ENEE626: Error-Correcting Codes

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Lecture 21 (11/19/05): Error probability of decoding for a linear code with known weight distribution

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Error probability of decoding

Let $C[n, k, d]$ be a binary linear code which can correct up to $t = \lfloor \frac{d-1}{2} \rfloor$ errors.
Let $\mathbf{c}$ be the transmitted vector. Recall from Theorem 3.5 that the set of correctable errors for a linear code does not depend on the vector being transmitted. Therefore, without loss of generality we will assume that $\mathbf{c} = 0$.

Error Events for a $BSC(p)$ with received vector $\mathbf{r} = \mathbf{c} + \mathbf{e}$:

1. If $\mathrm{wt}(\mathbf{e}) \le t$, then $\mathbf{r}$ is always decoded correctly. The probability $\Pr[\mathrm{wt}(\mathbf{e}) \le t]$ equals

$$P_{\text{corr}}(C) = \sum_{i=0}^{t} \binom{n}{i} p^i (1-p)^{n-i}$$

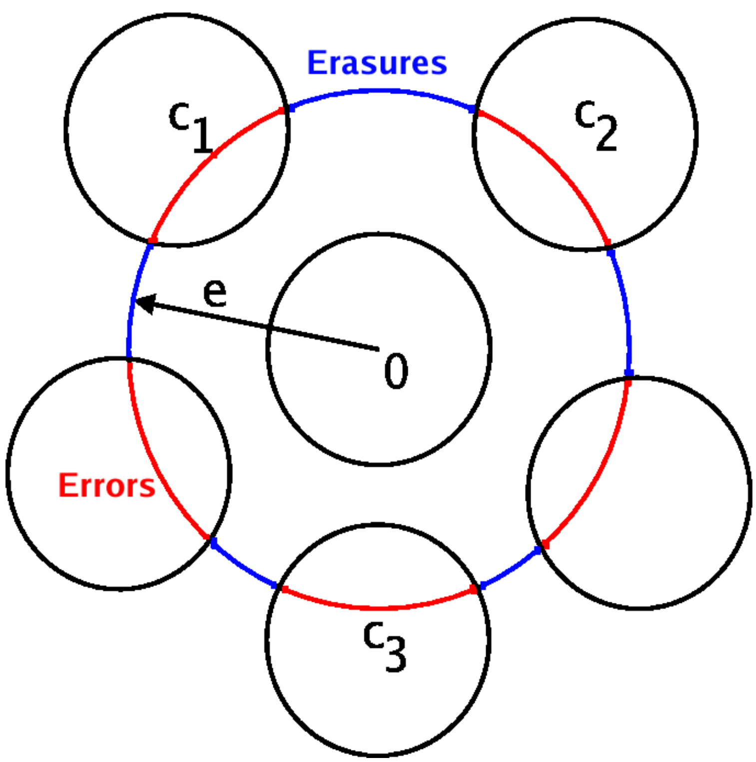

Figure 1: $P(\mathrm{wt}(\mathbf{r}) = e | \mathbf{c} = \mathbf{0}) = p^e(1-p)^{n-e}$ on a $BSC(p)$

2. If $\mathrm{wt}(e) > t$ then two situations are possible, as shown in Figure 2.

   (a) $\nexists \mathbf{c} \in C$ with $d(\mathbf{c}, \mathbf{r}) \le t$. In this case there is no decoding result – this is an erasure.

   (b) $\exists \mathbf{c} \in C$ with $d(\mathbf{c}, \mathbf{r}) \le t$. $\mathbf{c}$ is the decoding output – this is a decoding error/miscorrection.

Letting $P_e(C)$ be the probability that the decoding result is a miscorrection and $P_x(C)$ be the probability that the decoding result is an erasure, we see that:

$$P_e(C) + P_x(C) = 1 - P_{\text{corr}}(C) = \sum_{i=t+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Since erasures are not as serious as miscorrections in some cases, we would like to separate the erasure and miscorrection probabilities.

Let $\{A_0 = 1, A_d, A_{d+1}, ..., A_n\}$ be the weight distribution of C, where $A_i = |\{\mathbf{c} \in C, \text{wt}(\mathbf{c}) = i\}|$.

We would like to find $P_e(C)$, given by

$$P_e(C) = \sum_{w=d}^{n} \sum_{\substack{\mathbf{c} \in C \\ \text{wt}(\mathbf{c})=w}} \sum_{e=w-t}^{w+t} \mathbf{P}(S_e(\mathbf{0}) \cap B_t(\mathbf{c}))$$

where $S_e(0)$ is the sphere of radius t around $\mathbf{0}$ and $B_t(\mathbf{c})$ is the ball of radius t around a codevector $\mathbf{c}$ such that $\text{wt}(\mathbf{c}) = w$.

Computing $|S_e(\mathbf{0}) \cap B_t(\mathbf{c})|$ can be visualized as follows:

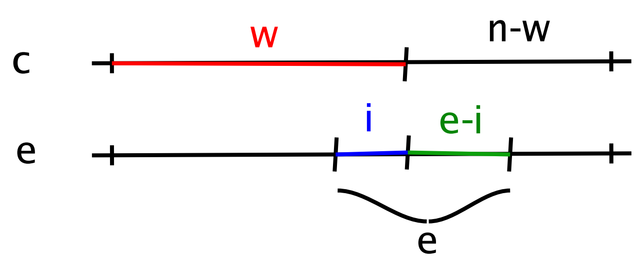

Figure 2: Evaluating $d(\mathbf{c}, \mathbf{e})$

Then we have that $|S_e(\mathbf{0}) \cap B_t(\mathbf{c})| = |\{\mathbf{e} : d(\mathbf{e}, \mathbf{c}) \leq t\}| = \binom{w}{i}\binom{n-w}{\mathbf{e}-i}$.
From the figure we see $t \geq w - i + e - i = w + e - 2i$, so $i \geq \lceil \frac{w+e-t}{2} \rceil$

This gives us the following theorem for computing the probability of miscorrection for the code C:
Theorem:

$$P_e(C) = \sum_{w=d}^{n} A_w \sum_{e=w-t}^{w+t} p^e (1-p)^{n-e} \sum_{i=\lceil \frac{w+e-t}{2} \rceil}^{w} \binom{w}{i} \binom{n-w}{e-i}$$