can show that all arithmetical statements can be translated into logical statements based on the Frege's definitions of the primitive vocabulary '0','successor', and 'being a natural number' of Peano arithmetic. The translation of arithmetical statements into purely logical ones gives an interpretation in the above sense: for any statement in the language of Peano arithmetic, in symbols $\varphi \in L_{PA}$, if $PA \vdash \varphi$ holds, we can show that $TT \vdash [\varphi(0, s, N)]^\tau$ also holds.[10] Arithmetic is thus reducible to type theory (possibly including Russell's axioms of choice and infinity) if the former is interpretable in the latter.[11]

3. Logical empiricism and conditional logicism

Logical empiricism in the 1920s and 1930s was strongly shaped by debates on the epistemological status of mathematics and by logicism in particular. Kant's traditional conception of mathematical principles as synthetic *apriori* truths was generally considered to be incompatible with a purely empiricist account of scientific knowledge. Logicism, in turn, provided twentieth-century empiricists with an alternative picture of the nature of mathematics which does not conflict with their general philosophical view. Compare Carnap in his "Intellectual Autobiography" on the general significance of the logicist thesis for the philosophers of the Vienna Circle:

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[10] Compare again Burgess (2005, pp. 50-51).

[11] We will return to a slightly different notion of interpretability in section 5.

But to the members of the Circle there did not seem to be a fundamental difference between elementary logic and higher logic, including mathematics. Thus we arrived at the conception that all valid statements of mathematics are analytic in the specific sense that they hold in all possible cases and therefore do not have any factual content. What was important in this conception from our point of view was the fact that it became possible for the first time to combine the basic tenet of empiricism with a satisfactory explanation of the nature of logic and mathematics. (Carnap 1963, p. 46)

Frege's logicism thus provided the logical empiricists with a strategy to establish the purely analytic status of mathematical knowledge. Nevertheless, the above passage already indicates that the understanding of Carnap and others was in several ways different from the classical program outlined above.

To characterize the logical empiricists' account of logicism, two aspects should be mentioned here. First, given the discussion in the previous section, it is not surprising that the standard logical system used by philosophers working in Vienna at the time was also a version of Russell's and Whitehead's type-theoretic logic of *Principia Mathematica*. Carnap's work from the 1920s and early 1930s contains important contributions to the simplification of type theory. In fact, his *Abriss der Logistik* (1929) can be viewed as one of the first textbooks of modern logic where a purely extensional version of type theory is presented in full detail. A second figure to mention in this respect is Hans Hahn, then head of mathematics department at the University of Vienna and one of the founders of the Vienna Circle. Like Carnap, Hahn was also an active proponent of Russell's type-theoretic logic.[12]

The second point to mention here is that the very understanding of logic changed radically in the period in question, mainly in reaction to Wittgenstein's *Tractatus Logico-Philosophicus* (1922). Wittgenstein's new conception of logic (already mentioned by Carnap in the passage above) can be characterized roughly as follows: logical laws are tautological in nature, that is, statements without factual content. Tautologies do not express facts about the world, but are statements true simply in virtue of their logical form. In Wittgenstein's own words:

The propositions of logic are tautologies. Therefore, the propositions of logic say nothing. (They are the

[12] Hahn taught a seminar on the logic of *Principia Mathematica* in Vienna in 1924/1925. See Uebel (2005) for a detailed study of Hahn's work on logic and philosophy of mathematics.

analytic propositions.) All theories that make a proposition of logic appear to have content are false. (1922, 6.1-6.111)

Philosophers such as Schlick, Carnap, and Hahn fully embraced Wittgenstein's new account of logic in their work from the 1920s and early 1930s.[13] How did this fact impact on their view of logicism? In several published texts from the time, logicism is officially described in the classical Fregean sense. Compare, for instance, Carnap's presentation of the thesis in his paper "Die Mathematik als Zweig der Logik":

> The basic idea of logicism can be formulated as [the claim that] mathematics is a branch of logic. That means: there are no specifically mathematical, extra-logical basic concepts and basic propositions. The concepts of mathematics can be derived from the logical concepts, i.e., from concepts which are indispensable for the development of logic even in the ordinary, non-mathematical sense; the propositions of mathematics form a part of the logical propositions. (Carnap 1930a, p. 298)

The general description of the program given here sounds very similar to Frege's original position. However, Carnap's account of the status of logical principles as tautologies clearly differs from Frege's and Russell's respective views.

This fact is particularly interesting given that both Hahn and Carnap were aware that the attempt to extend the tautological character of elementary (i.e. propositional) logic to higher mathematics was in itself deeply problematic.[14] Moreover, Wittgenstein himself not only rejected the logicist thesis in the *Tractatus*, there is also strong textual evidence that he did not take a higher-order system such as type theory to be properly logical in nature. For instance, he is clear on the point that Russell's "existential" axioms of infinity, choice, and reducibility as well as set theory more generally should not be seen as a part of logic (see, in particular, his 5.535, 6.031, and 6.1232 in

[13] Compare again Carnap in his autobiography on the significance of the *Tractatus* for his own work: "The most important insight I gained from his work was the conception that the truth of logical statements is based only on their logical structure and on the meaning of the terms. Logical statements are true under all conceivable circumstances; thus their truth is independent of the contingent facts of the world. On the other hand, it follows that these statements do not say anything about the world and thus have no factual content." (Carnap 1963, p. 25)

[14] Tarski, in his logical work from the 1930s, was also critical of the logical positivists' use of the notion of tautology. See, for instance, his article "*On the concept of logical consequence*" of 1936: "(…) the concept of *tautology* (i.e. of a statement which 'says nothing about reality'), a concept which to me personally seems rather vague, but which has been of fundamental importance for the philosophical discussions of L. Wittgenstein and the whole Vienna Circle." (Tarski 1983, pp. 419-420).

the *Tractatus*).[15]

One approach adopted by logical empiricists to defend type-theoretic logicism against this objection consists in a form of logical *if-thenism*. Roughly speaking, if-thenism is based on the reformulation of the theorems of a given mathematical theory as universally quantified conditional statements. Such statements contain a ramsified version of the relevant axioms in the antecedent and the ramsified theorem in the consequent. The general approach goes back to Russell and was first formulated systematically in *Principia Mathematica*. Compare Russell and Whitehead on the axiom of choice in volume I of the book:

We have not assumed its truth in the general [non-finite] case where it cannot be proved, but have included it in the hypotheses of all propositions which depend upon it. (Russell and Whitehead 1910-13, Vol.1, p. 504)

More generally, in his *Introduction to Mathematical Philosophy*, Russell famously claims that the if-thenist manoeuvre must be applied to any axiom which is problematic from a logical point of view:

(...) no principle of logic can assert "existence" except under a hypothesis (...) Propositions of this form, when they occur in logic, will have to occur as hypotheses or consequences of hypotheses, not as complete asserted propositions (...). (Russell 1919, p. 204)

The original motivation for the if-thenist reconstruction was thus to find a way of reducing mathematics to logic in a way in which one does not have to assert the logical truth of Russell's problematic axioms. The resulting conditional logicism was also embraced by several logical empiricists in order to address the problem of the non-tautological character of these axioms. In Carnap's *Abriss*, following a discussion of the status of the axiom of choice, he holds that:

If the axiom is not taken as a basic principle, these theorems can be formulated only as conditional propositions, as implications whose implicans is the axiom of choice. (Carnap 1929, §24b)

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[15] A similar view is also expressed in Carnap's work. For instance, in his *Abriss der Logistik*, he holds that: "The axiom of choice should not be included among the basic principles of logic, since its admissibility has been problematic. This is connected with its character as an existential assertion. However, the axiom is required in the proofs of certain theorems of set theory on transfinite powers (infinite cardinal numbers). (Carnap 1929, §24b) A similar verdict is adopted in the case of the axiom of infinity.

This Russellian approach is presented more explicitly in Carnap's "The logicist foundations of mathematics":

> [Russell] (...) transformed a mathematical sentence, say *S*, the proof of which required the axiom of infinity, *I*, or the axiom of choice, *C*, into a conditional sentence; hence *S* is taken to assert not *S*, but $I \supset S$ or $C \supset S$, respectively. This conditional sentence is then derivable from the axioms of logic. (Carnap 1931, pp. 96)

Interestingly, if-thenism was adopted by Carnap not only as a way to deal with the non-tautological character of several existential axioms of type theory. It was also used as a way to formulate a logicist thesis for non-arithmetical branches of pure mathematics. This is true, in particular, of Carnap's attempts from the 1920s to reconcile classical logicism with the structural approach underlying modern Hilbertian axiomatics.[16] Consider, for instance, Carnap's early monograph *Der Raum* of 1922. Carnap distinguishes between three concepts of spaces in the book, namely between formal, intuitive, and physical space. Formal space, i.e. the subject matter of pure geometry in the sense studied also by Russell, is described here as an abstract "relational system" that can be specified in two ways. The first one is in terms of an axiom system in the style of Hilbert. Compare Carnap on this account of axiomatic geometry as a "pure theory of relations or order theory":

> The object of this discipline is not space, i.e., the system of points, lines, and planes determined by *geometrical* axioms (...), but a "relational or structural system" determined by the *formal* axioms. As this represents the formal design of the spatial system, and turns into the spatial system again when spatial elements are substituted for indeterminate relata, it too will be called "space": "*formal space*". (Carnap 1922, p. 8)

An axiomatic theory does not describe a particular and independently accessible domain but rather an abstract structure shared by all systems that satisfy its axioms.

The second approach discussed in the book is more closely connected to the logicist approach. This is the idea to explicitly define a geometrical space in purely logical terms or based on a "logical construction". A formal space, according to Carnap, is what is definable in higher-order logic:

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[16] Carnap's approach is again strongly influenced by Russell's preceding work on an if-thenist reconstruction of geometrical theorems first presented in *Principles of Mathematics* (1903). See, in particular, Musgrave (1977) and Gandon (2012) for further details of Russell's position.

The construction of formal space can also be undertaken by a different path, however, not just by the above way of setting up certain axioms about classes and relations: by deriving (ordered) series and, as a special case, continuous series from *formal logic*, the general theory of classes and relations. (Carnap 1922, p. 8)

Thus, Carnap envisages two distinct but equivalent approaches to characterize the subject matter of theories of pure geometry, namely (i) in terms of implicit definitions through an axiom system and (ii) in terms of explicit constructions in a logical system. This duality of methods clearly reflects Carnap's above mentioned attempt to synthesize a Fregean (or Russellian) foundational stance with Hilbert's modern axiomatic approach (see, in particular, Awodey & Carus (2001) and Reck (2004)).

It should be noted here that the logical background system used for such constructions in geometry was not yet made precise in 1922. Nevertheless, the two approaches to study pure mathematics are still present in Carnap's later work on "general axiomatics" from the late 1920s. Here, the formalization of mathematical theories is expressed in a fully specified logical type theory. For instance, Carnap gives a type-theoretic formalization of several axiomatic theories in his *Abriss der Logistik*. He argues here that there are two ways in which axiom systems can be understood, namely as fully interpreted or as schematic. In the first case, mathematical primitives should be treated as non-logical constants with a fixed semantic interpretation. In the second reading, axiomatic theories are to be treated as formal (in roughly the modern sense of the term). Its primitive terms are thus non-interpreted and can be expressed by higher-order variables. More specifically, according to Carnap, axiom systems can be formalized in the language of simple type theory in the following way: the primitive terms of a theory are expressed as variables (of a given arity and type). The axioms, axiom systems and theorems, in turn, are expressed as sentential functions.

Given this approach, Carnap argues that an axiomatic theory gives an explicit definition of a higher-level concept, the "*Explizitbegriff*" of an axiom system. Put in modern terms, this is simply the class of models satisfying the theory. In Carnap's own terms:

For instance, if $x, y, \dots \alpha, \beta, \dots P, Q, \dots$. are the primitive variables of the AS and if we name the conjunction of axioms (that is a propositional function) $AS(x, y, \dots \alpha, \beta, \dots P, Q, \dots)$, then the definition of the explicit concept of this AS is:

$$\hat{x}, \hat{y}, \ldots \hat{\alpha}, \hat{\beta}, \ldots \hat{P}, \hat{Q}, \ldots \{AS(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots)\}$$ (Carnap 1929, p. 72)[17]

Two comments are in order here. First, Carnap's idea to describe the content of an axiom system in terms of an explicit concept (or rather its extension) is clearly motivated by Frege's critical analysis of Hilbert's axiomatics. Frege, in his correspondence with Hilbert and in subsequent writings, famously argued that an axiom system cannot be understood as an implicit definition of first-level concepts, but rather as an explicit definition of second-level or higher-level concepts. Precisely this idea, with which Carnap was well acquainted with from his time as a student of Frege's lectures in Jena, is presented in *Abriss* in the framework of a type-theoretic logic.[18]

Secondly, Carnap's formalization of axiomatic theories gives rise to an alternative, weakened form of the logicist thesis. In particular, given the ramsification of theories, the explicit concept corresponding to any axiom system is clearly a logical concept since it can be expressed in purely logical terms. It follows from this that any theory reconstructed in this way turns out to be logical in character. Compare again Carnap on this point:

> (...) the explicit concept of a geometrical AS, e.g. an AS of projective geometry presents the logical concept of the relevant type of space (e.g. the concept "projective space"). In this sense geometry can also be represented as a branch of logistic itself (as arithmetic) instead of being a case of application of logistics to a nonlogical domain. (Carnap 1929, p. 72)

In a related paper titled "Proper and improper concepts" (1927), Carnap further discusses how his understanding of mathematical theories is related to the conditional logicism described above. Given that the mathematical primitives of a theory are not defined explicitly, but only implicitly through an axiom system, they refer to so-called 'improper' concepts and should therefore be

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[17] Carnap discusses a number of mathematical axiom systems in his book to illustrate this account. His examples include formalizations of different projective geometries, of set theory, of Peano arithmetic, as well as of Hausdorff's topological neighborhood axioms.

[18] Compare, e.g., Carnap's notes based on Frege's lecture "Logic in mathematics" of 1914, in particular (Awodey & Reck 2014, pp.164-166). Carnap has also read and commented on Frege's two articles titled "The foundations of geometry" (1903) in which the understanding of axiom systems as definitions of second-level concepts is developed in closer detail. This can be seen from two shorthand notes with comments on Frege's papers written by Carnap in 1921 and documented in Carnap's *Nachlass* (ASP/RC 081-28-01).

symbolized by (free) variables. Mathematical axioms and theorems containing them are hence open formulas and not statements. However, again following Russell in this respect, Carnap argues that the real content of a theorem can be expressed in terms a quantified conditional statement that contains the 'explicit concept' of an axiom system in antecedent:

> Are the propositions of (Peano) arithmetic or (Hilbert) geometry then not sentences? After all, they contain symbols for improper concepts, thus variables. As they stand, indeed, they are not sentences, but rather functional expressions. But they serve as very effective abbreviations for proper sentences on the basis of an implicit convention. A sentence-like expression of this kind, in which variable symbols of a given AS occur, is to be taken as short for the sentence that looks like this (...): first comes a universal prefix containing all the variables of the AS and which applies to the entire implication, then comes the symbol for the logical product of the axioms of the AS as antecedent, and finally comes the sentence-like expression at issue as the consequent. The variables thus occur here only as apparent variables. (Carnap 1927, p. 371)

The "implicit convention" described here is precisely the if-thenist translation of mathematical statements indicated above. Theorems of mathematical theories such as Peano arithmetic or Hilbert's Euclidean geometry can thus be translated into purely logical statements of the following form:

$$\forall x, y, \dots \alpha, \beta, \dots P, Q, \dots [AS(x, y, \dots \alpha, \beta, \dots P, Q, \dots) \rightarrow \varphi(x, y, \dots \alpha, \beta, \dots P, Q, \dots)]$$

The variables $x, y, \dots \alpha, \beta, \dots P, Q, \dots$ present the "primitive signs" of the theory in question, AS presents the conjunction of the universally ramsified axioms, and φ the ramsified theorem considered.[19]

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[19] Interestingly, the same if-thenist reconstruction of mathematics is also present in philosophical writings of other logical empiricists. Compare, for instance, Hahn (1930) on a general "logization of geometry": "Every theorem of geometry thus appears as a (tautological) implication $P \rightarrow Q$ whose antecendent is the logical product of the axioms and whose consequent Q is the theorem in question. The axioms no longer appear here as self-evident but non-provable truths, but as stipulations from which one can deduce: the primitive concepts no longer appear as elements that cannot be further reduced by definition, but are immediately perceivable trough intuition, but rather as logical variables. Given that every single axiom is a relation between the variables representing the primitive concepts, it follows that geometry appears as a special chapter of the theory of relations, as an investigation of certain special relational systems." (Hahn 1930, p. 44) A similar discussion of this form of if-thenist reconstruction of mathematical theories is discussed in Hempel's classic article "On the nature of mathematical truth" from 1945.

Generally speaking, there are two important reasons for Carnap and other logical empiricists to adopt this form of if-thenism. The first one is to express in purely logical terms the structural character of axiomatic theories. As is well known, the development of modern axiomatics brought with it a model-theoretic conception of theories: axioms and mathematical theorems deducible from them are not merely true in an intended interpretation, but they hold in *any* model or structure that satisfies the primitive structural properties expressed in the axioms. Moreover, from a mathematical point of view, neither one of these structures is preferable to another one. This model-theoretic generality (i.e. the generalization over all possible interpretations of a theory) is certainly characteristic of modern axiomatic mathematics. In Carnap's work, it is expressed logically in terms of the symbolic representation of primitive mathematical signs in terms of variables.

Carnap's second motivation for his if-thenism was to develop an alternative to classical (i.e. arithmetical) logicism. This is established by the fact that a mathematical statement can always be translated into a purely logical statement by the methods of universal ramsification and conditionalization.[20] Generally speaking, we can understand this approach to reduce mathematical theories (including non-arithmetical ones) to a logical system as a kind of "if-thenist logicism". The position is aptly characterized by Musgrave in terms of two conditions:

(1*) All mathematical statements can be translated into purely logical ones, namely as quantified conditional statements with a conjunction of 'mathematical axioms' as antecedent and a 'mathematical theorem' as consequent.

(2*) All true mathematical statements can be deduced from logical axioms. (Musgrave 1977, pp. 117-118)

Taken for itself, the first condition expresses a kind of "language logicism" in the sense that all "mathematical sentences can be paraphrased in such a way that they contain no non-logical vocabulary" (Rayo 2005). Notice that this form of the logicist reduction can also be expressed in terms of the notion of translation presented in section 2. Thus, condition (1*) suggests a syntactic

[20] Consider another example given in Carnap's *Abriss*, namely Hausdorff's theory of topological spaces. As he shows, any theorem of that theory is thus best translated into purely logical statements of the form $\forall X(hausd(X) \rightarrow \varphi(X))$ (where $hausd(X)$ presents the logical product of the topological neighborhood axioms). Since both concept $hausd$ and the statement φ are universally ramsified here, the conditional statement is a purely logical statement in the language of type theory.

translation τ that maps each theorem φ of a mathematical theory A expressible in a mathematical language L (with a given mathematical signature $\vec{R}$) to purely logical statement of the form:

$$[\varphi(\vec{R})]^{\tau} = \forall\vec{X}(A(\vec{X}) \rightarrow \varphi(\vec{X}))$$

The translation $[\varphi]^{\tau}$ presents the universal ramsification, i.e. the result of uniformly substituting variables of the appropriate type for all non-logical primitives in conditional formula $(A \rightarrow \varphi)$.

Now, conditional logicism is usually considered to be a stronger thesis than language logicism. As pointed out by Musgrave and others, it is accompanied by a second thesis, namely that all mathematical statements so reconstructed are derivable within the logical system. This can be viewed as a form of "consequence logicism" (Rayo (2005)). Thesis (2*) thus states that the if-thenist translation is also theorem-preserving, i.e. that $[\varphi]^{\tau}$ is provable from the logical axioms if φ is deducible from theory A. The translation τ thus forms an *interpretation* of a mathematical theory A in type theory TT in the sense specified in section 2: for every φ such that $A \vdash \varphi$, we have $TT \vdash \varphi^{\tau}$.

Returning to Carnap's work: while a clear exposition of condition (2^{*}) is missing in his published work from the time, one can find indirect textual evidence in related unpublished work that he understood the logicist thesis precisely in this form of conditional logicism. In particular, Carnap's *Untersuchungen* manuscript (Carnap 2000) contains his most systematic discussion of the modern axiomatic method. Carnap presents here a general method of formalizing axiomatic theories as well as several metatheoretical concepts in a logical "basic system" ("*Grunddisziplin*"). What is relevant in the present context is that the manuscript contains also a version of if-thenism in the above sense. More specifically, the notions of theorems ("*Lehrsätze"*) and of "logical consequence" are introduced here in the following way: a sentence *g* is a consequence of an axiom system *f* if the purely logical statement $\forall X(f(X) \rightarrow g(X))$ holds in the type-theoretic basic system. Logical consequence is thus specified here in terms of the material conditional, or more precisely, in terms of a quantified conditional statement expressible in the purely logical language of the basic system. Notice that this logical reconstruction clearly suggests the kind of conditional logicism specified above. Not only are mathematical theorems expressible in purely logical terms. These logical translations should also be valid in the underlying logical system. This latter condition is clearly a version of condition (2*) stated above.

It is a matter of scholarly debate how the primitive notion of truth in the basic (type-theoretic) system was understood by Carnap *anno* 1928. One understanding explicitly mentioned by him in a related paper refers to Wittgenstein's notion of a tautology:

(…) '*consequence*' of f, if f generally implies g: $\forall R(fR \rightarrow gR)$, abbreviated: $f \ \rightarrow \ g$. The consequence is, as is the AS, not a sentence, but a propositional function; only the associated implication $f \rightarrow g$ is a sentence, namely a purely logical sentence, thus a tautology, since no nonlogical constants occur. (Carnap 1930b, p. 304)

As we saw above, it is notoriously unclear how Wittgenstein's notion of a tautological truth can be extended to apply also to statements of logical type theory. A second, more promising approach is to interpret the notion of "holding in type theory" purely syntactically, namely as being derivable from the logical axioms of the logical system. There is again indirect textual evidence for such as reading as well. In particular, after presenting his notion of logical consequence in (Carnap 2000), Carnap goes on to argue that the notion of logical consequence should not be conflated with Hilbert's notion of "derivability in a formal system". More specifically, Carnap holds that while "g follows from f" and "g is derivable from f in the basic system" are not identical, they are equivalent notions (Carnap 2000, p. 92). Now, in light of the incompleteness of higher-order logic, Carnap's argument and the intended equivalence result turned out to be false. However, his discussion is based on a correct version of the deduction theorem for such systems which is also relevant in the present context. In his own terminology, if a proposition g can be formally deduced from the axiom system f, then the statement $\forall X(fX \rightarrow gX)$ is deducible from the principle of type theory, and conversely.

This result allowed Carnap to assume a version of "consequence logicism" that is comparable to the standard logicist thesis that all arithmetical theorems are deducible from purely logical ones. At the same time, this account is certainly weaker than Frege's original logicism. There are two central differences: first, consequence logicism does not require the logicist definitions of the primitive terms of a mathematical theory in a pure higher-order language. As Carnap pointed out in his *Abriss*, the only thing that is explicitly defined by an axiomatic theory is a higher-level property, namely the property (or class) of its models. Second, what is also missing in the account is the requirement that mathematical axioms can be derived from purely logical principles. In contrast, what the present account of conditional logicism effectively shows is that all proofs of theorems can be expressed in a type-theoretic framework. Thus, for any mathematical theory S and every statement φ in the language L_S, the following equivalence can be established:

$$TT \cup \{S\} \vdash \varphi \iff TT \vdash \forall \vec{X}(S(\vec{X}) \rightarrow \varphi(\vec{X}))$$

Thus, for every theorem φ of theory S, the universal ramsification of the conditional statement $(S \rightarrow \varphi)$ can be derived from the logical principles alone.

4. The logicist thesis and semantic interpretability

While classical logicism lost much of its popularity as a foundational approach in the 1930s (mainly as a consequence of Gödel incompleteness results), it remained an important topic in subsequent work by philosophers affiliated with logical empiricism. This is true, in particular, of Carnap's contributions from the late 1930s.[21] His monograph *Foundations of Logic and Mathematics* (1939) contains a detailed discussion of the logicist reduction of mathematics to higher-order logic. The book is particularly interesting since it marks the starting point of Carnap's work on formal semantics, eventually culminating in three *Series in Semantics* volumes published in the course of the 1940s. What is characteristic of Carnap's account in 1939 is that the purely syntactic approach of *Logical Syntax of Language* (1934) is complemented by a semantic analysis of the languages of logic and mathematics (as well as of theoretical languages used in the physical sciences). Specifically, the scope of the metatheoretic study of mathematical languages is extended here from a pure "syntax theory" (as developed in detail in *Logical Syntax*) to a systematic exposition of different semantic systems used for the interpretation of such languages. Consequently, Carnap's central notion of analyticity (or L-truth) is defined now in a semantic way, based on the notion of truth relative to a semantic system.[22]

It is against the background of this new framework that the logicist thesis is addressed again by

[21] See, in particular, Bohnert (1975) for a detailed study of the different forms of logicism defended by Carnap throughout his intellectual career.

[22] Very roughly, a statement of a given semantical system *S* is logically true (or *L*-true) if its truth can be determined solely on the basis of the semantical rules of *S*. See Carnap (1939, §1.7). Compare also Koellner (unpublished) for more detailed discussions of Carnap's book.

Carnap, now in a decidedly semantic form. Sections 16 and 17 of the book deal with so-called "*Non-logical Calculi (Axiom systems)*", i.e. mathematical theories presented in axiomatic form. As Carnap argues, such theories usually consist of two parts, namely a logical calculus and a 'specific' mathematical calculus. The logical calculus presented here is a higher-order logic similar to the system of simple type theory discussed above. An example of an 'elementary' mathematical calculus that Carnap discusses is a version of second-order Peano arithmetic with a full induction (henceforth PA). As he points out, this theory has an intended or "customary" mathematical interpretation, which he describes as follows:

The *customary interpretation* of the Peano system may first be formulated in this way: 'b' designates the cardinal number 0; if '…' designates a cardinal number n, then '…′'designates the next one, i.e., $n+1$; 'N' designates the class of finite cardinal numbers. Hence in this interpretation the system concerns the progression of finite cardinal numbers, ordered according to magnitude. (Carnap 1939, p. 40)

Carnap's central contribution in the section is to show how Peano arithmetic can be reduced to the higher-order logic in a roughly Fregean sense. Interestingly, the logicist reduction is described here as an *interpretability* result (in the technical sense of the term) that is comparable to the kind of type-theoretic logicism outlined in section 2. However, in Carnap's *Foundations*, the notion of interpretability is not merely understood *syntactically*, i.e. in terms of the notion of formal provability, but also *semantically*, in terms of the construction of an interpretation (understood as a semantic system) based on the "translation" of the arithmetical calculus in the higher logical calculus.

The notion of a translation between calculi presented in Carnap (1939) corresponds roughly to the modern definition of the interpretation of a theory into another one specified above (see ibid., p. 40). As we saw, the latter notion is based on a translation function between the formulas of two formal languages that preserves their logical structure as well as the theorems of the interpreted theory. Interestingly, Carnap argues that such a theorem-preserving (in his terminology, a C-true) translation of this form also allows one to construct new 'interpretations' for the calculi in question. Compare his description of this translation-based method of model construction:

If we have an interpretation I_1, for the calculus K_1, then the translation of K_2 into K_1 determines in connection with I_1 an interpretation I_2 for K_2. I_2 may be called a *secondary interpretation.* If the translation is C-true and the (primary) interpretation I_1 is true, I_2 is also true. (ibid, p. 40)[23]

Paraphrased in modern terms, Carnap's idea seems to be roughly this: an interpretation (in the modern sense of the term) of theory T in theory S based on a translation of language L_T in language L_S allows one to construct a model of T based on a given model of S. More specifically, one can say that an L_T-structure M is interpretable in a L_S-structure N in this sense if M is *definable* (in the model-theoretic sense) in N, that is, if the domain, the relations, functions, and distinguished individuals of M are definable in N.[24]

Given this general method of model construction, Carnap shows how it can be applied to the program of reducing the arithmetic to higher-order logic. The logicist reduction is presented here in terms of the construction of a purely logical interpretation of PA based on the translation of the language of PA into a pure higher-order calculus. Compare again Carnap on this approach:

We shall now state rules of translation for the Peano system into the higher functional calculus and thereby give a secondary interpretation for that system. The logical basic calculus is translated into itself; thus we have to state the correlation only for the specific primitive signs. As correlates for 'b', '′', 'N', we take '0', '+', 'finite cardinal number', for any variable, a variable of two levels higher. (ibid., pp. 40-41)

The translation mentioned above is based on the well-known logicist definitions of the primitive arithmetical vocabulary in purely logical terms. These bridge definitions allow one to represent the axioms of arithmetic as purely logical statements. Moreover, as Carnap shows, the same syntactic translation also allows one to construct a purely logical version of the 'customary interpretation' of PA as a subsystem of the standard or "normal interpretation" of the logical calculus. Given this genuinely semantic approach of reinterpreting arithmetic in a 'secondary', purely logical interpretation, Carnap then specifies the logicist thesis in the following way:

If we assume that the normal interpretation of the logical calculus is true, the given secondary interpretation for the Peano system is shown to be true by showing that the correlates of the axioms are C-true. And it can

[23] To say that a translation is C-true means for Carnap that the translation determines an interpretation (in the modern sense of the term) of K_2 in K_1. See ibid, p. 40.

[24] See, e.g., Walsh (2014) for a more detailed discussion of the notion of semantic interpretability.

indeed be shown that the sentences P1-5 are provable in the higher functional calculus, provided suitable rules of transformation are established. As the normal interpretation of the logical calculus is logical and L-true, the given interpretation of the Peano system is also logical and L-true. (ibid, p. 41)

Carnap's line of reasoning can be recast in modern terms as follows: A mathematical theory can be shown to be reducible to logic if it is true in a purely logical model. Given that the higher-order logical calculus (henceforth HOC) is true in the intended logical universe, say V, and that there exists an interpretation of PA into HOC that allows the construction of a logical model M of PA as a model interpretable in V, it follows that PA is reducible to logic. More generally, what is shown here is an interpretability argument of the following form: if a mathematical theory T is interpretable in HOC, then the underlying translation function allows one to construct a model of T *within* the intended universe of HOC. Since this universe is purely logical and theory T is interpretable in HOC, it follows that T also has a purely logical interpretation.

Given Carnap's arithmetical logicism in *Foundations* of 1939, two further points of commentary should be made here. First, his technical presentation of the thesis obviously differs in several respects from the classical thesis of Frege and Russell. In particular, Carnap explicitly describes the logicist reduction of arithmetic to higher-order logic (or type theory) as an interpretability result, based on the notion of a theorem-preserving translation of one calculus into another one. Moreover, as we saw, his account is decidedly semantic in nature: as he points out, the interpretation of one axiomatic theory in another one is complemented by an additional semantic constraint, namely the fact that this interpretation also gives a uniform way to construct a model of the interpreted theory. Thus, according to this particular logicist thesis, a theory like PA is reducible to a logical theory such as HOC if (i) PA is interpretable in HOC and (ii) the standard model of PA is *semantically* interpretable in the logical universe of HOC.[25]

This form of 'interpretational' logicism clearly echos Carnap's general focus on axiomatic mathematics and can thus be traced back to his work on general axiomatics from the 1920s. In fact, one can identify a similar (however less explicit) form of interpretational logicism in his pre-semantical work from the time. For instance, in the discussion on the foundations of mathematics

[25] Compare again Walsh (2014) for a more systematic discussion of arithmetical logicism and different version of interpretability.

at the famous Königsberg meeting in 1930, based on his talk on the logicist foundations of mathematics, Carnap gives the following well-known remark on the "logical analysis of the formalistic system(s)":

(1) For every mathematical sign one or more interpretations are found, and in fact purely logical interpretations.
(2) If the axiom system is consistent, then upon replacing each mathematical sign by its logical interpretation (or one of its various interpretations), every mathematical formula becomes a tautology.
(3) If the axiom system is complete (...), then the interpretation is unique; every sign has exactly one interpretation, and with that the formalist construction is transformed into a logicist one. (Hahn and al. 1931, pp. 143-144)

This characterization of the logicist thesis based on the interpretation of axiomatically defined primitive terms already anticipates Carnap's position in *Foundations* of 1939. The basic idea already expressed in 1930 is that a mathematical theory, presented in axiomatic form, can be described as a branch of logic if one can construct a "purely logical interpretation" of it.

The second point to mention here concerns the scope of Carnap's logicism. While his discussion in *Foundations* is restricted primarily to the case of elementary arithmetic, he makes clear in later sections of the book that all other theories of pure mathematics can also be reduced to higher-order logic. In particular, he explicitly mentions in §18 that different "higher mathematical calculi", e.g. that of real analysis, can be reduced to Peano arithmetic and hence also to the logical calculus in question.[26] In §21, Carnap turns to a detailed discussion of "geometrical calculi and their interpretations". Geometrical theories are usually presented in axiomatic form according to him. While the customary interpretation of such systems is "descriptive" and thus empirical, Carnap points out that also purely logical interpretations can be constructed for them, based on the translation of geometrical terms into terms of real analysis. Given that real analysis can be reduced to arithmetic and thus to logic in the sense outlined above, it follows that a purely "logico-mathematical interpretation" can be given for geometry as well. Compare Carnap on this point:

Of especial importance for the development of geometry in the past few centuries has been a certain

[26] It should be noted here that Carnap's claim that real analysis can also be reduced to type-theoretic logic is contentious and currently under discussion. I would like to thank a reviewer for emphasizing this point.

translation of the geometrical calculus into the mathematical calculus. This leads, in combination with the customary interpretation of the mathematical calculus, to a logical interpretation of the geometrical calculus. The translation was found by Descartes and is known as analytic geometry or geometry of coordinates. 'P_1' (or, in ordinary formulation, 'point') is translated into 'ordered triple of real numbers'; 'P_3' ('plane') into 'class of ordered triples of real numbers fulfilling a linear equation', etc. The axioms, translated in this way, become C-true sentences of the mathematical calculus; hence the translation is C-true. On the basis of the customary interpretation of the mathematical calculus, the axioms and theorems of geometry become L-true propositions. (ibid, pp. 53-54)

Thus, as was originally shown in Hilbert's *Grundlagen der Geometrie* (1899), axiomatic Euclidean geometry can be interpreted in a purely analytic model. Given that analysis can be reduced to logic, it follows that a purely "logico-mathematical interpretation" can be given for geometry as well.

These remarks clearly show that Carnap's approach in *Foundations* is closely connected to his pre-*Syntax* work on logicism and general axiomatics. Specifically, his version of the logicist thesis given in 1939 corresponds closely to classical type-theoretic logicism, complemented by a semantic claim, namely, that the logicist translation of the language of arithmetic into a purely logical language also allows one to construct a purely logical model of PA. Now, one can view the semantic version of his interpretational logicism as Carnap's most systematic attempt to 'reconcile' the traditional logicist thesis with formalism or with the axiomatic approach in mathematics. At the same time, it is also evident that he upheld a more deflationist account of logicism in 1939 (that is also in spirit with his scattered remarks on the topic in his *Logical Syntax* of 1934). In particular, in §20 of the book, he argues that the former controversy between the foundational doctrines logicism and formalism "has at present lost much of its former appearance of importance" (ibid., p. 49). This is mainly due to the fact that both the axiomatic and the logicist approach are compatible with each other and should thus no longer be subject to philosophical dispute.

5. Conclusion

This article showed that Frege's classical logicism was subject to a number of transformations in the work of logical empiricists throughout the 1920s and 1930s. The focus here was on three contributions. Based on a brief account of the development of the classical type-theoretic logicism, we first surveyed how the logicist thesis was understood by Carnap in the course of the 1920s. As we saw, his contributions to the reduction of mathematics to logic were strongly influenced by Wittgenstein's *Tractatus*-view of the tautological nature of logical truths. This new conception of logic, paired with a sustained critique of the existential and thus non-logical character of the type-theoretic axioms of choice, infinity, and reducibility, led Carnap and others to develop a version of 'conditional' logicism, first outlined in Russell's and Whitehead's *Principia Mathematica*.

The second development analyzed in the article concerns the application of conditional logicism to non-arithmetical fields of mathematics. Carnap, in his work on general axiomatics, presented a precise account of a generalized logicist thesis based on two steps: (i) the type-theoretic formalization of axiomatic theories of different branches of mathematics and (ii) an *if-thenist* reconstruction of mathematical theorems. Concerning the latter, he argued that all theorems can be translated into quantified conditional statements where the mathematical primitives are substituted by variables of the correct type. As we saw, Carnap's if-thenism can also be viewed as a form of conditional logicism that aims to reconcile Frege's original thesis with a structuralist account of modern axiomatics.[27]

Finally, we surveyed how logicism was further developed in Carnap's work from the 1930s, that is, after his involvement in the Vienna Circle. Our focus here was on his re-adoption of classical logicism in *Foundations of Logic and Mathematics*, the first book belonging to his post-syntactic or semantic period. As we saw, Carnap explicitly formulated the reduction of arithmetic to higher-order logic in terms of an interpretability result in this book. Classical arithmetical logicism is usually expressed in terms of the syntactic interpretability of arithmetic in higher-order logic. However, as Carnap first showed in 1939, it can also be recast as a genuinely semantic result: arithmetic is reducible to logic if the standard model of the natural number can be constructed within the type theoretic universe.

6. Acknowledgements

[27] It would be interesting to further investigate the relation between Carnap's if-thenism and Hilbert's preceding views on logicism and axiomatics from the 1890s. See Ferreirós (2009) on Hilbert's early logicism.

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