MA4H7 Atmospheric Dynamics Support Handout 6 - Ekman Layers

1st March 2017 Jorge Lindley email: J.V.M.Lindley@warwick.ac.uk

1 Boundary Layers

In the inviscid ($\nu = 0$) case we have the free-slip boundary condition, that is the velocity normal to the boundary is zero, $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ but fluid is free to move along the boundary.

In reality there is a thin *boundary layer* where velocity undergoes a smooth but rapid adjustment to zero. This is because at the boundary the normal and tangential components of velocity must equal that of the boundary. If the boundary is at rest then $\boldsymbol{u} = 0$. The condition on the tangential component of velocity is known as the *no-slip* condition.

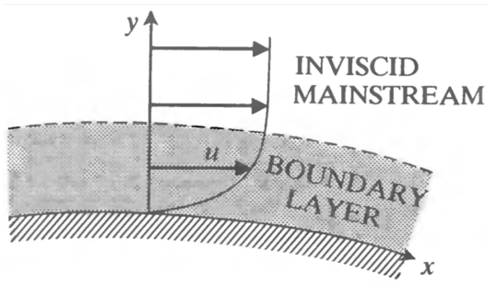

Figure 1: Boundary layers form near the boundary of a body in a flow, here viscous effects are very important. The mainstream can be considered as inviscid for low viscosity flows.

Velocity gradients in the boundary layer are much larger than in the main part of the flow, so viscous effect become significant no matter how small ν is.

In (Newtonian[1]) viscous fluids, shear stress τ is proportional to the velocity gradient $\frac{\partial u}{\partial y}$,

$$\tau = \mu \frac{\partial u}{\partial y}, \tag{1.1}$$

where μ is called the coefficient of viscosity. We define the *kinematic viscosity* as $\nu = \frac{\mu}{\rho}$.

Boundary layers may separate from the boundary causing dramatic differences in the flow from that of inviscid theory. Wakes behind objects will form causing drag.

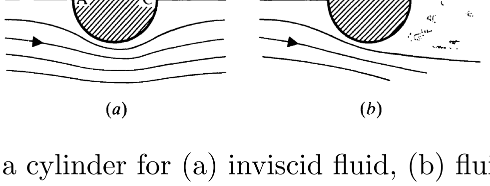

Figure 2: Flow past a cylinder for (a) inviscid fluid, (b) fluid of small viscosity.

[1] A fluid is said to be Newtonian if the viscous stresses that arise from its flow, at every point, are proportional to the local strain rate - the rate of change of its deformation over time.

Referring to Figure 2, inviscid theory says the pressure p has a local maximum at A, a minimum at B and a local maximum at C. So between B and C there is a substantial increase in pressure along the boundary in the direction of the flow. This severe adverse pressure gradient along the boundary causes the boundary layer to separate. We can see how this adverse pressure gradient drives a reversed flow in Figure 3 on the right hand side. This pushes the boundary layer (typically of thickness $O(R^{-\frac{1}{2}})$) into the mainstream. Here $R = UL/\nu$ is the *Reynolds number*, where U is a typical flow speed and L is a characteristic length scale. This is a ratio of inertia/viscosity, a high Reynolds number means viscous effects are negligible. Note that near the boundary U becomes very small so viscous effects become important even for small ν. At high R reattachment can occur, this is where the boundary layer reattaches to the surface so the boundary layer doesn't fully separate until further towards the rear end of the object reducing the size of the wake, hence reducing drag. This reattachment principle is behind the design of aerofoils, the dimples on golf balls and the fur on tennis balls.

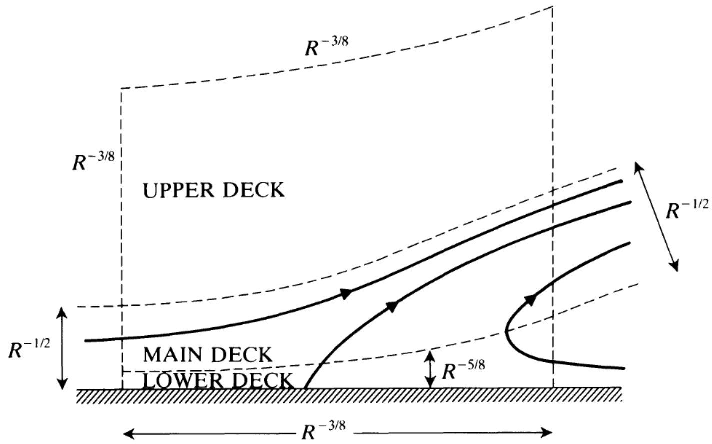

Figure 3: Separation point of a boundary layer on a rigid boundary.

2 Taylor-Proudman Theorem

Recall that for rapid rotation $\boldsymbol{\Omega} = (0, 0, \Omega)$, the Euler equation becomes a balance between Coriolis and pressure forces,

$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p. \tag{2.1}$$

This is called a *geostrophic flow* and the *geostrophic wind* is

$$(u_g, v_g) = \frac{1}{2\Omega\rho}\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right). \tag{2.2}$$

The Taylor-Proudman Theorem shows how rapidly rotating fluids exhibit strong two-dimensional behaviour.

Theorem 1. *(Taylor-Proudman) Assume we have a rapidly rotating flow, then all the components of velocity are independent of z, that is $\partial_z \boldsymbol{u} = 0$.*

Proof. Use (2.1) and $\nabla \times \nabla p = 0$ to get $\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = 0$. Expanding this we get

$$\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} + \boldsymbol{u}(\nabla \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\nabla \cdot \boldsymbol{u}) = 0$$

Assuming that $\boldsymbol{\Omega}$ is constant and using incompressibility we get $\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0 \Rightarrow \partial_z \boldsymbol{u} = 0$. □

3 Taylor Columns

Consider an object in a rotating flow, then due to the Taylor-Proudman Theorem the fluid above the object will act as a single rotating body.

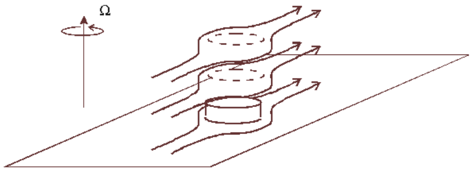

Figure 4: Flow is deflected around a column of fluid above the object.

Any flow through the Taylor column would require a change in the flow with height hence violating $\partial_z \boldsymbol{u} = 0$.

4 Ekman Boundary Layers

An *Ekman spiral* is a structure of currents or winds near a horizontal boundary in which the flow rotates as one moves away from the boundary.

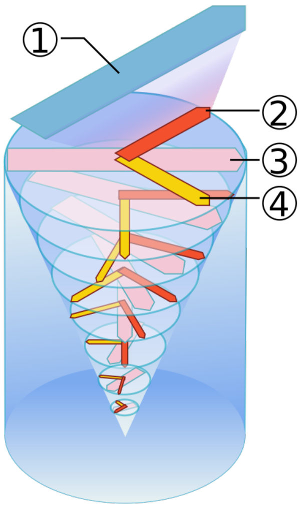

Figure 5: Ekman spiral induced in surface of the ocean due to wind. (1) Wind direction. (2) Force from above. (3) Effective direction of flow. (4) Coriolis force. The average transport of the fluid layer ends up going at 90° to right (left) of the wind direction in the Northern (Southern) Hemisphere.

Here we find the solution for the flow of an Ekamn spiral induced by a bulk flow near a rigid bottom boundary. Assume the main body flow is geostrophic so that the velocities are

$$u_g = -\frac{1}{2\Omega\rho}\frac{\partial p}{\partial y}, \quad v_g = \frac{1}{2\Omega\rho}\frac{\partial p}{\partial x}. \tag{4.1}$$

Near the boundary, within the spiral, there will be an ageostrophic component of velocity $\boldsymbol{u}_a$ varying slowly in (x, y) with $\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a$ and $\nabla \cdot (\boldsymbol{u}_g + \boldsymbol{u}_a) \approx 0$, $w_a = 0$. Then the equations in this boundary layer are (remember viscosity cannot be ignored here)

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \tag{4.2}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \tag{4.3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2} \tag{4.4}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.5}$$

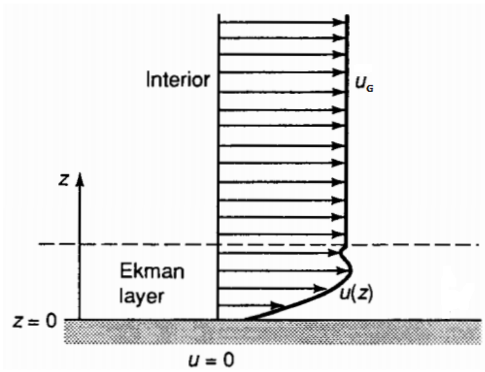

Figure 6: Ekman layer with rigid bottom boundary. This could be the atmosphere near the surface of the Earth or the bottom of the ocean. Note that the velocity vectors are also rotating within the Ekman layer.

The vertical velocity component w will be much smaller in comparison to u and v near the boundary so from the equations we deduce $\partial_z p \approx 0$. This pressure is a function of x and y so retains its inviscid value from the main body of the fluid throughout the boundary layer, which is given by (4.1). Then noting that $\partial_{zz}\boldsymbol{u}_g = 0$ the boundary layer equations become

$$-2\Omega(v - v_g) = -2\Omega v_a = \nu\frac{\partial^2 u_a}{\partial z^2}$$

$$-2\Omega(u - u_g) = -2\Omega u_a = \nu\frac{\partial^2 v_a}{\partial z^2}$$

Multiply the second equation by i and add them together to get

$$2\Omega i\mathcal{U}_a = \nu\frac{\partial^2 \mathcal{U}_a}{\partial z^2}$$

where $\mathcal{U}_a = u_a + iv_a$. This has general solution

$$\mathcal{U}_a = Ae^{-(1+i)z_*} + Be^{(1+i)z_*} \tag{4.6}$$

where $z_* = (\Omega/\nu)^{1/2} z$. Let $u_* = u + iv = u_g + \mathcal{U}_a$ (assume $v_g = 0$ for simplicity). To match with the interior flow we need $u_* \to u_g$ as $z_* \to \infty$ so $B = 0$. We also need $u_* = 0$ at $z_* = 0$ which gives $A = -u_g$ so

$$u = u_g(1 - e^{-z_*}\cos(z_*)), \quad v = u_g e^{-z_*}\sin(z_*). \tag{4.7}$$

Now we observe that $\lim_{z_*\to 0} u/v = 1$. This limit can be calculated by observing that for small z_* we have (from Taylor expansions) $e^{-z_*} \approx 1 - z_*$ and $\cos(z_*) \approx 1 - \frac{1}{2}z_*^2$ and $\sin(z_*) \approx z_*$. Alternatively one may use L'Hôpital's Rule. Therefore, very close to the boundary the flow (4.7)

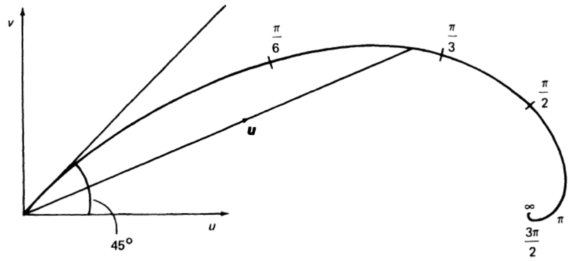

Figure 7: Ekman spiral: polar diagram of velocity vector $\boldsymbol{u}$ in Ekman layer. Numbers along the spiral are values of z_*.

is at 45° to the interior flow $(\mathcal{U}_g, 0)$.

Ekman layers can occur at the bottom of the atmosphere near the surface of the Earth or ocean (like in the example above). They can also occur at the bottom of the ocean (also like the example), or at the top of the ocean near the air-water interface. The thickness of the Ekamn layer is $(\nu/\Omega)^{1/2}$.

5 Ekman Pumping

Ekman pumping refers to the inward or outward flow around a pressure cell that induces a vertical velocity that communicates the boundary layer effects towards the interior of the fluid. This vertical velocity is $w_I = 1/2(\nu\Omega)^{1/2}\epsilon$, where $\epsilon\Omega$ is the geostrophic component of vorticity. We have seen before that (in the Northern Hemisphere) the flow around a low pressure cell is cyclonic so $\epsilon > 0$ and $w_I > 0$, therefore the ageostrophic flow points in at the bottom, dragging pressure lines in and steepening pressure gradients.

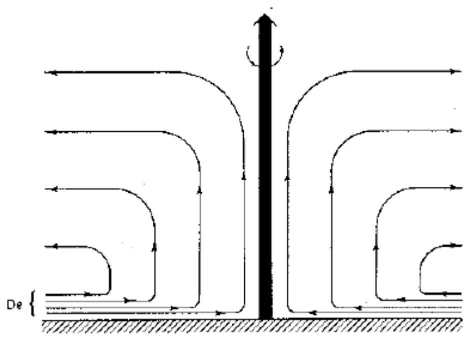

Figure 8: Ekman pumping.

Flow around a high pressure cell is anti-cyclonic so $\epsilon < 0$ and $w_I < 0$, this causes subsidence, and pushes pressure lines out decreasing pressure gradients.