Tutorial on Hertz Contact Stress

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Abstract

In mechanical engineering and tribology, Hertzian contact stress is a description of the stress within mating parts. This kind of stress may not be significant most of the time, but may cause serious problems if not take it into account in some cases. This tutorial provides a brief introduction to the Hertzian contact stress theory, five types of the classical solutions for non-adhesive elastic contact are illustrated, and the applications of the Hertz contact stress theory on optomechanical engineering are also addressed.

Introduction

Theoretically, the contact area of two spheres is a point, and it is a line for two parallel cylinders. As a result, the pressure between two curved surfaces should be infinite for both of these two cases, which will cause immediate yielding of both surfaces. However, a small contact area is being created through elastic deformation in reality, thereby limiting the stresses considerable. These contact stresses are called Hertz contact stresses, which was first studies by Hertz in 1881. The Hertz contact stress usually refers to the stress close to the area of contact between two spheres of different radii.

After Hertz's work, people do a lot of study on the stresses arising from the contact between two elastic bodies. An improvement over the Hertzian theory was provided by Johnson et al. (around 1970) with the JKR (Johnson, Kendall, Roberts) Theory. In the JKR-Theory the contact is considered to be adhesive. And then a more involved theory (the DMT theory) also considers Van der Waals interactions outside the elastic contact regime, which give rise to an additional load. Nowadays, the study of two contact bodies and the applications of the theory have become a new discipline "Contact Mechanics".

Hertzian theory of non-adhesive elastic contact

In Hertz's classical theory of contact, he focused primarily on non-adhesive contact where no tension force is allowed to occur within the contact area. The following assumptions are made in determining the solutions of Hertzian contact problems:

i. The strains are small and within the elastic limit.

ii. Each body can be considered an elastic half-space, i.e., the area of contact is much smaller than the characteristic radius of the body.

iii. The surfaces are continuous and non-conforming.

iv. The bodies are in frictionless contact.

Classical solutions for non-adhesive elastic contact

The theory of contact between elastic bodies can be used to find contact areas and indentation depths for simple geometries. There are five types of commonly used solutions, which are listed in the following part.

A. Contact between a sphere and an elastic half-space

As showed in Figure 1, an elastic sphere of radius R indents an elastic half-space to depth u, and thus creates a contact area of radius $a \cong (\frac{3}{2}\frac{RF}{E_*})^{1/3}$.
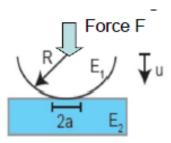

Fig. 1. Contact stress

The depth of indentation u is given by: $u \cong (\frac{2F^2}{E_*^2 R})^{1/3}$

where

$$\frac{1}{E_*} = \frac{1}{2}(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2})$$

And E_1, E_2 are the elastic moduli and v_1, v_2 the Poisson's ratios associated with each body.

Stiffness: $k = \frac{dF}{du} \cong (E_*^2 RF)^{1/3}$

Stress: $(\sigma_c)_{max} \cong \frac{3}{2}\frac{F}{\pi a^2} = 0.4 \left(\frac{E_*^2 F}{R^2}\right)^{\frac{1}{3}} = 0.4\frac{k}{R}$

$\tau_{max} \cong \frac{(\sigma_c)_{max}}{3}$

Example

For a ½" steel tooling on a steel flat with 100 lb loading, E_1 and E_2 are 30 MPa, v_1 and v_2 are 0.3, according to the equations, we have:

$a \cong (\frac{3}{2}\frac{RF}{E_*})^{1/3} = 0.13572$ in

$\frac{1}{E_*} = \frac{1}{2}(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}) \approx 32.967$ Msi

Stiffness: $k = \frac{dF}{du} \cong (E_*^2 RF)^{\frac{1}{3}} \approx 378771 \; lb/in$

Stress: $(\sigma_c)_{max} \cong \frac{3}{2}\frac{F}{\pi a^2} = 0.4 \left(\frac{E_*^2 F}{R^2}\right)^{\frac{1}{3}} = 0.4\frac{k}{R} \approx 260 \; Ksi$

$$\tau_{max} \cong \frac{(\sigma_c)_{max}}{3} \approx 87 \ Ksi$$

B. Contact between two spheres

For contact between two spheres of radii R_1 and R_2, the area of contact is a circle of radius a, as showed in Figure 2.
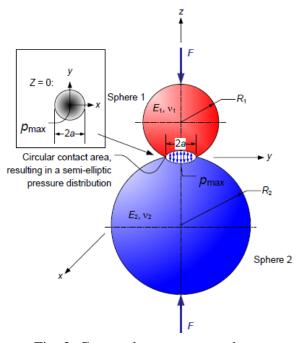

Fig. 2. Contact between two spheres

The radius of the contact area is given by:

$$a = \sqrt[3]{\frac{3F[\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}]}{4(\frac{1}{R_1} + \frac{1}{R_2})}}$$

Where E_1 and E_2 are the moduli of elasticity for spheres 1 and 2 and v_1 and v_2 are the Poisson's ratios, respectively.

The maximum contact pressure at the center of the circular contact area is:

$$P_{max} = \frac{3F}{2\pi a^2}$$

The depth of indentation d is related to the maximum contact pressure by

$$d = \frac{a^2}{R} = \sqrt[3]{\frac{9F^2}{16RE^{*2}}}$$

Where R is the effective radius defined as $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

For ν = 0.33, the maximum shear stress occurs in the interior at $z \approx 0.49a$.

C. Contact between two cylinders with parallel axes

Figure 3 showed the contact between two cylinders with the radii of R_1 and R_2 with parallel axes. In contact between two cylinders, the force is linearly proportional to the indentation depth.
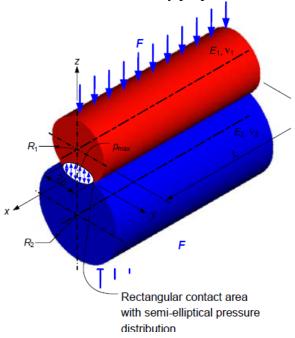

Fig. 3. Contact between two cylinders

The half-width b of the rectangular contact area of two parallel cylinders is found as:

$$b = \sqrt{\frac{4F[\dfrac{1-v_1^2}{E_1} + \dfrac{1-v_2^2}{E_2}]}{\pi L (\dfrac{1}{R_1} + \dfrac{1}{R_2})}}$$

Where E_1 and E_2 are the moduli of elasticity for cylinders 1 and 2 and v_1 and v_2 are the Poisson's ratios, respectively. L is the length of contact.

The maximum contact pressure along the center line of the rectangular contact area is:

$$P_{max} = \frac{2F}{\pi b L}$$

D. Contact between a rigid cylinder and an elastic half-space

If a rigid cylinder is pressed into an elastic half-space, as showed in Figure 4,
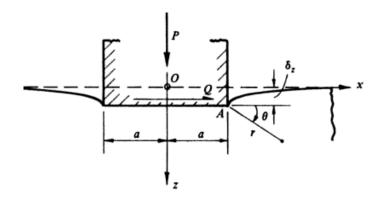
Fig. 4. Contact between a rigid cylinder and an elastic half-space

it creates a pressure distribution described by:

$$p(r) = p_0 \left(1 - \frac{r^2}{a^2} \right)^{-\frac{1}{2}}$$

where a is the radius of the cylinder and

$$p_0 = \frac{1}{\pi} E^* \frac{d}{a}$$

The relationship between the indentation depth and the normal force is given by

$$P = 2aE^* \delta_z$$

E. Contact between a rigid conical indenter and an elastic half-space

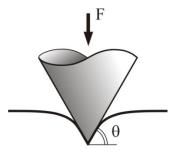In the case of indentation of an elastic half-space using a rigid conical indenter, as showed in Figure 5,

Fig. 5. Contact between a rigid conical indenter and an elastic half-space

the indentation depth and contact radius are related by

$$a = \frac{2}{\pi} d \tan\theta$$

with θ defined as the angle between the plane and the side surface of the cone.

The pressure distribution takes on the form

5

$$p(r) = -\frac{Ed}{\pi a(1-v^2)} \ln\left[\frac{a}{r} + \sqrt{\left(\frac{a}{r}\right)^2 - 1}\right]$$

The stress has a logarithmic singularity on the tip of the cone. The total force is

$$F_N = \frac{2}{\pi} E^* \frac{d^2}{\tan\theta}$$

Application

The stresses and deflections arising from the contact between two elastic solids have practical application in hardness testing, wear and impact damage of engineering ceramics, the design of dental prostheses, gear teeth (Fig. 6), and ball and roller bearings (Fig. 7). For optomechanics, contact stress can cause fretting of the surface, which is showed in Figure 8. This effect could lead to the degradation of accuracy over time and need to be considered during the design procedure. All the calculation can be done using the five classical solutions given above.
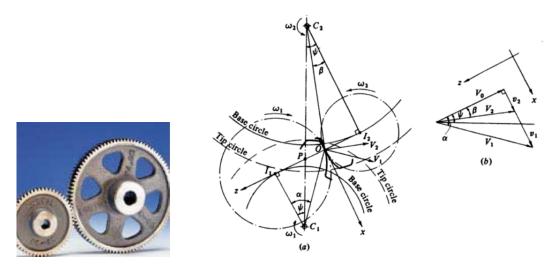

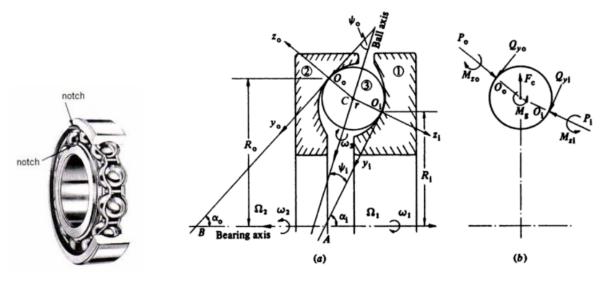
Fig. 6 Contact of involute spur gear teeth
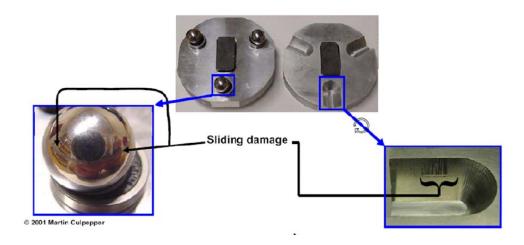
Fig. 7. Angular contact ball bearing

Fig. 8. Effect of contact stress in kinematic constraint using spheres

Summary

Hertz contact stress theory has been applied to many practical applications. The elastic stress fields generated by an indenter, whether it is a sphere, cylinder, or diamond pyramid are well defined in this tutorial. Certain aspects of an indentation stress field make it an ideal tool for investigating the mechanical properties of engineering materials. Although a full mathematical derivation is not given in this tutorial, it presents an overall picture of how these stresses are calculated from first principles. Considering the effects of the contact stress it may cause, it is necessary to consider the stresses and deformation which arise when the surfaces of two solid bodies are brought into contact.

7

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