Siegel's Main Theorem for Quadratic Forms

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§ 1

A classical question in the Theory of Numbers is one of expressing a positive integer as a sum of squares of integers. The qualitative aspects of this problem require at times no more than rudimentary congruence considerations e.g. a prime number leaving remainder 3 on division by 4 cannot be a sum of two squares of integers; however, in general, subtle arguments are called for. Fermat's Principle of Descent needs to come into play for a proof of the (Euler-Fermat-) Lagrange theorem that every positive integer is a sum of four squares of integers. Skillful use of elliptic theta functions was made by Jacobi to obtain a quantitative refinement of that assertion, viz. according as n is an odd or even natural number, the number of ways of expressing n as a sum of four squares of integers is $8\sigma^*(n)$ or $24\sigma^*(n)$, where $\sigma^*(t)$ for any natural number t is the sum of all the odd natural numbers dividing t; Jacobi's famous identity linking θ_3^4 with other theta constants θ_2, θ_4 and their derivatives is an analytic encapsulation of all these formulae for varying n. An analytic formulation of similar nature arises also as a *special case* of the *Siegel formula* (extended suitably to cover the boundary case of quaternary quadratic forms as well) which connects theta series associated with quadratic forms to Eisenstein series: for complex z with positive imaginary part,

$$\left(\sum_{n\in\mathbb{Z}} \exp(\pi i\, n^2 z)\right)^4 = 1 + \sum_{q\in\mathbb{N}} \sum_{\substack{p\in\mathbb{Z} \\ (p,q)=1,\ p+q\equiv 1 (\mathrm{mod}\ 2)}} (p-qz)^{-2}$$

where the sum over p, q giving the Eisenstein series on the right hand side is only conditionally convergent and can be realized from an absolutely convergent Eisenstein series by analytic continuation via Hecke's Grenzprozess (The inner sum over p is over all integers which are coprime to q and are of opposite parity to q).

§ 2

More generally, let us consider a positive definite quadratic form $f(x_1, \ldots, x_m) := \sum_{1\le i,j\le m} s_{ij}x_i x_j$, in m variables $x_1, \ldots, x_m$ with the associated symmetric positive definite integral (coefficient) matrix $S := (s_{ij})$ and then the number $r(f; t) = r(S; t)$ of ways of representing an integer t as $f(a_1, \ldots, a_m)$ with integers $a_1, \ldots, a_m$ (instead of representing

t merely as the sum $a_1^2 + \cdots + a_m^2$ of m squares of integers $a_1, \ldots, a_m$). We know from Minknowski's reduction theory that there are only finitely many integral positive definite quadratic forms, say $f_1 = f, f_2, \ldots, f_h$ such that any integral positive definite form g which is equivalent to f over the ring $\mathbb{Z}_p$ of p-adic integers for every prime p is equivalent to one of $f_1, \ldots, f_h$ over $\mathbb{Z}$ while no two distinct ones among $f_1, \ldots, f_h$ are equivalent over $\mathbb{Z}$. In other words, the *genus* of the positive definite integral quadratic form f splits into h different *classes* each containing precisely one of $f_1, \ldots, f_h$. (We recall here that for two quadratic forms g_1, g_2 with coefficients in a commutative ring R with unit element, g_1 is said to *represent* g_2 over R if there exists a linear transformation of the variables with coefficients from R taking g_1 to g_2 and moreover g_1 is called *equivalent to* g_2 over R (or said to be in the same R equivalence class as g_2) if g_1 and g_2 represent each other mutually over R. The number $r(f; t)$ above is thus the number of representations of the quadratic form ty^2 by $f(x_1, \ldots, x_m)$ over $\mathbb{Z}$). Analogously, let for a power p^s of any given prime number p, $r(f, t; p^s)$ denote the number of representations of t by f over the quotient ring $\mathbb{Z}/p^s\mathbb{Z}$. Then $d_p(f, t)$, the *p-adic density of representations* of t by f is defined as $c_m \lim_{s\to\infty} p^{-s(m-1)} r(f, t; p^s)$ with $c_m := 2$ or 1 according as $m = 1$ or $m > 1$; this p-adic density $d_p(f, t)$ is clearly non-negative and indeed it is a rational number which equals 0 precisely when f fails to represent t over the p-adic ring $\mathbb{Z}_p$. The infinite product $\prod_p d_p(f, t)$ extended over all the prime numbers p converges and is equal to 0 exactly when f fails to represent t over $\mathbb{Z}_p$ for at least one prime i.e. when at least one $d_p(f, t)$ is 0. The *real density* $d_\infty(f, t)$ for measuring the representability of t by f over the field $\mathbb{R}$ of real numbers is defined as $\lim_U$ vol $(f^{-1}(U)/\text{vol } U)$, where the limit is taken over measurable neighbourhoods U of t shrinking to $\{t\}$ with vol($\bullet$) denoting Lebesgue measure in the respective spaces; it is known [17] that $d_\infty(f, t) = \pi^{m/2} t^{(m-2)/2} / \{\Gamma(m/2)\ (\det f)^{1/2}\}$, where Γ is Euler's gamma function and det f is just the determinant of f. Finally let, for $1 \le i \le h$, e_i denote the number of linear transformations over $\mathbb{Z}$ which preserve the form. We can now state *Siegel's main theorem for positive definite integral quadratic forms* f (in this special case, for $t \in \mathbb{N}$):

$$\left\{ \sum_{1\le i\le h} r(f_i, t)/e_i \right\} \Big/ \left\{ \sum_{1\le i\le h} 1/e_i \right\} = \frac{1}{1+\delta_{m,2}}\, d_\infty(f, t) \prod_p d_p(f, t) \qquad (*)$$

with $\delta_{m,2}$ denoting the Kronecker delta function.

For f to represent t over $\mathbb{Z}$, it is clearly necessary that f represents t over $\mathbb{R}$ and over the p-adic ring $\mathbb{Z}_p$ for every prime p. But even all the latter infinitely many conditions put together can not ensure that f represents t over $\mathbb{Z}$ although by the Hasse principle, the representation of t by f over $\mathbb{Q}$ is then assured. Siegel's main theorem guarantees in such an event that at least one of $f_1, \ldots, f_h$ represents t over $\mathbb{Z}$ (even if f does not do so) and is thus a far more subtle refinement and a quantitative one too. The remarkable string of papers ([17], [18], [19]) by Siegel deal with the general situation of quadratic forms f representing quadratic forms g (instead of numbers t), again not necessarily over the ring $\mathbb{Z}$ but over the ring of algebraic integers in a totally real algebraic number field and furthermore with f not having to be a definite quadratic form.

§ 3

For $m > 4$, Siegel reformulated the main theorem stated in $(*)$ as an *analytic identity* between theta series associated with $f_1, \ldots, f_h$ and Eisenstein series:

$$\left\{\sum_{1\le j\le h} \theta(f_j, z)/e_j\right\} \Big/ \left\{\sum_{1\le j\le h} 1/e_j\right\} = 1 + \sum_{\substack{(a,b)=1 \\ b>0}}{}' H(f, b, a)(bz-a)^{-m/2} \qquad (**)$$

where, for complex z with positive imaginary part, the theta-series

$$\theta(f_j, z) := \sum_{a_1,\ldots,a_m\in\mathbb{Z}} \exp(\pi i z\, f_j(a_1, \ldots, a_m)),$$

$$H(f, b, a) := (\sqrt{-1/b})^{m/2}(\det S)^{-1/2} \sum_{a_1,\ldots,a_m \bmod b} \exp(\pi i a\, f(a_1, \ldots, a_m)/b)$$

are generalized Gaussian sums and the summation over a, b in the Eisenstein series is carried out over all coprime integer pairs a, b with $b > 0$ and the accent requires ab to be even whenever f represents some odd integer over $\mathbb{Z}$. The Eisenstein series on the right hand side of the analytic identity $(**)$ converges absolutely for $m > 4$. For $m = 4$, one invokes Hecke's limiting process, to suitably define the Eisenstein series via analytic continuation. The significance of the identity $(**)$ can be fully realized only when we notice that it is analytically impossible perhaps to distinguish between the various theta series $\theta(f_i, \mathbb{Z})$ individually in view of their exhibiting the same kind of behaviour as the variable z approaches the 'rational points' a/b or infinity (so that for $i \neq j$, $\theta(f_i, z) - \theta(f_j, z)$ tends to 0 as z tends to a/b or ∞); thus the individual theta series seems to resist being expressible in terms of explicit objects like Eisenstein series. It is on the other hand remarkable that the arithmetical main theorem correctly points to the right weighted mean of theta series $\theta(f_i, z)$ being expressible as an Eisenstein series!

§ 4

For considering the problem of representing forms g in n variables by the given form f (instead of merely representing numbers by f) over $\mathbb{Z}$, the complex variable z above is to be replaced by a point Z of the Siegel upper half plane $\mathbb{H}_n := \{Z = {}^tZ \in \mathfrak{M}_n(\mathbb{Z}) \mid \frac{1}{2\sqrt{-1}}(Z - \overline{Z})$ is positive definite$\}$ and the associated theta series $\theta(S, Z) := \sum_G \exp(\pi i tr({}^tGSGZ))$, where now G runs over all (m, n) integral matrices and tr denotes matrix trace. The general term in the relevant Eisenstein series that features in an analogue of $(**)$ takes the form $H(S, C, D)(\det(CZ + D))^{-m/2}$, where $H(S, C, D)$ are generalized Gauss sums and the summation in the Eisenstein series is over all n-rowed coprime symmetric pairs (C, D) i.e. with (n, n) integral matrices C, D satisfying the conditions that (i) $C^tD = D^tC$ and (ii) GC, GD integral for any (n, n) rational matrix imply that G is integral. Such n-rowed coprime symmetric pairs C, D make up precisely the last n rows of elements

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the Siegel modular $\Gamma_n := \Big\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_{2n}(\mathbb{Z}) \big| A, B, C, D \in \mathfrak{M}_n(\mathbb{Z}), A^t B = B^t A, C^t D = D^t C$ and $A^t D - B^t C = E_n$, the n-rowed identity matrix in $\mathfrak{M}_n(\mathbb{C}) \Big\}$. The Siegel modular group Γ_n acts on $\mathbb{H}_n$ via the modular transformations $Z \to (AZ + B)(CZ + D)^{-1}$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Under modular transformations, the behaviour of the Eisenstein series on the right hand side of an analogue of (**) is quite similar to that of the h theta series $\theta(S_i, Z)$. For $n = 1$, as remarked earlier, $\theta(S_i, z) - \theta(S_j, z)$ is a so-called cusp form for an appropriate subgroup of the elliptic modular group Γ_1; this is no longer true, in general, for $n > 1$. For $n = 1$ this phenomenon however leads immediately to an asymptotic formula for $r(S; t)$ with the main term in the asymptotic formula being given by the Fourier coefficient corresponding to the index t in the Fourier expansion of the Eisenstein series on the right hand side of the identity (*) above.

§ 5

In the proof of Siegel's main theorem (*) and more general formulations covering the representation of forms by f or the case of f being indefinite as well or when $\mathbb{Z}$ is replaced by rings of algebraic integers, a crucial step eventually is to show that a certain number $\rho(S)$ equals 2 where, for S positive,

$$\rho(S) := 2(\det S)^{(m+1)/2} \left(\sum_{1 \le i \le h} 1/e_i \right)^{-1} \prod_{1 \le j \le m} \pi^{-j/2} \Gamma(j/2)$$
$$\lim_{q \to \infty} 2^{\omega(q)} q^{(m+1)/2} / e_q(S)$$

where the limit is taken over natural numbers q tending to infinity, through the sequence of factorials $n!$, $\omega(q)$ is the number of prime factors of q and $e_q(S)$ is the number of linear transformations preserving the quadratic form f associated with S over the quotient ring $\mathbb{Z}/q\mathbb{Z}$. A group-theoretic interpretation of this definition of $\rho(S)$ by bringing in the orthogonal group G of f (defined over $\mathbb{Q}$) and choosing the rings $\mathbb{Z}$, $\mathbb{Z}_p$ for all primes p and $\mathbb{R}$ for the base domain, makes the above crucial step equivalent to proving that $\rho(S)$ is just the (Weil-) Tamagawa number $\tau(G)$ for the orthogonal group G corresponding to f; it was actually shown by Tamagawa that $\tau(G) = 2$. By considering the 'adelic zeta function' attached to orthogonal groups of quadratic forms f and its residues at poles (aptly generalizing Siegel's zeta functions associated with f when indefinite too), Weil proved in [25], for orthogonal groups G of (non-degenerate) quadratic forms in $m \ge 3$ variables, by induction on m, the "Siegel-Tamagawa theorem that $\tau(G) = 2$" which "by purely formal calculations" might be seen to be equivalent to "Siegel's main theorem" (on representation by quadratic forms); Weil's generalization of the Siegel formula for classical groups [27] yields again that $\tau(G)$ is the same for all $m \ge 4$.

§ 6

A nice and surprising proof for Siegel's main theorem for representation of integral quadratic forms (or equivalently (n, n) symmetric matrices T) by unimodular positive definite even quadratic forms $f(x_1, \ldots, x_m)$ for $m > 2n+2$ has been given by Andrianov ([1], [4, Ch. IV, §6]), by explicit determination of the effect of Hecke operators on theta series attached to such f and invoking properties of Eisenstein series. If S is the integral m-rowed symmetric (m, m) matrix associated with any such given form f, then S has even positive integers as diagonal entries and is of determinant 1; it is well-known that m is then necessarily a multiple of 8. Let $S_1(= S), S_2, \ldots, S_h$ be the matrices corresponding to the representatives $f_1, f_2, \ldots, f_h$ of the h different ($\mathbb{Z}$-equivalence) classes in the genus of f. Let $r(S_i, T)$ be the number of integral (m, n) matrices G such that ${}^tGS_iG = T$. Then with notation as in §3, we have for $Z \in \mathbb{H}_n$ and any $S_i (1 \leq i \leq h), \theta(S_i, Z) = \sum_T r(S_i, T) \exp(\pi i\, tr(TZ))$, where the summation in the Fourier expansion on the right hand side is over all (n, n) symmetric non-negative definite even integral matrices T and tr denotes matrix trace as before. For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, we have $\theta(S_i; Z)|M :=$ $\theta(S_i, (AZ + B)(CZ + D)^{-1}) \det (CZ + D)^{-m/2}$ equals $\theta(S_i, Z)$ and thus $\theta(S_i, Z)$ is a Siegel modular form of degree n and weight $m/2$. For any given prime number p, the double coset $\Gamma_n\begin{pmatrix} E_n & 0 \\ 0 & pE_n \end{pmatrix}\Gamma_n$ of $(2n, 2n)$ matrices is a finite union of left cosets $\Gamma_n N_j$ with $N_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$. The Hecke operator $T(p)$ acting on Siegel modular forms $f(Z)$ of degree n and weight $m/2$ is given by

$$f(Z)\big|T(p) = p^{(mn-n^2-n)/2} \sum_j f(A_jZ + B_j)(C_jZ + D_j)^{-1} \times (\det C_jZ + D_j)^{-m/2}.$$

Andrianov determined explicitly the effect of the Hecke operator $T(p)$ on $\theta(S_i; Z)$ and showed that the genus invariant $F(S, Z) := \sum_{1 \leq i \leq h} e_i^{-1} \theta(S_i; Z) / \sum_{1 \leq i \leq h} 1/e_i$ which is the weighted mean of $\theta(S_i; Z)$ generalizing the left hand side of the identity (**) in §3 is actually an eigen form of the Hecke operator $T(p)$ for every prime p; the constant term in the Fourier expansion of $F(S, Z)$ is clearly equal to 1. On the other hand, it is known that any such Siegel modular form (i.e. of degree n and weight $m/2$) which has constant term 1 and is further an eigen form of an infinite number of Hecke operators $T(p)$ has to coincide, for $m/2 > n + 1$, with the Eisenstein series $E_{m/2}(Z) = \sum_{C,D} \det (CZ + D)^{-m/2}$, the summation being over a complete set of n-rowed coprime symmetric pairs (C, D) such that no two distinct ones among them, say $(C_1, D_1), (C_2, D_2)$ are related by the condition $C_1 = UC_2, D_1 = UD_2$ with some integral (n, n) matrix U of determinant ± 1. The condition $m/2 > n + 1$ ensures the absolute convergence of this series. Its Fourier coefficients are well-known from Siegel's fundamental paper [20]; comparison of Fourier coefficients on both sides of the relation $F(S, Z) = E_{m/2}(Z)$ yields Siegel's main theorem on representation of (n, n) positive definite integral matrices T by an even (m, m) positive definite integral S of determinant 1 under the condition $m > 2n + 2$. These restrictions on S may seem to be rather severe since the (arithmetical) main theorem of Siegel is true without such 'stringent' conditions; still the function theoretic proof above for Siegel's main theorem even for a special case of positive definite S does represent a spin off for Arithmetic, especially when, as remarked earlier, it seems impossible to distinguish between the various $\theta(S_i, Z)$ analytically. Let us also recall that, in the other direction, Siegel's main theorem for quadratic forms and the resultant analytic identity (linking theta series with totally disparate objects such as Eisenstein series) was the starting point for Siegel's full fledged analytic theory of modular functions of degree n. Siegel modular functions of degree n have the same significance for algebraic function fields of genus n as the usual (elliptic) modular functions for function fields of genus 1; they represent the contribution from Arithmetic towards the study of an important problem of the theory of algebraic functions. Of course, for the development of the theory of authomorphic forms on semi simple Lie groups, the theory of Siegel modular forms has been the driving force and the touchstone, as well.

§ 7

A Siegel modular form of degree n and weight k is a holomorphic (complex-valued) function f on $\mathbb{H}_n$ such that for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, $f((AZ+B)(CZ+D)^{-1}) \det(CZ+D)^{-k} = f(Z)$ and further f is regular at infinity for $n = 1$. Given a Siegel modular form f of degree n and weight k, the Siegel operator ϕ associates a Siegel modular form ϕf of degree $n-1$ and weight k by the prescription $(\phi f)(Z_1) := \lim_{t\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & it \end{pmatrix}$ for $Z_1 \in \mathbb{H}_{n-1}$. Cusp forms of degree n are those belonging to the kernel of ϕ. If k is *large enough* in relation to n, the Siegel operator is surjective as shown by Maass [12] with the use of his Poincaré series and by Klingen [10] through his Eisenstein series 'lifting' cusp forms g of degree $j \le n$ to Siegel modular forms of degree n. The latter Eisenstein series of degree n are series $G(Z, g)$ of the form $\sum g(\pi(AZ+B)(CZ+D)^{-1}) \det(CZ+D)^{-k}$ with g being any given cusp form of degree $j \le n$, $\pi(W)$ represents the principal (j, j) minor for $W \in \mathbb{H}_n$ and the summation over $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs through representatives of Γ_n modulo an appropriate subgroup. This series converges absolutely under the condition that $k > n + j + 1$ and is mapped to the cusp form g of degree j under the iterated operator ϕ^{n-j}. For $k \le 2n$, one needs to insert in the general summand a Hecke convergence factor $\left(\frac{\det(\mathrm{Im}(\pi(M\langle Z\rangle)))}{\det(\mathrm{Im}(M\langle Z\rangle))}\right)^{-s}$ with $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$, $\mathrm{Im}(W) := \frac{1}{2\sqrt{-1}}(W - \bar{W})$ and a complex parameter s, while real part of s being large clearly ensures absolute convergence of the series; thus the analytic continuation of the Eisenstein series with convergence factors as a function of s has to be studied. For $n = 1$, the first such (vector-valued) Dirichlet series in s arising from Eisenstein series (with Hecke convergence factors) associated to Jacobi theta constants $\theta_1, \theta_2, \theta_3$ or theta series attached to even quadratic forms (of given signature) were thoroughly investigated by Siegel [22, 23] as regards their continuation as a function of s and their functional equation involving s. For general n, the analytic continuation and functional equation for similar Eisenstein series with suitable Hecke convergence factors are subsumed in the general framework occurring in Langlands' theory of Eisenstein series on semi simple Lie groups [11].

It is not clear, on the face of it, if, even for a simple looking Eisenstein series $E_k^{(n)}(Z, s)$ with Hecke convergence factors defined by $E_k^{(n)}(Z, s) := (\det \mathrm{Im}(Z))^s \sum_{C,D} \det (CZ + D)^{-k} \left| \det (CZ + D) \right|^{-2s}$ convergent absolutely for $k > n + 1$ and complex s with $Re(s) \geq 0$, we have, in the 'boundary case' of $k = n + 1 \in 2\mathbb{N}$, a holomorphic function of Z as limit when s tends to 0 say, through positive real values. For $n = 1$ and $k = 2$, we know Hecke [7] that the limit as s tends to 0 exists but is not holomorphic as a function of Z; the first example for higher degrees n, of the Hecke limiting process yielding a non-zero holomorphic modular form of degree n is for the case $n = 3$ and $k = 4$ and may be found in [13]. For general $n \geq 1$ with the 'boundary case' $k = n + 1 \in 2\mathbb{N}$, we know from the comprehensive researches of Weissauer [28] and results obtained independently by Shimura [16], when precisely the limiting process yields, holomorphic modular forms; Weissauer shows in particular that Hecke's Grenzprozess yields non-zero holomorphic modular forms for (even) $k > (n + 3)/2$ or if $4|k$ with $k \leq (n + 1)/2$. Weissauer [28] also determines the obstruction to ϕ-lifting i.e. lifting a cusp form g of degree j and weight k to a holomorphic limit of series resembling Eisenstein series $G(Z, g)$ with appropriate Hecke convergence factors in the case of 'boundary weights' k. The study of Eisenstein series and results on ϕ-lifting are moreover applied in [28] to obtain, in particular, theorems on representation of Siegel modular forms as linear combinations of theta series $\theta(S, Z)$ as above, generalizing Bŏcherer's notable results [2]. Actually Bŏcherer proved that every Siegel modular form (respectively cusp form) of degree n and weight k with $4|k$ and $k > 2n$ is a linear combination of theta series (respectively with spherical harmonics) associated to even unimodular positive definite quadratic forms, by first determining the Fourier expansion of Klingen's Eisenstein series $G(Z, g)$ and then applying important results of Garrett [6], difficult work of Andrianov [1] on Euler products and Siegel's main theorem for quadratic forms. The 'basis problem' for elliptic modular forms (not necessarily of level 1) is one of expressing them as linear combinations of theta series associated with quadratic forms and it is Waldspurger's noteworthy paper [24] that made use of Siegel's main theorem to tackle the 'basis problem', for the first time. "At the other end of the spectrum", when the weight k of Siegel modular forms of degree n is much smaller in relation to n (say, k does not satisfy the condition $k > 2n$ but actually $k < n/2$), we have precisely the singular Siegel modular forms which can also be characterized by having all their Fourier coefficients indexed by (n, n) positive symmetric matrices T (in the Fourier expansion) equal to 0; we know ([4, Ch. IV, §5], [14]) that every singular Siegel modular form of degree n (for Γ_n) is a linear combination of theta series associated with even unimodular positive definite quadratic forms.

§ 8

In two powerful papers ([26], [27]), Weil used the fascinating setting of adelic analysis to present the analytic formulation of Siegel's main theorem for quadratic forms (see also [21]) as an identity between two 'invariant tempered distributions' - the 'theta distribution' and the 'Eisenstein distribution' defined in the context of not just orthogonal groups but more generally classical groups arising from algebras with involution. A vital step in his proof is a very general Poisson formula involving transforms F_Φ^* of Schwartz-Bruhat functions Φ on locally compact abelian groups G and the Fourier transform F_Φ of F_Φ^* and the recognition of the measures 'making up' F_Φ. Suitable specialization of Φ and G leads one to recover Siegel's analytic formulation of his main theorem. Using tools from analysis and deep algebraic geometry such as the Hironaka resolution of singularities in the case of hypersurfaces arising from forms f in n variables and of degree $m \geq 2$, Igusa [9] obtained under appropriate restrictions (e.g. $n > 2m \geq 4$ and the variety defined by $f = 0$ is irreducible and normal) a Poisson formula generalizing that of Weil to the case of forms of higher degree. A local-global theorem of Birch [2] generalizing, to forms f of higher degree, Davenport's results on cubic forms and concerning the existence of rational points on hypersurfaces defined by f, results as an application of Igusa's Poisson formula (see [16] for a nice survey on this).

§ 9

Siegel's main theorem for positive definite quadratic forms f as stated in (*) of §2 is valid without restriction on m while the analytic formulation in (**) of §3 valid for $m > 4$ can be extended to cover the boundary case $m = 4$ as well by an appropriate method viz. Hecke's limiting process to overcome the problem of the Eisenstein series not converging absolutely, for $m = 4$. On the other hand, when f is indefinite, the numbers $r(f, t)$ are no longer finite in general and have to be replaced by 'measures of representation' which are essentially volumes of relevant fundamental domains for discrete subgroups of the orthogonal group of f. Siegel [21] proved an analogue of (*) for indefinite f with $m \geq 4$ and similarly of (**) except when f is a quaternary integral quadratic form with square determinant and represents 0 over $\mathbb{Z}$ non-trivially; the left hand side of this analogue of (**) is given as an integral of a theta series over a fundamental domain for the group of ($\mathbb{Z}-$) automorphisms of f while the right hand side is still an Eisenstein series. Siegel's generalization of all these results of his including those covering representations over algebraic number fields are to be found in [21] while Weil's Siegel formula with its several distinctive features surprisingly unifies the discussion of the definite and indefinite quadratic forms (also with rings of algebraic integers in place of $\mathbb{Z}$) under conditions (e.g. $m > 4$) ensuring absolute convergence of the Eisenstein series. For indefinite ternary or binary quadratic forms, the analogue of Siegel's main theorem fails to be valid [21]; distinct classes in a given genus of binary indefinite forms represent integrally essentially distinct sets of prime numbers.

§ 10

In [15], the Siegel-Weil formula in a range of cases not covered by [27] is established for representation of quadratic forms in n variables by 'anisotropic' (i.e. not representing 0 non-trivially over $\mathbb{Q}$) quadratic forms in m variables over $\mathbb{Q}$ with even integral $m \geq n$ or with all $m \geq 1 = n$. The problem here is far more than just taking care of non-absolute convergence of Eisenstein series (via an analogue of Hecke's procedure)! The proof makes use of Hecke intertwining maps and singular automorphic forms in the sense of Howe [8].

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