A CONJECTURE ON COMPACT FRÉCHET SPACES

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ABSTRACT. Let X be a compact Hausdorff space. X is Fréchet if every feebly compact subset is closed in X. Under MA, the converse is false.

A space X is called *Fréchet* if $A \subseteq X$ and $x \in \overline{A}$ imply that there is a sequence $\{a_i; i < \omega\}$ in A which converges to x. Here and in the sequel, spaces are always assumed to be regular. A space is called *feebly compact* if every discrete family of open subsets is finite, or equivalently, for every countable open cover there is a finite subcollection such that the closures of its members cover the space. For more characterizations,. see [BCM]. Note that any countably compact space is feebly compact, and feeble compactness is equivalent to countable compactness for normal spaces, or equivalent to pseudocompactness for completely regular spaces.

Recently, Ismail and Nyikos [IN] showed that under (MA or $2^\omega < 2^{\omega_1}$), a compact Hausdorff space is sequential if and only if every countably compact subset is closed. Since Fréchet spaces are sequential, a reasonable conjecture is that a compact Hausdorff space is Fréchet if and only if every feebly compact subset is closed. One direction is right, i.e. Y. Tanaka [T₂] suggested the following.

THEOREM 1. *Let X be a countably compact space. If every feebly compact subset of X is closed, then X is Fréchet.*

PROOF. Suppose $A \subseteq X$, $x \in \overline{A} \setminus A$ and there are no sequences in A converging to x. Obviously, $\overline{A} \setminus \{x\}$ is not closed. If we could show $\overline{A} \setminus \{x\}$ is feebly compact, the contradiction will lead to the conclusion. Indeed, if $\mathcal{U}$ is a discrete family of nonempty (relatively) open subsets in $\overline{A} \setminus \{x\}$, $\mathcal{V} = \{U \cap A; U \in \mathcal{U}\}$ is clearly a discrete family of (relatively) open subsets in A. Suppose $|\mathcal{U}| \geqslant \omega$, hence $|\mathcal{V}| = |\mathcal{U}| \geqslant \omega$. Take $V_i \in \mathcal{V}$ and $x_i \in V_i \subseteq A$ ($i < \omega$). Since $\mathcal{U}$ is discrete, $\{x_i; i < \omega\}$ has no cluster points in $\overline{A} \setminus \{x\}$. Therefore x is the only cluster point of $\{x_i; i < \omega\}$ in $\overline{A}$, which means x is the limit of $\{x_i; i < \omega\}$.

A Fréchet space is called *fairly Fréchet* if $x \in \overline{A} \setminus A$ implies there is a discrete (relative to A) sequence $\{U_i; i < \omega\}$ of relatively open subsets U_i of A with $x \in \overline{\bigcup_{i < \omega} U_i}$. Note that any hereditarily normal Fréchet space is fairly Fréchet. The next corollary follows directly from the above proof.

COROLLARY. *A countably compact space X is fairly Fréchet if and only if every feebly compact subset is closed.*

Tanaka asked whether the other direction of Theorem 1 is true for countably compact spaces [T_1]. Theorem 2 shows consistently that it is not, even for compact spaces, i.e. there is a compact Fréchet, nonfairly-Fréchet space. I do not know whether it is absolutely false. In the following, we assume readers to be familiar with the fundamental theory of MA and some of its corollaries, for example, the materials in [R].

THEOREM 2 (MA). *There is a compact, Fréchet, Hausdorff space X with a feebly compact nonclosed subset Y.*

PROOF. Let C be D^ω minus one point where $D = \{0, 1\}$.

Let $A_\alpha (\alpha < c)$ be all countable subsets of C. Let $\mathcal{B}$ be a countable basis consisting of compact clopen subsets, and $\mathcal{U}_\alpha$ ($\alpha < c$) be all infinite countable disjoint families of members of $\mathcal{B}$. Now, we are going to inductively define subsets D_α of C and new spaces X_α with topologies τ_α satisfying:

(a) $D_\alpha \subseteq A_\alpha$, and if D_α is nonempty, then it is an infinite discrete closed subset of C;

(b) $X_\alpha = C \cup \{x_\beta; \beta \leqslant \alpha\}$, where the x_β's may be new points for $\beta > 0$;

(c) τ_α is locally compact, zero dimensional, first countable;

(d) for $\alpha_1 < \alpha_2, \tau_{\alpha_1} \subseteq \tau_{\alpha_2}$;

(e) $\mathcal{U}_\alpha$ is not discrete in $X_{\alpha+1}$;

(f) all D_β ($\beta < \alpha$) are still discrete in X_α if they are not empty.

If A_0 is not relatively compact in C, i.e. there is an infinite subset without cluster points in C, let D_0 be an arbitrary infinite discrete closed subset of A_0. Otherwise, let $D_0 = \varnothing$. Let $X_0 = C$ and $x_0 \in C$ be any fixed point. τ_0 is the usual topology on C.

Suppose $X_\beta, x_\beta, D_\beta, \tau_\beta$ have been defined for all $\beta < \alpha$ with properties (a)–(f). Let $X'_\alpha = \bigcup_{\beta < \alpha} X_\beta$ and τ'_α be generated by $\bigcup_{\beta < \alpha} \tau_\beta$. If α is a limit, or $\alpha = \gamma + 1$ and $\mathcal{U}_\gamma$ is not discrete in $X'_\alpha = X_\gamma$, define $X_\alpha = X'_\alpha, \tau_\alpha = \tau'_\alpha$ and $x_\alpha = x_0$.

Now assume $\alpha = \gamma + 1$ and $\mathcal{U}_\gamma$ is discrete in X_γ. Let $\mathcal{U}_\gamma = \{B_i; i < \omega\}$ and let the B_i's be members of $\mathcal{B}$. Note that since each B_i is a CCC compact Hausdorff subspace, and each D_β is nowhere dense in C, so is $D_\beta \cap B_i$ in B_i. By Theorem 2 of [R], $B_i \backslash \bigcup_{\beta < \alpha} D_\beta \neq \varnothing$ for all $i < \omega$. Pick $d_i \in B_i \backslash \bigcup_{\beta < \alpha} D_\beta$. Clearly, $D = \{d_i; i < \omega\}$ is an infinite closed discrete subset of C.

Consider any nonempty D_β. Let the decreasing family $\{G_{i,n}; n < \omega\}$ be a local basis at d_i in C, where all $G_{i,n} \in \mathcal{B}$ and $G_{i,n} \subseteq B_i$. Since $D_\beta \cap D = \varnothing$, there exists a function $f_\beta: \omega \to \omega$ and neighborhoods $G_{i,f_\beta(i)}$ of d_i for $i < \omega$ such that $D_\beta \cap \bigcup_{i < \omega} G_{i,f_\beta(i)} = \varnothing$. By [R, Corollary 11], there is an upper bound g of $\{f_\beta; \beta < \alpha\}$ such that for each $\beta < \alpha$, there is $n < \omega$ such that $g(k) > f_\beta(k)$ for all $k > n$.

Add a new point x_α to X'_α and define $\{\{x_\alpha\} \cup \bigcup_{i > k} G_{i,g(i)}; k < \omega\}$ as its basic neighborhoods. Let $X_\alpha = X'_\alpha \cup \{x_\alpha\}$ and τ_α be defined in an obvious way. It is easy to see that τ_α is locally compact, zero dimensional, first countable and Hausdorff. Moreover, each nonempty D_β ($\beta < \alpha$) is still discrete in X_α by the above definition of g because D_β intersects at most finitely many $G_{i,g(i)}$'s. Besides, (e) is clearly satisfied.

Consider A_α for any $\alpha < c$. If A_α is relatively compact, i.e. A_α is contained in a finite union of compact open subsets in X_α, let $D_\alpha = \varnothing$. In the other case,

$A_\alpha \setminus \bigcup_{k \leq n} U_k$ is infinite for any finitely many compact basic open subsets U_0, $U_1, \ldots, U_n$ in X_α. By Corollary 9 of [R], there is an infinite $D_\alpha \subseteq A_\alpha$ such that $D_\alpha \cap U$ is finite for all compact open subsets U of X_α; hence D_α is a closed discrete subset of X_α. The induction is completed.

Let $Y = \bigcup_{\alpha < c} X_\alpha$ with a basis $\bigcup_{\alpha < c} \tau_\alpha$ and let $X = Y \cup \{e\}$ be the one-point compactification of Y. Finally, we end the proof with two claims.

Claim 1. X is Fréchet.

Obviously, it suffices to check the cases $e \in \overline{A} \setminus A$ for some $A \subseteq X$. If $A \setminus C$ is infinite, since any infinite sequence in $Y \setminus C$ is convergent to e by definition, any infinite sequence in $A \setminus C$ is as desired. Without loss of generality, assume $A \subseteq C$. Since C is hereditarily separable, there is a countable dense subset A' of A. It follows that $e \in \overline{A'} \setminus A'$. Suppose $A' = A_\alpha$ for some $\alpha < c$, then D_α is the desired convergent sequence because A_α could not be relatively compact in X_α, and any compact subset of Y only contains finitely many points of D_α by (f).

Claim 2. Y is feebly compact.

Let $\mathfrak{U}$ be any infinite disjoint family of open subsets of Y. We may assume $\mathfrak{U}$ is countable and each member of $\mathfrak{U}$ is in $\mathfrak{B}$, or a basic neighborhood of the form $\{x_\beta\} \cup \bigcup_{i > n} B_i$ at a new point x_β. Substitute the basic neighborhood by B_i in $\mathfrak{U}$ to get a new family $\mathfrak{U}'$. It will not affect our discussion if we turn to the consideration of $\mathfrak{U}'$. Let $\mathfrak{U}' = \mathfrak{U}_\alpha$ for some $\alpha < c$. (e) says $\mathfrak{U}_\alpha$ is not discrete in $X_{\alpha+1}$, hence in Y.

REMARKS. (a) Note that the nonclosed feebly compact subset Y in Theorem 2 has cardinality c. By using the Čech-Pospišil technique [K], it is not hard to prove that every feebly compact subset of cardinality less than c is closed in any Fréchet space.

(b) After this paper was accepted, E. van Douwen kindly informed the author that for Theorem 2 Berner [B] had obtained another counterexample under MA. But my proof seems simpler and more direct than his.

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