Ch 10.1: Two-Point Boundary Value Problems

- In many important physical problems there are two or more independent variables, so the corresponding mathematical models involve partial differential equations.
- Chapter 10 treats one important method for solving partial differential equations, known as **separation of variables**.
- Its essential feature is the replacement of a partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions.
- Section 10.1 deals with some basic properties of boundary value problems for ordinary differential equations.
- The solution of the partial differential equation is then a sum, usually an infinite series, formed from the solutions to the ordinary differential equations, as we see later in the chapter.

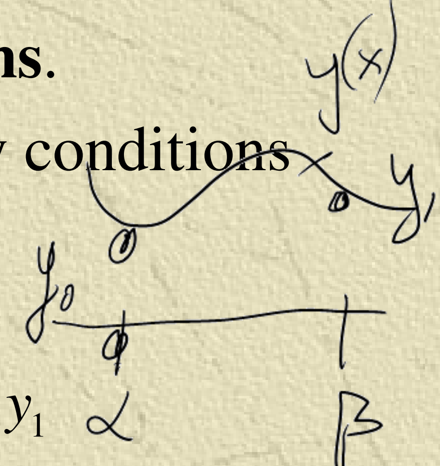
Boundary Value Problems—Ch. 10.1

- Up to this point we have dealt with initial value problems, consisting of a differential equation together with suitable initial conditions at a given point.
- A typical example is

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0,\ y'(t_0) = y_0'$$

- Physical applications often require the dependent variable y or its derivative y' to be specified at two different points.
- Such conditions are called **boundary conditions**.
- The differential equation and suitable boundary conditions form a **two-point boundary value problem**.
- A typical example is

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$



Homogeneous Boundary Value Problems

- The natural occurrence of boundary value problems usually involves a space coordinate as the independent variable, so we use x instead of t in the boundary value problem

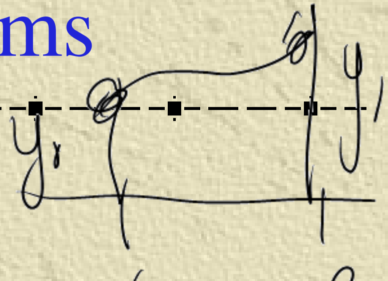
$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0, \; y(\beta) = y_1$$

- Boundary value problems for nonlinear equations can be posed, but we restrict ourselves to linear equations only.
- If the above boundary value problem has the form

$$y'' + p(x)y' + q(x)y = 0, \quad y(\alpha) = 0, \; y(\beta) = 0$$

then it is said to be **homogeneous**. Otherwise, the problem is **nonhomogeneous**.

Solutions to Boundary Value Problems



- To solve the boundary value problem,

  $$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$

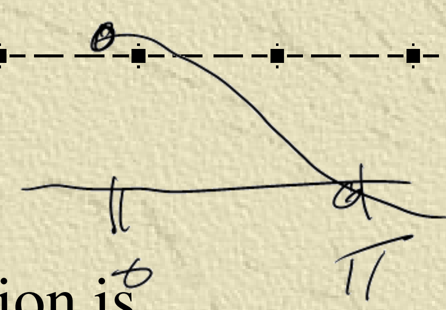
  we need to find a function $y = \phi(x)$ that satisfies the differential equation on the interval $\alpha < x < \beta$ and that takes on the specified values y_0 and y_1 at the endpoints.
- Initial value and boundary value problems may superficially appear similar, but their solutions differ in important ways.
- Under mild conditions on the coefficients, an initial value problem is certain to have a unique solution.
- Yet for similar conditions, boundary value problems may have a unique solution, no solution, or infinitely many solutions.
- In this respect, linear boundary value problems resemble systems of linear algebraic equations.

Linear Systems

- Consider the system **Ax** = **b**, where **A** is an *n* x *n* matrix, **b** is a given *n* x 1 vector, and **x** is an *n* x 1 vector to be determined.
- Recall the following facts (see Section 7.3): $X = A^{-1} b$
  - If **A** is nonsingular, then **Ax** = **b** has unique solution for any **b**.
  - If **A** is singular, then **Ax** = **b** has no solution unless **b** satisfies a certain additional condition, in which case there are infinitely many solutions.
  - The homogeneous system **Ax** = **0** always has the solution **x** = **0**.
  - If **A** is nonsingular, then this is the only solution, but if **A** is singular, then there are infinitely many (nonzero) solutions.
- Thus the nonhomogeneous system has a unique solution iff the homogeneous system has only the solution **x** = **0**, and

  the nonhomogeneous system has either no solution or infinitely many solutions iff homogeneous system has nonzero solutions.

Example 1



- Consider the boundary value problem

$$y'' + 2y = 0, \quad y(0) = 1, \; y(\pi) = 0$$

- The general solution of the differential equation is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$

$y = \cos(\sqrt{2}x)$

- The first boundary condition requires that $c_1 = 1$. $\Rightarrow$ $+ c_2 \sin(\sqrt{2}x)$
- From the second boundary condition, we have

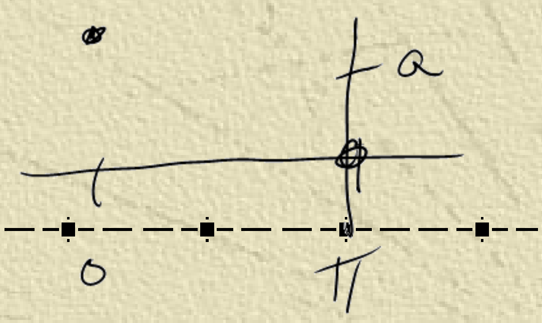
$$c_1 \cos\sqrt{2}\pi + c_2 \sin\sqrt{2}\pi = 0 \quad \Rightarrow \quad c_2 = -\cot\sqrt{2}\pi \cong -0.2762$$

- Thus the solution to the boundary value problem is

$$y = \cos\sqrt{2}x - \cot\sqrt{2}\pi \sin\sqrt{2}x$$

- This is an example of a nonhomogeneous boundary value problem with a unique solution.

Example 2



- Consider the boundary value problem

$$y'' + y = 0, \quad y(0) = 1, \; y(\pi) = a, \qquad a > 0 \text{ arbitrary.}$$

- The general solution of the differential equation is

$$y = c_1 \cos x + c_2 \sin x$$

$$c_1 \underline{\cos\pi} + c_2 \overset{0}{\sin\pi} = a$$

- The first boundary condition requires that $c_1 = 1$, while the second requires $c_1 = -a$. Thus there is no solution.
- However, if $a = -1$, then there are infinitely many solutions:

$$y = \cos x + c_2 \sin x, \quad c_2 \text{ arbitrary}$$

- This example illustrates that a nonhomogeneous boundary value problem may have no solution, and also that under special circumstances it may have infinitely many solutions.

Nonhomogeneous Boundary Value Problem and Corresponding Homogeneous Problem

- Corresponding to a nonhomogeneous boundary value problem

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$

  is the homogeneous problem

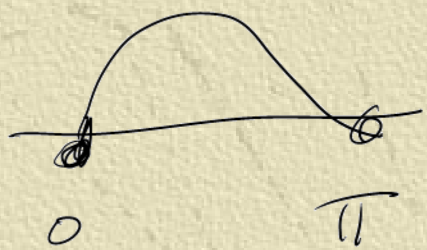
$$y'' + p(x)y' + q(x)y = 0, \quad y(\alpha) = 0,\ y(\beta) = 0$$

- Observe that this problem has the solution $y = 0$ for all x, regardless of the coefficients $p(x)$ and $q(x)$.
- This solution is often called the trivial solution and is rarely of interest.
- What we would like to know is whether the problem has other, nonzero solutions.

later $\begin{pmatrix} y(0)=0, & y(L)=0 \\ y'(0)=0, & y'(L)=0 \end{pmatrix}$

Example 3

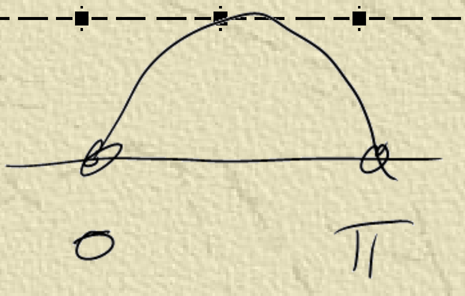
- Consider the boundary value problem
$$y'' + 2y = 0, \quad y(0) = 0, \; y(\pi) = 0$$
- As in Example 1, the general solution is
$$y = \cancel{c_1 \cos\sqrt{2}x} + c_2 \sin\sqrt{2}x$$



- The first boundary condition requires that $c_1 = 0$.
- From the second boundary condition, we have $c_2 = 0$.
- Thus the only solution to the boundary value problem is $y = 0$.
- This example illustrates that a homogeneous boundary value problem may have only the trivial solution $y = 0$.

Example 4

- Consider the boundary value problem
$$y'' + y = 0, \quad y(0) = 0, \; y(\pi) = 0$$



- As in Example 2, the general solution is
$$y = c_1 \cos x + c_2 \sin x$$

$y(0)=0 \Rightarrow y = c_2 \sin x$

- The first boundary condition requires $c_1 = 0$, while the second boundary condition is satisfied regardless of the value of c_2.
- Thus there are infinitely many solutions of the form
$$y = c_2 \sin x, \quad c_2 \text{ arbitrary}$$
- This example illustrates that a homogeneous boundary value problem may have infinitely many (nontrivial) solutions.

Linear Boundary Value Problems

- Thus examples 1 through 4 illustrate that there is a relationship between homogeneous and nonhomogeneous linear boundary value problems similar to that between homogeneous and nonhomogeneous linear algebraic systems.
- A nonhomogeneous boundary value problem (Example 1) has a unique solution, and the corresponding homogeneous problem (Example 3) has only the trivial solution.
- Further, a nonhomogeneous problem (Example 2) has either no solution or infinitely many solutions, and the corresponding homogeneous problem (Example 4) has nontrivial solutions.

$L(y) = 0$, $y(\alpha) = 0$, $y(\beta) = 0$

or $y'(\alpha) = 0$ $y'(\beta) = 0$

$L(y) = 0 \quad = \quad L(y) = p(x)y'' + q(x)y' + r(x)y = 0$

Eigenvalue Problems (1 of 8)

$Ax + \lambda x = 0$

- Recall from Section 7.3 the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$.
- Note that $\mathbf{x} = \mathbf{0}$ is a solution for all λ, but for certain λ, called eigenvalues, there are nonzero solutions, called eigenvectors.
- The situation is similar for boundary value problems.
- Consider the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

$$\boxed{L(y) = -\lambda y}$$

- This is the same problem as in Example 3 if $\lambda = 2$, and is the same problem as in Example 4 if $\lambda = 1$.
- Thus the above boundary value problem has only the trivial solution for $\lambda = 2$, and has other, nontrivial solutions for $\lambda = 1$.

Eigenvalues and Eigenfunctions (2 of 8)

- Thus our boundary value problem

$\lambda = 2$ not e.v.

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

$\lambda = 1$ was e.v.

  has only the trivial solution for $\lambda = 2$, and has other, nontrivial solutions for $\lambda = 1$.
- By extension of the terminology for linear algebraic systems, the values of λ for which nontrivial solutions occur are called **eigenvalues**, and the nontrivial solutions are **eigenfunctions**.
- Thus $\lambda = 1$ is an eigenvalue of the boundary value problem and $\lambda = 2$ is not.

$\lambda = 1$ eigenfuns $= c_2 \sin(x)$

- Further, any nonzero multiple of $\sin x$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

$c_2 \neq 0$

Boundary Value Problem for $\lambda > 0$ (3 of 8)

- We now seek other eigenvalues and eigenfunctions of

- We consider separately the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.
- Suppose first that $\lambda > 0$. To avoid the frequent appearance of radical signs, let $\lambda = \mu^2$, where $\mu > 0$.
- Our boundary value problem is then

$$y'' + \mu^2 y = 0, \quad y(0) = 0,\ y(\pi) = 0$$

- The general solution is

$$\underline{y = c_1 \cos \mu x + c_2 \sin \mu x}$$

- The first boundary condition requires $c_1 = 0$, while the second is satisfied regardless of c_2 as long as $\mu = n$, $n = 1, 2, 3, \ldots$.

$y'' + \lambda y = 0$

$y(0) = 0$

$y(\pi) = 0$

$\Rightarrow \lambda$ e.v. ✓

$\Rightarrow \lambda > 0$

$\underline{y = c_2 \sin(\mu x)}$, $y(\pi) = 0$

$c_2 \sin(\mu \pi) = 0$

$\Rightarrow \mu = n$, $n = 1, 2, 3$ | e.f
e.v. are $1, 2^2, 3^2, \ldots\ n^2 = $ e.v | $\sin(nx)$

Eigenvalues, Eigenfunctions for $\lambda > 0$

- We have $\lambda = \mu^2$ and $\mu = n$. Thus the eigenvalues of

$$y'' + \lambda y = 0, \quad y(0) = 0,\ y(\pi) = 0$$

are

$$\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9,\ \ldots,\ \lambda_n = n^2,\ \ldots$$

with corresponding eigenfunctions

$$y_1 = a_1 \sin x,\ y_2 = a_2 \sin 2x,\ y_3 = a_3 \sin 3x,\ \ldots,\ y_n = a_n \sin nx,\ \ldots$$

where $a_1, a_2, \ldots, a_n, \ldots$ are arbitrary constants. Choosing each constant to be 1, we have

$$y_1 = \sin x,\ y_2 = \sin 2x,\ y_3 = \sin 3x,\ \ldots,\ y_n = \sin nx,\ \ldots$$

Boundary Value Problem for $\lambda < 0$

- Now suppose $\lambda < 0$, and let $\lambda = -\mu^2$, where $\mu > 0$.
- Then our boundary value problem becomes

$$y'' - \mu^2 y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- The general solution is

$$y = c_1 \cosh \mu x + c_2 \sinh \mu x$$

- We have chosen $\cosh\mu x$ and $\sinh\mu x$ instead of $e^{\mu x}$ and $e^{-\mu x}$ for convenience in applying the boundary conditions.
- The first boundary condition requires that $c_1 = 0$, and from the second boundary condition, we have $c_2 = 0$.
- Thus the only solution is $y = 0$, and hence there are no negative eigenvalues for this problem.

Boundary Value Problem for $\lambda = 0$

- Now suppose $\lambda = 0$. Then our problem becomes
$$y'' = 0, \quad y(0) = 0, \; y(\pi) = 0$$
- The general solution is
$$y = c_1 x + c_2$$
- The first boundary condition requires that $c_2 = 0$, and from the second boundary condition, we have $c_1 = 0$.
- Thus the only solution is $y = 0$, and $\lambda = 0$ is not an eigenvalue for this problem.

Eigenvalue Problems

continuous
$(p(x), q(x), r(x))$

Consider the ~~operator~~ $\mathcal{M}(y) = p(x)y'' + q(x)y' + r(x)y$

let $0 \le \alpha < \beta$. An eigenvalue problem of type $(\mathcal{M}, \alpha, \beta)$ is the problem of finding all λ s.t. there is a non-zero function y_λ satisfying certain boundary conditions at α and β s.t. $\mathcal{M}(y) + \lambda y = 0$ for all $x \in [\alpha, \beta]$

Any such λ is called an <u>eigenvalue</u> of the problem and any such $y_\lambda(x)$ is called an associated eigenfunction of the problem

We will focus on the case when $\mathfrak{M} = \frac{d^2 y}{dx^2}$, and $\alpha = 0$. Thus, we consider eigenvalue problems of type $(y'', 0, L)$.

We consider 4 particular boundary conditions which we label as

$BC(0,0)$: $y(0) = 0$, $y(L) = 0$

$BC(0,1)$: $y(0) = 0$, $y'(L) = 0$

$BC(1,0)$: $y'(0) = 0$, $y(L) = 0$

$BC(1,1)$: $y'(0) = 0$, $y'(L) = 0$

We call these of type $(y'', 0, L, 00)$, $(y'', 0, L, 01)$, $(y'', 0, L, 10)$, $(y'', 0, L, 11)$, respectively.

Consider $y'' + \lambda y = 0$, $y(0) = 0$

$y(L) = 0$

An e.v for is a scalar λ (real or complex) s.t. $\exists\ y_\lambda$ not $=$ 0 fcn

s.t. $y'' + \lambda y_\lambda = 0$ and

$y_\lambda(0) = 0$

$y_\lambda(L) = 0$

Fact: Any e.v. of type $(y'', 0, L)$ has only real e.v.s.

Claim: Any real e.v λ of type $(y'', 0, L)$ must be >0.

Pf: Case 1: Consider $\lambda < 0$ (assume it is an e.v.)

Can write $\lambda = -\mu^2$ $\exists\, \mu > 0$.

Gen soln: $y'' - \mu^2 y = 0$

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

BC's: $y(0)=0$, $c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$$y = c_1 e^{\mu x} - c_1 e^{-\mu x}$$

$y(L) = 0$, $L > 0$

$$\Longrightarrow c_1 e^{\mu L} - c_1 e^{-\mu L} = 0$$

know $c_1 \neq 0 \Rightarrow e^{\mu L} = e^{-\mu L}$

$$\Longrightarrow e^{2\mu L} = 1$$

$$\Longrightarrow 2\mu L = 0 \Rightarrow L = 0 \quad X$$

$\Rightarrow \nexists$ neg e.v's

Case 2: $\lambda = 0$.

$$y'' + \lambda y = 0, \quad y'' = 0$$

$$\Longrightarrow \quad y = c_1 x + c_2$$

$$y(0) = 0 \Rightarrow c_2 = 0$$

$$y(L) = 0 \Rightarrow c_1 L = 0$$

$$\Rightarrow c_1 = 0$$

$\Longrightarrow$ 0 is not an e.v.

As above, any eigenvalue of type $(y'', 0, L, ij)$ with $i = 0, 1$, $j = 0, 1$ has only positive real eigenvalues. So, we may write the problem as

$$\boxed{y'' + \lambda^2 y = 0, \quad \text{Boundary conds.}}$$

Let us consider these one at a time

type $(y'', 0, L, 00)$:

This is the eigenvalue problem

$$\underline{y'' + \lambda^2 y = 0}, \quad y(0) = 0, \quad y(L) = 0$$

$$y = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

$$y(0) = 0 \Longrightarrow C_1 = 0, \text{ so } y = C_2 \sin(\lambda x)$$

Since $y \neq 0$, we need $C_2 \neq 0$

Then, $y(L)=0 \Rightarrow \sin(\lambda L)=0$

$\Rightarrow \lambda L = n\pi$ for $n=1,2,3$.

Hence $\lambda = \frac{n\pi}{L}$ is the only possibility.

Thus, the eigenvalues are

$$\lambda_n^2 = \left(\frac{n\pi}{L}\right)^2$$

and the associated eigenfunctions are $y_n = \sin\left(\frac{n\pi x}{L}\right)$ (and non-zero multiples).

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type $(y'', 0, L, 01)$: $y = c_1\cos(\lambda x) + c_2\sin(\lambda x)$

$y(0)=0 \Rightarrow c_1 = 0 \Rightarrow \underline{y = c_2\sin(\lambda x)}$

$$y'(L)=0 \Rightarrow \lambda \cos(\lambda L)=0$$

$$\Rightarrow \lambda_n L = \frac{(2n-1)\pi}{2}, \quad n=1,2,\ldots$$

eigenvalues $\lambda_n^2 = \left(\frac{(2n-1)\pi}{2L}\right)^2$

eigenfunctions: $y_n = \sin(\lambda_n x)$

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types $(y'', 0, L, 10)$, $(y'', 0, L, 11)$ - exercises

Example problems:

2) $y'' + 2y = 0$, $y'(0) = 1$, $y'(\pi) = 0$

Question: Is there a soln?
If so, how many

Gen soln: $y = c_1 \cos(\sqrt{2}x) +$
$c_2 \sin(\sqrt{2}x)$

BC: $y' = -\sqrt{2}\, c_1 \sin(\sqrt{2}x) + \sqrt{2}\, c_2 \cos(\sqrt{2}x)$

$BC\big|_{x=0}$ $\sqrt{2}\, c_2 = 1 \Rightarrow c_2 = \frac{1}{\sqrt{2}}$

$$y'(\pi) = 0, \quad -\sqrt{2}\, C_1 \sin(\sqrt{2}\pi) + \cancel{\sqrt{2}}\, \frac{1}{\cancel{\sqrt{2}}} \cos(\sqrt{2}\pi) = 0$$

$$\Rightarrow \quad C_1 = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi)$$

Ans:

$$y = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi) \cos(\sqrt{2}x) + \frac{1}{\sqrt{2}} \sin(\sqrt{2}x)$$

4) $y'' + y = 0$, $y'(0) = 1,\ y(L) = 0$

Gen soln: $y = c_1 \cos(x) + c_2 \sin(x)$

$y(L) = 0 \Rightarrow c_1 \cos(L) + c_2 \sin(L) = 0$

$y'(x) = -c_1 \sin(x) + c_2 \cos(x)$

$y'(0) = 1 \Rightarrow c_2 = 1$

$c_1 \cos(L) + \sin(L) = 0$

$\Rightarrow c_1 = -\dfrac{\sin(L)}{\cos(L)} = -\tan(L)$

if $\cos(L) \neq 0$

$\Rightarrow$ unique soln if $\cos(L) \neq 0$

Soln is $\underbrace{c_1}_{\text{determined}} \cos(x) + c_2 \sin(x)$

$\downarrow$

$y = -\tan(L)\cos(x) + \sin(x)$ if $\cos(L) \neq 0$

If $\cos(L) = 0 \Rightarrow \sin(L) = 0$ also.

$\Downarrow$ $\qquad \Downarrow$

$L = \left(\frac{2n-1}{2}\right)\pi$ $\qquad n\pi$

different

so $\cos(L) = 0$ $\Rightarrow$ no solns