CYCLOID, BRACHISTOCHRONE, TAUTOCHRONE

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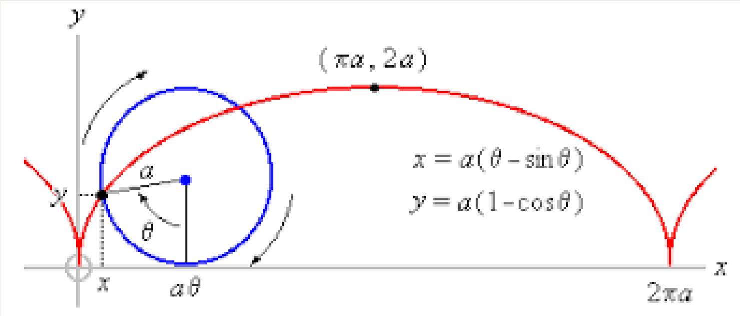
Problem?

- **Brachistochrone Problem**: “What is the shape of the curve so that an object starting at rest and moving along the curve, without friction under uniform gravity, will fall to it’s lowest point in the shortest time?”
- **Tautochrone Problem:** “What is the shape of the curve so that the time taken by an object sliding without friction in uniform gravity to its lowest point is same no matter what starting height is?”
- **Solution?:** The inverted cycloid is the solution to both of the problems.

What is a Cycloid?

- it is a curve generated by a point on the circumference of a circle that rolls along a straight line
- The circle $x^2 + (y-1)^2 = 1$ is rolled with out slipping along the x-axis so that the position of the center at time θ is $(\theta, 1)$
  - $P = (\theta - \sin\theta, 1-\cos\theta)$



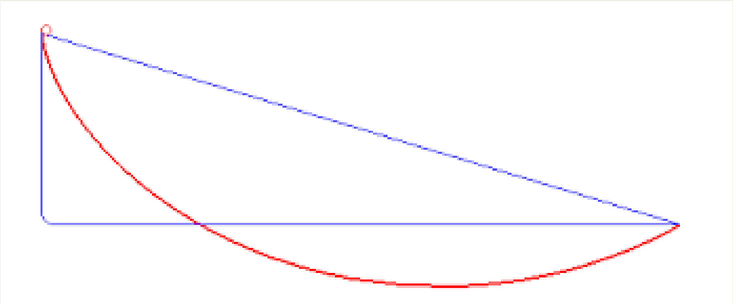
Brachistochrone Problem

- It is one of the earliest problems first proposed in calculus by John Bernoullis, “the planar curve on which a body subjected only to the force of gravity will slide (without friction) between two points in the least possible time” (Britannica, 2017).
- Late 17^{th} century, Newton was challenged to solve the problem, in fact did so, along side Leibniz, L’Hospital, and the two Bernoullis, the solution which is a segment of a cycloid (Weisstein).

Curve

- Time to travel from point P1 to another point P2 is given by the integral:
- s is the arc length and v is the speed

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v},$$



- The brachistochrone curve is similar to the tautochrone curve; both are cycloids. However, the portion of the cycloid used for each of the two varies.

Solution

Brachistochrone Problem

* Integral: $t_{12} = \int_{P_1}^{P_2} \frac{ds}{v}$

s = arc length

v = speed

$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$

* Plugging: $ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx$

$$t_{12} = \int_{P_1}^{P_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \int_{P_1}^{P_2} \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

$f = (1+y'^2)^{1/2}(2gy)^{-1/2}$

Beltrami Identity: $\Rightarrow f - y'\frac{\partial f}{\partial y'} = C \Rightarrow \frac{\partial f}{\partial y'} = y'(1+y'^2)^{-1/2}(2gy)^{-1/2}$

$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = C \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right] y = \frac{1}{2gC^2} = k^2$$

$T = \frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y}$ $N = -\frac{dy}{ds}\hat{x} + \frac{dx}{ds}\hat{y}$

$F_{gravity} = mg\hat{y}$

$F_{friction} = -u\,(F_{gravity} N)\,T = -umg\frac{dx}{ds}T$

$F_{gravity}T = mg\frac{dy}{ds}$ $F_{friction}T = -umg\frac{dx}{ds}$

* Newton's 2nd Law: $m\frac{dv}{dt} = mg\frac{dy}{ds} - umg\frac{dx}{ds}$

$\frac{dv}{dt} = v\frac{dv}{ds} = \frac{1}{2}\frac{d}{ds}(v^2)$

$\frac{1}{2}v^2 = g(y-ux) \Rightarrow v = \sqrt{2g(y-ux)}$

so, $t = \int \sqrt{\frac{1+(y')^2}{2g(y-ux)}}\,dx$

* Euler-Lagrange differential equation:

$$[1+y'^2](1+uy') + 2(y-ux)y'' = 0$$

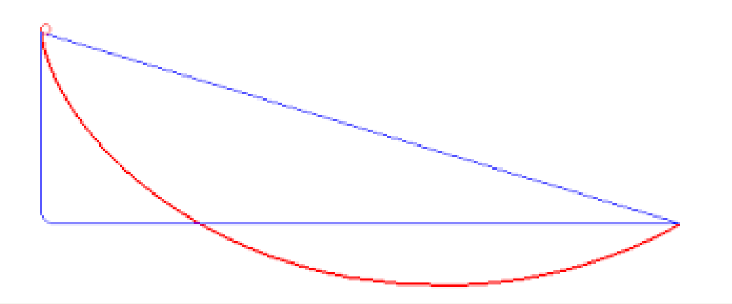
$$\Rightarrow \frac{1+(y')^2}{(1+uy')^2} = \frac{C}{y-ux} \Rightarrow y' = \cot\left(\tfrac{1}{2}\theta\right)$$

* Solution: $x = \frac{1}{2}k^2[(\theta - \sin\theta) + u(1-\cos\theta)]$

$y = \frac{1}{2}k^2[(1-\cos\theta) + u(\theta + \sin\theta)]$

Tautochrone Problem

- while Brachistochrone is the path between two points that takes the shortest time period to travel (without friction), the Tautochrone is the curve where no matter at what height you start, any mass will reach the lowest point in equal time
- First discovered by Huygens, constructed first pendulum clock, making sure it is isochronous by forcing the pendulum to swing in an arc of a cycloid



Solution

Tautochrone Problem

Parametric Equations of Cycloid:

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

Derivatives:

$$x' = a(1-\cos\theta)$$
$$y' = a\sin\theta$$

$$x'^2 + y'^2 = a^2\left[(1 - 2\cos\theta + \cos^2\theta) + \sin^2\theta\right]$$
$$= 2a^2(1-\cos\theta)$$

* Now:

$$\frac{1}{2}mv^2 = mgy \qquad v = \frac{ds}{dt} = \sqrt{2gy} \qquad dt = \frac{ds}{\sqrt{2gy}}$$

$$= \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}} = \frac{a\sqrt{2(1-\cos\theta)}\,d\theta}{\sqrt{2ga(1-\cos\theta)}} = \sqrt{\frac{a}{g}}\,d\theta$$

* time required to travel from top to bottom:

$$* \; T = \int_0^{\pi} dt = \sqrt{\frac{a}{g}}\,\pi$$

However, from an immediate point θ_0

$$v = \frac{ds}{dt} = \sqrt{2g(y-y_0)}$$

$$T = \int_{\theta_0}^{\pi}\sqrt{\frac{2a^2(1-\cos\theta)}{2ag(\cos\theta_0 - \cos\theta)}}\,d\theta = \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\,d\theta$$

* Integrate: half-angle formulas

$$* \; \sin\left(\tfrac{1}{2}x\right) = \sqrt{\frac{1-\cos x}{2}} \qquad * \; \cos\left(\tfrac{1}{2}x\right) = \sqrt{\frac{1+\cos x}{2}}$$

$$= \cos\theta = 2\cos^2\left(\tfrac{1}{2}\theta\right) - 1 \qquad T = \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\frac{\sin\left(\frac{1}{2}\theta\right)d\theta}{\sqrt{\cos^2\left(\frac{1}{2}\theta_0\right) - \cos^2\left(\frac{1}{2}\theta\right)}}$$

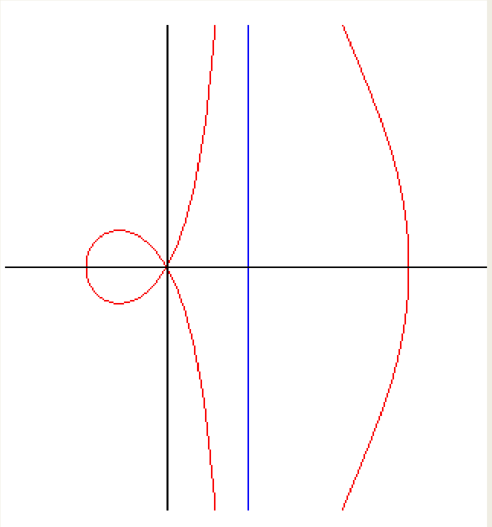
$$u = \frac{\cos\left(\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta_0\right)} \qquad du = -\frac{\sin\left(\frac{1}{2}\theta\right)d\theta}{2\cos\left(\frac{1}{2}\theta_0\right)}$$

$$T = -2\sqrt{\frac{a}{g}}\int_1^0 \frac{du}{\sqrt{1-u^2}} = 2\sqrt{\frac{a}{g}}\left[\sin^{-1}u\right]_0^1 = \pi\sqrt{\frac{a}{g}}$$

Conchoid of Nicomedes Exploration

- Studied by Greek mathematician, Nicomedes, favorite amongst 17^{th} century mathematicians
- Why? – used to solve problems of cube duplication, angle trisection, heptagon construction, and etc.
- It is a curve with polar coordinates: locus of points a fixed distance away from a line as measured along a line from the focus point
  - $r = b + a \sec \theta$

  Three distinct forms:
  - $0 < a/b < 1$
  - $a/b = 1$
  - $a/b > 1$



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