# Vorticity

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February 22, 2017

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# 1 Introduction

In this section we will introduce the concept of **vorticity**, which is formally defined as the curl of the velocity field, but can be thought of as the 'spininess' of a parcel in a fluid.

$$\boxed{\boldsymbol{\omega} = \nabla \times \mathbf{U}} \tag{1}$$

This means that if the flow is two dimensional, the vorticity will be a vector in the vertical direction. As we will later see, both vorticity and potential vorticity play a central role in large scale dynamics. But first a few more definitions.

## 1.1 Definitions

**Divergence -** the divergence of a fluid is defined as $D = \nabla \cdot \mathbf{U}$

**Stokes theorem** relates the surface integral of the curl of a vector field ($\mathbf{F}$) over a surface (A) to the line integral of the vector field over its boundary.

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

So if we apply this

**Circulation, C,** - around a closed path is the integral of the tangential velocity around that path:

$$C = \oint \mathbf{U} \cdot d\mathbf{l} = \int \int_S (\nabla \times \mathbf{U}) \cdot d\mathbf{S} = \int \int_S \boldsymbol{\omega} \cdot d\mathbf{S}$$

Where we applied stokes theorem to relate the closed path integral to the vorticity. In words, this means that the ciruclation around a closed path is equal to the integral of the normal component of vorticity over *any* surface bounded by that path. To consider the vorticity of a single point, one can imagine shrinking the bounding path smaller and smaller until it is an infinitesimal point. Or, alternatively, consider dropping a flower into a draining sink. If you drop the flower into the outer edges of the sink, it will be carried around the drain by the flow but will not itself spin. If, however, you drop it on the water directly above the drain, it will spin in place. By this example we can infer that the point of the drain has vorticity, while the parcels circulating around the drain do not.

**Relative vorticity** ($\boldsymbol{\zeta}$) - Vorticity as viewed in the rotating reference frame of earth. In cartesian coordinates $\boldsymbol{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

**Planetary vorticity** ($\boldsymbol{\omega}_p$) - Vorticity associated with the rotation of the earth ($\boldsymbol{\omega}_p = 2\Omega$)

**Absolute vorticity** ($\boldsymbol{\omega}_a$) - Vorticity as viewed in an inertial reference frame $\boldsymbol{\omega}_a = \boldsymbol{\zeta} + \boldsymbol{\omega}_p$

Where $\Omega$ is the rotation of the earth. Because we are concerned with horizontal motion on the earth's surface, we can make use of the tangent plane approximation. And if we remember that in the case of two-dimensional flow the vorticity is normal to the surface, then we can rewrite the planetary vorticity in terms of the component of the earths rotation that is normal to our tangent plane as $f = 2\Omega sin\phi$. We can then rewrite the absolute vorticity as the sum of the absolute and planetary vorticity

$$\boldsymbol{\omega}_a = \boldsymbol{\zeta} + f$$

Similarly, we can now rewrite the absolute circulation

$$\mathbf{C}_a = \mathbf{C}_r + 2\Omega cos(\theta_0)A$$

where A is the area enclosed by the circulation, so $cos(\theta_0)A$ is the projection of that area onto a plane perpendicular to the axis of rotation of the earth.

### 1.2 Conventions

**Northern Hemisphere**
Low pressure systems (cyclones): anti-clockwise flow, $C > 0, \boldsymbol{\zeta} > 0$
High pressure systems (anticyclones): clockwise flow, $C < 0, \boldsymbol{\zeta} < 0$

**Southern Hemisphere**
Low pressure systems (cyclones): clockwise flow, $C > 0, \boldsymbol{\zeta} > 0$
High pressure systems (anticyclones): anti-clockwise flow,$C < 0, \boldsymbol{\zeta} < 0$

## 2 Vorticity and circulation

Here we will explore *Kelvin's Circulation theorem*, which is one of the most fundamental conservation laws in fluid mechanics. The theorem provides a constraint on the rate of change of a circulation, and is intimately related to the *potential vorticity.*

So beginning again with the absolute circulation

$$\mathbf{C}_a = \mathbf{C}_r + 2\Omega cos(\theta_0)A$$

where A is the area enclosed by the circulation, so $cos(\theta_0)A$ is the projection of that area onto a plane perpendicular to the axis of rotation of the earth. We will explore the implications of the above formula by first considering a closed loop around a fluid parcel as it travels toward the pole (see below figure). If the parcel begins with no relative circulation, then as it travels towards the pole its projection onto a surface normal to the rotation of the earth will increase. In order to conserve absolute circulation, the relative circulation will go from zero to negative (anticyclonic). We have thereby induced a circulation by decreasing the relative term as the $2\Omega cos(\theta_0)A$ term increases

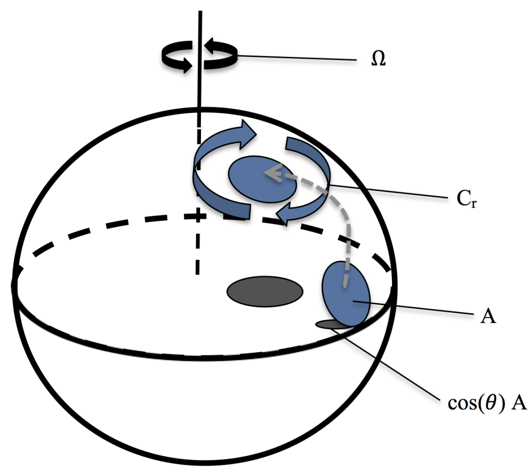


Figure 1: circulation induced by moving a parcel polewards

Now that we've described the behavior of this system, let's explicitly define the time rate of change of the circulation of a fluid parcel (the material derivative of the circulation)

$$\frac{D\mathbf{C}}{Dt} = \frac{D}{Dt}\oint \mathbf{U}\cdot d\mathbf{l} = \oint \frac{D\mathbf{U}}{Dt}\cdot d\mathbf{l} + \oint \frac{Dd\mathbf{l}}{Dt}\cdot \mathbf{U}$$

And we can rewrite the last term as:

$$\oint \frac{Dd\mathbf{l}}{Dt}\cdot \mathbf{U} = \oint \mathbf{U}\cdot (d\mathbf{l}\cdot \nabla \mathbf{U}) = \oint d\mathbf{l}\cdot \nabla\left(\frac{1}{2}|\mathbf{U}|^2\right) = 0$$

The term goes to zero because it is the integral of a gradient around a closed curve. We can then rewrite the remaining term as a momentum equation. Let's first remind ourselves of one form of the momentum equation:

$$\frac{D\mathbf{U}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$

where $\Phi$ represents conservative body forces (i.e. the Coriolis force). Now let's consider this equation in our context. Here we will neglect viscosity but include friction:

$$\boxed{\frac{D\mathbf{C}}{Dt} = \oint (-2\Omega \times \mathbf{U})\cdot d\mathbf{l} - \oint \frac{\nabla p}{\rho}\cdot d\mathbf{l} + \oint \mathbf{F}\cdot d\mathbf{l}} \tag{2}$$

So here we can see that there are three main terms that can alter the circulation. The first is the *Coriolis force*, the second is the *baroclinic term* and the third is the *friction term.* We will now explore each of these in greater detail.

**1. The Coriolis term** If we consider the circulation around a divergent flow, the Coriolis force will act on the flow field to induce a circulation.

**2. The baroclinic term** Let's begin by rewriting this term in a more helpful form using Stokes Theorem

$$-\oint \frac{\nabla p}{\rho} \cdot d\mathbf{l} = -\int\int_S \nabla \times \left(\frac{\nabla p}{\rho}\right) \cdot d\mathbf{l} = \int\int_S \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot d\mathbf{l}$$

From this form we can see that the numerator (and therefore the entire term) will be zero when the surfaces of constant pressure are also surfaces of constant density. We can define a fluid as either *Barotropic* or *Baroclinic.* A fluid is barotropic when the density depends *only* on pressure, which implies that temperature does not vary along a pressure surface. This furthermore implies – via thermal wind – that the geostrophic flow of the fluid does not vary with height. When a fluid is baroclinic $\nabla\rho \times \nabla p \neq 0$, so temperature is allowed to vary along a pressure surface, and therefore the geostrophic wind will vary with height.

To visually see how the baroclinic term can induce a circulation, consider the case in which a fluid is initially at rest such that two fluids of different densities are side by side. Here we have a pressure gradient in the vertical and a density gradient in the horizontal. This means that $\nabla\rho \times \nabla p$ induces a circulation such that the denser fluid flows beneath the less dense fluid until the system comes to equilibrium with the lighter fluid sitting atop the denser fluid as pictured in the final panel.

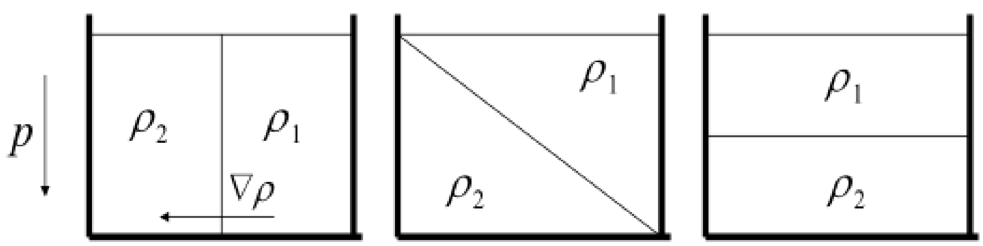


**Figure 3:** An extremely baroclinic situation in pressure coordinates. Two fluids of different densities $\rho_1$ and $\rho_2$ are side by side ($\rho_1 < \rho_2$). The baroclinic term generates a circulation which causes the denser fluid to slump under the lighter one until eventually an equilibrium is reached with the lighter fluid layered on top of the denser fluid and the baroclinic term =0.

Figure 2: Credit: Isla Simpson's notes

**3. the Friction term** The friction is often simply considered to be a linear drag on velocity such that it acts to damp the circulation.

# 3 The vorticity equation

Now that we've described how circulation changes around a parcel, let's walk through the same exercise for a single point by considering vorticity. We will again begin with the momentum equation.

$$\frac{D\mathbf{U}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F}$$

Here we'll use a few vector identities. First remember that

$$\mathbf{U} \times (\nabla \times \mathbf{U}) = \frac{1}{2}\nabla(\mathbf{U} \cdot \mathbf{U}) - (\mathbf{U} \cdot \nabla)\mathbf{U}$$

Now substitute in the definition of vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{U}$), expand the material derivative and make use of the above identity to get

$$\frac{\partial \mathbf{U}}{\partial t} + (\boldsymbol{\omega} \times \mathbf{U}) = -\frac{1}{\rho}\nabla p + \mathbf{F} - \nabla\big(\Phi + \frac{1}{2}|U|^2\big)$$

now take the curl of this field, again keeping in mind the definition of vorticity and that $\nabla \times (\nabla\mathbf{A}) = 0$, where $\mathbf{A}$ is any twice differentiable scalar field

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{U}) = -\frac{\nabla\rho \times \nabla p}{\rho^2} + \nabla \times \mathbf{F}$$

Now make use of one more vector identity

$$\nabla \times (\mathbf{U} \times \mathbf{V}) = \mathbf{U}\nabla \cdot \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{U} - \mathbf{V}(\cdot\nabla\mathbf{U}) - (\mathbf{U} \cdot \nabla)\mathbf{V}$$

and note that the divergence of vorticity is zero, such that we are left with

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{U} - \boldsymbol{\omega}(\nabla \cdot \mathbf{U}) + \frac{\nabla\rho \times \nabla p}{\rho^2} + \nabla \times \mathbf{F}} \tag{3}$$

As with the time rate of change of the circulation (equation 2), the last two terms are the baroclinic term and the frictional term. The first two terms on the left hand side are the vortex tilting ($(\boldsymbol{\omega} \cdot \nabla)\mathbf{U}$) and vortex stretching term ($\boldsymbol{\omega}(\nabla \cdot \mathbf{U})$), respectively.

## 3.1 Vortex stretching and tilting

A useful property of vorticity in a barotropic, inviscid (having negligible viscosity), unforced field the lines of vorticity follow material lines, meaning the two are joined together as the fluid evolves (they are 'frozen in'). Let's first expand the stretching and tilting terms to see more clearly what they describe

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{U} - \boldsymbol{\omega}(\nabla \cdot \mathbf{U}) = \omega \frac{\partial}{\partial z}(ui + vj + wk) - \omega k\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{U} - \boldsymbol{\omega}(\nabla \cdot \mathbf{U}) = \left(\omega i \frac{\partial u}{\partial z} + \omega j \frac{\partial v}{\partial z}\right) - \omega k\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

So because of the 'frozen in' property of vorticity, the vortex tilting term tells us that when advection acts to tilt the material lines, vorticity in one direction (e.g. x-direction) may be generated from vorticity in either of the orthogonal directions (e.g. y- or z-directions). The stretching term tells us that if the material lines are stretched, then the coincident vorticity component is intensified proportionally to the stretching.

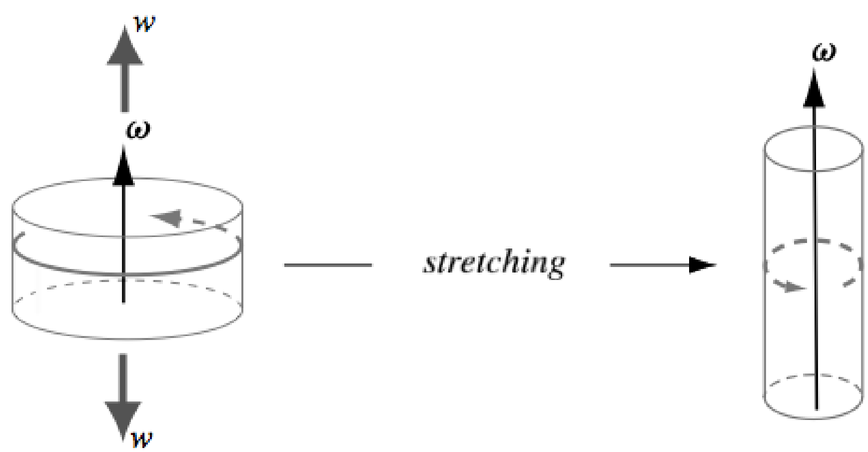


**Fig. 4.5** Stretching of material lines distorts the cylinder of fluid as shown. Vorticity is tied to material lines, and so is amplified in the direction of the stretching. However, because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge and the integral of vorticity over a material surface (the circulation) remains constant, as discussed in section 4.3.2.

Figure 3: Credit: Vallis (2006)

# 4 Potential vorticity

So far we have shown that Kelvin's circulation theorem is, in fact, a general statement about the conservation of vorticity. But there are two constraints on our derivations thus far. (1) Kelvin's circulation theorem is only applicable to barotropic flow but the motion in the atmosphere and the ocean is often baroclinic and (2) it is a statement about flow around a parcel, not what is happening at any individual point. While equation 3 is a statement about a point, it provides no constraint (i.e. the right hand side of equation 3 could be anything). So what we want is to combine these two concepts to provide a constraint at each point in a flow field.

To do this we can tweak the concept of vorticity to form a conservation law that holds for baroclinic flow. This is the conservation of *potential vorticity.* The idea here is to formulate a scalar field that is advected by the fluid and which describes the evolution of fluid elements. As we will see, potential vorticity is a consequence of the 'frozen in' property of vorticity. Below we examine potential vorticity in the case of both bartropic and baroclinic flow.

## 4.1 Barotropic flow

In the absence of friction and viscosity, Kelvin's circulation theorem holds for barotropic flow

$$\frac{D\mathbf{C}_a}{Dt} = 0- > \frac{D}{Dt}\oint \mathbf{U}\cdot d\mathbf{l} = \frac{D}{Dt}\int\int_S \boldsymbol{\omega}\cdot d\mathbf{S} = 0$$

Now consider two isosurfaces of a conserved tracer ($\chi$). Imagine an infinitesimal volume element bounded by these two isosurfaces, as depicted below.

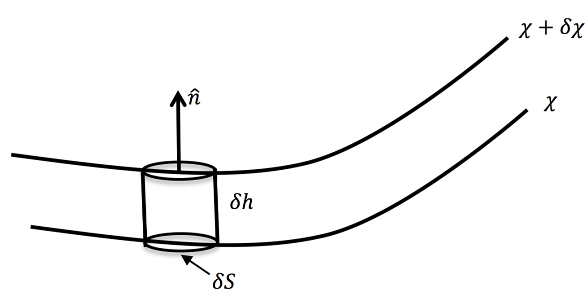


Figure 4: A fluid element confined between two isosurfaces of a conserved tracer $\chi$

Because we have defined $\chi$ to be materially conserved, $\frac{D\chi}{Dt} = 0$. So if we apply Kelvin's circulation theorem to this fluid element

$$\frac{D}{Dt}\boldsymbol{\omega_a}\cdot d\mathbf{S} = \frac{D}{Dt}(\boldsymbol{\omega}_a\cdot\mathbf{n})dS$$

where $\mathbf{n}$ is the unit vector in the direction normal to the isosurfaces of $\chi$. $\mathbf{n}$ can be defined as

$$\mathbf{n} = \frac{\nabla\chi}{|\nabla\chi|}$$

And we can define the volume of the infinitesimal element using the spacing between isosurfaces and the surface area of the top/bottom of the fluid element ($\partial V = \partial h \partial S$). Therefore we have

$$(\boldsymbol{\omega_a} \cdot \mathbf{n})dS = \boldsymbol{\omega_a} \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\partial V}{\partial h}$$

Now we make use of the fact that we defined $\partial h$ as the separation between isosurfaces ($\partial\chi|$). So because $\partial\chi = \partial\boldsymbol{\chi} \cdot \nabla\chi = \partial h|\nabla\chi$, we can substitute this in

$$(\boldsymbol{\omega_a} \cdot \mathbf{n})dS = \boldsymbol{\omega_a} \cdot \frac{\nabla\chi}{\partial\chi}\partial V$$

So substituting the above equation into Kelvin's circulation theorem

$$\frac{D}{Dt}\left[\frac{(\boldsymbol{\omega_a} \cdot \nabla\chi)\partial V}{\partial\chi}\right] = \frac{1}{\partial\chi}\frac{D}{Dt}\left[(\boldsymbol{\omega_a} \cdot \nabla\chi)\partial V\right] = \frac{\partial M}{\partial\chi}\frac{D}{Dt}\left[\frac{(\boldsymbol{\omega_a} \cdot \nabla\chi)}{\rho}\right] = 0$$

where we have made use of the fact that $\chi$ and therefore $\partial\chi$ are conserved scalars, so we can move them outside of the material derivative. Rewriting this result in a more compact form, we have

$$\boxed{\frac{Dq}{Dt} = 0, \quad where \quad q = \frac{\boldsymbol{\omega_a} \cdot \nabla\chi}{\rho}} \tag{4}$$

Here we have defined the conservation of potential vorticity, where q is potential vorticity and $\chi$ is any materially conserved quantity (e.g. potential temperature ($\theta$) for adiabatic motion of an ideal gas).

## 4.2 Baroclinic flow

Kelvin's circulation theorem applies only to barotropic motion, but throughout much of the atmosphere the baroclinic term will be nonzero (particularly in the midlatitudes). However, we can make the baroclinic term zero if we are clever about how we choose our $\chi$. We need to choose a $\chi$ that will both make the baroclinic term zero, and will be materially conserved. So let's look at the baroclinic term:

$$\int\int_S \left(\frac{\nabla\rho \times \nabla p}{\rho^2}\right) \cdot d\mathbf{S} = -\int\int_S (\nabla ln\theta \times \nabla T) \cdot d\mathbf{S}$$

From the above equation we can see that if we choose isosurfaces of $\theta, T, \rho$ or $p$, then the baroclinic term will go to zero. But out of these only $\theta$ will be materially conserved in an ideal gas. So using $\theta$ as our tracer, we can write potential vorticity as

$$\frac{Dq}{Dt} = 0, \quad where \quad q = \frac{\boldsymbol{\omega_a} \cdot \nabla\theta}{\rho} = 0$$

This is an important expression of the relation between potential vorticity and potential temperature in a baroclinic atmosphere. In words, the potential vorticity, which is materially conserved, is related to the absolute vorticity ($\boldsymbol{\omega_a}$) and the stratification ($\nabla\theta$) of the atmosphere. Remember, however, that we have assumed friction and viscosity are zero. If we included these sink terms, they would appear on the right hand side of the equations above

## 4.3 Physical interpretation

In atmospheric science, potential vorticity (PV) often shows up at the very foundation of our understanding of the dynamics of a system. Because PV is related to the velocity and stratification of a fluid and is materially conserved (i.e. it is advected with the mean flow), we can use it to both diagnose large-scale dynamics and to predict the evolution of the flow in the future.

In many instances, we will be concerned with the vertical component of the vorticity:

$$q = \frac{\boldsymbol{\omega_{a,z}} \frac{\partial\theta}{\partial z}}{\rho}$$

and using hydrostatic balance, we can rewrite this

$$\frac{Dq}{Dt} = 0, \quad q = \frac{(f+\zeta)}{\frac{\partial p}{\partial \theta}} \quad where \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

From these equations, we can tell that PV is the product of absolute vorticity and a term that accounts for the stratification of the atmosphere (i.e. the thickness of the layer between isentropes of $\theta$). The vorticity described here is not quite with respect to the vertical z, but rather normal to isentropes of $\theta$. This will often be nearly the same in the absence of strong horizontal gradients of $\theta$ (and by thermal wind strong vertical wind shear). If $\frac{\partial\theta}{\partial p}$ is constant, then temperature isn't varying on pressure surfaces (the atmosphere is barotropic) and absolute vorticity is conserved following the flow.

To understand how $\frac{\partial\theta}{\partial p}$ (i.e. the thickness between isentropes) affects potential vorticity, we will revisit the concept of vortex stretching in the figure below.

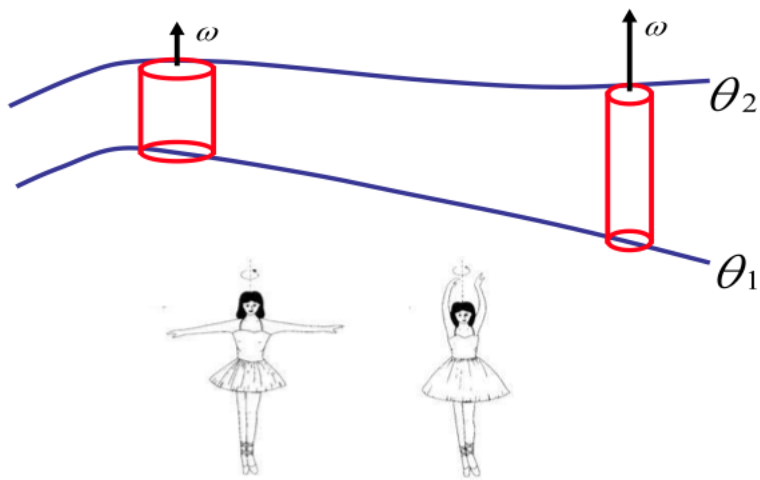


**Figure 8:** Illustration of potential vorticity conservation. A fluid column is bounded by two surfaces of potential temperature. As potential temperature is conserved following the motion of the fluid column it stretches or compresses as the thickness between the potential temperature surfaces varies. Since the mass of the fluid column is conserved the stretching reduces the surface area of the column and vice versa. Therefore, via the circulation theorem the vorticity must increase/decrease if the thickness increases/decreases. Therefore, it is the ratio of the vorticity to the thickness that is conserved.

Figure 5: Credit: Isla Simpson's notes. Available through her website at NCAR

Above a parcel is stretched between two isentropes of potential temperature. As the area of the parcel projected onto each isentrope shrinks when the column is stretched, the vorticity must increase to conserve the circulation. This can be thought of as a conservation of angular momentum. When a ballerina moves from a pirouette in a crouched position with her arms extended to a standing position in which her arms are extended, she greatly increases her spin. Similarly, when the thickness between isentropes $\frac{\partial\theta}{\partial p}$ increases, the sum of the absolute and planetary vorticities $(f+\zeta)$ must also increase to conserve PV.

In the example of the ballerina, it is $\zeta$ that changes. However, we can also change $f$ when height in the PV equation changes. Consider the Taylor Proudman effect on a sphere, as demonstrated in the ocean. In the north Atlantic water masses mix, become more dense than their surroundings, and sink (or they are 'pumped' downward as a result of the wind stress curl forcing Ekman pumping). In either case, water sinks and is compressed (h decreases). To conserve total PV, rather than inducing relative vorticity, the water column moves equatorward so that although the physical height of the column decreases, the projection of the height of the water column onto the axis of rotation (axis of the earth) remains constant. Mathematically, the balance between vertical descent and meridional advection of planetary vorticity can be expressed as:

$$\beta v = f\frac{w_{ek}}{h} \qquad or, \quad equivalently \qquad \beta v_g = f\frac{\partial w}{\partial z}$$

where $w_{ek} < 0$ is Ekman pumping (i.e. deep water formation) that, by conservation of potential vorticity, leads to equatorward flow ($v < 0$). So to review: Because PV is a conserved quantity, the compression of a water column either (a) generates negative relative vorticity if the water column remains stationary or (b) forces the water column to move to a location of lower planetary vorticity (towards the equator). This explains why deep-water formation in the North Atlantic (compression of the column) leads to deep western boundary currents (equatorward flow). Figure 6 illustrates the Taylor Proudman effect.

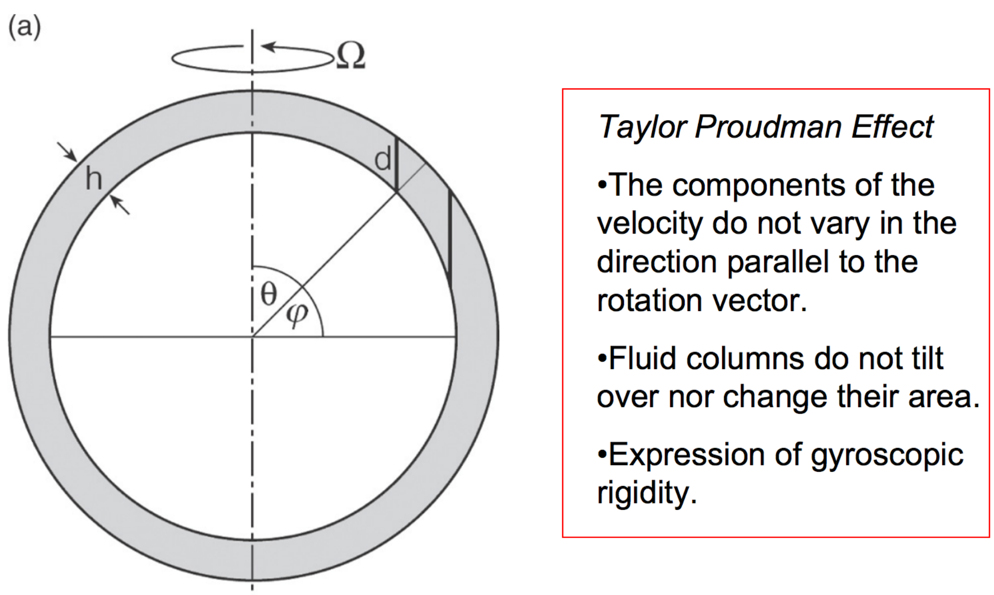


Figure 6:

Figure source: `https://pangea.stanford.edu/courses/EESS146Bweb/Lecture%206.pdf`

# References

Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation.* Cambridge University Press, 2006.