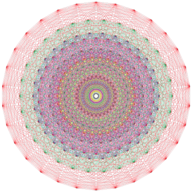
Jordan algebras, non-linear PDE's and integrability

With warm wishes to Dmitri Prokhorov on his 65 birthday

Vladimir Tkachev

Åsö Vuxenutbildning / KTH, Stockholm, Sweden



Why Jordan algebras?

One of reasons is because

$$65 = 5 \cdot 13 = \mathbf{5} \cdot (\mathbf{14} - 1)$$

Some other reasons: applications of Jordan algebras

- Origins in Quantum mechanics (27-dimensional Albert's exceptional algebra)
- Non-associative algebras (Zelmanov's theory)
- Self-dual homogeneous cones (Vinberg's and Koecher's theory)
- Lie algebras (Lie algebra functor, Exceptional Lie algebras, Freudental's magic square)
- Non-linear PDE's (integrable hierarchies, Generalized KdV equation, regularity of Hessian equations, higherdimensional minimal surface equation)
- Extremal black hole, supergravity
- Operator theory (JB-algebras)
- Differential geometry (symmetric spaces, projective geometry, isoparametric hypersurfaces)
- Statistics (Wishart distributions on Hermitian matrices and on Euclidean Jordan algebras)

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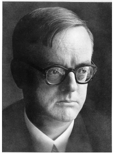
Origins in Quantum mechanics: The Jordan program

Kevin McCrimmon, Taste of Jordan algebras, 2004:

> *I am unable to prove Jordan algebras were known to Archimedes, or that a complete theory has been found in the unpublished papers of Gauss. Their first appearance in recorded history seems to be in the early 1930's when the theory bursts forth full-grown from the mind, not of Zeus, but of*



Pascual Jordan John von Neumann Eugene Wigner

> *in their 1934 paper on an algebraic generalization of the quantum mechanical formalism.*

The Jordan program

The usual matrix operations are not ‘observable’.

Matrix operations:		**Observable operations:**	
λx	multiplication by a $\mathbb{C}$-scalar	αx	multiplication by a $\mathbb{R}$-scalar
$x+y$	addition	$x+y$	addition
xy	multiplication of matrices	x^k	powers of matrices
x^*	complex conjugate	x	identity map

The matrix interpretation was philosophically unsatisfactory because it derived the observable algebraic structure from an unobservable one.

The Jordan program

In 1932 Jordan proposed a program to discover a new algebraic setting for quantum mechanics:

- it would be freed from dependence on an invisible but all-determining metaphysical matrix structure,
- yet would enjoy all the same algebraic benefits as the highly successful Copenhagen model;
- to capture intrinsic algebraic properties of Hermitian matrices, and then to see what other possible non-matrix systems satisfied these axioms.

Jordan multiplication
By linearizing the quadratic squaring operation, to replace the usual matrix multiplication by the anticommutator product (called also the Jordan product)

$$x \bullet y = \frac{1}{2}(xy + yx);$$

The Jordan program

A Jordan algebra J (over $\mathbb{F}$) is vector space defined equipped with a bilinear product $\bullet : J \times J \to J$ satisfying the

$$x \bullet y = y \bullet x \qquad \text{Commutativity}$$

$$x^2 \bullet (x \bullet x) = x \bullet (x^2 \bullet y) \qquad \text{the Jordan identity}$$

In other words, the multiplication operator $L_x y = x \bullet y$ satisfies

$$[L_x, L_{x^2}] = 0.$$

The Jordan quest: *Can quantum theory be based on the commutative and non-associative product* $x \bullet y = \frac{1}{2}(xy + yx)$ *alone, or do we need the associative product* xy *somewhere in the background?*

A positivity condition (comes from Artin-Schreier theory): an algebra is called *formally real* if

$$x_1^2 + \ldots + x_k^2 = 0 \quad \Rightarrow \quad x_1 = \ldots = x_k = 0. \tag{1}$$

The Jordan program: Special algebras

"Jordanization":
Given an associative algebra A with product xy, the linear space A (denoted A^+) with the Jordan product

$$x \bullet y = \frac{1}{2}(xy + yx)$$

becomes a Jordan algebra. Such a Jordan algebra is called special.

Compare with the Lie algebra construction from an associative algebra A: the skew-symmetric product is defined by

$$[x, y] = \frac{1}{2}(xy - yx)$$

such that $(A, [\ ])$ becomes a **Lie algebra**.

Observe that **all** Lie algebras are special.

The Jordan program: Special algebras

The classical (associative) division algebras:

- the reals $\mathbb{F}_1 = \mathbb{R}$,
- the complexes $\mathbb{F}_2 = \mathbb{C}$
- the quaternions $\mathbb{F}_4 = \mathbb{H}$.

Two basic examples of Jordan algebras builded of $\mathbb{F}_d$:

- $M(n, \mathbb{F}_d)$, $n \times n$ matrices over $\mathbb{F}_d$,
- a Jordan subalgebra of $M(n, \mathbb{F}_d)$ consisting of hermitian matrices:

$$\mathfrak{h}_n(\mathbb{F}_d) = \{x \in M(n, \mathbb{F}_d) : \overline{x^t} = x\}$$

Classification of formally real Jordan algebras

P. Jordan, J. von Neumann, E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, *Annals of Math.*, **1934**

Any (finite-dimensional) formally real Jordan algebra is a direct sum of the following simple ones:

- Three 'invited guests' (special algebras)...
  - $\mathfrak{h}_n(\mathbb{R})$;
  - $\mathfrak{h}_n(\mathbb{C})$;
  - $\mathfrak{h}_n(\mathbb{H})$;
- ...and two new structures which met the Jordan axioms but were not themselves hermitian matrices:
  - $\mathfrak{J}_n(|x|^2)$, the spin factors (not to be confused with *spinors*);
  - $\mathfrak{h}_3(\mathbb{O})$, hermitian matrices of size 3×3 over the octonions $\mathbb{O}$, (also known as the Albert algebra)

More about the spin-factor $\mathfrak{J}_n(Q)$

Definition of $\mathfrak{J}_n(Q)$

Given a non-degenerate quadratic form Q, one defines a multiplication on $\mathbf{1}\mathbb{R} \oplus \mathbb{R}^n$ by making the distinguished element $\mathbf{1}$ acting as unit, and the product of two vectors $v, w \in \mathbb{R}^n$ to be given by

$$(x_0, x) \bullet (y_0, y) = (x_0 y_0 + Q(x, y), x_0 y + y_0 x), \qquad \mathbf{1} = (1, 0).$$

- If Q is positive definite then $\mathfrak{J}_n(Q)$ is formally real
- $\mathfrak{J}_n(Q)$ can be realized as a certain subspace of $\mathfrak{h}_{2^n}(\mathbb{R}) \Rightarrow$ is special
- The hermitian 2×2 matrices are actually a spin factors:

$$\mathfrak{h}_2(\mathbb{F}_d) = \mathfrak{J}_{1+d}(-|x|^2), \qquad d = 1, 2, 4, 8.$$

Though Jordan algebras were invented to study quantum mechanics, the spin factors are also deeply related to special relativity.

The exceptional Albert algebra $\mathfrak{h}_3(\mathbb{O})$

Hermitian matrices over **octonions**:

- For $n = 2$, $\mathfrak{h}_2(\mathbb{O}) \cong \mathfrak{J}_9(-|x|^2)$, hence special.
- For $n \geq 4$, $\mathfrak{h}_n(\mathbb{O})$ is not a Jordan algebra at all.
- For $n = 3$:

$$\mathfrak{h}_3(\mathbb{O}) = \text{ the 27-dim space of matrices } \begin{pmatrix} t_1 & z_3 & \overline{z}_2 \\ \overline{z}_3 & t_2 & z_1 \\ z_2 & \overline{z}_1 & t_3 \end{pmatrix}, \quad t_i \in \mathbb{R},\ z_i \in \mathbb{O}$$

Adrian Albert (1934)

$\mathfrak{h}_3(\mathbb{O})$ is an exceptional Jordan algebra, i.e. it cannot be imbedded in any associative algebra.

But this lone exceptional algebra $\mathfrak{h}_3(\mathbb{O})$ was too tiny to provide a home for quantum mechanics, and too isolated to give a clue as to the possible existence of infinite-dimensional exceptional algebras.

The exceptional Albert algebra $\mathfrak{h}_3(\mathbb{O})$

In 1979, Efim Zelmanov (a Fields medal, 1994) quashed all remaining hopes for such an exceptional systems. He showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than Albert algebras.

> *...and there is no new thing under the sun especially in the way of exceptional Jordan algebras; unto mortals the Albert algebra alone is given.*
>
> (McCrimmon, Taste of Jordan algebras, 2004)

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

The five exceptional Lie algebras and groups G_2, F_4, E_6, E_7 and E_8 appeared mysteriously in the nineteenth-century Cartan-Killing classification and were originally defined in terms of multiplication tables.

In 1930s – '50s N. Jacobson, C. Chevalley, R.D. Schafer, J. Tits, H. Freudenthal, E. Vinberg obtained in an intrinsic coordinate-free representations using the Albert algebra and octonions.

- G_2 is the the automorphism group of $\mathbb{O}$;
- F_4 is the automorphism group of $\mathfrak{h}_3(\mathbb{O})$;
- E_6 is the isotopy group of $\mathfrak{h}_3(\mathbb{O})$;
- E_7 is the superstructure Lie algebra of $\mathfrak{h}_3(\mathbb{O})$;
- E_8 is connected to $\mathfrak{h}_3(\mathbb{O})$ in a more complicated manner.

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

The **Freudenthal-Tits-Vinberg magic square** (a symmetric version)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$sp(3)$	F_4
$\mathbb{C}$	$su(3)$	$su(3) \oplus su(3)$	$su(6)$	E_6
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

answer to the question how to construct explicitly these representations ("the exceptional Lie groups all exist because of the octonions").

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

Main ingredients: Triality and Tits-Kantor-Koecher construction.

Duality. If $\dim V_1 = \dim V_2 < \infty$ then there exists a nondegenerate bilinear form

$$\phi : V_1 \times V_2 \to \mathbb{R}.$$

Triality comes from Èlie Cartan's investigations (1925) of a trilateral symmetry in the Lie group D_4. Algebraically, for real vector spaces V_1, V_2, V_3, **triality** is a nondegenerate trilinear form

$$\psi : V_1 \times V_2 \times V_3 \to \mathbb{R}.$$

Triality comes from division algebras and exist only if

$$\dim V_1 = \dim V_2 = \dim V_3 = d, \quad d \in \{1, 2, 4, 8\},$$

Specifically, if V is a division algebra then by dualizing and identifying $V \cong V^*$ one gets

$$\text{multiplication} : V \times V \to V = V^*.$$

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

The Tits-Kantor-Koecher construction

In the matrix algebra $M_n(\mathbb{R})$,

$$xy = \frac{1}{2}(xy - yx) + \frac{1}{2}(xy + yx) \equiv [x, y] + x \bullet y,$$

i.e. the first term leads to the **Lie algebra** $\mathfrak{gl}(n, \mathbb{R})$, and the second term to the **Jordan algebra** $M_n^+(\mathbb{R})$.

The **TKK construction** allows one to construct a Lie algebra from a Jordan algebra, introduced by Tits (1962), Kantor (1964), and Koecher (1967).

Triple products and systems:

$$\begin{aligned} [x, y, z] &= [[x, y,]z]], && \text{for a Lie algebra} \\ (x, y, z) &= (x \bullet y) \bullet z + x \bullet (y \bullet z) - y \bullet (x \bullet z) && \text{for a Jordan algebra} \end{aligned}$$

The exceptional Lie algebras, $\mathfrak{h}_3(\mathbb{O})$, and triality

If $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a graded Lie algebra with involution $\tau : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ then it gives rise to a triple Jordan system $J = \mathfrak{g}_{-1}$ via

$$(x, y, z) = [[x, \tau(y)], z].$$

Conversely, given a Jordan triple system J one can construct a graded Lie algebra $\mathfrak{g} = \bigoplus_{i=-1}^{1} \mathfrak{g}_i$ with $\mathfrak{g}_{-1} = J$.

The TKK construction is a powerfull instrument that allows one to transfer results in the Lie algebras theory to the theory of Jordan algebras, and vice versa.

Jordan algebras and symmetric cones

Self-dual homogeneous cones

$V = \mathbb{R}^n$ equipped with the scalar product $\langle x, y\rangle$;
$\Omega \subset V$ is an open convex cone;
$\Omega^* = \{y \in V : \langle x, y\rangle > 0, \forall x \in \overline{\Omega} \setminus 0\}$ is the open dual cone
$G(\Omega) = \{g \in GL(V) : \ g\Omega = \Omega\}$ is the automorphism group of Ω

Definition

Ω is called **homogeneous** if $G(V)$ acts on it transitively, and Ω is called **symmetric** if Ω is homogeneous and self-dual ($\Omega = \Omega^*$);

Examples:

- The Lorentz cone $\Lambda_n = \{x \in \mathbb{R}^{n+1} : \ x_0^2 - x_1^2 - \ldots - x_n^2 > 0, \ x_0 > 0\}$.
- The cone of positive definite symmetric matrices.

Self-dual homogeneous cones

Example (the Lorentz cone Λ_n)

$$\Lambda_n = \{x \in V : m(x,x) > 0,\ x_0 > 0\},$$

where $m(x,y) = x_0 y_0 - \sum_{i=1}^{n} x_i y_i$ is the Minkowski scalar product on V. One can show that the group

$$G = \mathbb{R}_+ \times \mathrm{SO}_0(1,n) = \text{dilatations} \times \text{rotations}$$

acts transitively on Λ_n and by using Schwarz's inequality derive that Λ_n is self-dual. If one defines a multiplication on V by

$$x \bullet y = (\langle x, y\rangle,\ x_0\bar{y} + y_0\bar{x}), \qquad \bar{x} = (x_1, \ldots, x_n),$$

then

$$x^{\bullet 2} = x \bullet x = (x_0^2 + |\bar{x}|^2,\ 2x_0\bar{x}).$$

and

$$m(x^{\bullet 2}, x^{\bullet 2}) = (x_0^2 + |\bar{x}|^2)^2 - (2x_0\bar{x})^2 = (x_0^2 - |\bar{x}|^2)^2 > 0$$

readily implies that

$$\Lambda_n = V^{\bullet 2} := \{x^{\bullet 2} : x \in V\}.$$

Self-dual homogeneous cones

M. Koecher (1957), **Rothhaus** (1960), **E. Vinberg** (1963):
Given a symmetric cone Ω, one can naturally induce on V a structure of formally real Jordan algebra with identity $\mathbf{1} \in \Omega$ such that

$$\text{closure}(\Omega) = V^{\bullet 2} := \{x^{\bullet 2} : x \in V\}.$$

Koecher-Vinberg theory

The self-dual cones with homogeneous interior in real Hilbert spaces of finite dimension are precisely (up to linear equivalence) the cones of squares in formally real Jordan algebras

The correspondence:

- The Lorentz cone
- Positive definite symmetric matrices in $\mathbb{R}^n$

- The spin-factor $\mathfrak{J}_{n-1}(|x^2|)$
- $\mathfrak{h}_n(\mathbb{R})$

Self-dual homogeneous cones

Recent applications in:

- **Tube domains over symmetric cones** (Bergman kernel, Hardy spaces, special functions);
- **Wishart distributions** on Hermitian matrices and on Euclidean Jordan algebras;
- **Scorza** and **Severi varieties, prehomogeneous vector spaces** (F. Zak, P. Etingof, D.Kazhdan, P.-E. Chaput)
- The geometry of **Maxwell-Einstein supergravity** (M. Gűnaydin, G. Sierra, P. Townsend, M. Duff, etc)

Jordan algebras and non-linear PDE's

Jordan algebras and integrable systems

A prototypical example is the Korteweg-de Vries equation

$$u_t = u_{xxx} - 6uu_x.$$

V. Sokolov and **V. Drinfel'd** (1986) studied the integrability of the system

$$u^i_x = \lambda^i_j u^j_{xxx} + a^i_{jk} u^j u^k_x, \qquad i,j,k = 1,\ldots,N, \quad a^i_{jk} = a^i_{kj}$$

in context of Kac-Moody algebras.

C. Athorne and **A. Fordy** (1987), associated a similar class of generalized KdV and MKdV equations, and the associated Miura transformations to (Hermitian) symmetric spaces.

S. Svinolupov and **V. Sokolov** in 1990's established the integrability of

$$w^i_x = w^j_{xxx} - 6a^i_{jk} w^j w^k_x, \qquad a^i_{jk} = a^i_{kj} \tag{2}$$

by means of Jordan algebras (Jordan triple systems).

Integrability question: How to determine a^k_{jk} such that (2) possesses higher symmetries or conservation laws?

Jordan algebras and integrable systems

Svinolupov's approach. The system

$$w^i_x = w^j_{xxx} - 6a^i_{jk} w^j w^k_x, \qquad w : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}^n \tag{2}$$

is called a generalization of the scalar KdV equation if $a^i_{jk} = a^i_{kj}$ and (2) it possesses generalized symmetries and local conservation laws.

- Associate to (2) the N-dimensional commutative algebra J with structural constants a^i_{jk}, i.e. the multiplication in J is defined by

  $$e_j \bullet e_k = e_k \bullet e_j = a^i_{jk} e_i,$$

  where e_i is some basis in J, and set $W = e_i w^i \in J$. (The multiplication law is actually independent of the choice of basis e_i and reduces to an equivalent system).
- (2) becomes a differential equation in the algebra J:

  $$W_t = W_{xxx} - 6W \bullet W_x$$

Jordan algebras and integrable systems

The existence of local conservation laws or generalized symmetries of (2) implies rather restrictive conditions on the algebra J. In fact, one has

Svinolupov, 1991

- System (2) possesses nondegenerate generalized symmetries or conservation laws if and only if the corresponding commutative algebra J is a Jordan algebra.
- The system (2) splits into a sum of two systems if and only if J is reducible.

Example. If $J = \mathfrak{J}_1(|x|^2)$ is a spin-factor then one obtains the Jordan KdV-system

$$\begin{aligned} u_t &= u_{xxx} - 3(u^2 + v^2)_x \\ v_t &= v_{xxx} - 6(uv)_x \end{aligned} \tag{3}$$

Jordan algebras and integrable systems

It is well-known that

$$\begin{aligned} w_t &= w_{xxx} - 6ww_x, && \text{the scalar KdV} \\ u_t &= u_{xxx} - 6u^2u_x, && \text{the scalar mKdV} \end{aligned} \tag{4}$$

are related by the Miura transformation $w = u_x + u^2$

The generalized Miura transformation (Svinolupov, 1991)

The system

$$W_t = W_{xxx} - 6W \bullet W_x, \quad \alpha \in \mathbb{R}, \; W \in J, \tag{5}$$

admits a (generalized Miura) substitution

$$W = U_x + U^{\bullet 2}$$

and the modified system related to (5) by the generalized Miura transformation is

$$U_t = U_{xxx} - 6(U^{\bullet 2} \bullet U_x).$$

Singular solutions of fully-nonlinear PDE's

We consider the Dirichlet problem

$$\begin{aligned} F(D^2u) &= 0, \ \text{ in } \quad \Omega \\ u &= \phi \ \text{ on } \quad \partial\Omega \end{aligned}$$

where F is Lipschitz, defined on an open subset of $n \times n$ symmetric matrices.

(For instance, Monge-Ampère, Pucci, Bellman equations.)

Basic question: is the viscosity solution of the Dirichlet problem going to be C^2?

Singular solutions of fully-nonlinear PDE's

- **Evans, Crandall, Lions, Jensen, Ishii**: If $\Omega \subset \mathbb{R}^n$ is bounded with C^1-boundary, ϕ continuous on $\partial\Omega$, F uniformly elliptic operator then the Dirichlet problem

$$\begin{aligned} F(D^2u) = 0, \ \ &in \quad \Omega \\ u = \phi \ \ &on \quad \partial\Omega \end{aligned}$$

  has a unique viscosity solution $u \in C(\Omega)$;
- **Krylov, Safonov, Trudinger, Caffarelli**, early 80's: the solution is always $C^{1,\varepsilon}$
- **Nirenberg**, 50's: if $n = 2$ then u is classical (C^2) solution
- **Nadirashvili, Vlăduţ**, 2007: if $n = 12$ then there are solutions which are not C^2

In 2005–2011, N. Nadirashvili and S. Vlăduţ constructed (uniformly elliptic) Hessian equations with $F(S)$ being smooth, homogeneous, depending only on the eigenvalues of S, and such that they have singular $C^{1,\delta}$-solutions (not $C^{1,\delta+\epsilon}$).

The function

$$u(x) = \frac{\operatorname{Re} z_1 z_2 z_3}{|x|}, \qquad x = (z_1, z_2, z_3),$$

where $z_i \in \mathbb{F}_d$, $d = 4, 8$ (quaternions, octonions), is a viscosity solution of a fully nonlinear uniformly elliptic equation.

In fact, the numerator is the generic determinant of a generic diagonal free element in the Jordan algebra $\mathfrak{h}_3(\mathbb{F}_d)$,

$$\operatorname{Re} v_1 v_2 v_3 = \frac{1}{2} \det \begin{pmatrix} 0 & z_3 & \overline{z}_2 \\ \overline{z}_3 & 0 & z_1 \\ z_2 & \overline{z}_1 & 0 \end{pmatrix}$$

Cubic minimal cones

In 1969, **E. Bombieri**, **E. De Giorgi** and **E. Giusti** found the first non-affine entire solution of the minimal surface equation

$$(1+|Du|^2)\Delta u - \frac{1}{2}Du\cdot(D|Du|^2) = 0, \qquad x \in \mathbb{R}^8.$$

The construction heavily depends on certain properties of the quadratic minimal (Clifford–Simons) cones over $S^{p-1}\times S^{q-1}$.

A search and characterization of cubic and higher order minimal cones is a long-standing problem. Algebraically, this reduces to finding homogeneous solutions of

$$|Du|^2\Delta u - \frac{1}{2}Du\cdot(D|Du|^2) \equiv 0 \mod u.$$

- **H.B. Lawson, R.Osserman, W. Hsiang**, early 70's: examples of cones of deg $=3,4,6$ sporadically distributed in $\mathbb{R}^n$.
- **L.Simon, B. Solomon**, 80's: construction of higherdimensional minimal graphs based on isoparametric hypersurfaces in S^n;

Cubic minimal cones

V.T. [2010], we obtained a particular classification of cubic minimal cones in the case

$$|Du|^2\Delta u - \frac{1}{2}Du\cdot(D|Du|^2) = \lambda|x|^2\cdot u^2$$

Some known examples ($d = 1, 2, 4, 8$):

- Four Cartan's isoparametric cubics in $n = 5, 8, 14$ and 26, based on $\mathfrak{h}_3(\mathbb{F}_d)$:

$$u = \det X, \qquad X\in\mathfrak{h}_3(\mathbb{F}_d), \quad \operatorname{tr} X = 0.$$

- Lawson cubic (an instance of a Clifford type eigencubic)

$$u = \operatorname{Re}(z^2\bar{w}), \qquad z, w\in\mathbb{C}$$

- 'Mutates'

$$u = \operatorname{Re} z_1z_2z_3 \equiv \det X, \qquad X\in\mathfrak{h}_3(\mathbb{F}_d),\ \operatorname{diagonal}(X) = 0,$$

where $z_i\in\mathbb{F}_d$.

Cubic minimal cones

Two key ingredients: **the cubic trace identity**

$$\operatorname{tr}(\operatorname{Hess} u)^3 = \alpha u,$$

and **The eiconal cubic theorem**, V.T. (2011), preprint

Cubic minimal cones

The eiconal cubic theorem provides a generalization of an È. Cartan result on isoparametric eigencubics.

Recall that a submanifold of the Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ is called isoparametric if it has constant principal curvatures. A celebrated result due to H.F. Münzner (1987) asserts that *any isoparametric hypersurface is algebraic and its defining polynomial u is homogeneous of degree $g = 1, 2, 3, 4$ or 6, where g is the number of distinct principal curvatures.*

In the late 1930-s, Élie Cartan classified isoparametric hypersurfaces with $g = 3$ different principal curvatures and proved that any such a hypersurface is a level set of a **harmonic** cubic polynomial solution of (6). Moreover, Cartan showed that there are exactly four (congruence classes of) such solutions, expressed in terms of Jordan algebras as

$$u(x) = \sqrt{2} \cdot \det x, \qquad x \in \mathfrak{h}_3(\mathbb{F}_d), \ \operatorname{tr} x = 0 \ \ d = 1, 2, 4, 8.$$

If u is supposed to be an **arbitrary** cubic polynomial then there is exactly one additional infinite family of solutions (**V.T.**, *Proc. Amer. Math. Soc.*, 2010).

Cubic minimal cones

The eiconal cubic theorem, V.T., 2011, preprint

There is a natural one-to-one correspondence between the isomorphism classes of formally real Jordan algebras of rank 3 and the congruence classes cubic solutions of the eiconal equation

$$|\nabla u|^2 = 9|x|^4, \quad x \in \mathbb{R}^n. \tag{6}$$

Namely, any solution of (6) is given by by

$$u = \sqrt{2}\det X, \quad X \in J_u^0,$$

where $J = J_u$ is a Jordan algebra of rank 3, and $J_u^0 = \{X \in J : \operatorname{tr} X = 0\}$.

Cubic minimal cones

The Jordan structure can be described explicitly as follows.

Let u be a cubic polynomial solution of

$$|\nabla u|^2 = 9|x|^4, \quad x \in \mathbb{R}^n.$$

Define a multiplication on $J_f = \mathbb{R} \oplus \mathbb{R}^n$ by

$$(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle,\ x_0 y + y_0 x + \frac{1}{6\sqrt{2}} \operatorname{Hess}_x(f) y). \tag{7}$$

Then $(J_f, \bullet)$ is a formally real Jordan algebra of rank 3 with the unit element $c = (1, 0)$ and the following holds:

(a) any element $X = (x_0, x) \in J_f$ satisfies the (minimal) cubic relation

$$X^3 - T(X)X^2 + S(X)X - N(X)c = 0,$$

(b) the cubic eiconal f is recovered from the generic norm by

$$f(x) = \sqrt{2}\, \det(i(x)) \tag{8}$$

where $i(x) = (0, x) : \mathbb{R}^n \to J_f^0 = \{\operatorname{tr} X = 0\}$ is the standard embedding.

(c) if f_1 and f_2 are two congruent cubic solutions then the Jordan algebras J_1 and J_2 are isomorphic.

According to the mentioned above classification of P. Jordan, von Neumann and E. Wigner, any formally real Jordan algebra is a direct sum of simple ones, and the only simple formally real Jordan algebras are

1. rank $m = 1$: the field of reals $\mathbb{F}_1 = \mathbb{R}$;
2. rank $m = 2$: the spin factors $\mathfrak{J}_n(|x|^2)$, $n \geq 2$;
3. rank $m = 3$: the Hermitian algebras $\mathfrak{h}_3(\mathbb{F}_d)$, $d = 1, 2, 4$, and, additionally, the Albert exceptional algebra $\mathcal{H}_3(\mathbb{F}_8)$;
4. rank $m \geq 4$: the Hermitian algebras $\mathfrak{h}_m(\mathbb{F}_d)$, $d = 1, 2, 4$.

This yields the complete list of all (isomorphic classes of) formally real rank 3 Jordan algebras:

(i) $\mathfrak{h}_3(\mathbb{F}_d)$, $d = 1, 2, 4, 8$;

(ii) the reduced spin algebra $\mathbb{R} \oplus \mathfrak{J}_n$, $n \geq 2$;

(iii) the reduced algebra $\mathbb{F}_1^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with the coordinate-wise multiplication.

More explicitly: the only (cubic) solutions of

$$|\nabla u|^2 = 9|x|^4, \quad x \in \mathbb{R}^n,$$

are:

- The Cartan polynomials

$$\frac{\sqrt{3}}{2} \cdot \det \begin{pmatrix} x_{3d+1} - \frac{x_{3d+2}}{\sqrt{3}} & z_3 & \overline{z}_2 \\ \overline{z}_3 & -x_{3d+1} - \frac{x_{3d+2}}{\sqrt{3}} & z_1 \\ z_2 & \overline{z}_1 & \frac{2x_{3d+2}}{\sqrt{3}} \end{pmatrix}$$

- or reducible cubics

$$r_n(x) = x_n^3 - 3x_n(x_1^2 + \ldots x_{n-1}^2), \quad x \in \mathbb{R}^n, n \geq 2.$$

Thank you!