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Finite Groups whose Cyclic Subnormal Subgroups Satisfy Certain Permutability Conditions [1]

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Dedicated to Professor Hermann Heineken on his 80th birthday

Abstract

Finite groups in which each cyclic subnormal subgroup is semipermutable, S-semipermutable or seminormal are investigated.

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1 Introduction and statement of results

All groups considered in the paper are finite.

There are many articles in the literature (for instance, [1], [5], [8] to name just the three classical ones) where global information about

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a group G is obtained by assuming that some members of relevant families of subgroups of G are either normal or satisfy a sufficiently strongly embedding property extending normality. In many cases, the subgroups are the subnormal subgroups of G, and the embedding assumptions are that they are permutable or S-permutable in G.

Recall that a subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. H is said to be *permutable* (respectively, *S-permutable*) in G if H permutes with all subgroups (respectively, Sylow subgroups) of G. Examples of permutable subgroups include the normal subgroups of G. Non-Dedekind modular groups and non-modular nilpotent groups show that S-permutablilty, permutability and normality are quite different subgroup embedding properties. However, according to a result of Kegel [6], every S-permutable subgroup of G is always subnormal.

A group G is a *PST-group* if every subnormal subgroup of G is S-permutable in G. In the same way classes of *PT-groups* and *T-groups* are defined, in which every subnormal subgroup is permutable or normal respectively. Since normal subgroups are permutable and obviously permutable subgroups are S-permutable then it follows that T is a proper subclass of PT and PT is a proper subclass of PST.

Soluble PST, PT and T-groups were studied and characterised by Agrawal [1], Zacher [8] and Gaschütz [5] respectively.

Theorem 1.1

1. *A soluble group* G *is a PST-group if and only if the nilpotent residual* L *of* G *is an abelian Hall subgroup of* G *on which* G *acts by conjugation as power automorphisms.*

2. *A soluble PST-group* G *is a PT-group (respectively T-group) if and only if* G/L *is a modular (respectively Dedekind) group.*

Note that if G is a soluble T, PT or PST-group then every subgroup and every quotient of G inherits the same properties.

We mention that in [2, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and S-permutability are semipermutability and S-semipermutability introduced by Chen in [4]: a subgroup X of a group G is said to be *semipermutable* (respectively, *S-semipermutable*) in G provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|X|, |K|) = 1$. A semipermutable subgroup of a group need not be subnormal. For example a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (respectively, S-semipermutable) subgroup of a group G must be normalised by every subgroup (respectively Sylow subgroup) P of G such that $\gcd(|X|, |P|) = 1$. This observation was the basis for Beidleman and Ragland [3] to introduce the following subgroup embedding properties.

A subgroup X of a group G is said to be *seminormal* (respectively, *S-seminormal*)[2] in G if it is normalised by every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|X|, |K|) = 1$.

By [3, Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal: it is enough to consider a non-subnormal subgroup H of a group G such that $\pi(H) = \pi(G)$. The following result is an interesting characterisation of soluble PST-groups.

Theorem 1.2 ([3, Theorem 1.5]) *Let* G *be a soluble group. Then the following statements are pairwise equivalent:*

1. G *is a PST-group.*
2. *All the subnormal subgroups of* G *are seminormal in* G.
3. *All the subnormal subgroups of* G *are semipermutable in* G.
4. *All the subnormal subgroups of* G *are S-semipermutable in* G.

Robinson [7] introduced classes of groups in which cyclic subnormal subgroups are S-permutable, permutable or normal.

Definition 1.3 *A group* G *is called* PST_c*-group if every cyclic subnormal subgroup of* G *is S-permutable in* G.

Similarly, classes PT_c and T_c are defined, by requiring cyclic subnormal subgroups to be permutable or normal respectively. Robinson [7] provided characterisations for both soluble and insoluble cases. Here we mention only the soluble case.

Theorem 1.4 ([7]) *Let* G *be a group and* $F = F(G)$*, the Fitting subgroup of* G.

1. G *is a soluble* PST_c*-group if and only if there is a normal subgroup* L *such that*

[2] Note that the term *seminormal* has different meanings in the literature

a) L *is abelian and* G/L *is nilpotent.*

b) p'*-elements of* G *induce power automorphisms in the Sylow* p*-subgroup* L_p *of* L *for all primes* p.

c) $\pi(L) \cap \pi(F/L) = \emptyset$.

2. G *is a soluble PT*$_c$ *(T*$_c$*)-group if and only if* G *is a soluble PST*$_c$*-group such that all the elements of* G *induce power automorphisms in* L *and* F/L *is a modular (Dedekind) group, where* L *is the subgroup described in (1).*

Note that the important distinction between soluble PST- and PST$_c$-groups is that the nilpotent residual is Hall subgroup of the Fitting subgroup whereas the nilpotent residual of a soluble PST-group is a Hall subgroup of the entire group. In fact, Robinson in [7] showed that the sets of primes $\pi(L)$ and $\pi(G/L)$ can have a large intersection, even when G is a soluble T$_c$-group.

It is clear that a soluble PST$_c$-group such that the nilpotent residual is a Hall subgroup of G is a PST-group. Also, note that the class of all soluble PST$_c$-groups is neither subgroup-closed nor quotient closed as proved in [7, Theorems 2.5 and 2.6]. In addition, a PST$_c$-group is a PT$_c$ (T$_c$)-group if all of its Sylow subgroups are modular (Dedekind), respectively [7].

Our interest lies in developing similar connections as in Theorems 1.2 and 1.4 with classes PST$_c$, PT$_c$ and T$_c$.

Theorem A *Let* G *be a soluble group. Then the following statements are pairwise equivalent:*

1. G *is a PST*$_c$*-group.*

2. *All the cyclic subnormal subgroups of* G *are seminormal in* G.

3. *All the cyclic subnormal subgroups of* G *are semipermutable in* G.

4. *All the cyclic subnormal subgroups of* G *are S-semipermutable in* G.

Applying Theorem 1.4 and Theorem A, we have:

Theorem B *Let* G *be a soluble group with abelian nilpotent residual* L. *Then:*

1. G *is a PT*$_c$ *(T*$_c$*)-group if and only if every cyclic subnormal subgroup of* G *is seminormal in* G*, all the elements of* G *induce power automorphisms in* L *and* F/L *is a modular (Dedekind) group.*

2. G *is a PT*$_c$ *(T*$_c$*)-group if and only if every cyclic subnormal subgroup of* G *is semipermutable in* G*, all the elements of* G *induce power automorphisms in* L *and* F/L *is a modular (Dedekind) group.*

3. G *is a PT*$_c$ *(T*$_c$*)-group if and only if every cyclic subnormal subgroup of* G *is S-semipermutable in* G*, all the elements of* G *induce power automorphisms in* L *and* F/L *is a modular (Dedekind) group.*

4. G *is a PT*$_c$ *(T*$_c$*)-group if and only if* G *is an PST*$_c$*-group such that all the elements of* G *induce power automorphisms in* L *and* F/L *is a modular (Dedekind) group.*

2 Proof of Theorem A

The proof of Theorem A depends on the following lemmas.

Lemma 2.1 *Let* G *be a soluble PST*$_c$*-group and let* M *be a cyclic subnormal subgroup of* G*. Then* M *is seminormal in* G*.*

Proof — Assume first that M is a p-group for some prime p. Let L be the nilpotent residual of G and $F = F(G)$ be the Fitting subgroup of G. Then M is contained in F and, by Theorem 1.4, $\pi(L) \cap \pi(F/L) = \emptyset$. Thus L is a Hall subgroup of F which is complemented by $Z_\infty(G)$, the hypercentre of G. It follows that $M \leqslant L$ or $M \leqslant Z_\infty(G)$. Assume that M is contained in L. Then M is a subgroup of L_p. By Theorem 1.4, M is normalised by all p'-elements of G. In particular, M is normalised by all Sylow q-subgroups Q of G for all $q \neq p$. This means that M is seminormal in G.

Suppose that M is a subgroup of $Z_\infty(G)$. In this case, M normalises every Sylow subgroup of G. Let Q be a Sylow q-subgroup of G, where $q \neq p$. Then M is a subnormal Sylow p-subgroup of MQ and so M is normal in MQ. Consequently, M is S-seminormal in G.

Now assume that M has not a prime power order and let M_p be the Sylow p-subgroup of M for a prime $p \in \pi(M)$ and $q \notin \pi(M)$. Let Q be a Sylow q-subgroup of G. Since M_p is a cyclic subnormal S-seminormal subgroup of G, it follows that Q normalises M_p. This implies that Q normalises M and then M is S-seminormal in G. Applying [3, Theorem 1.3], M is a seminormal subgroup of G. □

Lemma 2.2 *Let* G *be a soluble group with every cyclic subnormal subgroup seminormal. Then* G *is a PST*$_c$*-group.*

PROOF — Let X be a cyclic subnormal subgroup of G. We have to show that X is S-permutable in G. Let $p \in \pi(X)$. Then the Sylow p-subgroup X_p of X is seminormal in G and so it is normalised by every Sylow q-subgroup of G for all $q \neq p$. Since X_p is subnormal in G, it follows that X_p is S-permutable in G. This implies that X is S-permutable in G. Thus G is a PST_c-group. □

Lemma 2.3 *Let* G *be a soluble group. Then* G *is a* PST_c*-group if and only if every cyclic subnormal subgroup is S-semipermutable in* G*.*

PROOF — If G is a PST_c-group, then every cyclic subnormal subgroup is seminormal in G by Lemma 2.1. Thus G has all the subnormal cyclic subgroups S-semipermutable.

Assume now that every cyclic subnormal subgroup of G is S-semipermutable in G. If X is cyclic and subnormal in G and $\pi = \pi(X)$, then X is contained in every Hall π-subgroup of G and normalised by every Hall π'-subgroup of G. Consequently, X is S-permutable in G and G is a PST_c-group. □

PROOF OF THEOREM A — Let G be a soluble group. By Lemma 2.1 and Lemma 2.2, G is a PST_c-group if and only if every cyclic subnormal subgroup of G is seminormal in G so that (1) is equivalent to (2). By Lemma 2.3, (1) is equivalent to (4). Finally, (2) implies (3) and (3) implies (4). Therefore statements (1)-(4) of Theorem A are equivalent. □

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