# Algebraic semantics for propositional superposition logic

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**Abstract**

We provide a new semantics and a slightly different formalization for the propositional logic with superposition (PLS) introduced and studied in [6]. PLS results from Propositional Logic (PL) by adding a new binary connective | construed as the "superposition operation" and a few axioms about it. The original semantics used in the above paper was the so called sentence choice semantics (SCS), based on choice functions for all pairs of classical formulas of PL. In contrast, the algebraic or Boolean-value choice semantics (BCS) developed in this paper is based on choice functions for pairs of elements of a Boolean algebra $\mathcal{B}$ in which the classical sentences take truth values. The Boolean-value choice functions can be subject to similar constraints as those imposed on sentence choice functions. The new axiomatization is based on the same set of axioms as the previous one and a new inference rule, called Rule of Analogy ($RA$), in place of the rule Salva Veritate ($SV$) of the previous systems. The Deduction Theorem fails in the systems containing the new rule. As a consequence the completeness theorems for them hold conditionally again, namely the systems are complete with respect to BCS if and only if every consistent set of sentences is extended to a consistent and complete set. Finally connections are established between tautologies with respect to the semantics SCS and BCS.

*Mathematics Subject Classification (2010)*: 03B60, 03G12

*Keywords:* Propositional superposition logic. Sentence choice semantics. Boolean-value choice semantics. Choice function for pairs of

elements of a Boolean algebra. Associative and $-$-decreasing choice functions/total orderings over a Boolean algebra.

# 1 Introduction

This paper is a sequel to [6] and its aim is to offer a new semantics for the systems of propositional superposition logics (PLS) introduced in [6], together with new completeness theorems. For the reader's convenience we shall give in this introduction a brief account of the systems PLS and the main facts established in [6].

Before coming to technical notions and facts let us first explain the motivating idea behind the introduction and investigation of PLS. That was the attempt to grasp the *purely logical content* of the phenomenon of superposition as the latter presents itself in quantum systems. The question was roughly this: if $|$ is a new binary logical connective and $\varphi|\psi$ denotes the "superposition of two states" (or, more precisely, the propositions expressing these states), what can we say about the *truth* of $\varphi|\psi$ without leaving the ground of classical logic? The basic intuition is that $\varphi|\psi$ expresses a "strange conjunction" of properties before the measurement, but also a "strange disjunction" after the measurement, i.e., after the "collapsing" of the superposed states. This collapsing can be formalized by the help of a choice function that acts on pairs of sentences $\{\varphi, \psi\}$, turning each formula $\varphi|\psi$ into a classical one. Such functions formed the basis of a semantics for the new logic that allows $\varphi|\psi$ to present simultaneously conjunctive and disjunctive characteristics, which are nicely exemplified in the "interpolation property", i.e., that $\varphi|\psi$ is strictly logically interpolated between $\varphi \wedge \psi$ and $\varphi \vee \psi$ (see also Fact 2.3 below).

In general a Propositional Superposition Logic (PLS) will consist, roughly, of a pair $(X, K)$, where $X$ is its *semantical* part and $K$ is its *syntactic* part. Actually $K$ is a *formal system* in the usual sense of the word, and $X$ is a set of functions that provides meaning to sentences in a way described below. The notation PLS$(X, K)$ will denote the propositional superposition logic with semantical part $X$ and syntactic part $K$.

## 1.1 Overview of PLS with sentence choice semantics

In this subsection we give an overview of the main notions and facts contained in [6]. Although the semantical part is the most intuitively appealing we start with the description of the syntactic part $K$. The language of $K$ (or the language of PLS), $L_s$, is that of standard Propositional Logic (PL) $L = \{p_0, p_1, \ldots\} \cup \{\wedge, \vee, \rightarrow, \leftrightarrow, \neg\}$ augmented with the new binary connective "|". That is, $L_s = L \cup \{|\}$. The set of sentences of $L_s$, $Sen(L_s)$, is defined by induction as usual, with the additional inductive step that $\varphi|\psi$ is a sentence whenever $\varphi$ and $\psi$ are so.

Throughout the letters $\alpha$, $\beta$, $\gamma$ range exclusively over the set of sentences of $L$, $Sen(L)$, while $\varphi$, $\psi$, $\sigma$ range over elements of $Sen(L_s)$ in general.

A formal system $K$ consists of a set of axioms $\mathsf{Ax}(K)$ and a set of inference rules $\mathsf{IR}(K)$. The axioms of $K$ always include the axioms of PL, while $\mathsf{IR}(K)$ includes the inference rule of PL. So let us first fix the axiomatization for PL consisting of the following axiom schemes (for the language $L_s$).

(P1) $\varphi \to (\psi \to \varphi)$
(P2) $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))$
(P3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$,

together with the inference rule Modus Ponens ($MP$). So for every $K$, $\{\text{P1}, \text{P2}, \text{P3}\} \subset \mathsf{Ax}(K)$ and $MP \in \mathsf{IR}(K)$. In addition each $K$ contains axioms for the new connective |. These are some or all of the following schemes.

$(S_1)$ $\varphi \wedge \psi \to \varphi|\psi$
$(S_2)$ $\varphi|\psi \to \varphi \vee \psi$
$(S_3)$ $\varphi|\psi \to \psi|\varphi$
$(S_4)$ $(\varphi|\psi)|\sigma \to \varphi|(\psi|\sigma)$
$(S_5)$ $\varphi \wedge \neg\psi \to (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi)$

Provability (à la Hilbert) in $K$, denoted $\vdash_K \varphi$, is defined as usual. It is clear that

$$\Sigma \vdash \alpha \ \Leftrightarrow\ \Sigma \vdash_K \alpha,$$

where $\vdash$ denotes provability in PL. $\Sigma$ is said to be $K$*-consistent,* if $\Sigma \nvdash_K \bot$.

Let $K_0$ denote the formal system described as follows.

$$\mathsf{Ax}(K_0) = \{\text{P1}, \text{P2}, \text{P3}\} + \{S_1, S_2, S_3\}, \qquad \mathsf{IR}(K_0) = \{MP\}.$$

Extensions of $K_0$ defined below will contain also the rule $SV$ (from *salva veritate*) defined as follows.

$$(SV) \qquad \textit{from}\ \ \varphi \leftrightarrow \psi\ \ \textit{infer}\ \varphi|\sigma \leftrightarrow \psi|\sigma,$$

if $\varphi \leftrightarrow \psi$ is provable in $K_0$.

The rule $SV$ guarantees that if $\alpha$, $\beta$ are classical logically equivalent sentences, then truth is preserved if $\alpha$ is substituted for $\beta$ in expressions containing $|$ (just as in the case with the standard connectives). Let the formal systems $K_1$, $K_2$ and $K_3$ be defined as follows.

$$\mathsf{Ax}(K_1) = \mathsf{Ax}(K_0) = \mathsf{Ax}(\mathrm{PL}) + \{S_1, S_2, S_3\}, \qquad \mathsf{IR}(K_1) = \{MP, SV\},$$

$$\mathsf{Ax}(K_2) = \mathsf{Ax}(K_1) + S_4, \qquad \mathsf{IR}(K_2) = \{MP, SV\},$$

$$\mathsf{Ax}(K_3) = \mathsf{Ax}(K_2) + S_5, \qquad \mathsf{IR}(K_3) = \{MP, SV\}.$$

A consequence of $SV$ is that if $\vdash_{K_0} (\varphi \leftrightarrow \psi)$ then, for any $\sigma$, $\vdash_{K_i} (\varphi|\sigma \leftrightarrow \psi|\sigma)$, for $i = 1, 2, 3$.

So much for the syntax of PLS. We now turn to the semantics. The axioms $S_i$ were motivated by the intended meaning of $|$, that was briefly described in the introduction, and the corresponding semantics for sentences of $L_s$ based on choice functions. This semantics consists of pairs $\langle v, f\rangle$, where $v : Sen(L) \to \{0,1\}$ is a usual two-valued assignment of the sentences of $L$, and $f$ is a choice function for pairs of elements of $Sen(L)$, i.e., $f : [Sen(L)]^2 \to Sen(L)$ such that $f(\{\alpha, \beta\}) \in \{\alpha, \beta\}$. $f$ is defined also for singletons with $f(\{\alpha\}) = \alpha$. We simplify notation by writing $f(\alpha, \beta)$ instead of $f(\{\alpha, \beta\})$, thus by convention $f(\alpha, \beta) = f(\beta, \alpha)$ and $f(\alpha, \alpha) = \alpha$. $f$ gives rise to a function $\overline{f} : Sen(L_s) \to Sen(L)$, defined inductively as follows.

(i) $\overline{f}(\alpha) = \alpha$, for $\alpha \in Sen(L)$,
(ii) $\overline{f}(\varphi \wedge \psi) = \overline{f}(\varphi) \wedge \overline{f}(\psi)$,
(iii) $\overline{f}(\neg\varphi) = \neg\overline{f}(\varphi)$,
(iv) $\overline{f}(\varphi|\psi) = f(\overline{f}(\varphi), \overline{f}(\psi))$.

We refer to $\overline{f}$ as the *collapsing function* induced by $f$. Then we define the truth of $\varphi$ in $\langle v, f\rangle$, denoted $\langle v, f\rangle \models_s \varphi$, as follows.

$$\langle v, f\rangle \models_s \varphi :\Leftrightarrow v(\overline{f}(\varphi)) = 1. \tag{1}$$

(In [6] we denote by $M$ the two-valued assignments of sentences of $L$ and write $\langle M, f\rangle$ instead of $\langle v, f\rangle$. Also we write $M \models \alpha$ instead $M(\alpha) = 1$.)

We shall refer to the semantics defined by (1) as *sentence choice semantics,* or SCS for short. A remarkably similar notion of choice function for pairs of sentences, and its interpretation as a "conservative" binary connective, was given also independently in [3] (see Example 3.24.14, p. 479).

The reason that we used four formal systems $K_0$-$K_3$, in increasing strength, is that they correspond to four different classes of choice functions defined below.

**Definition 1.1** Let $\mathcal{F}$ denote the set of all choice functions for $Sen(L)$ and let $X \subseteq \mathcal{F}$.

(i) For a set $\Sigma \subseteq Sen(L_s)$ and $X \subseteq \mathcal{F}$, $\Sigma$ is said to be *$X$-satisfiable* if there are $v$ and $f \in X$ such that $\langle v, f\rangle \models_s \Sigma$.

(ii) For $\Sigma \subseteq Sen(L_s)$ and $\varphi \in Sen(L_s)$, $\varphi$ is an *$X$-logical consequence of* $\Sigma$, denoted $\Sigma \models_X \varphi$, if for every $v$ and every $f \in X$, $\langle v, f\rangle \models_s \Sigma \Rightarrow \langle v, f\rangle \models_s \varphi$.

(iii) $\varphi$ is an *$X$-tautology,* denoted $\models_X \varphi$, if $\emptyset \models_X \varphi$.

iv) $\varphi$ and $\psi$ are *$X$-logically equivalent,* denoted $\varphi \sim_X \psi$, if $\models_X (\varphi \leftrightarrow \psi)$. Also let

$$Taut(X) = \{\varphi \in Sen(L_s) : \models_X \varphi\}.$$

Now while the axioms of $K_0$ are easily seen to be $\models_{\mathcal{F}}$-tautologies, this is not the case with the axioms $S_4$ and $S_5$. They correspond to some special subclasses of $\mathcal{F}$ described below.

**Definition 1.2** 1) A choice function is said to be *associative* if for all $\alpha, \beta, \gamma \in Sen(L)$

$$f(f(\alpha, \beta), \gamma) = f(\alpha, f(\beta, \gamma)).$$

2) An $f \in \mathcal{F}$ is said to be *regular* if for all $\alpha, \alpha', \beta \in Sen(L)$,

$$\alpha \sim \alpha' \Rightarrow f(\alpha, \beta) \sim f(\alpha', \beta)$$

where $\alpha \sim \beta$ denotes logical equivalence in PL.

Let

$$Asso = \{f \in \mathcal{F} : f \text{ is asociative}\},$$

$$Reg = \{f \in \mathcal{F} : f \text{ is regular}\},$$

Both properties of associativity and regularity are strongly desirable and would be combined. This is because (a) $f \in Asso$ iff for every $v$, $\langle v, f\rangle$ satisfies associativity for $|$, i.e., $\langle v, f\rangle_s \models S_4$ (see [6, Th. 2.19]), and (b) for every $X$, $X \subseteq Reg$ iff the relation $\sim_X$ is logically closed, i.e., $\varphi \sim_X \varphi[\sigma'/\sigma]$ if $\sigma' \sim_X \sigma$, when $\sigma$ is a subformula of $\varphi$ and $\varphi[\sigma'/\sigma]$ is the result of substitution of $\sigma'$ for $\sigma$ in $\varphi$ (see [6, Th. 2.28]).

We have the following simple and nice characterization of the functions in *Asso*.

**Lemma 1.3** *(*[6, Corollary 2.18]*) A choice function $f$ is associative, i.e., $f \in Asso$, if and only if there is a total $<$ ordering of $Sen(L)$ such that $f = \min_<$, i.e., $f(\alpha, \beta) = \min(\alpha, \beta)$ for all $\alpha, \beta \in Sen(L)$.*

(Actually 1.3 holds for associative choice functions on an arbitrary set $A$, see Theorem 2.14 of [6].) In view of the above characterization of associative functions through total orderings, the following definition is natural.

**Definition 1.4** A total ordering $<$ of $Sen(L)$ is *regular* if the corresponding choice function $f = \min_<$ is regular or, equivalently, if for all $\alpha$, $\beta$ in $Sen(L)$

$$\alpha \not\sim \beta \;\&\; \alpha < \beta \;\Rightarrow\; [\alpha] < [\beta],$$

where $[\alpha]$ is the $\sim$-equivalence class of $\alpha$.

Let

$$Reg^* = Reg \cap Asso.$$

Clearly $f \in Reg^*$ iff $f = \min_<$ for a regular total ordering $<$ of $Sen(L)$.

**Definition 1.5** Let $<$ be a total ordering of $Sen(L)$. $<$ is said to be $\neg$-*decreasing* if for all $\alpha, \beta \in Sen(L)$ such that $\alpha \not\sim \beta$,

$$\alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha.$$

If $f \in Reg^*$, $f$ is said to be $\neg$-*decreasing* if $f = \min_<$ for some $\neg$-decreasing $<$.

Let

$$Dec = \{f \in Reg^* : f \text{ is } \neg\text{-decreasing}\}.$$

Since $Dec \subseteq Reg^* \subseteq Reg \subseteq \mathcal{F}$, it follows that

$$Taut(\mathcal{F}) \subseteq Taut(Reg) \subseteq Taut(Reg^*) \subseteq Taut(Dec).$$

**Definition 1.6** Given a set $X \subseteq \mathcal{F}$, and a set $\mathsf{Ax}(K) \subseteq Taut(X)$, PLS$(X, K)$ is the logic w.r.t. to $X$ and $K$, where $K$ is the syntactic part, while $\models_X$ is the consequence relation determined by the structures $\langle v, f\rangle$, with $f \in X$.

Given a logic PLS$(X, K)$, the soundness and completeness theorems for it refer as usual to the connections between the relations $\models_X$ and $\vdash_K$, or between $X$-satisfiability and $K$-consistency.

At this point a word of caution is needed. As is well-known the soundness theorem (ST) and completeness theorem (CT) of a logic have two distinct formulations which are equivalent for classical logic, but need not be so in general. For the logic PLS$(X, K)$ these two forms, ST1 and ST2 for Soundness and CT1 and CT2 for Completeness, are the following.

(ST1) $$\Sigma \vdash_K \varphi \;\Rightarrow\; \Sigma \models_X \varphi,$$

(ST2) $$\Sigma \text{ is } X\text{-satisfiable} \;\Rightarrow\; \Sigma \text{ is } K\text{-consistent}$$

(CT1) $$\Sigma \models_X \varphi \;\Rightarrow\; \Sigma \vdash_K \varphi,$$

(CT2) $$\Sigma \text{ is } K\text{-consistent} \;\Rightarrow\; \Sigma \text{ is } X\text{-satisfiable}.$$

ST1 and ST2 are easily shown to be equivalent for every system PLS$(X, K)$. Moreover the Soundness Theorem for each one of the logics PLS$(\mathcal{F}, K_0)$, PLS$(Reg, K_1)$, PLS$(Reg^*, K_2)$ and PLS$(Dec, K_3)$ is easily established. But the equivalence of CT1 and CT2 is based on the *Deduction Theorem* (DT) which is not known to be true for every PLS$(X, K)$, when $K$ contains the inference rule $SV$. Recall that DT is the following implication. For all $\Sigma$, $\varphi$, $\psi$,

$$\Sigma \cup \{\varphi\} \vdash_K \psi \;\Rightarrow\; \Sigma \vdash_K \varphi \rightarrow \psi. \tag{2}$$

Concerning the relationship between CT1 and CT2 for PLS$(X, K)$ the following holds.

**Fact 1.7** CT1 $\Rightarrow$ CT2 *holds for every* PLS$(X, K)$. *If* $\vdash_K$ *satisfies DT, then the converse holds too, i.e.,* CT1 $\Leftrightarrow$ CT2.

The system PLS$(\mathcal{F}, K_0)$, whose only inference rule is $MP$, satisfies CT1 $\Leftrightarrow$ CT2 as a consequence of DT. So we can just say it is "complete" instead of "CT1-complete" and "CT2-complete". The following is shown in [6, §3.1].

**Theorem 1.8** PLS$(\mathcal{F}, K_0)$ *is complete.*

However in the systems over $K_i$, for $i > 0$, that contain the extra rule $SV$, the status of DT is open, so the distinction between CT1 and CT2 remains. So concerning the logics PLS$(Reg, K_1)$, PLS$(Reg^*, K_2)$ and PLS$(Dec, K_3)$ it is reasonable to try to prove the weaker of the two forms of completeness, namely CT2-completeness. But even this will be proved only conditionally. Because there is still another serious impact of the lack of DT. This is that we don't know if every consistent set of sentences can be extended to a consistent and *complete* set (i.e., one that contains one of the $\varphi$ and $\neg\varphi$, for every $\varphi$). Of course every consistent set $\Sigma$ can be extended (e.g. by Zorn's Lemma) to a *maximal* consistent set $\Sigma' \supseteq \Sigma$. But maximality of $\Sigma'$ does not guarantee completeness without DT. Because $\Sigma'$ may be maximal consistent and yet there is a $\varphi$ such that $\varphi \notin \Sigma'$ and $\neg\varphi \notin \Sigma'$, so $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both inconsistent. This property of extendibility of a consistent set to a consistent and complete one, for a formal system $K$, is crucial for the proof of completeness of $K$ (with respect to a given semantics), so we isolate it as a property of $K$ denoted $cext(K)$. It reads as follows.

($cext(K)$) *Every $K$-consistent set of sentences can be extended to a $K$-consistent and complete set.*

Then the following conditional CT2-completeness results are shown in [6, §3.2]).

**Theorem 1.9** *(i)* PLS$(Reg, K_1)$ *is* CT2*-complete if and only if* $cext(K_1)$ *is true.*

*(ii)* PLS$(Reg^*, K_2)$ *is* CT2*-complete if and only if* $cext(K_2)$ *is true.*

*(iii)* PLS$(Dec, K_3)$ *is* CT2*-complete if and only if* $cext(K_3)$ *is true.*

### 1.2 Summary of contents

In section 2 we introduce the Boolean-value choice semantics (BCS) with respect to a Boolean algebra $\mathcal{B}$ and the class $\mathcal{F}(\mathcal{B})$ of choice functions on $[B]^2$. The basic truth relation is $\langle \mathcal{B}, v, \pi\rangle \models_a \varphi$, where $v : Sen(L) \to \mathcal{B}$ is a homomorphism and $\pi \in \mathcal{F}(\mathcal{B})$. Further, given $X \subseteq \mathcal{F}(\mathcal{B})$, the logical consequence relation $\Sigma \models_X \varphi$, and the induced notion of $X$-tautology, denoted $\models_X \varphi$, are defined when $\pi$ are restricted to $X$. The main classes $X$ of choice functions considered, besides $\mathcal{F}(\mathcal{B})$, are the classes of associative functions $Asso(\mathcal{B})$ and of $-$-decreasing functions $Dec(\mathcal{B})$. In section 2.1 we show the existence of $-$-decreasing functions on every algebra $\mathcal{B}$. In section 2.2 we show some peculiarities of BCS that distinguish it from SCS. In section 2.3 we examine in particular BCS with respect to the two-element algebra **2** and some special properties of this semantics.

In section 3 we axiomatize the relations $\models_{\mathcal{F}(\mathcal{B})}$, $\models_{Asso(\mathcal{B})}$ and $\models_{Dec(\mathcal{B})}$ by the formal systems $K_0^a$, $K_1^a$ and $K_2^a$, respectively, consisted of certain axioms and inference rules. The axioms are roughly the same as those of the formal systems $K_0$-$K_3$ of [6]. However a different inference rule, $RA$, is used in place of the rule $SV$ of [6]. The presence of the extra rule makes the Deduction Theorem (provably) fail, and this again necessitates referring to two forms of the completeness theorem, as well as the use of the extendibility condition $cext(K)$ for a formal system $K$ mentioned in section 1.1., already used in [6]. Section 3.1 contains the main technical result of this section, and perhaps of the paper, Theorem 3.8, concerning the conditional CT2-completeness of all three systems with respect to the algebra **2**.

In section 4 we establish connections between tautologies with respect to the semantics BCS and SCS. Specifically the connections concern tautologies based on classes of choice functions in the sense of SCS on the one hand, and tautologies based on the corresponding classes of choice functions in the sense of BCS, especially those over the Lindenbaum Boolean algebra $\mathcal{L}$ for the set of sentences of $L$.

## 2 Superposition as a binary "modality" and its algebraic semantics

Given a Boolean algebra $\mathcal{B} = \langle B, +, \cdot, -, 0, 1\rangle$, a $\mathcal{B}$*-valuation* of (or $\mathcal{B}$-*assignment* to) the sentences of the language $L$ of PL is a mapping

$v : Sen(L) \to \mathcal{B}$ such that:

$v(\alpha \wedge \beta) = v(\alpha) \cdot v(\beta)$,

$v(\alpha \vee \beta) = v(\alpha) + v(\beta)$,

$v(\neg\alpha) = -v(\alpha)$.

It is well-known that $\alpha$ is a tautology of PL if and only if for every $\mathcal{B}$ and every $\mathcal{B}$-valuation $v : Sen(L) \to \mathcal{B}$, $v(\alpha) = 1$.

Replacing **2** with a general Boolean algebra $\mathcal{B}$ as the set of truth values for the sentences of $L$, the semantics for PLS defined in (1) of the previous section generalizes in the obvious way, namely we interpret sentences of $L_s$ in triples $\langle \mathcal{B}, f, v\rangle$, where $f$ is again a choice function for pairs of $Sen(L)$, $v$ is a $\mathcal{B}$-valuation and

$$\langle \mathcal{B}, v, f\rangle \models_s \varphi :\Leftrightarrow v(\overline{f}(\varphi)) = 1. \tag{3}$$

Although (3) itself does not make a big difference compared to (1), the shift from **2** to $\mathcal{B}$ prompts us to see and treat $|$ as a binary "modality" and interpret it algebraically by a Boolean algebra expansion (BAE), namely an expansion $\langle \mathcal{B}, \pi\rangle$ of $\mathcal{B}$, where $\pi$ is a choice function for pairs of elements of $B$, i.e.,

$$\pi : [B]^2 \to B$$

such that $\pi(\{a, b\}) \in \{a, b\}$. However we must be careful with the terms "modality" and "Boolean expansion", since these terms possess a specific meaning related to their historic origin.

On the one hand, Boolean expansions originated with the work of Jónsson and Tarski [4]. By definition such expansions are pairs $\langle \mathcal{B}, m\rangle$, where $m : B^n \to B$ is an $n$-ary "additive operator", i.e., preserves $+$ and 0 in each of its arguments. For a binary $m$ in particular, $m(x, y+z) = m(x, y) + m(x, z)$, and similarly with the other argument, as well as $m(0, y) = m(x, 0) = 0$. For example in the algebra $\mathcal{P}(X^2)$ of all binary relations on a set $X$, the composition operator $R \circ S$ and the inverse operator $R^{-1}$ are additive. Such functions are called today Boolean algebra operators (or BAO for short, see [1] for a survey of this topic). However a choice function $\pi$ on $[B]^2$ need not satisfy additivity. One easily give examples of choice functions on $B$ such that $\pi(x, y + z) \neq \pi(x, y) + \pi(x, z)$, and also $\pi(x, 0) \neq 0$. Thus choice functions are not BAO's.

On the other hand, the connective $|$ cannot be viewed as a modality in the standard sense of the word either. The reason is not that $|$ is binary, whereas the standard modalities $\Diamond$ and $\Box$ are unary. In fact there are exist also polyadic, i.e., $n$-ary, modalities. What makes an $n$-connective $\nabla$ to be a polyadic modality is adequately explained in [2, p. 420] as follows:

> "What *are* polyadic modal operators? Syntactically, an $n$-ary modal operator is just an $n$-ary connective; what makes

it modal is its intended interpretation, which uses accessibility relations of arity $n+1$. Generalizing the definition of ordinary modal logic, the truth condition for an $n$-ary operator $\nabla$ reads as follows:

$$\mathcal{M}, w \Vdash \nabla(\varphi_1, \ldots \varphi_n) \text{ iff there are } v_1, \ldots, v_n \text{ such that}$$

$$R_\nabla(w, v_1, \ldots, v_n) \text{ and } \mathcal{M}, v_j \Vdash \varphi_j \text{ for each } j,$$

where $R_\nabla$ is the $n+1$-ary relation associated with $\nabla$."

Does the superposition connective $|$ meet the above criterion in order to be characterized as a modality? If the answer were Yes, there should exist a ternary accessibility relation $R$ on a set of nodes such that $\mathcal{M}, w \Vdash \varphi|\psi$ iff there are $v_1, v_2$ such that $R(w, v_1, v_2)$ and $\mathcal{M}, v_1 \Vdash \varphi$ and $\mathcal{M}, v_2 \Vdash \psi$. But one cannot see how such a condition (or its dual) might be related to the intuitive meaning of $|$.

After these explanations, we can still consider pairs $\langle \mathcal{B}, \pi \rangle$ as BAE's, but of a very special, rather sui generis, nature. Let us stress the fact that the letter $\pi$ will denote throughout choice functions for pairs of elements of a Boolean algebra, since $f$ has already been used to denote choice functions for pairs of sentences. The notational distinction will be necessary in the last section where the two semantics are compared. As usual we write $\pi(a, b)$ instead of $\pi(\{a, b\})$, with the proviso that $\pi(a, b) = \pi(b, a)$ and $\pi(a, a) = a$. Given $\mathcal{B}$ and $\pi$ as above, and a $\mathcal{B}$-valuation $v$ of the sentences of $L$ of PL, we shall define the truth of sentences of the language of PLS, $L_s$, in the triple $\langle \mathcal{B}, \pi, v \rangle$ in the more or less standard algebraic way. Specifically, first we extend $v$ by the help of $\pi$ to a $\mathcal{B}$-valuation of the sentences of $L_s$

$$\overline{v}_\pi : Sen(L_s) \to \mathcal{B}$$

defined by the following recursion:

(a) $\overline{v}_\pi(\alpha) = v(\alpha)$, for $\alpha \in Sen(L)$.
(b) $\overline{v}_\pi(\varphi \wedge \psi) = \overline{v}_\pi(\varphi) \cdot \overline{v}_\pi(\psi)$.
(c) $\overline{v}_\pi(\varphi \vee \psi) = \overline{v}_\pi(\varphi) + \overline{v}_\pi(\psi)$.
(d) $\overline{v}_\pi(\neg\varphi) = -\overline{v}_\pi(\varphi)$.
(e) $\overline{v}_\pi(\varphi|\psi) = \pi(\overline{v}_\pi(\varphi), \overline{v}_\pi(\psi))$.

Then the *algebraic truth relation* $\langle \mathcal{B}, \pi, v \rangle \models_a \varphi$, for a $\mathcal{B}$, $f$ and a $v$ as above, is defined as follows.

$$\langle \mathcal{B}, \pi, v \rangle \models_a \varphi :\Leftrightarrow \overline{v}_\pi(\varphi) = 1. \tag{4}$$

More generally, for every $\Sigma \subset Sen(L_s)$, we define

$$\langle \mathcal{B}, \pi, v \rangle \models_a \Sigma :\Leftrightarrow \overline{v}_\pi(\varphi) = 1, \text{ for every } \varphi \in \Sigma. \tag{5}$$

Sometimes we shall write simply $\overline{v}$ instead of $\overline{v}_\pi$ if $\pi$ is understood and there is no danger of confusion.

Sometimes we refer to the semantics defined by (4) as *Boolean-value choice semantics,* or BCS for short, in contradistinction to sentence-choice semantics (SCS) described in section 1.1.

For any Boolean algebra $\mathcal{B} = \langle B, \ldots \rangle$ let us set

$$\mathcal{F}(\mathcal{B}) = \{\pi : \pi \text{ is a choice function for } [B]^2\}.$$

In analogy to definitions 1.1 of the previous section we have the following ones.

**Definition 2.1** Let $\mathcal{B}$ be a Boolean algebra and let $X \subseteq \mathcal{F}(\mathcal{B})$. Then the following hold.

(i) $\varphi$ is $X$*-satisfiable* if there are $v : Sen(L) \to \mathcal{B}$ and $\pi \in X$ such that $\langle \mathcal{B}, \pi, v \rangle \models_a \varphi$.

(ii) For $\Sigma \subseteq Sen(L_s)$ and $\varphi \in Sen(L_s)$, $\varphi$ is an $X$*-logical consequence of* $\Sigma$, denoted $\Sigma \models_X \varphi$, if for every $\mathcal{B}$-valuation $v : Sen(L) \to \mathcal{B}$ and every $\pi \in X$, $\langle \mathcal{B}, \pi, v \rangle \models_a \Sigma \Rightarrow \langle \mathcal{B}, v, \pi \rangle \models_a \varphi$.

(iii) $\varphi$ is a $X$*-tautology,* denoted $\models_X \varphi$, if $\emptyset \models_X \varphi$. Also let

$$Taut(X) = \{\varphi \in Sen(L_s) : \models_X \varphi\}.$$

Clearly all schemes of tautologies of PL are also $\mathcal{F}(\mathcal{B})$-tautologies for the sentences of $L_s$, for every structure $\langle \mathcal{B}, \pi \rangle$. More generally the following holds.

**Fact 2.2** *Let $\alpha(p_1, \ldots, p_n)$ be a sentence of $L$, made of the atomic sentences $p_1, \ldots, p_n$, let $\psi_1, \ldots, \psi_n$ be any sentences of $L_s$ and let $\alpha(\psi_1, \ldots, \psi_n)$ be the sentence of $L_s$ resulting from $\alpha$ if we replace each $p_i$ by $\psi_i$. Then:*

$$\models \alpha(p_1, \ldots, p_n) \ \Rightarrow \models_{\mathcal{F}(\mathcal{B})} \alpha(\psi_1, \ldots, \psi_n).$$

Let us recall that one of the first simple, highly attractive and motivating results of PLS with respect to semantics SCS, was the "interpolation property" of $\varphi|\psi$ relative to $\varphi \wedge \psi$ and $\varphi \vee \psi$. That is, for all $\varphi$, $\psi$, $\varphi \wedge \psi \models_s \varphi|\psi \models_s \varphi \vee \psi$, while the converse relations are false (see Theorem 2.8 of [6]). This result is still true with respect to BCS. Its proof is a straightforward consequence of the above definitions.

**Fact 2.3** *Let $\mathcal{B}$ be a Boolean algebra and $X \subseteq \mathcal{F}(\mathcal{B})$. Then for all $\varphi, \psi \in Sen(L_s)$,*

$$\varphi \wedge \psi \models_X \varphi|\psi \models_X \varphi \vee \psi,$$

*while in general*

$$\varphi \vee \psi \not\models_X \varphi|\psi \not\models_X \varphi \wedge \psi.$$

*Proof.* Assume $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \wedge \psi$, with $\pi \in X$. Then $\overline{v}(\varphi) = \overline{v}(\psi) = 1$. Thus $\overline{v}(\varphi|\psi) = \pi(\overline{v}(\varphi), \overline{v}(\psi)) = \pi(1,1) = 1$, so $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi|\psi$. Now if the latter is true, then $\pi(\overline{v}(\varphi), \overline{v}(\psi)) = 1$, so either $\overline{v}(\varphi) = 1$ or $\overline{v}(\psi) = 1$. It follows that $\overline{v}(\varphi) + \overline{v}(\psi) = 1$, and hence $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \vee \psi$.

On the other hand to see that $\varphi \vee \psi \not\models_X \varphi|\psi$ in general, pick classical sentences $\alpha$, $\beta$ and $v$ such that $v(\alpha) = 1$ and $v(\beta) = 0$, and $\pi$ such that $\pi(0,1) = 0$. Then $\overline{v}_\pi(\alpha \vee \beta) = v(\alpha) + v(\beta) = 1$, while $\overline{v}_\pi(\alpha|\beta) = \pi(0,1) = 0$. Thus $\langle \mathcal{B}, \pi, v\rangle \models_a \alpha \vee \beta$ while $\langle \mathcal{B}, \pi, v\rangle \not\models_a \alpha|\beta$. Similarly, to see that $\varphi|\psi \not\models_X \varphi \wedge \psi$, pick $\alpha$, $\beta$ and $v$ such that $v(\alpha) = 1$ and $v(\beta) = 0$, and $\pi$ such that $\pi(0,1) = 1$. ⊣

Choice functions for $[B]^2$ can have properties quite similar or analogous to those of choice functions for $[Sen(L)]^2$ considered in the previous section. Our aim is to identify classes $X \subseteq \mathcal{F}(\mathcal{B})$ suitable to implement the axioms $S_1$-$S_5$ cited in the Introduction. Axioms $S_1$, $S_2$ and $S_3$ do not require any further conditions for $\pi$ to hold. Concerning $S_4$, the extra condition for $\pi$ is *associativity,* where associative functions $\pi : [B]^2 \to B$ are defined exactly as the corresponding functions $f : [Sen(L)]^2 \to Sen(L)$ in Definition 1.2. So we set

$$Asso(\mathcal{B}) = \{\pi \in \mathcal{F}(\mathcal{B}) : \pi \text{ is associative}\}.$$

By the general Theorem 2.14 of [6], we have in particular the following.

**Lemma 2.4** $\pi \in Asso(\mathcal{B})$ *if and only if there is total* $<$ *ordering of* $B$ *such that* $\pi = \min_<$*, i.e.,* $\pi(a,b) = \min(a,b)$ *for all* $a, b \in B$*.*

Finally for $S_5$ we need the analogue of $\neg$-decreasing orderings of $Sen(L)$ which are defined for $B$ in the expected way, that is, using the complement operation $-$ of $\mathcal{B}$ in place of $\neg$.

**Definition 2.5** Let $<$ be a total ordering of $B$. $<$ is said to be *complement-decreasing* or just $-$*-decreasing,* if for all $a, b \in B$

$$a < b \Leftrightarrow -b < -a.$$

If $\pi \in Asso(\mathcal{B})$, $\pi$ is said to be $-$*-decreasing* if $\pi = \min_<$ for some $-$-decreasing ordering.

## 2.1 Existence of complement-decreasing orderings and functions

In this section we show how to construct $-$-decreasing total orderings on every algebra $\mathcal{B}$. For any algebra $\mathcal{B}$, let $\preccurlyeq_{\mathcal{B}}$ denote throughout the natural ordering of $\mathcal{B}$. Recall that $\preccurlyeq_{\mathcal{B}}$ is defined as follows:

$$a \preccurlyeq_{\mathcal{B}} b :\Leftrightarrow a \cdot b = a \ (\Leftrightarrow a + b = b).$$

We often write $\preccurlyeq$ instead of $\preccurlyeq_{\mathcal{B}}$, if there is no danger of confusion. $\preccurlyeq$ is already a $-$-decreasing and, by Zorn's Lemma, can be extended to a total such function. But in general the $-$-decreasing orderings we deal with below need not extend $\preccurlyeq$.

There is a general method to construct such orderings. Given $\mathcal{B}$, let $P = \{\{a, -a\} : a \in B\}$, and let $F$ be a choice function for $P$. Let $X = F[P]$ and $Y = B - X$. Then clearly $a \in X \Leftrightarrow -a \in Y$. Pick any total ordering $<_1$ of $X$ and define $<_2$ of $Y$ by setting for all $a, b \in Y$,

$$a <_2 b :\Leftrightarrow -b <_1 -a.$$

Then define $<$ on $B$ by setting $a < b$ iff:
$a \in X$ and $b \in Y$, or
$a \in X$ and $b \in X$ and $a <_1 b$, or
$a \in Y$ and $b \in Y$ and $a <_2 b$.
Obviously $<$ is $-$-decreasing and moreover $X, Y$ form a partition of $B$ such that $X < Y$. Let us call a set $X \subseteq B$ *selective* if for every pair of opposite elements $\{u, -u\}$ of $B$, $X$ contains exactly one. (Note that in particular, every ultrafilter, as well as every prime ideal, on $\mathcal{B}$ is a selective set, but the converse is not true.) The sets $X, Y$ defined above are selective and what we have shown is the following.

**Proposition 2.6** *Let $\mathcal{B}$ be a Boolean algebra. Then for every selective set $X \subset B$, there is a $-$-decreasing ordering $<$ of $B$ such that if $Y = B - X$, then $X < Y$.*

We can moreover see that every $-$-decreasing ordering on $B$ is generated in the above way. Namely the following holds.

**Proposition 2.7** *Let $<$ be a $-$-decreasing ordering on $\mathcal{B}$. Then $B$ is partitioned into two selective sets $X, Y$ such that $X < Y$.*

*Proof.* Let $<$ be a $-$-decreasing ordering of $B$. Call a set $Z \subset B$ weakly selective if for every $a \in B$, $Z$ contains at most one element of the pair $\{a, -a\}$. We claim that there are weakly selective initial segments of $B$ (and consequently, weakly selective final segments of $B$). Indeed, for every $a \in B$ at least one of the initial segments $\{b : b \leq a\}$ and $\{b : b \leq -a\}$ is weakly selective. Otherwise, there are $a$, $b$ and $c$ such that $b, -b \leq a$ and $c, -c \leq -a$. From the first relation and the fact that $<$ is $-$-decreasing, we have $b \leq a$ and $-b \leq a$, hence $-a \leq -b$, therefore $-a \leq a$. From the second relation we have $c \leq -a$ and $-c \leq -a$, so $a \leq c$, therefore $a \leq -a$. Thus $-a \leq a$ and $a \leq -a$, a contradiction.

Let $X$ be the union of all weakly selective initial segments of $B$. Then using the previous argument we easily see that $X$ is a selective initial segment and hence $Y = B - X$ is a selective final segment such that $X < Y$. $\dashv$

Since the natural ordering $\preccurlyeq$ of $B$ is $-$-decreasing, it is natural to ask whether there are $-$-decreasing total orderings of $B$ that extend $\preccurlyeq$ of $\mathcal{B}$. The answer is yes and follows by an easy application of Zorn's Lemma.

**Proposition 2.8** *For every Boolean algebra $\mathcal{B}$, with natural ordering $\preccurlyeq$, there is a $-$-decreasing total ordering $<$ such that $\prec \subset <$.*

*Proof.* Let $S = \{R : \prec \subseteq R \ \&\ R$ is a $-$-decreasing ordering of $B\}$. By Zorn's Lemma, $S$ has a maximal element $<$. Clearly $<$ is a $-$-decreasing ordering. So it suffices to see that $<$ is total. Assume not. Then there are $a, b$ such that $a \not< b$ and $b \not< a$. Since $<$ is $-$-decreasing it follows that also $-a \not< -b$ and $-b \not< -a$. Then clearly we can extend $<$ to an ordering $<'$ such that $a <' b$ or $b <' a$. Suppose $a <' b$. But then it is not hard to see that if we set $<'' = <' \cup \{\langle -y, -x\rangle : x <' y\}$, then $<''$ is a $-$-decreasing ordering that properly extends $<$. But this contradicts the maximality of $<$. $\dashv$

Let us set

$$Dec(\mathcal{B}) = \{\pi \in Asso(\mathcal{B}) : \pi \text{ is } -\text{-decreasing}\}.$$

We have for any $\mathcal{B}$,

$$Dec(\mathcal{B}) \subseteq Asso(\mathcal{B}) \subseteq \mathcal{F}(\mathcal{B}),$$

therefore

$$Taut(\mathcal{F}(\mathcal{B})) \subseteq Taut(Asso(\mathcal{B})) \subseteq Taut(Dec(\mathcal{B})). \qquad (6)$$

**Proposition 2.9** *Let $\mathcal{B} \subseteq \mathcal{B}_1$. Every total ordering $<$ of $\mathcal{B}$ can be extended to total ordering $<_1$ of $\mathcal{B}_1$. Moreover if $<$ is $-$-decreasing, then $<_1$ can be $-$-decreasing too.*

*Proof.* The first claim is straightforward since every total ordering on a set $X$ can be extended to a total ordering on a set $Y \supseteq X$. Concerning the second claim, let $<$ be a $-$-decreasing ordering on $B$. By Proposition 2.7 $B$ is partitioned into two selective sets $X, Y$ for $B$ such that $X < Y$. Let $A = B_1 - B$. Then for every $a \in A$, $\{a, -a\} \subset A$. Using a choice function for the pairs $\{a, -a\} \subset A$, we can extend $X, Y$ to two selective subsets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ for $B_1$ that partition $B_1$. Let us extend $<\restriction X$ to a total ordering $<_1'$ of $X_1$. Then define a total ordering $<_1''$ on $Y_1$ by setting

$$a <_1'' b \Leftrightarrow -b <_1' -a.$$

It is clear that $<_1''$ extends $<\restriction Y$ and if $<_1 = <_1' \cup <_1''$, then $<_1$ is a $-$-decreasing total ordering of $B_1$ that extends $<$. $\dashv$

For each $\mathcal{B}$ and each $X \subseteq \mathcal{F}(\mathcal{B})$, we have a logical consequence relation denoted $\models_X$. Our next goal is to formalize the relations $\models_{\mathcal{F}(\mathcal{B})}$, $\models_{Asso(\mathcal{B})}$ and $\models_{Dec(\mathcal{B})}$. This will be done in section 3 below. Before that however, in the following two subsections we point out certain peculiarities of algebraic semantics as well as some special features of it when the underlying Boolean algebra is the trivial one.

## 2.2 Some peculiarities of the algebraic semantics

To facilitate writing interpretations of sentences of $L_s$ in $\langle \mathcal{B}, \pi, v\rangle$ we fix some notation. For $a, b \in B$ let us set

$$a \lhd b := -a + b, \text{ and } a \bigtriangledown b := (a \lhd b) \cdot (b \lhd a) = (-a+b) \cdot (-b+a).$$

Note that $a \bigtriangledown b$ is the complement of the symmetric difference $a \bigtriangleup b$ (see [5, p. 18]), i.e., $a \bigtriangledown b = -(a \bigtriangleup b)$. The operations $\lhd$ and $\bigtriangledown$ are used for the interpretation of sentences $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$.

**Fact 2.10** *For any $\mathcal{B}$, $v$, $\pi$ and $a, b \in B$, the following hold.*
*(i) $a \lhd b = 1 \Leftrightarrow a \preccurlyeq b$.*
*(ii) $a \bigtriangledown b = 1 \Leftrightarrow a = b$.*
*(iii) $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \rightarrow \psi$ iff $\overline{v}_\pi(\varphi) \lhd \overline{v}_\pi(\psi) = 1$ iff $\overline{v}_\pi(\varphi) \preccurlyeq \overline{v}_\pi(\psi)$*
.
*(iv) $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \leftrightarrow \psi$ iff $\overline{v}_\pi(\varphi) \bigtriangledown \overline{v}_\pi(\psi) = 1$ iff $\overline{v}_\pi(\varphi) = \overline{v}_\pi(\psi)$.*

**Lemma 2.11** *(i) For every algebra $\mathcal{B}$ and any sentences $\varphi$, $\psi$, $\sigma$, $\tau$ of $Sen(L_s)$*

$$\{\varphi, \neg\psi, \sigma, \neg\tau\} \models_{\mathcal{F}(\mathcal{B})} (\varphi|\psi) \leftrightarrow (\sigma|\tau). \tag{7}$$

*(ii) On the other hand for every $\mathcal{B} \neq \mathbf{2}$ and any distinct atoms $p_1$, $p_2$, $p_3$, $p_4$ of $L$,*

$$\not\models_{Dec(\mathcal{B})} (p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \rightarrow (p_1|p_2 \leftrightarrow p_3|p_4). \tag{8}$$

*Proof.* (i) To show (7) assume $\langle \mathcal{B}, \pi, v\rangle \models_{\mathcal{F}(\mathcal{B})} \{\varphi, \neg\psi, \sigma, \neg\tau\}$. Then $\overline{v}_\pi(\varphi) = \overline{v}_\pi(\sigma) = 1$ and $\overline{v}_\pi(\psi) = \overline{v}_\pi(\tau) = 0$. Therefore $\overline{v}_\pi(\varphi|\psi) = \overline{v}_\pi(\varphi|\psi) = \pi(1,0)$. Hence $\overline{v}_\pi(\varphi|\psi \leftrightarrow \sigma|\tau) = 1$, i.e., $\langle \mathcal{B}, \pi, v\rangle \models_{\mathcal{F}(\mathcal{B})} (\varphi|\psi \leftrightarrow \sigma|\tau)$.

(ii) Fix an algebra $\mathcal{B} \ncong \mathbf{2}$. Without loss of generality we may assume that $\mathcal{B}$ is the smallest nontrivial algebra with a four-element set $B = \{a, -a, 0, 1\}$. Let $p_1, p_2, p_3, p_4$ be atoms of $L$. To show (8), we have to find $v : Sen(L) \rightarrow \mathcal{B}$ and $\pi \in Dec(\mathcal{B})$ such that

$$\overline{v}_\pi(p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \not\preccurlyeq \overline{v}_\pi(p_1|p_2 \leftrightarrow p_3|p_4). \tag{9}$$

Take an assignment $v : Sen(L) \rightarrow \mathcal{B}$ such that $v(p_1) = a$, $v(p_2) = -a$, $v(p_3) = 1$, and $v(p_4) = 0$. Then $\overline{v}_\pi(p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) = a \cdot 1 = a$ and $\overline{v}_\pi(p_1|p_2 \leftrightarrow p_3|p_4) = \pi(a, -a) \bigtriangledown \pi(1, 0)$. So (9) becomes

$$a \not\preccurlyeq \pi(a, -a) \bigtriangledown \pi(1, 0). \tag{10}$$

Now if $<$ is one of the total orderings of $B$

$$0 < a < -a < 1, \quad \text{or} \quad a < 0 < 1 < -a,$$

then clearly $<$ is $-$-decreasing, so if $\pi = \min_<$, then $\pi \in Dec(\mathcal{B})$ and $\pi(a, -a) = a$ and $\pi(1, 0) = 0$. Therefore $\pi(a, -a) \bigtriangledown \pi(1, 0) = a \bigtriangledown 0 = -a$. So (10) becomes $a \not\leq -a$, which is true. $\dashv$

**Remark 2.12** *Notice that clause (i) of 2.11 fails in SCS. That is, for any of the classes $X \subseteq \mathcal{F}$ considered in section 1.1, e.g. for $X = Dec$, in general*

$$\{\varphi, \neg\psi, \sigma, \neg\tau\} \not\models_X (\varphi|\psi \leftrightarrow \sigma|\tau).$$

*Indeed, pick atoms $p, q, r, s$ and a valuation $v$ such that $v \models p \wedge \neg q \wedge r \wedge \neg s$. Pick also a $\neg$-decreasing total ordering $<$ of $Sen(L)$ such that $p < q$ and $s < r$. If $f = \min_<$, then $f \in Dec$ and $\langle v, f\rangle \models_s \{p, \neg q, r, \neg s, \}$, while $\langle v, f\rangle \not\models_s (p|q \leftrightarrow r|s)$, since $v \models f(p, q)$ and $v \not\models f(r, s)$.*

A consequence of Lemma 2.11 is the failure of the Semantic Deduction Theorem (SDT) for the relations $\models_X$. The latter is the implication

$$\Sigma \cup \{\varphi\} \models_X \psi \ \Rightarrow \ \Sigma \models_X \varphi \to \psi. \tag{11}$$

**Proposition 2.13** *For every algebra $\mathcal{B} \ncong \mathbf{2}$, the Semantic Deduction Theorem fails for $\models_X$, where $X$ is any of the sets $\mathcal{F}(\mathcal{B})$, $Asso(\mathcal{B})$ and $Dec(\mathcal{B})$.*

*Proof.* Fix an algebra $\mathcal{B} \ncong \mathbf{2}$. Since $Dec(\mathcal{B}) \subset Asso(\mathcal{B}) \subset \mathcal{F}(\mathcal{B})$, in order to show the failure of SDT for all these sets of choice functions, it suffices to show that there are $\varphi$, $\psi$ such that

$$\varphi \models_{\mathcal{F}(\mathcal{B})} \psi \ \& \ \not\models_{Dec(\mathcal{B})} (\varphi \to \psi).$$

Pick distinct propositional atoms $p_1, p_2, p_3, p_4$. By clause (i) of Lemma 2.11

$$\{p_1, \neg p_2, p_3, \neg p_4\} \models_{\mathcal{F}(\mathcal{B})} (p_1|p_2, \leftrightarrow p_3|p_4),$$

while by clause (ii) of this Lemma

$$\not\models_{Dec(\mathcal{B})} (p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \to (p_1|p_2 \leftrightarrow p_3|p_4).$$

This proves the claim. $\dashv$

**Failure of monotonicity and distributivity properties** Besides the axioms $S_1$-$S_5$ that were shown to axiomatize completely the basic properties of $|$ with respect to SCS, some other principles expressing reasonable properties of $|$ have been tested in [6] and proved false. Such are the following principles expressing the distributivity of $\wedge$ and $\vee$ with respect to $|$ and conversely.

$$\begin{array}{rl}
(Mon_\wedge) & \varphi \wedge (\psi|\sigma) \leftrightarrow (\varphi \wedge \psi)|(\varphi \wedge \sigma), \\
(Mon_\vee) & \varphi \vee (\psi|\sigma) \leftrightarrow (\varphi \vee \psi)|(\varphi \vee \sigma), \\
(Dis_\wedge) & \varphi|(\psi \wedge \sigma) \leftrightarrow (\varphi|\psi) \wedge (\varphi|\sigma), \\
(Dis_\vee) & \varphi|(\psi \vee \sigma) \leftrightarrow (\varphi|\psi) \vee (\varphi|\sigma).
\end{array}$$

**Proposition 2.14** *Let $\mathcal{B}$ be a Boolean algebra containing at least three independent elements $a, b, c$, i.e., such that $\pm a \cdot \pm b \cdot \pm c \neq 0$. Then there is no $X \subseteq Asso(\mathcal{B})$ for which some of the above schemes is a $X$-tautology.*

*Proof.* Fix a $\mathcal{B}$ with the aforementioned property. It suffices to show that there is no total ordering $<$ of $B$ such that if $\pi = \min_<$, then for every $\mathcal{B}$-valuation $v : Sen(L) \to \mathcal{B}$, $\langle \mathcal{B}, \pi, v\rangle$ satisfies all instances of some of the above schemes. We shall show this for the schemes $Mon_\wedge$ and $Dis_\wedge$. For the other two schemes we work analogously.

(i) Falsity of $Mon_\wedge$: Towards reaching a contradiction assume that there is a total ordering $<$ of $B$ such that if $\pi = \min_<$, then for every $v$ and all $\varphi, \psi, \sigma$,

$$\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \wedge (\psi|\sigma) \leftrightarrow (\varphi \wedge \psi)|(\varphi \wedge \sigma).$$

It follows from the definitions that this holds if and only if for all $a, b, c \in B$,

$$(a \cdot \pi(b, c)) \bigtriangledown \pi(a \cdot b, a \cdot c) = 1,$$

or $a \cdot \pi(b, c) = \pi(a \cdot b, a \cdot c)$. The latter again holds if for all $a, b, c$, such that $b \neq c$ and $a \cdot b \neq a \cdot c$,

$$b < c \Leftrightarrow a \cdot b < a \cdot c. \tag{12}$$

By our assumption there are independent $a, b, c$. Suppose without loss of generality that $a < b < c$ and consider the element $u = a \cdot c \cdot -b$. Then by independence $a \cdot (c \cdot -b) \neq b \cdot (c \cdot -b) = 0$, therefore by (12), $a < b$ implies $a \cdot (c \cdot -b) < b \cdot (c \cdot -b)$, or $u < 0$. But also for the same reasons $b < c$ implies similarly $b \cdot (a \cdot -b) < c \cdot (a \cdot -b)$, or $0 < u$, a contradiction.

(ii) Falsity of $Dis_\wedge$: Now assume that there is a total ordering $<$ of $B$ such that if $\pi = \min_<$, then for every $v$ and all $\varphi, \psi, \sigma$,

$$\langle \mathcal{B}, \pi, v\rangle \models_a \varphi|(\psi \wedge \sigma) \leftrightarrow (\varphi|\psi) \wedge (\varphi|\sigma).$$

In particular, for $\sigma = \neg\psi$ we have

$$\langle \mathcal{B}, \pi, v\rangle \models_a \varphi|\top \leftrightarrow (\varphi|\psi) \wedge (\varphi|\neg\psi).$$

Clearly the latter holds iff for all $a, b$,

$$\pi(a, 0) \bigtriangledown (\pi(a, b) \cdot \pi(a, -b)) = 1,$$

or

$$\pi(a, 0) = \pi(a, b) \cdot \pi(a, -b). \qquad (13)$$

Fix two of the independent elements $a, b$. Clearly $a \neq 0, 1$. We examine below some consequences of $a < 0$ and $a > 0$.

(a) Let $a < 0$. By independence of $a, b$, $a \cdot b \neq a$ and $a \cdot -b \neq a$ (otherwise $a \cdot -b = 0$ or $a \cdot b = 0$). Then $\pi(a, 0) = a$, so (13) clearly implies $\pi(a, b) = \pi(a, -b) = a$, i.e., $a < b$ and $a < -b$. It means that

$$(\forall a)(\forall b)(a < 0 \wedge \ a, b \text{ indepenedent} \Rightarrow a < b, -b). \qquad (14)$$

(b) Let $0 < a$. Then $\pi(0, a) = 0$. Take again $b$ such that $a$, $b$ are independent, so $a \cdot b \neq a$ and $a \cdot -b \neq a$. Then (13) implies that $\pi(a, b) = b$ and $\pi(a, -b) = -b$, for otherwise we would have $0 = a \cdot b$, or $0 = a$, or $a \cdot -b =$ which contradict the previous assumptions. Therefore $b, -b < a$. It means that

$$(\forall a)(\forall b)(0 < a \wedge \ a, b \text{ indepenedent} \Rightarrow a < b, -b). \qquad (15)$$

By assumption there are three independent elements $a, b, c$. Since $a, b, c, 0$ are linearly ordered, there are two of them on the left of 0 or on the right of 0. That is, there are independent elements $a, b$ such that $a, b < 0$ or $0 < a, b$. If $a, b < 0$, (14) implies $a < b$, $a < -b$, $b < a$ and $b < -a$, a contradiction. If $0 < a, b$, (15) implies $b < a$, $-b < a$, $a < b$ and $-a < b$, a contradiction again. $\dashv$

## 2.3 The special case of the algebra 2

The Boolean-value choice semantics based on the trivial algebra **2** has some special properties not shared by the other algebras. This is because there are only two choice functions $\pi_0, \pi_1 : [\mathbf{2}]^2 \to \mathbf{2}$ such that:

1) $\pi_0(0) = 0, \pi_0(1) = 1$, $\pi_0(0, 1) = 0$, i.e., $\pi_0 = \min_{\preccurlyeq}$, and

2) $\pi_1(0) = 0, \pi_1(1) = 1$, $\pi_1(0, 1) = 1$, i.e., $\pi_1 = \max_{\preccurlyeq}$,

where $\preccurlyeq$ is the natural (total) ordering of **2**. So both $\pi_0$ and $\pi_1$ are associative. Moreover they are $-$-decreasing, since $0 \prec 1 \Leftrightarrow -1 \prec -0$. Therefore $\mathcal{F}(\mathbf{2}) = Dec(\mathbf{2})$. Below we cite some points on which truth with respect to **2** differs from the one with respect to an arbitrary $\mathcal{B}$.

1) **SDT is true with respect to 2**.

In contrast to Proposition 2.13, we have the following.

**Proposition 2.15** *The Semantical Deduction Theorem holds with respect to $\models_{\mathcal{F}(\mathbf{2})}$. That is, for all $\Sigma$, $\varphi$, $\psi$*

$$\Sigma \cup \{\varphi\} \models_{\mathcal{F}(\mathbf{2})} \psi \ \Rightarrow \ \Sigma \models_{\mathcal{F}(\mathbf{2})} \varphi \to \psi.$$

*Proof.* Since for every $v$ and $\pi$, and every $\varphi$, $\overline{v}_\pi(\varphi) \in \{0,1\}$, the proof is trivial, and quite similar to that of the SDT for classical logic. $\dashv$

2) **The monotonicity and distributivity schemes hold with respect to $\mathbf{2}$**.

Further, in contrast to Proposition 2.14, the following holds.

**Proposition 2.16** *The schemes $Mon_\wedge$, $Mon_\vee$, $Dis_\wedge$, $Dis_\vee$, mentioned in the previous subsection, are $\mathcal{F}(\mathit{2})$-tautologies.*

*Proof.* We verify that $Mon_\wedge$ and $Dis_\wedge$ are $\mathcal{F}(\mathbf{2})$-tautologies, leaving the other ones to the reader.

(i) $Mon_\wedge$: As we saw in the disproof of $Mon_\wedge$ in 2.14, $Mon_\wedge$ is a $\models_{\mathcal{F}(\mathbf{2})}$-tautology iff for all $a, b, c \in \{0,1\}$ and all $\pi \in \mathcal{F}(\mathbf{2})$,

$$a \cdot \pi(b,c) \bigtriangledown \pi(a \cdot b, a \cdot c) = 1,$$

or

$$a \cdot \pi(b,c) = \pi(a \cdot b, a \cdot c).$$

If $a = 0$, both sides of the equation equal 0. If $a = 1$, both sides equal $\pi(b,c)$, and we are done.

(ii) $Dis_\wedge$: This is a $\models_{\mathcal{F}(\mathbf{2})}$-tautology iff for all $a, b, c \in \{0,1\}$, and all $\pi \in \mathcal{F}(\mathbf{2})$,

$$\pi(a, b \cdot c) \bigtriangledown (\pi(a,b) \cdot \pi(a,c)) = 1,$$

or

$$\pi(a, b \cdot c) = \pi(a,b) \cdot \pi(a,c).$$

If $b = c = 0$ both sides of the equation equal $\pi(a,0)$. If $b = c = 1$, both sides equal $\pi(a,1)$. Assume $b \neq c$. Then the equation becomes $\pi(a,0) = \pi(a,0) \cdot \pi(a,1)$, which is true since clearly $\pi(a,0) \preccurlyeq \pi(a,1)$. $\dashv$

**Proposition 2.17** *The scheme*

*(†)* $(\varphi|\neg\varphi) \leftrightarrow (\psi|\neg\psi)$

*is a $\mathcal{F}(\mathit{2})$-tautology. However (†) is not a $\mathcal{F}(\mathcal{B})$-tautology for any $\mathcal{B} \neq \mathit{2}$.*

*Proof.* The truth of (†) in $\langle \mathbf{2}, \pi, v \rangle$ amounts to showing that

$$\pi(a, -a) \bigtriangledown \pi(b, -b) = 1,$$

for any $a, b \in \{0,1\}$, which is obvious, since $\pi(a,-a) = \pi(b,-b) = \pi(0,1)$. On the other hand, given $\mathcal{B} \neq \mathbf{2}$, let $a \in \mathcal{B}$ such that $a \neq 1, 0$ and $v(\alpha) = a$ and $v(\beta) = 1$, for two sentences $\alpha, \beta$ of $L$ ,and let

$\pi(a,-a)=a$ and $\pi(0,1)=0$. Then $\overline{\pi}v^*(\alpha|\neg\alpha \leftrightarrow \beta|\neg\beta) = a \bigtriangledown 0 = -a \neq 1$. Therefore $\langle \mathcal{B}, \pi, v\rangle \not\models_a (\alpha|\neg\alpha \leftrightarrow \beta|\neg\beta)$. $\dashv$

Of course (†) by no means expresses some intuitively natural or even reasonable property of the connective $|$. It is easy to see that (†) fails also in SCS, namely (†) $\notin Taut(Dec)$, hence also (†) $\notin Taut(\mathcal{F})$.

# 3 Axiomatization of algebraic semantics

In this section we shall formalize the sets of choice functions $\mathcal{F}(\mathcal{B})$, $Asso(\mathcal{B})$ and $Dec(\mathcal{B})$ for an arbitrary $\mathcal{B}$, by formal systems denoted $K_0^a$, $K_1^a$ and $K_2^a$, respectively. These systems are quite analogous to $K_0$, $K_1/K_2$[1] and $K_3$ that axiomatize the corresponding classes of SCS (the superscript "a" stands for algebraic). As with $K_i$, $K_i^a$ consist of axioms and inference rules. The axioms are again among $S_1$-$S_5$, as before. However a new inference rule will replace the rule $SV$ of SCS. This is necessitated by the peculiarities of BCS exposed in Lemma 2.11. Namely, if the provability relation $\vdash_K$ is to formalize $\models_{\mathcal{F}(\mathcal{B})}$, then on the one hand we should have $\{\varphi, \neg\psi, \sigma, \neg\tau\} \vdash_K \varphi|\psi \leftrightarrow \sigma|\tau$ (because of (7)), and on the other (for $\mathcal{B} \neq \mathbf{2}$) $\not\vdash_K (\varphi\wedge\neg\psi\wedge\sigma\wedge\neg\tau) \rightarrow (\varphi|\psi \leftrightarrow \sigma|\tau)$ (because of (8)). This situation can be captured not by an axiom but only by a rule. The rule is called *rule of analogy* (abbreviated $RA$) and reads as follows.

$$\textit{From} \quad \varphi, \neg\psi, \sigma, \neg\tau, \ \textit{infer} \ \ \varphi|\psi \leftrightarrow \sigma|\tau,$$

or formally,

$$(RA) \qquad \frac{\varphi, \neg\psi, \sigma, \neg\tau}{\varphi|\psi \leftrightarrow \sigma|\tau}.$$

The formal systems $K_0^a$, $K_1^a$ and $K_2^a$ are defined as follows.

$$\mathsf{Ax}(K_0^a) = \mathsf{Ax}(K_0), \quad \mathsf{IR}(K_0^a) = \{MP, RA\}$$

$$\mathsf{Ax}(K_1^a) = \mathsf{Ax}(K_0) + S_4, \qquad \mathsf{IR}(K_1^a) = \{MP, RA\},$$

$$\mathsf{Ax}(K_2^a) = \mathsf{Ax}(K_1^a) + S_5, \qquad \mathsf{IR}(K_2^a) = \{MP, RA\}.$$

For an algebra $\mathcal{B}$, a set $X \subseteq \mathcal{F}(\mathcal{B})$ and a formal system $K$, PLS$(\mathcal{B}, X, K)$, or more simply,

$$\mathrm{PLS}(X, K),$$

denotes the superposition logic with syntax $K$ and semantics $X$. As usual $\vdash_{K_i^a}$ denotes the provability relation of $K_i^a$.

[1]The distinction between "regular" and "non-regular" choice functions for $[Sen(L)]^2$ used in SCS does not make sense for choice functions for $[B]^2$. That is why the four formal systems of SCS reduce to three in BCS.

A natural question concerning the rule $RA$, might be why we need to postulate that $\varphi|\psi \leftrightarrow \sigma|\tau$ follows from $\{\varphi, \neg\psi, \sigma, \neg\tau\}$ and not, e.g., from $\{\varphi, \psi, \sigma, \tau\}$. The answer is that indeed $\varphi|\psi \leftrightarrow \sigma|\tau$ follows from $\{\varphi, \psi, \sigma, \tau\}$, but this inference does not need an extra rule. It follows from the axioms. Namely, $\varphi|\psi$ follows from $\{\varphi, \psi\}$ by $S_1$, and similarly $\sigma|\tau$ follows from $\{\sigma, \tau\}$. Therefore $\{\varphi, \psi, \sigma, \tau\}$ proves $(\varphi|\psi) \wedge (\sigma|\tau)$, and hence $(\varphi|\psi \leftrightarrow \sigma|\tau)$.

As we saw in section 1.1 concerning the systems $K_i$, the existence of the rule $SV$ undermines the truth of the Deduction Theorem (DT). Here we shall see that the introduction of the rule $RA$ *provably falsifies* DT, so we must distinguish again between the two forms ST1, ST2 of the Soundness Theorem, and the two forms CT1, CT2 of the Completeness Theorem, that were defined in section 1.1.

**Lemma 3.1** *(i)* ST1 $\Rightarrow$ ST2.
*(ii)* CT1 $\Rightarrow$ CT2.

*Proof.* (i) Assume ST1 and let $\Sigma$ be $K$-inconsistent. Then $\Sigma \vdash_K \bot$. By ST1 $\Sigma \models_X \bot$, thus $\Sigma$ cannot be $X$-satisfiable.

(ii) Assume CT1 and let $\Sigma$ be not $X$-satisfiable. Then $\Sigma \models_X \bot$. By CT1 $\Sigma \vdash_K \bot$, thus $\Sigma$ is $K$-inconsistent. ⊣

Two remarks concerning the converse implications of the preceding proposition.

1) For $\mathcal{B} \neq \mathbf{2}$ the proof of the converse of (i) above does not seem to go through. Indeed assume ST2 and $\Sigma \not\models_X \varphi$. If $\mathcal{B} \neq \mathbf{2}$, the latter does not imply that $\Sigma \cup \{\neg\varphi\}$ is $X$-satisfiable. It only implies that there are $v$ and $\pi \in X$ such that $\overline{v}_\pi(\Sigma) = 1$ and $\overline{v}_\pi(\varphi) \neq 1$, or equivalently $\overline{v}_\pi(\neg\varphi) \neq 0$. It may well be $\overline{v}_\pi(\varphi) = a \neq 0$. Thus we cannot infer that $\Sigma \cup \{\neg\varphi\}$ is satisfiable.

2) As remarked in Fact 1.7, direction CT2 $\Rightarrow$ CT1 holds if DT is true, but is open otherwise.

**Lemma 3.2** *For every Boolean algebra $\mathcal{B}$ the following hold.*
*(i)* $\mathsf{Ax}(K_0^a) \subseteq Taut(\mathcal{F}(\mathcal{B}))$.
*(ii)* $\mathsf{Ax}(K_1^a) \subseteq Taut(Asso(\mathcal{B}))$.
*(iii)* $\mathsf{Ax}(K_2^a) \subseteq Taut(Dec(\mathcal{B}))$.

*Proof.* Fix an algebra $\mathcal{B}$, a $\mathcal{B}$-valuation $v : Sen(L) \to \mathcal{B}$ and a choice function $\pi \in \mathcal{F}(\mathcal{B})$ and let $\varphi, \psi \in Sen(L_s)$.

(i) $\langle \mathcal{B}, \pi, v\rangle \models_a S_1$: Let $\sigma := (\varphi \wedge \psi \to \varphi|\psi)$. We have to show that $\overline{v}_\pi(\sigma) = 1$. By the definitions we have that

$$\overline{v}_\pi(\sigma) = (\overline{v}_\pi(\varphi) \cdot \overline{v}_\pi(\psi)) \lhd \pi(\overline{v}_\pi(\varphi), \overline{v}_\pi(\psi))$$

or, setting $\overline{v}_\pi(\varphi) = a$ and $\overline{v}_\pi(\psi) = b$,

$$\overline{v}_\pi(\sigma) = a \cdot b \lhd \pi(a, b).$$

Obviously $a \cdot b \preccurlyeq \pi(a,b)$, so, by Fact 2.10 (i), $a \cdot b \lhd \pi(a,b) = \overline{v}_\pi(\sigma) = 1$.

$\langle \mathcal{B}, \pi, v\rangle \models_a S_2$: Quite similar to the previous one and using the fact that for any $a, b$, $\pi$, $\pi(a,b) \preccurlyeq a + b$.

$\langle \mathcal{B}, \pi, v\rangle \models_a S_3$: With the same conventions as before, $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi|\psi \to \psi|\varphi$ amounts to $\pi(a,b) \lhd \pi(b,a) = 1$, which follows immediately from the fact that by definition $\pi(a,b) = \pi(b,a)$.

(ii) Since $Taut(\mathcal{F}(\mathcal{B})) \subseteq Taut(Asso(\mathcal{B}))$, it suffices to show that $S_4 \in Taut(Asso(\mathcal{B}))$, i.e., for every $v$ and $\pi \in Asso(\mathcal{B})$, $\langle \mathcal{B}, \pi, v\rangle \models_a (\varphi|\psi)|\sigma \to \varphi(\psi|\sigma)$. With the previous conventions it amounts to showing that for all $a, b, c \in B$ $\pi(\pi(a,b),c) \lhd \pi(a, \pi(b,c)) = 1$, which holds since actually by the associativity of $\pi$, $\pi(\pi(a,b),c) = \pi(a,\pi(b,c))$.

(iii) Since $Taut(Asso(\mathcal{B})) \subseteq Taut(Dec(\mathcal{B}))$, it suffices to show that $S_5 \in Taut(Dec(\mathcal{B}))$, i.e., for every $v$ and $\pi \in Dec(\mathcal{B})$ $\langle \mathcal{B}, \pi, v\rangle \models_a S_5$, i.e., $\overline{v}_\pi(\sigma) = 1$ for any

$$\sigma := \varphi \wedge \neg\psi \to (\varphi|\psi \leftrightarrow \neg\varphi|\neg\psi).$$

With the preceding conventions we have to show that for $\pi \in Dec(\mathcal{B})$ and all $a, b \in B$

$$(a \cdot -b) \lhd (\pi(a,b) \bigtriangledown \pi(-a,-b)) = 1 \tag{16}$$

Since $\pi \in Dec(\mathcal{B})$, $\pi = \min_<$ for a $-$-decreasing total ordering $<$ of $Sen(L)$, i.e., $a < b \Leftrightarrow -b < -a$. If $a = b$, then $a \cdot -b = 0$, so (16) is true. So assume $a \neq b$. We consider the following two cases.

• $\pi(a,b) = a$. Then $a < b$, hence $-b < -a$, so $\pi(-a,-b) = -b$. Therefore

$$\pi(a,b) \bigtriangledown \pi(-a,-b) = a \bigtriangledown -b = (-a + -b) \cdot (a + b).$$

Since obviously $a \cdot -b \preccurlyeq (-a + -b) \cdot (a + b)$, by Fact 2.10 (16) is true.

• $\pi(a,b) = b$. Then $b < a$ and hence $-a < -b$, so $\pi(-a,-b) = -a$. Therefore $\pi(a,b) \bigtriangledown \pi(-a,-b) = b \bigtriangledown -a$. But

$$b \bigtriangledown -a = -b \bigtriangledown a = a \bigtriangledown -b = (-a + -b) \cdot (a + b),$$

so we are reduced to the previous case. This completes the proof. $\dashv$

In addition to the axioms of $K_2^a$, the relation $\models_{Dec(\mathcal{B})}$ satisfies also the distributivity of the dual connective of $|$ with respect to $|$ (and vice versa), a result completely analogous to what holds for the $\neg$-decreasing functions of SCS (see Proposition 2.46 of [6]). The dual connective of $|$ is

$$\varphi \circ \psi := \neg(\neg\varphi|\neg\psi).$$

**Proposition 3.3** *For any Boolean algebra $\mathcal{B}$, the equivalences*

$$\varphi \circ (\psi|\sigma) \leftrightarrow (\varphi \circ \psi)|(\varphi \circ \sigma),$$

*and*

$$\varphi|(\psi \circ \sigma) \leftrightarrow (\varphi|\psi) \circ (\varphi|\sigma)$$

*are schemes of $Dec(\mathcal{B})$-tautologies.*

*Proof.* We prove the claim for the first scheme, the other being proved similarly. Fix sentences $\varphi$, $\psi$, $\sigma$. Let $v$ be any $\mathcal{B}$-valuation of $Sen(L)$ and let $\pi \in Dec(\mathcal{B})$, i.e., $\pi = \min_<$ where $<$ is a $-$-decreasing total ordering of $B$. It suffices to show that

$$\overline{v}_\pi(\varphi \circ (\psi|\sigma) \leftrightarrow (\varphi \circ \psi)|(\varphi \circ \sigma)) = 1,$$

or

$$\overline{v}_\pi(\varphi \circ (\psi|\sigma)) = \overline{v}_\pi((\varphi \circ \psi)|(\varphi \circ \sigma))). \tag{17}$$

Putting $\overline{v}_\pi(\varphi) = a$, $\overline{v}_\pi(\psi) = b$ and $\overline{v}_\pi(\sigma) = c$, and taking into account the definition of $\circ$ and the fact that for any $x, y \in B$, $\pi(x, y) = \min(x, y)$, (17) is equivalent to

$$-\min(-a, -\min(b, c)) = \min(-\min(-a, -b), -\min(-a, -c)). \tag{18}$$

By $-$-decreasingness, clearly $-\min(-a, -b) = \max(a, b)$, so (18) becomes

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)). \tag{19}$$

We verify (19) by cases, although we guess that there is some more elegant direct proof.

*Case 1.* Assume $a \leq \min(b, c)$. Then $\max(a, \min(b, c)) = \min(b, c)$. Besides $a \leq \min(b, c)$ implies $\max(a, b) = b$ and $\max(a, c) = c$. Therefore both sides of (19) are equal to $\min(b, c)$.

*Case 2.* Assume $\min(b, c) < a$. Then $\max(a, \min(b, c)) = a$. To decide the right-hand side of (19), suppose $b \leq c$ so we have the following subcases.

(2$i$) $b < a \leq c$: Then $\max(a, b) = a$, $\max(a, c) = c$, therefore, $\min(\max(a, b), \max(a, c)) = a$, thus (19) holds.

(2$ii$) $b \leq c < a$: Then $\max(a, b) = \max(a, c) = a$. So

$$\min(\max(a, b), \max(a, c)) = a,$$

thus (19) holds again.

*Case 3.* Assume again $\min(b, c) < a$, so $\max(a, \min(b, c)) = a$, but suppose now $c < b$. Then we have the subcases:

(3$i$) $c < a \leq b$: Then $\max(a, b) = b$ and $\max(a, c) = a$. Thus $\min(\max(a, b), \max(a, c)) = a$, that is, (19) holds.

(3$ii$) $c < b \leq a$: Then $\max(a,b) = a$ and $\max(a,c) = a$. So

$$\min(\max(a,b), \max(a,c)) = a.$$

This completes the proof. $\dashv$.

Let $\mathcal{B}$ be a subalgebra of $\mathcal{B}_1$ (notation $\mathcal{B} \subseteq \mathcal{B}_1$). If $v : Sen(L) \to \mathcal{B}$ is a $\mathcal{B}$-valuation, then $v$ can be considered also a $\mathcal{B}_1$-valuation $v : Sen(L) \to \mathcal{B}_1$, since the operations $+, \cdot$ and $-$ are preserved between $\mathcal{B}$ and $\mathcal{B}_1$. The following simple general fact is easy but useful.

**Lemma 3.4** *Let $\mathcal{B} \subseteq \mathcal{B}_1$, $\Sigma$ be a set of sentences of $L_s$, and $v$ be a $\mathcal{B}$-valuation. Then the following hold.*

*(i) For every $\pi \in \mathcal{F}(\mathcal{B})$ such that $\langle \mathcal{B}, \pi, v \rangle \models_a \Sigma$, there is a $\pi_1 \in \mathcal{F}(\mathcal{B}_1)$ such that $\langle \mathcal{B}_1, \pi_1, v \rangle \models_a \Sigma$. Thus every $\mathcal{F}(\mathcal{B})$-satisfiable $\Sigma$ is $\mathcal{F}(\mathcal{B}_1)$-satisfiable.*

*(ii) For every $\pi \in Asso(\mathcal{B})$ such that $\langle \mathcal{B}, \pi, v \rangle \models_a \Sigma$, there is a $\pi_1 \in Asso(\mathcal{B}_1)$ such that $\langle \mathcal{B}_1, \pi_1, v \rangle \models_a \Sigma$. Thus every $Asso(\mathcal{B})$-satisfiable $\Sigma$ is $Asso(\mathcal{B}_1)$-satisfiable.*

*(iii) For every $\pi \in Dec(\mathcal{B})$ such that $\langle \mathcal{B}, \pi, v \rangle \models_a \Sigma$, there is a $\pi_1 \in Asso(\mathcal{B}_1)$ such that $\langle \mathcal{B}_1, \pi_1, v \rangle \models_a \Sigma$. Thus every $Dec(\mathcal{B})$-satisfiable $\Sigma$ is $Dec(\mathcal{B}_1)$-satisfiable.*

*Proof.* (i) Since $\mathcal{B} \subseteq \mathcal{B}_1$, clearly $v : Sen(L_s) \to \mathcal{B} \subseteq \mathcal{B}_1$. Further, using $\pi$ for $[B]^2$ we have $\overline{v}_\pi(\varphi) \in \mathcal{B} \subseteq \mathcal{B}_1$, for every $\varphi \in \Sigma$. Therefore, since $\langle \mathcal{B}, v, \pi \rangle \models_a \Sigma$, it follows that $\langle \mathcal{B}_1, v, \pi \rangle \models_a \Sigma$. It remains only to extend $\pi$ to the set $[B_1]^2$. But this can be done arbitrarily, because the elements of $B_1 - B$ are not involved in the range of $v$ or $\overline{v}$, by setting $\pi_1(a,b) =$ whatever if $(a,b) \in [B_1]^2 - [B]^2$. Thus $\langle \mathcal{B}_1, \pi_1, v \rangle \models_a \Sigma$.

(ii) and (iii) are shown similarly by the help of Proposition 2.9, since for every if $\pi = \min_<$, for a total ordering $<$ of $B$, we can take $<_1$ extending $<$ and $\pi_1 = \min_{<_1}$. Moreover if $<$ is $-$-decreasing, $<_1$ can be taken to be $-$-decreasing too. $\dashv$

**Lemma 3.5** *Let $\mathcal{B} \subseteq \mathcal{B}_1$. The following hold.*

*(i) $Taut(\mathcal{F}(\mathcal{B}_1)) \subseteq Taut(\mathcal{F}(\mathcal{B}))$.*

*(ii) $Taut(Asso(\mathcal{B}_1)) \subseteq Taut(Asso(\mathcal{B}))$.*

*(i) $Taut(Dec(\mathcal{B}_1)) \subseteq Taut(Dec(\mathcal{B}))$.*

*Proof.* (i) Let $\varphi \in Taut(\mathcal{F}(\mathcal{B}_1))$ and let $\langle \mathcal{B}, \pi, v \rangle$ be a $\mathcal{B}$-structure. It suffices to see that $\langle \mathcal{B}, \pi, v \rangle \models_a \varphi$. $v$ is a $\mathcal{B}$-valuation of the sentences of $L$, hence it is also a $\mathcal{B}_1$-valuation. Also $\pi \in \mathcal{F}(\mathcal{B})$ and, by Proposition 2.9, $\pi$ is extended to a function $\pi_1 \in \mathcal{F}(\mathcal{B}_1)$. Since $\varphi \in Taut(\mathcal{F}(\mathcal{B}_1))$, $\langle \mathcal{B}_1, v, \pi_1 \rangle \models_a \varphi$. But obviously $\overline{v}_{\pi_1}(\varphi) = \overline{v}_\pi(\varphi) = 1$, therefore $\langle \mathcal{B}, \pi, v \rangle \models_a \varphi$. (ii) and (iii) are shown similarly because by 2.9, every $\pi \in Asso(\mathcal{B})$ can be extended to a $\pi_1 \in Asso(\mathcal{B}_1)$ and every $\pi \in Dec(\mathcal{B})$ can be extended to a $\pi_1 \in Dec(\mathcal{B}_1)$. $\dashv$

**Theorem 3.6** (Soundness) *For every algebra $\mathcal{B}$, the logics*

$$\mathrm{PLS}(\mathcal{F}(\mathcal{B}), K_0^a),\ \mathrm{PLS}(Asso(\mathcal{B}), K_1^a),\ \mathrm{PLS}(Dec(\mathcal{B}), K_2^a)$$

*satisfy* ST1 *(hence also* ST2*, by 3.1). Namely, for every $\Sigma \cup \{\varphi\} \subseteq Sen(L_s)$ the following hold.*
*(i)* $\Sigma \vdash_{K_0^a} \varphi \ \Rightarrow \Sigma \models_{\mathcal{F}(\mathcal{B})} \varphi$.
*(ii)* $\Sigma \vdash_{K_1^a} \varphi \ \Rightarrow \Sigma \models_{Asso(\mathcal{B})} \varphi$.
*(iii)* $\Sigma \vdash_{K_2^a} \varphi \ \Rightarrow \Sigma \models_{Dec(\mathcal{B})} \varphi$.

*Proof.* Pick some algebra $\mathcal{B}$. Clearly we can verify ST1 for all the above systems in one move by showing that $\Sigma \vdash_{K_2^a} \varphi$ implies $\Sigma \models_{\mathcal{F}(\mathcal{B})} \varphi$. Let $\Sigma \vdash_{K_2^a} \varphi$ and let $\sigma_1, \ldots, \sigma_n = \varphi$ be a $K_2^a$-proof of $\varphi$ from $\Sigma$. We show by induction that $\Sigma \models_{\mathcal{F}(\mathcal{B})} \sigma_i$, for $i \leq n$. In view of Lemma 3.2, the only nontrivial step of the induction is the one involving the rule $RA$. That is, given $i > 1$, assume the claim holds for all $j < i$ and let $\sigma_i$ follows from previous sentences by the help of $RA$. It means that $\sigma_i = (\psi_1|\psi_2) \leftrightarrow (\tau_1|\tau_2)$ and there are $i_1, i_2, i_3, i_4 < i$ such that $\sigma_{i_1} = \psi_1$, $\sigma_{i_2} = \neg\psi_2$, $\sigma_{i_3} = \tau_1$ and $\sigma_{i_4} = \neg\tau_2$. The formulas $\{\psi_1, \neg\psi_2, \tau_1, \neg\tau_2\}$ all occur before $\sigma_i$, so by the induction assumption,

$$\Sigma \models_{\mathcal{F}(\mathcal{B})} \psi_1 \wedge \neg\psi_2 \wedge \tau_1 \wedge \neg\tau_2.$$

By Lemma 2.11 (i), we have

$$\psi_1 \wedge \neg\psi_2 \wedge \tau_1 \wedge \neg\tau_2 \models_{\mathcal{F}(\mathcal{B})} (\psi_1|\psi_2 \leftrightarrow \tau_1|\tau_2).$$

Thus $\Sigma \models_{\mathcal{F}(\mathcal{B})} (\psi_1|\psi_2 \leftrightarrow \tau_1|\tau_2)$, or $\Sigma \models_{\mathcal{F}(\mathcal{B})} \sigma_i$. This completes the proof. $\dashv$

Note that since ST1 $\Rightarrow$ ST2 and the systems above satisfy ST1, we may say that they are "sound" instead of "ST1-sound".

**Proposition 3.7** *The Deduction Theorem (DT) fails in the formal systems $K_i^a$, for $i \in \{0, 1, 2\}$.*

*Proof.* Each $K_i^a$ contains the rule $RA$, and in view of this we have that for all $\varphi$, $\psi$, $\sigma$, $\tau$,

$$\{\varphi, \neg\psi, \sigma, \neg\tau\} \vdash_{K_i^a} (\varphi|\psi) \leftrightarrow (\sigma|\tau).$$

Suppose

$$\vdash_{K_i^a} (\varphi \wedge \neg \wedge \psi \wedge \sigma \wedge \neg\tau) \rightarrow (\varphi|\psi \leftrightarrow \sigma|\tau).$$

Then by Theorem 3.6 we should have

$$\models_X (\varphi \wedge \neg \wedge \psi \wedge \sigma \wedge \neg\tau) \rightarrow (\varphi|\psi \leftrightarrow \sigma|\tau),$$

for every $\mathcal{B}$ and $X = \mathcal{F}(\mathcal{B})$ (if $i = 0$), or $X = Asso(\mathcal{B})$ (if $i = 1$), or $X = Dec(\mathcal{B})$ (if $i = 2$). But for $\mathcal{B} \neq \mathbf{2}$ and $X = Dec(\mathcal{B})$, this is false by Lemma 2.11 (ii). Moreover the failure of the above sentence to be a tautology with respect to $\models_{Dec(\mathcal{B})}$ implies the failure also with respect to $\models_{Asso(\mathcal{B})}$ and $\models_{\mathcal{F}(\mathcal{B})}$. Therefore

$$\nvdash_{K_i^a} (\varphi \wedge \neg \wedge \psi \wedge \sigma \wedge \neg\tau) \to (\varphi|\psi \leftrightarrow \sigma|\tau),$$

and thus DT fails in $K_i^a$. ⊣

## 3.1 Completeness

We come to the completeness of the logics based on the systems $K_0^a$-$K_2^a$. In view of the failure of DT shown above, the situation is analogous to that of the logics based on the systems $K_1$-$K_3$ outlined in section 1. The failure of DT on the one hand does not allow identifying the two forms of completeness CT1 and CT2, because of Fact 1.7 (ii). After all, we shall prove below (Theorem 3.11) that CT1 is false. So we can only hope to prove CT2. On the other hand, by the failure of DT, we cannot infer that every consistent set of sentences can be extended to a consistent and *complete* set, hence we must invoke again the property $cext(K)$ of "complete extendibility", for a formal system $K$. Recall that this is the property

($cext(K)$) *Every $K$-consistent set of sentences can be extended to a*

*$K$-consistent and complete set.*

The justification given immediately after Theorem 1.8 for the introduction of $cext(K)$ is valid also here. So we shall prove again *conditional* versions of CT2-completeness for these systems. Satisfiability of a consistent system will be shown with respect to the algebra **2**.

**Theorem 3.8** (Conditional CT2-completeness with respect to **2**) *The logics* PLS($\mathcal{F}(2), K_0^a$), PLS($Asso(2), K_1^a$) *and* PLS($Dec(2), K_2^a$) *are* CT2*-complete if and only if $cext(K_i^a)$ are true for $i = 0, 1, 2$, respectively.*

*Proof.* Recall first that $\mathcal{F}(\mathbf{2}) = Asso(\mathbf{2}) = Dec(\mathbf{2})$, so we shall refer to all of them as $\mathcal{F}(\mathbf{2})$. One direction is easy. Assume that $cext(K_i^a)$ is false for some $i \in \{0, 1, 2\}$. Then there is a maximal $K_i^a$-consistent set $\Sigma$ of sentences non-extendible to and a $K_i^a$-consistent and complete set, so there is $\varphi$ such that $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both $K_i^a$-inconsistent. But then $\Sigma$ cannot be $\mathcal{F}(\mathbf{2})$-satisfiable. For if there is a binary valuation $v$ and a choice function $\pi \in [2]^2$ such that $\langle \mathbf{2}, \pi, v\rangle \models_a \Sigma$, then $\langle \mathbf{2}, \pi, v\rangle \models_a \varphi$ or $\langle \mathbf{2}, \pi, v\rangle \models_a \neg\varphi$. So either $\Sigma \cup \{\varphi\}$ or $\Sigma \cup \{\neg\varphi\}$ would be $\mathcal{F}(\mathbf{2})$-satisfiable. But this is false since

$\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are inconsistent and the above logics are sound, by Theorem 3.6. Thus $\Sigma$ is $K_i^a$-consistent and not $\mathcal{F}(\mathbf{2})$-satisfiable, i.e., PLS$(\mathcal{F}(\mathbf{2}), K_i^a)$ is not CT2-complete.

For the main direction assume $cext(K_i^a)$ is true for all $i$. Since $\mathcal{F}(\mathbf{2}) = Asso(\mathbf{2}) = Dec(\mathbf{2})$, we can prove CT2-completeness of all three systems in one move by showing that if $\Sigma$ is $K_0^a$-consistent, then it is $\mathcal{F}(\mathbf{2})$-satisfiable. So fix a $\Sigma \subset Sen(L_s)$ which is $K_0^a$-consistent.

By $cext(K_0^a)$ we can take $\Sigma$ to be complete, i.e., $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$, for every $\varphi \in Sen(L_s)$. Let $\Sigma_1 = \Sigma \cap Sen(L)$, be the set of classical sentences of $\Sigma$. Clearly $\Sigma_1$ is a complete consistent subset of $Sen(L)$, so by the completeness and soundness of PL there is a binary assignment $v : Sen(L) \to \mathbf{2}$ such that for all $\alpha \in Sen(L)$,

$$\alpha \in \Sigma_1 \iff v(\alpha) = 1. \tag{20}$$

It suffices to determine a $\pi \in \mathcal{F}(\mathbf{2})$ such that $\langle \mathbf{2}, \pi, v\rangle \models_a \Sigma$, or equivalently, for all $\varphi \in Sen(L_s)$,

$$\varphi \in \Sigma \iff \overline{v}_\pi(\varphi) = 1. \tag{21}$$

Since there are only two nontrivial choice functions, we simply need to determine whether $\pi(0,1) = 0$ or $\pi(0,1) = 1$. Let us call a pair of sentences $\{\varphi, \psi\}$ of $L_s$ *homonymous for* $\Sigma$ if $\{\varphi, \psi\} \subset \Sigma$ or $\{\neg\varphi, \neg\psi\} \subset \Sigma$, and *heteronymous for* $\Sigma$ if either $\{\varphi, \neg\psi\} \subset \Sigma$ or $\{\neg\varphi, \psi\} \subset \Sigma$. Due to the completeness of $\Sigma$, every pair is either homonymous or heteronymous. The reason for giving this definition is the following two facts.

*Fact 1.* If $\{\varphi, \psi\}$ and $\{\sigma, \tau\}$ are heteronymous pairs for $\Sigma$, then $(\varphi|\psi) \in \Sigma \Leftrightarrow (\sigma|\tau) \in \Sigma$.

*Proof.* This is a consequence of the $K_0^a$-consistency and completeness of $\Sigma$, together with the action of rule $RA$ in $K_0^a$-proofs. For if $\{\varphi, \psi\}$ and $\{\sigma, \tau\}$ are heteronymous, then we can assume without loss of generality (due to axiom $S_3$, i.e., the commutativity of $|$) that $\{\varphi, \neg\psi, \sigma, \neg\tau\} \subset \Sigma$. By $RA$, $\{\varphi, \neg\psi, \sigma, \neg\tau\} \vdash_{K_0^a} (\varphi|\psi) \leftrightarrow (\sigma|\tau)$. So by completeness and consistency $\Sigma \vdash_{K_0^a} (\varphi|\psi) \leftrightarrow (\sigma|\tau)$, and for the same reason the latter implies $(\varphi|\psi) \in \Sigma \Leftrightarrow (\sigma|\tau) \in \Sigma$.

*Fact 2.* Let $\{\alpha, \beta\}$ be a heteronymous pair for $\Sigma$, where $\alpha, \beta \in Sen(L)$, say with $\{\alpha, \neg\beta\} \subset \Sigma$. If $v$ is the assignment for which $v \models \Sigma_1$, then $\langle \mathbf{2}, \pi, v\rangle \models_a \{\alpha, \neg\beta, \alpha|\beta\}$ if and only if $\pi(0,1) = 1$. And $\langle \mathbf{2}, \pi, v\rangle \models_a \{\alpha, \neg\beta, \neg(\alpha|\beta)\}$ if and only if $\pi(0,1) = 0$.

*Proof.* Suppose $\langle \mathbf{2}, \pi, v\rangle \models_a \{\alpha, \neg\beta, \alpha|\beta\}$. This simply means that $v(\alpha) = 1$, $v(\beta) = 0$ and $\pi(v(\alpha), v(\beta)) = 1$, thus necessarily, $\pi(1,0) = 1$. While $\langle \mathbf{2}, \pi, v\rangle \models_a \{\alpha, \neg\beta, \neg(\alpha|\beta)\}$ means that $v(\alpha) = 1$, $v(\beta) = 0$ and $-\pi(v(\alpha), v(\beta)) = 1$, i.e., $\pi(1,0) = 0$.

*Fact 3.* If the pair $\{\varphi,\psi\}$ is homonymous for $\Sigma$, then either $\{\varphi,\psi,\varphi|\psi\} \subset \Sigma$, or $\{\neg\varphi,\neg\psi,\neg(\varphi|\psi)\} \subset \Sigma$. In particular, if $\{\alpha,\beta\}$ is homonymous then $\langle \mathbf{2},\pi,v\rangle \models_a \{\alpha,\beta,\alpha|\beta\}$, for every $\pi$ or $\langle \mathbf{2},\pi,v\rangle \models_a \{\neg\alpha,\neg\beta,\neg(\alpha|\beta)\}$ for every $\pi$.

*Proof.* Just note that the remaining cases $\{\varphi,\psi,\neg(\varphi|\psi)\} \subset \Sigma$ and $\{\neg\varphi,\neg\psi,\varphi|\psi\} \subset \Sigma$ contradict the consistency of $\Sigma$. The first one because then $\Sigma \vdash_{K_0^a} \varphi \wedge \psi \wedge \neg(\varphi|\psi)$, while by $S_1$, $\varphi \wedge \psi \vdash_{K_0^a} \varphi|\psi$, and the second one because then $\Sigma \vdash_{K_0^a} \neg\varphi \wedge \neg\psi \wedge \varphi|\psi$, while by $S_2$, $\neg\varphi \wedge \neg\psi \vdash_{K_0^a} \neg(\varphi|\psi)$. The other claim follows from the fact that $\pi(1,1) = 1$ and $\pi(0,0)$ for every $\pi$.

In view of Fact 2, to decide whether we shall use the $\pi$ for which $\pi(0,1) = 0$ or the $\pi$ for which $\pi(0,1) = 1$, fix some heteronymous pair $\{\alpha,\beta\}$ of sentences of $L$, say such that $\{\alpha,\neg\beta\} \subset \Sigma$. (Clearly we can always find such a pair.) For this pair, either $(\alpha|\beta) \in \Sigma$ or $\neg(\alpha|\beta) \in \Sigma$. If $(\alpha|\beta) \in \Sigma$ we choose the $\pi$ for which $\pi(0,1) = 1$. If $\neg(\alpha|\beta) \in \Sigma$ we choose the $\pi$ for which $\pi(0,1) = 0$. We claim that for the $\pi$ chosen this way, (21) holds for every $\varphi \in Sen(L_s)$. Let us prove this for the two cases just mentioned.

*Case 1.* Suppose $(\alpha|\beta) \in \Sigma$, i.e., $\{\alpha,\neg\beta,\alpha|\beta\} \subset \Sigma$, and let $\pi(0,1) = 1$. We prove (21) by induction on the length of $\varphi$. For $\varphi \in Sen(L)$, $\overline{v}_\pi(\varphi) = v(\varphi)$, so (21) holds because of (20). The only nontrivial clause of the induction is the one concerning the connective $|$. So assume $\varphi = \varphi_1|\varphi_2$ and that the claim holds for $\varphi_1$ and $\varphi_2$, i.e.,

$$\varphi_1 \in \Sigma \iff \overline{v}_\pi(\varphi_1) = 1 \tag{22}$$

and

$$\varphi_2 \in \Sigma \iff \overline{v}_\pi(\varphi_2) = 1. \tag{23}$$

It suffices to prove

$$(\varphi_1|\varphi_2) \in \Sigma \iff \overline{v}_\pi(\varphi_1|\varphi_2) = 1. \tag{24}$$

We examine a few subcases.

(a) Suppose first that $\{\varphi_1,\varphi_2\}$ is heteronymous. Then by Fact 1 and the initial assumption of this case, it follows that $(\varphi_1|\varphi_2) \in \Sigma$, i.e., the left-hand side of (24) is true. We consider two subcases.

(a1) Let $\{\varphi_1,\neg\varphi_2\} \subset \Sigma$. By the induction assumptions (22) and (23) we have $\overline{v}_\pi(\varphi_1) = 1$ and $\overline{v}_\pi(\varphi_2) = 0$. Therefore

$$\overline{v}_\pi(\varphi_1|\varphi_2) = \pi(\overline{v}_\pi(\varphi_1),\overline{v}_\pi(\varphi_2)) = \pi(1,0) = 1.$$

So the right-hand side of (24) is true, and hence (24) itself is true.

(a2) If $\{\neg\varphi_1,\varphi_2\} \subset \Sigma$, we conclude similarly that $\overline{v}_\pi(\varphi_1|\varphi_2) = 1$, so (24) holds again.

(b) Suppose next that $\{\varphi_1, \varphi_2\}$ is homonymous. By Fact 3, either $\{\varphi_1, \varphi_2, \varphi_1|\varphi_2\} \subset \Sigma$, or $\{\neg\varphi_1, \neg\varphi_2, \neg(\varphi_1|\varphi_2)\} \subset \Sigma$. We have the following subcases.

(b1) $\{\varphi_1, \varphi_2, \varphi_1|\varphi_2\} \subset \Sigma$. Then the left-hand side of (24) is true. Also by the induction assumptions (22) and (23) we have $\overline{v}_\pi(\varphi_1) = \overline{v}_\pi(\varphi_2) = 1$. Then $\overline{v}_\pi(\varphi_1|\varphi_2) = \pi(\overline{v}_\pi(\varphi_1), \overline{v}_\pi(\varphi_2)) = \pi(1,1) = 1$. Thus the right-hand side of (24) is also true, so (24) is true.

(b2) $\{\neg\varphi_1, \neg\varphi_2, \neg(\varphi_1|\varphi_2)\} \subset \Sigma$. Then $(\varphi_1|\varphi_2) \notin \Sigma$, i.e., the left-hand side of (24) is false. Since also $\varphi_1 \notin \Sigma$ and $\varphi_2 \notin \Sigma$, by the induction assumptions (22) and (23) we have $\overline{v}_\pi(\varphi_1) = \overline{v}_\pi(\varphi_2) = 0$. Thus $\overline{v}_\pi(\varphi_1|\varphi_2) = \pi(\overline{v}_\pi(\varphi_1), \overline{v}_\pi(\varphi_2)) = \pi(0,0) = 0$, i.e., the right-hand side of (24) is also false, hence (24) is true. This completes the proof of (24) in Case 1.

*Case 2.* Suppose $\neg(\alpha|\beta) \in \Sigma$, i.e., $\{\alpha, \neg\beta, \neg(\alpha|\beta)\} \subset \Sigma$. In this case we choose $\pi$ so that $\pi(0,1) = 0$. It suffices as before to assume (22) and (23) and prove (24).

(a) Suppose $\{\varphi_1, \varphi_2\}$ is heteronymous. Then by Fact 1 and the initial assumption of this case, it follows that $\neg(\varphi_1|\varphi_2) \in \Sigma$. So the left-hand side of (24) is false. We consider two subcases.

(a1) Let $\{\varphi_1, \neg\varphi_2\} \subset \Sigma$. By the induction assumptions we have $\overline{v}_\pi(\varphi_1) = 1$ and $\overline{v}_\pi(\varphi_2) = 0$. Therefore

$$\overline{v}_\pi(\varphi_1|\varphi_2) = \pi(\overline{v}_\pi(\varphi_1), \overline{v}_\pi(\varphi_2)) = \pi(1,0) = 0.$$

So the right-hand side of (24) is false, and hence (24) is true.

(a2) If $\{\neg\varphi_1, \varphi_2\} \subset \Sigma$, we conclude similarly that $\overline{v}_\pi(\varphi_1|\varphi_2) = 0$, so (24) holds again.

(b) Suppose that $\{\varphi_1, \varphi_2\}$ is homonymous. This subcase is identical to subcase (b) of Case 1 above. (24) holds in view of the induction assumptions and the fact that $\pi(1,1) = 1$ and $\pi(0,0) = 0$ for all $\pi$. This completes the proof of the theorem. $\dashv$

A natural question is whether Theorem 3.8 is still true with an arbitrary algebra $\mathcal{B}$ in place of $\mathbf{2}$. We can only establish one direction of the equivalences.

**Corollary 3.9** *For any Boolean algebra $\mathcal{B}$ the following hold.*

*(i) If $cext(K_0^a)$ is true, then* PLS$(\mathcal{F}(\mathcal{B}), K_0^a)$ *is* CT2*-complete.*

*(ii) If $cext(K_1^a)$ is true, then* PLS$(Asso(\mathcal{B}), K_1^a)$ *is* CT2*-complete.*

*(iii) If $cext(K_2^a)$ is true, then* PLS$(Dec(\mathcal{B}), K_2^a)$ *is* CT2*-complete.*

*Proof.* (i) Let $cext(K_0^a)$ hold true and assume $\Sigma$ is $K_0^a$-consistent. By 3.8 PLS$(\mathcal{F}(\mathbf{2}), K_0^a)$ is CT2-complete, so $\Sigma$ is $\mathcal{F}(\mathbf{2})$-satisfiable. But then by Lemma 3.4 (i) $\Sigma$ is also $\mathcal{F}(\mathcal{B})$-satisfiable. Thus PLS$(\mathcal{F}(\mathcal{B}), K_0^a)$ is CT2-complete. Clauses (ii) and (iii) follow similarly using clauses (ii) and (iii) of Lemma 3.4. $\dashv$

The reason that the converse implications of 3.9 are open is the following. Assume $cext(K_i^a)$ is false. Then as before there is a maximal $K_i^a$-consistent set $\Sigma$ not extendible to a $K_i^a$-consistent and complete set. Thus there is a $\varphi$ such that $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both $K_i^a$-inconsistent, hence not $\mathcal{F}(\mathcal{B})$-satisfiable. But from this we cannot infer that $\Sigma$ is not $\mathcal{F}(\mathcal{B})$-satisfiable. For it may be that $\langle \mathcal{B}, \pi, v\rangle \models_a \Sigma$ and yet $\varphi$ is neither true nor false in $\langle \mathcal{B}, \pi, v\rangle$, that is, $0 \prec \overline{v}_\pi(\varphi) \prec 1$, where $\preccurlyeq$ is the natural ordering of $\mathcal{B}$. So $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are not satisfied in $\langle \mathcal{B}, \pi, v\rangle$ and no contradiction arises as before.

Here is also a conditional converse to Lemma 3.4.

**Corollary 3.10** *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be any Boolean algebras.*

*(i) Suppose $cext(K_0^a)$ is true. Then for every $\Sigma \subset Sen(L_s)$, $\Sigma$ is $\mathcal{F}(\mathcal{B}_1)$-satisfiable iff it is $\mathcal{F}(\mathcal{B}_2)$-satisfiable.*

*(ii) Suppose $cext(K_1^a)$ is true. Then for every $\Sigma \subset Sen(L_s)$, $\Sigma$ is $Asso(\mathcal{B}_1)$-satisfiable iff it is $Asso(\mathcal{B}_2)$-satisfiable.*

*(iii) Suppose $cext(K_2^a)$ is true. Then for every $\Sigma \subset Sen(L_s)$, $\Sigma$ is $Dec(\mathcal{B}_1)$-satisfiable iff it is $Dec(\mathcal{B}_2)$-satisfiable.*

*Proof.* Clearly it suffices to show these conditional equivalences taking one of the algebras to be $\mathbf{2}$. That is, for (i) it suffices to show, assuming $cext(K_0^a)$, that $\Sigma$ is $\mathcal{F}(\mathcal{B})$-satisfiable iff it is $\mathcal{F}(\mathbf{2})$-satisfiable. By Lemma 3.4, if $\Sigma$ is $\mathcal{F}(\mathbf{2})$-satisfiable then it is $\mathcal{F}(\mathcal{B})$-satisfiable. For the converse assume $\Sigma$ is not $\mathcal{F}(\mathbf{2})$-satisfiable. By condition $cext(K_0^a)$ and Theorem 3.8, $\Sigma$ is $K_0^a$-inconsistent, i.e., $\Sigma \vdash_{K_0^a} \bot$. But then by Theorem 3.6 (i) we have $\Sigma \models_{\mathcal{F}(\mathcal{B})} \bot$. So $\Sigma$ is cannot be $\mathcal{F}(\mathcal{B})$-satisfiable. Clauses (ii) and (iii) are shown similarly. $\dashv$

In contrast to Theorem 3.8 we have the following.

**Theorem 3.11** CT1 *is false for the logics* PLS$(\mathcal{F}(\mathbf{2}), K_i^a)$.

*Proof.* Assume on the contrary that CT1 holds for PLS$(\mathcal{F}(\mathbf{2}), K_i^a)$. It suffices to see that this implies the truth of the Deduction Theorem (DT), which contradicts Proposition 3.7. Let $\Sigma \cup \{\varphi\} \vdash_{K_i^a} \psi$. By Soundness Theorem 3.6 $\Sigma \cup \{\varphi\} \models_{\mathcal{F}(\mathbf{2})} \psi$. By Proposition 2.15 the Semantical Deduction Theorem (SDT) holds in PLS$(\mathcal{F}(\mathbf{2}), K_i^a)$. Therefore $\Sigma \models_{\mathcal{F}(\mathbf{2})} (\varphi \to \psi)$. Then CT1 for PLS$(\mathcal{F}(\mathbf{2}), K_i^a)$ implies $\Sigma \vdash_{K_i^a} (\varphi \to \psi)$. Thus DT is true in $K_i^a$, a contradiction.

For another proof by contradiction, recall that by Proposition 2.17 the scheme (†) $(\varphi|\neg\varphi) \leftrightarrow (\psi|\neg\psi)$ is a $\mathcal{F}(\mathbf{2})$-tautology, i.e. $\models_{\mathcal{F}(\mathbf{2})} (\varphi|\neg\varphi) \leftrightarrow (\psi|\neg\psi)$. CT1 would imply $\vdash_{K_0^a} (\varphi|\neg\varphi) \leftrightarrow (\psi|\neg\psi)$. By Theorem 3.6, it follows that $\models_{\mathcal{F}(\mathcal{B})} (\varphi|\neg\varphi) \leftrightarrow (\psi|\neg\psi)$, for every Boolean algebra $\mathcal{B}$. But this is false as we show in Proposition 2.17. In the same argument instead of (†) we could use some of the monotonicity or distributivity schemes that hold with respect to **2**, but fail with respect to other algebras as we saw in Propositions 2.14 and 2.16. $\dashv$

# 4 Connections between sentence choice and Boolean-value choice semantics

Let $\mathcal{L}$ be the Lindenbaum algebra of classical propositional logic. That is, $\mathcal{L} = \langle Sen(L)/\!\sim, +, \cdot, -, 1, 0\rangle$, where $Sen(L)/\!\sim = \{[\alpha] : \alpha \in Sen(L)\}$ is the set of equivalence classes of sentences of $L$ and $+, \cdot, -$ are the transfers of $\vee$, $\wedge$, $\neg$ to the classes, so $1 = \top$ and $0 = \bot$. For the connection between SCS and BCS the following two simple facts play a key-role:

1) The natural embedding $e : Sen(L) \to \mathcal{L}$ for which $e(\alpha) = [\alpha]$, is obviously a Boolean valuation for $Sen(L)$, specifically a $\mathcal{L}$-valuation.

2) If $f : [Sen(L)]^2 \to Sen(L)$ is a *regular* sentence choice function, then it induces a Boolean-value choice function $\pi_f : [Sen(L)/\!\sim]^2 \to Sen(L)/\!\sim$, defined by $\pi_f([\alpha], [\beta]) = [f(\alpha, \beta)]$. (Recall Definitions 1.2 and 1.4 of regular choice function and ordering for $Sen(L)$.)

**Lemma 4.1** *(i) Let $f$ be a regular choice function for $Sen(L)$, i.e., $f \in Reg$. Then setting*

$$\pi_f([\alpha], [\beta]) = [f(\alpha, \beta)],$$

*or equivalently, $\pi_f([\alpha], [\beta]) = [\alpha] \Leftrightarrow f(\alpha, \beta) \sim \alpha$, $\pi_f$ is a well-defined choice function for pairs of elements of $\mathcal{L}$, i.e., $\pi_f \in \mathcal{F}(\mathcal{L})$.*

*(ii) If in addition $f$ is associative, i.e., $f \in Reg^*$, then $\pi_f \in Asso(\mathcal{L})$.*

*(iii) If further $f$ is $\neg$-decreasing, i.e., $f \in Dec$, then $\pi_f \in Dec(\mathcal{L})$.*

*Proof.* (i) If $f \in Reg$, then the definition of $\pi_f$ above is good, i.e., independent of the class representatives. Indeed, by regularity, $\alpha \sim \alpha'$ implies $f(\alpha, \beta) \sim f(\alpha', \beta)$, therefore, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $f(\alpha, \beta) \sim f(\alpha', \beta')$, hence $\pi_f([\alpha], [\beta]) = \pi_f([\alpha'], [\beta'])$.

(ii) Let $f \in Reg^*$ and $f = \min_<$ for a regular total ordering $<$ of $Sen(L)$. Regularity induces a total ordering $<_1$ of $Sen(L)/\sim$. Then clearly $\pi_f = \min_{<_1}$, so $\pi_f \in Asso(\mathcal{L})$.

(iii) Let $f \in Dec$ with $f = \min_<$, i.e., $\alpha \not\sim \beta$ implies $\alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha$. Then , if $<_1$ is the induced ordering on $Sen(L)/\sim$, by (ii) $\pi_f = \min_{<_1}$ and we have for all $\alpha \not\sim \beta$,

$$[\alpha] <_1 [\beta] \Leftrightarrow \alpha < \beta \Leftrightarrow \neg\beta < \neg\alpha \Leftrightarrow [\neg\beta] <_1 [\neg\alpha] \Leftrightarrow -[\beta] <_1 -[\alpha],$$

which means that $\pi_f \in Dec(\mathcal{L})$. $\dashv$

The connection between $\overline{f}$, $\pi_f$ and $e$ is as follows.

**Lemma 4.2** *For every $f \in Reg$ and every $\varphi \in Sen(L_s)$,*

$$\overline{e}_{\pi_f}(\varphi)) = [\overline{f}(\varphi)]. \tag{25}$$

*Proof.* Pick some $f \in Reg$. We show (25) by induction on the length of $\varphi$.

(a) For $\varphi = \alpha \in Sen(L)$, we have

$$\overline{e}_{\pi_f}(\alpha) = e(\alpha) = [\alpha] = [\overline{f}(\alpha)].$$

(b) Suppose the claim holds for $\varphi$, $\psi$, i.e., $\overline{e}_{\pi_f}(\varphi)) = [\overline{f}(\varphi)]$ and $\overline{e}_{\pi_f}(\psi)) = [\overline{f}(\psi)]$. It suffices to show that it holds for $\varphi|\psi$ (the other cases being trivial), i.e. $\overline{e}_{\pi_f}(\varphi|\psi)) = [\overline{f}(\varphi|\psi)]$. Now in view of the inductive assumptions and the definitions we have

$$\overline{e}_{\pi_f}(\varphi|\psi)) = \pi_f(\overline{e}_{\pi_f}(\varphi), \overline{e}_{\pi_f}(\psi)) = \pi_f([\overline{f}(\varphi)], [\overline{f}(\psi)]) =$$

$$[f(\overline{f}(\varphi), \overline{f}(\psi))] = [\overline{f}(\varphi|\psi)].$$

**Proposition 4.3** *(i) $Taut(\mathcal{F}(\mathcal{L})) \subseteq Taut(Reg)$.*

*(ii) $Taut(Asso(\mathcal{L})) \subseteq Taut(Reg^*)$.*

*(iii) $Taut(Dec(\mathcal{L})) \subseteq Taut(Dec)$.*

*Proof.* (i) Let $\varphi \in Taut(\mathcal{F}(\mathcal{L}))$. Then $\langle \mathcal{L}, \pi, v\rangle \models_a \varphi$ for all $\pi$ and $v$. Pick some $f \in Reg$. Then in particular, $\langle \mathcal{L}, \pi_f, v\rangle \models_a \varphi$, i.e., $\overline{e}_{\pi_f}(\varphi) = 1$. By (25) above it means that $[\overline{f}(\varphi)] = 1 = \top$, therefore, for every two-valued assignment $v$ for $Sen(L)$, $v(\overline{f}(\varphi)) = 1$ or, equivalently, $\langle v, f\rangle \models_s \varphi$ for all $v$ and $f \in Reg$, which means that $\varphi \in Taut(Reg)$.

(ii) Suppose $\varphi \in Taut(Asso(\mathcal{L}))$. It means that $\langle \mathcal{L}, \pi, v\rangle \models_a \varphi$ for all $\pi \in Asso(\mathcal{L})$ and $v$. Pick $f \in Reg^*$. By Lemma 4.1 (ii), $\pi_f \in Asso(\mathcal{L})$, so in particular $\langle \mathcal{L}, \pi_f, e\rangle \models_a \varphi$. Then as in (i) above we obtain $\langle v, f\rangle \models_s \varphi$ for all $v$ and $f \in Reg^*$, so $\varphi \in Taut(Reg^*)$.

(iii) Suppose $\varphi \in Taut(Dec(\mathcal{L}))$, i.e., $\langle \mathcal{L}, \pi, v\rangle \models_a \varphi$ for all $\pi \in Dec(\mathcal{L})$ and $v$. Picking an $f \in Dec$, we have by 4.1 (iii) that $\pi_f \in Dec(\mathcal{L})$, so $\langle \mathcal{L}, \pi_f, e\rangle \models_a \varphi$. Thus we deduce as before that $\langle v, f\rangle \models_s \varphi$ for all $v$ and $f \in Dec$, so $\varphi \in Taut(Dec)$. ⊣

There is a kind of converse to Proposition 4.3.

**Lemma 4.4** *Let $\mathcal{B}$ be an algebra and $v : Sen(L) \to \mathcal{B}$ be a $\mathcal{B}$-valuation. Then:*

*(i) For every $\pi \in \mathcal{F}(\mathcal{B})$ there is a choice function $f_{\pi,v} \in Reg$ on $Sen(L)$ such that for all $\varphi \in Sen(L_s)$,*

$$v\overline{f}_{\pi,v}(\varphi) = \overline{v}_\pi(\varphi). \tag{26}$$

*Therefore for these $\mathcal{B}$, $v$, and $\pi$, and for every $\varphi$,*

$$\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \;\Leftrightarrow\; \langle \mathcal{B}, v, f_{\pi,v}\rangle \models_s \varphi.$$

*(ii) If $\pi \in Asso(\mathcal{B})$, then $f_{\pi,v} \in Reg^*$.*
*(iii) If $\pi \in Dec(\mathcal{B})$, then $f_{\pi,v} \in Dec$.*

*Proof.* (i) Fix $\mathcal{B}$ and $v$ and let $\pi \in \mathcal{F}(\mathcal{B})$. Consider the fibers $v^{-1}(a)$ of $v$. Obviously every fiber is closed under logical equivalence, i.e., $\alpha \in v^{-1}(a)$ implies $[\alpha] \subseteq v^{-1}(a)$. Using AC, pick a representative $\xi_\alpha$ from each equivalence class $[\alpha]$ and let $D = \{\xi_\alpha : \alpha \in Sen(L)\}$. Every $\alpha \in Sen(L)$ is logically equivalent to $\xi_\alpha \in D$. Take a choice function $g_{\pi,v} : [D]^2 \to D$ subject to the following conditions:

- If $v(\alpha) \neq v(\beta)$, then $g_{\pi,v}(\xi_\alpha, \xi_\beta) = \xi_\alpha \iff \pi(v(\alpha), v(\beta)) = v(\alpha)$.
- If $v(\alpha) = v(\beta)$, $g_{\pi,v}(\xi_\alpha, \xi_\beta)$ is defined arbitrarily.

By the help of $g_{\pi,v}$ we define a choice function $f_{\pi,v} \in [Sen(L)]^2 \to Sen(L)$ as follows.

- If $\alpha \not\sim \beta$, then $f_{\pi,v}(\alpha, \beta) = \alpha \iff g_{\pi,v}(\xi_\alpha, \xi_\beta) = \xi_\alpha$.
- If $\alpha \sim \beta$, $f_{\pi,v}(\alpha, \beta)$ is defined arbitrarily.

It is easy to see that $f_{\pi,v}$ is regular. Indeed let $\alpha \sim \alpha'$ and $\beta \in Sen(L)$. We have to show $f_{\pi,v}(\alpha, \beta) \sim f_{\pi,v}(\alpha', \beta)$. Now if $\beta \sim \alpha$, then $\beta \sim \alpha'$ and the claim holds trivially. If $\beta \not\sim \alpha$, then also $\beta \not\sim \alpha'$, so $f_{\pi,v}(\alpha, \beta)$ and $f_{\pi,v}(\alpha', \beta)$ are defined via $g_{\pi,v}(\xi_\alpha, \xi_\beta)$ and $g_{\pi,v}(\xi_{\alpha'}, \xi_\beta)$, respectively. But $\xi_\alpha = \xi_{\alpha'}$ since $\alpha \sim \alpha'$, i.e., $g_{\pi,v}(\xi_\alpha, \xi_\beta) = g_{\pi,v}(\xi_{\alpha'}, \xi_\beta)$. Therefore $f_{\pi,v}(\alpha, \beta) \sim f_{\pi,v}(\alpha', \beta)$.

It follows from the above definition of $f_{\pi,v}$ that for all $\alpha, \beta \in Sen(L)$ such that $v(\alpha) \neq v(\beta)$ (hence also $\alpha \not\sim \beta$),

$$f_{\pi,v}(\alpha, \beta) = \alpha \iff \pi(v(\alpha), v(\beta)) = v(\alpha). \tag{27}$$

Next we show (26) by induction on the length of $\varphi$. Clearly it suffices to assume that

$$v\overline{f}_{\pi,v}(\varphi) = \overline{v}_\pi(\varphi), \quad v\overline{f}_{\pi,v}(\psi) = \overline{v}_\pi(\psi), \tag{28}$$

and prove

$$v\overline{f}_{\pi,v}(\varphi|\psi) = \overline{v}_\pi(\varphi|\psi). \tag{29}$$

For simplicity we write $f$ instead of $f_{\pi,v}$. If $v\overline{f}(\varphi) = v\overline{f}(\psi)$, then by the inductive assumptions (28) also $\overline{v}_\pi(\varphi) = \overline{v}_\pi(\psi)$, so (29) follows immediately. So assume that $v\overline{f}(\varphi) \neq v\overline{f}(\psi)$ and let $f(\overline{f}(\varphi), \overline{f}(\psi)) = \overline{f}(\varphi)$. Then on the one hand

$$v\overline{f}(\varphi|\psi) = vf(\overline{f}(\varphi), \overline{f}(\psi)) = v\overline{f}(\varphi), \tag{30}$$

and on the other it follows by (27) that

$$\pi(v\overline{f}(\varphi), v\overline{f}(\psi)) = v\overline{f}(\varphi).$$

In view of the last equality and the induction assumptions (28) we have

$$\overline{v}_\pi(\varphi|\psi) = \pi(\overline{v}_\pi(\varphi), \overline{v}_\pi(\psi)) = \pi(v\overline{f}(\varphi), v\overline{f}(\psi)) = v\overline{f}(\varphi). \qquad (31)$$

Thus (30) and (31) entail (29). ⊣

(ii) Let $\pi \in Asso(\mathcal{B})$ and let $<$ be the total ordering of $B$ for which $\pi = \min_<$. Then $<$ naturally induces a total ordering $<_1$ of the fibers $v^{-1}(a)$ of $v$, for $a \in B$, which partition $Sen(L)$. For $a \neq b \in B$,

$$v^{-1}(a) <_1 v^{-1}(b) \iff a < b.$$

Since $\alpha \in v^{-1}(a)$ implies $[\alpha] \subseteq v^{-1}(a)$, clearly $<_1$ can be refined (not uniquely) to a *regular* total ordering $<_{v,\pi}$ of $Sen(L)$ such that for all $\alpha, \beta \in Sen(L)$ such that $v(\alpha) \neq v(\beta)$,

$$\alpha <_{v,\pi} \beta \iff v(\alpha) < v(\beta). \qquad (32)$$

If we set $f_{\pi,v} = \min <_{\pi,v}$, then $f_{\pi,v}$ is associative and regular, i.e., $f_{\pi,v} \in Reg^*$. Also (32) becomes equivalent to (27), and so it follows by the previous argument that $f_{\pi,v}$ satisfies (26).

(iii) Finally let $\pi = \min_<$ and $<$ be $-$-decreasing. Let again $<_1$ be the total ordering of fibers, defined in (ii). Every total ordering $<'$ of $Sen(L)$ that refines $<_1$ satisfies $\neg$-decreasingness for $\alpha, \beta$ such that $v(\alpha) \neq v(\beta)$. Because for such pairs we have by (32),

$$\alpha <' \beta \Leftrightarrow v(\alpha) < v(\beta) \Leftrightarrow -v(\beta) < -v(\alpha) \Leftrightarrow$$

$$v(\neg\beta) < v(\neg\alpha) \Leftrightarrow \neg\beta <' \neg\alpha.$$

Now by the same method as in the proof of Theorem 2.43 of [6], we can easily construct an ordering $<_{\pi,v}$ which is $\neg$-decreasing and also satisfies condition (32). Setting $f_{\pi,v} = \min <_{\pi,v}$, then $f_{\pi,v} \in Dec$ and satisfies 27, which guarantees that (26) holds. ⊣

**Proposition 4.5** *For every algebra $\mathcal{B}$ we have:*
*(i) $Taut(Reg) \subseteq Taut(\mathcal{F}(\mathcal{B}))$.*
*(ii) $Taut(Reg^*) \subseteq Taut(Asso(\mathcal{B}))$.*
*(iii) $Taut(Dec) \subseteq Taut(Dec(\mathcal{B}))$.*

*Proof.* (i) Let $\varphi \in Taut(Reg)$, and let $v : Sen(L) \to \mathcal{B}$ and $\pi \in \mathcal{F}(\mathcal{B})$. It suffices to show that $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi$. Given $v$ and $\pi$ there is, by Lemma 4.4 (i), an $f_{\pi,v} \in Reg$ such that

$$\langle \mathcal{B}, \pi, v\rangle \models_a \varphi \;\Leftrightarrow\; \langle \mathcal{B}, f_{\pi,v}, v\rangle \models_s \varphi.$$

Since $\varphi \in Taut(Reg)$, the right-hand side of this equivalence is true, hence so is the left-hand side. Thus $\langle \mathcal{B}, \pi, v\rangle \models_a \varphi$ for all $v$ and $\pi$, so $\varphi \in Taut(\mathcal{F}(\mathcal{B})$. Clauses (ii) and (iii) follow similarly from clauses (ii) and (iii) of 4.4. ⊣

**Theorem 4.6** *If $\mathcal{L}$ is the Lindenbaum algebra, then:*
*(i)* $Taut(Reg) = Taut(\mathcal{F}(\mathcal{L}))$.
*(ii)* $Taut(Reg^*) = Taut(Asso(\mathcal{L}))$.
*(iii)* $Taut(Dec) = Taut(Dec(\mathcal{L}))$.

*Proof.* Immediate consequence of Propositions 4.3 and 4.5. $\dashv$

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