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PRIME IDEALS IN UNIFORM ALGEBRAS

WILLIAM E. DIETRICH, JR.

ABSTRACT. A uniform algebra on a compact metric space has infinite Krull dimension and exactly 2^c nonmaximal prime ideals.

A subalgebra of the continuous, complex-valued functions $C(X)$ on a compact Hausdorff space X which separates the points of X, contains the constant functions and is closed in the uniform norm $\| \ \|_\infty$ is called a uniform algebra. This topic has been elaborated for more than two decades (e.g. [**4**]), but with little attention to algebraic questions. In the classical disc algebra, for example, 0 is a prime ideal and no nonzero prime ideal can lie properly in any maximal ideal determined by an interior point of the disc: does the disc algebra even have a nonzero, nonmaximal prime? An interpolation method used here implies that each boundary maximal ideal will contain 2^c such ideals, arranged in 2^c nonoverlapping infinite chains [Theorem 1]. Even more, every uniform algebra on a first countable space has at least one maximal ideal with this property [Theorem 2]; in particular, its Krull dimension is infinite.

For a subset B of a uniform algebra A, let $Z(B)$ stand for the set of common zeros of the functions in B and for $p \in X$, let I_p denote the maximal ideal of functions in A which vanish at p. Denote the Choquet boundary of A by ∂A [**1**, p. 81].

THEOREM 1. *Suppose A is a uniform algebra and J is an ideal of A. If p is a peak point for A which is not isolated in $Z(J) \cap \partial A$, there are 2^c pairwise disjoint, infinite ascending chains of prime ideals of A with each prime containing J and densely contained in I_p. In particular*, krull dim $A/J = \infty$.

PROOF. Choose $f \in A$ with $f(p)=1$ and $|f(x)|<1$ if $x \neq p$. Inductively select $p_n \in Z(J) \cap \partial A$ so that $|1-f(p_n)|<\min\{1/n, |1-f(p_{n-1})|\}$. p is the only possible accumulation point q of the set $\{p_n\}$. For $f(q)$ is an accumulation point of the distinct points $\{f(p_n)\}$; since $f(p_n) \to 1$, $f(q)=1$ so actually $q=p$. Thus $K=\{p_n\} \cup \{p\}$ is compact, f is a homeomorphism of K onto $f(K)$ and $p_n \to p$.

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$A|K$ is dense in $C(K)$. Take a Borel measure μ on K which annihilates $A|K$; since $f^n \to \chi_{\{p\}}$ boundedly, $0 = \int_K f^n \, d\mu \to \mu(\{p\})$. Also because $p_n \to p$, each p_n has a neighborhood V_n which misses $K - \{p_n\}$; since p_n is a strong boundary point of A [**1**, 2.3.4], there is some $k_n \in A$ with $\|k_n\|_\infty = 1 = k_n(p_n)$ and $|k_n| < 1$ off V_n. Given any $\varepsilon > 0$ we can take a high enough power of k_n to obtain some $g_n \in A$ with $g_n(p_n) = 1$ and $|g_n| < \varepsilon$ on $K - \{p_n\}$.

$$0 = \left| \int_K g_n \, d\mu \right| \geqq |\mu(\{p_n\})| - \varepsilon \, \|\mu\|,$$

and letting $\varepsilon \to 0$ we see that $\mu(\{p_n\}) = 0$. Thus

$$|\mu| \, (K) = |\mu(\{p\})| + \sum_{n=1}^{\infty} |\mu(\{p_n\})| = 0;$$

by the Hahn-Banach theorem, $A|K$ is dense in $C(K)$.

K is a closed set which is a countable union of peak points in the weak sense, so that Glicksberg's peak set theorem [**4**, II. 12.7, p. 58] implies K is an intersection of peak sets. Thus $A|K$ is closed in $C(K)$; in fact then, $A|K = C(K)$.

K is homeomorphic to N_∞, the one point compactification of the natural numbers, and composing the induced isomorphism $C(K) \cong C(N_\infty)$ with restriction $A \to C(K)$, we obtain an algebra homomorphism Φ of A onto $C(N_\infty)$ such that $J \subset \ker \Phi$ and $\Phi(I_p) = M_\infty = \{f \in C(N_\infty) : f(\infty) = 0\}$. According to [**5**, 14G, p. 213] there are 2^c maximal chains of prime ideals of $C_r(N_\infty)$ [the real-valued continuous functions on N_∞] contained in $M_\infty^r = M_\infty \cap C_r(N_\infty)$, and any two chains have only M_∞^r in common. For any such chain $\mathscr{C}$, set $\mathscr{C}^* = \{\Phi^{-1}(P + iP) : P \in \mathscr{C}, P \neq M_\infty^r\}$. Since $P \to P + iP$ is a lattice preserving one-to-one correspondence between the primes of $C_r(N_\infty)$ and those of $C(N_\infty)$ [**3**, 1.1], $\mathscr{C}^*$ is a chain of prime ideals of A contained in $\Phi^{-1}(M_\infty) = I_p$ and containing J; plainly if $\mathscr{D}^*$ is any other such chain, $\mathscr{D}^* \cap \mathscr{C}^* = \varnothing$. Each chain $\mathscr{C}^*$ is infinite ascending since $P \to \Phi^{-1}(P + iP)$ is a lattice preserving bijection and $\mathscr{C}$ is infinite ascending. For otherwise there is some largest $P \in \mathscr{C}$ properly contained in M_∞^r and because $\mathscr{C}$ is maximal, there is no prime ideal of $C_r(N_\infty)$ strictly between P and M_∞^r: a violation of [**2**, 3.2, p. 71].

Finally each $Q \in \mathscr{C}^*$ is dense in I_p. Indeed the prime $P = \Phi(Q)$ is dense in M_p [**2**, 1.5, 1.8]: given $f \in I_p$, there are $g_n \in P$ with $\|g_n - \Phi(f)\|_\infty \to 0$. Since K is a peak interpolation set, there are $h_n \in A$ with $\|h_n\|_X = \|g_n - \Phi(f)\|_\infty$ and $\Phi(h_n) = g_n - \Phi(f)$. Thus $h_n + f \in \Phi^{-1}(P) = Q$ and $\|h_n + f - f\|_X \to 0$.

Of course none of the nonmaximal primes constructed above is closed. This is to be expected since in the disc algebra, for example, Rudin's characterization of the closed ideals [**6**, p. 85] implies that 0 is the only nonmaximal closed prime.

In practice a uniform algebra may only have peak points isolated in its Šilov boundary, or because no point has a countable neighborhood base, even none at all. Nevertheless we have

THEOREM 2. *A uniform algebra A on an infinite first countable space X has a maximal ideal which contains 2^c pairwise disjoint infinite chains of prime ideals.*

PROOF. Suppose ∂A is a discrete subspace of the Šilov boundary Γ of A. Then each $p \in \partial A$ is a peak point [**1**, 2.3.1] which is open in Γ: if f peaks at p, $f^n \to \chi_{\{p\}}$ uniformly on Γ so $1-\chi_{\{p\}} \in A$ peaks (in Γ) on $\Gamma-\{p\}$. Since Γ is infinite [otherwise $A \cong A|\Gamma$ is finite-dimensional and because A separates point, X is finite], $\mathscr{F}=\{\Gamma-F: F\subset \partial A \text{ finite}\}$ is a family of nonvoid closed subsets of Γ with the finite intersection property, so $P=\bigcap \mathscr{F}$ is a generalized peak set for $A|\Gamma$. By a theorem of Bishop [**1**, 2.4.6, p. 105] P contains a generalized peak point $p \in \partial A|\Gamma=\partial A$. Thus $p \in \Gamma-\{p\}$, a contradiction.

We conclude that ∂A contains a point p nonisolated in ∂A. p is a peak point which is the limit of an infinite sequence on ∂A; the result follows from the proof of Theorem 1 with $J=0$.

Notice if X is actually metric (and hence separable), A will have cardinality c, so that Theorem 2 implies

COROLLARY 1. *A uniform algebra on an infinite metric space has infinite Krull dimension and exactly 2^c nonmaximal prime ideals.*

The following answers a question of M. Weiss [**8**, p. 94].

COROLLARY 2. *In a uniform algebra on an infinite first countable space, not every finitely generated ideal is principal.*

PROOF. Otherwise, the primes contained in a fixed maximal ideal form a chain [**5**, 14L, p. 214] in violation of Theorem 2.

EXAMPLE. H^∞, the bounded analytic functions on the open unit disc Δ, considered as a uniform algebra on its maximal ideal space M has no peak points and no point of $M-\Delta$ has a countable neighborhood base: Theorems 1 and 2 do not apply. Nonetheless, suppose $q \in M-\Delta$ lies in the closure of a Carleson-Newman interpolating sequence $S\subset\Delta$: $H^\infty|S=l^\infty=C^*(S)$. Then cl S is homeomorphic to βS, the Stone-Čech conpactification of S [**6**, p. 205], so that actually $H^\infty|\beta S=C(\beta S)$. Since S is realcompact, there is some $f \in C_r(\beta S)$ with $f(q)=0$ and $|f|>0$ on S [**5**, p. 119]. f cannot vanish on any neighborhood of q and assuming the continuum hypothesis, it follows that the maximal ideal M_q of $C_r(\beta X)$ determined by q contains a chain of at least 2^c prime ideals [**5**, 14.19, p. 204]. Because restriction is a homomorphism of H^∞ onto $C(\beta S)$ which takes I_q onto M_q+iM_q, I_q contains a chain of 2^c prime ideals of H^∞; in particular, krull dim $H^\infty=\infty$.

Although interpolating sequences exist in profusion [**6**, p. 204], not every $q \in M-\Delta$ lies in the closure of such a set; in fact Hoffman has shown this happens exactly when the Gleason part for q is nontrivial [**7**, 5.5, p. 101]. Since H^∞ is logmodular on the maximal ideal space X of L^∞, each point of X is a one point part; and there are others [**4**, Example 3, p. 162]. The prime structure at these points is not known, but things are clear elsewhere: if $q \in \Delta$, a routine order of zero argument shows that 0 is the only nonmaximal prime of H^∞ in $I_q=(z-q)H^\infty$; if $q \in M-\Delta$ has a nontrivial part, I_q contains *exactly* 2^c nonmaximal primes. Indeed, Carleson's corona theorem makes M separable, so that H^∞ has cardinality c: even H^∞ has exactly 2^c nonmaximal primes.

For $|\lambda|=1$ the fiber $M_\lambda=\{\phi \in M:\phi(z)=\lambda\}$ is a peak set for H^∞, so that $A_\lambda=H^\infty|M_\lambda$ is a uniform algebra. There is an embedding $\psi:\Delta\to M_\lambda$ so that $A_\lambda|\psi(\Delta)\cong H^\infty$ [**6**, p. 168], and the composite $A_\lambda\to A_\lambda|\psi(\Delta)\to H^\infty\to C(\beta S)$ makes $C(\beta S)$ a homomorphic image of A_λ. Therefore A_λ will also contain exactly 2^c nonmaximal prime ideals and will have infinite Krull dimension.

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Department of Mathematics, University of Texas, Austin, Texas 78712