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ON PRIME RINGS WITH COMMUTING NILPOTENT ELEMENTS

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ABSTRACT. Let R be a prime ring in which the nilpotent elements commute. If R has finite right uniform dimension or its maximal right quotient ring is Dedekind finite, then R contains no nonzero nilpotent elements.

At his 1961 AMS Hour Talk, Herstein suggested that the results on the Lie and Jordan structures can be applied to study more purely associative questions. More precisely, he asked the following questions [6, Questions 8 (a-e)]:

(a) If R is a simple ring with a nonzero zero-divisor and if T is a subring of R invariant under all the automorphisms of R, is $T \subseteq Z$ (the center of R) or $T = R$?

(b) If R is a simple ring, and if the nilpotent elements of R form a subring W of R, is $W = (0)$ or $W = R$?

(c) In the special case of (b), wherein any two nilpotent elements of R commute, is it true that R must have no nilpotent elements?

(d) If R is a simple ring with a nontrivial zero-divisor, is every $ab - ba$ in R a sum of nilpotent elements?

(e) If R is a simple ring and has a nonzero nil right-ideal, is R itself nil? This is a special case of the Koethe problem.

It seems that among these questions the first one was studied the most. In particular, the answer to Question (a) for simple rings with nontrivial idempotents was obtained by Amitsur [1] and extended by Herstein [8] for prime rings with nontrivial idempotents. Some nice generalizations for invariant subgroups were later obtained by Lanski [12] and Chuang [3, 4].

To the best of our knowledge all of the questions listed above are still open. In 1980 Herstein mentioned in an unpublished paper that Sasiada had constructed a semiprimitive simple ring in which the set of nilpotent elements forms a nonzero commutative subring. Such an example would give a negative answer to Question (c) (and hence to Question (b)). Unfortunately Sasiada did not publish his result either and nobody today is sure about what it was. The purpose of this paper is to show that Question (c) has an affirmative answer in the case that the rings satisfy certain finiteness conditions. We shall investigate this question not only for simple rings but also for prime rings.

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For a prime ring R the maximal right quotient ring of it will be denoted by $Q_{mr}(R)$. (See [2, Section 2.1] for a detailed study of $Q_{mr}(R)$.)

Theorem 1. *Let R be a prime ring in which any two nilpotent elements commute. Then the subring N of all nilpotent elements in R is a ring with zero multiplication. Moreover, $rN = 0$ for any left zero-divisor $r \in R$ and $Nr = 0$ for any right zero-divisor $r \in R$.*

Proof. Assume that $N \neq 0$; otherwise there is nothing to prove. Let a be a nonzero nilpotent element of R and n the positive integer such that $a^n = 0$ but $a^{n-1} \neq 0$. Note that, for arbitrary $x, y \in R$, the elements axa^{n-1} and $a^{n-1}ya$ are nilpotent so that $a^{n-1}ya^2xa^{n-1} = axa^{n-1}a^{n-1}ya = 0$. Hence $a^2 = 0$ since R is a prime ring. This means that all nilpotent elements in R are square-zero.

Let a, b be two nilpotent elements in R. Then $a^2 = b^2 = 0$ and bRb consists of nilpotent elements. Since the nilpotent elements commute with each other, we have $abRab = abRba = a^2bRb = 0$. Hence $ab = 0$ by the primeness of R. Therefore, the product of any two nilpotent elements of R must be zero.

Now we prove the last statement. Suppose that $rs = 0$ for some nonzero $s \in R$ and let $a \in N$. Note that $sxr \in N$ and so $sx(ra) = (sxr)a = 0$ for all $x \in R$. Since R is prime we get $ra = 0$ as desired. That $Nr = 0$ for any right zero-divisor $r \in R$ can be proved symmetrically. □

Theorem 2. *Let R be a prime ring in which every regular element is invertible in $Q_{mr}(R)$. If any two nilpotent elements of R commute, then R has no nonzero nilpotent elements.*

Proof. Assume on the contrary that R contains a nonzero nilpotent element a; then $a^2 = 0$ by Theorem 1. Let r be a regular element in R. By [2, Proposition 2.1.7 (ii)] there exists a dense right ideal D of R such that $r^{-1}D \subseteq R$. Since D is an essential right ideal by [2, Remark 2.1.3], the intersection $D \cap aR$ is not zero. We claim that $r^{-1}b \in R$ for some nonzero nilpotent element $b \in R$. Indeed, we may take a nonzero element ax in $D \cap aR$ and set $b = ax$ if $axa = 0$ or $b = axa$ if $axa \neq 0$. Observe that $r^{-1}br$ is a nonzero nilpotent element in R, so we have $r^{-1}bra = 0$ by Theorem 1 and consequently $bra = 0$. Then ra is a right zero-divisor in R and so $ara = 0$ by Theorem 1. On the other hand, for an element $r \in R$ which is not regular in R, we have $ra = 0$ or $ar = 0$ by Theorem 1, and so $ara = 0$. Thus $ara = 0$ for all $r \in R$, contradicting the primeness of R. Therefore R does not contain any nonzero nilpotent element. □

For a ring R we set $Z_r(R) = \{x \in R \mid x\rho = 0$ for some essential right ideal ρ of $R\}$. Then $Z_r(R)$ is an ideal of R [2, Lemma 2.1.13 (ii)] and is called the right singular ideal of R. In our situation we need only consider the nonsingular case, as the following lemma shows.

Lemma 3. *Let R be a prime ring in which any two nilpotent elements commute. Then either $Z_r(R) = 0$ or R has no nonzero nilpotent elements.*

Proof. Let N be the subring of all nilpotent elements in R. Since every element in $Z_r(R)$ is a left zero-divisor in R, we have $Z_r(R)N = 0$ by Theorem 1. Then it follows from the primeness of R that either $Z_r(R) = 0$ or $N = 0$. □

For a ring R with identity we say that R is Dedekind finite if $rs = 1$ implies $sr = 1$ for $r, s \in R$.

Theorem 4. *Let R be a prime ring such that $Q_{mr}(R)$ is Dedekind finite. If any two nilpotent elements of R commute, then R has no nonzero nilpotent elements.*

Proof. By Lemma 3 we may assume that $Z_r(R) = 0$. Let r be a regular element in R. Since $Q = Q_{mr}(R)$ is a von Neumann regular ring by [2, Theorem 2.1.15], we have $Qr = Qe$ for some idempotent $e \in Q$. Then $Qr(1-e) = 0$ and so $r(1-e) = 0$ by [2, Lemma 2.1.9 (i)]. If $e \neq 1$, then there exists a dense right ideal D of R such that $0 \neq (1-e)D \subseteq R$ by [2, Proposition 2.1.7 (ii); Proposition 2.1.1 (v)]. Thus $r(1-e)D = 0$, contradicting the regularity of r in R. Hence $e = 1$ and so $Qr = Q$. Consequently, $r'r = 1$ for some $r' \in Q$. By the Dedekind finiteness of Q we have $rr' = 1$ too. That is, every regular element in R is invertible in Q and so the conclusion follows from Theorem 2. □

Recall that a ring R is called a right Goldie ring if the right annihilators in R satisfy the ascending chain condition and R has finite right uniform dimension; that is, R contains no infinite direct sum of nonzero right ideals.

Theorem 5. *Let R be a prime ring of finite right uniform dimension. If any two nilpotent elements of R commute, then R has no nonzero nilpotent elements.*

Proof. By Lemma 3 we may assume that $Z_r(R) = 0$. Then R is a right Goldie ring by [13, Theorem 2.3.6], so it has a classical right quotient ring by [7, Theorem 7.2.1] which can be embedded in $Q_{mr}(R)$ by [10, Proposition 4.6.1]. In particular, every regular element in R is invertible in $Q_{mr}(R)$ and so the conclusion follows from Theorem 2. □

Our next goal is to show that the conclusion of Theorem 4 (or Theorem 5) does not hold for general prime rings, so some additional assumptions such as finiteness conditions are necessary.

Example 6. Let F be a field, $A = F\langle X, Y\rangle$ the free algebra in noncommuting indeterminates X, Y over F, and $R = A/(X^2)$ the factor algebra of A modulo the ideal generated by X^2. Then R is a prime ring with nontrivial nilpotent elements, and any two nilpotent elements of R commute.

Proof. First observe that $Yf \in XA + (X^2)$ or $fY \in AX + (X^2)$ implies $f \in (X^2)$ for $f \in A$ and that $(XA + (X^2)) \cap (AX + (X^2)) = XAX + (X^2)$.

Suppose that $f, g \in A$ are such that $f \notin (X^2)$ and $g \notin (X^2)$. Then $fY \notin AX + (X^2)$ and $Yg \notin XA + (X^2)$ and so $fY^2g \notin (X^2)$ by [5, Theorem]. Hence $fAg \not\subseteq (X^2)$. Therefore, R is a prime ring.

Let $f \in A$ be nilpotent modulo (X^2); that is, $f^n \in (X^2)$ for some positive integer n. Then it follows from [5, Theorem] again that $f \in XA+(X^2)$ and $f \in AX+(X^2)$. Hence $f \in XAX + (X^2)$. Therefore xRx is the set of all nilpotent elements in R where x is the image of X in R under the canonical epimorphism $A \to A/(X^2)$ and so $rs = sr (= 0)$ for all $r, s \in xRx$. □

As to general simple rings we do not yet know the answer to Herstein's question (c) even in the unital case. One possible approach is to try to construct some nontrivial idempotents to get a contradiction to the following auxiliary lemma concerning the maximal symmetric quotient ring $Q_{ms}(R)$ of a prime ring. See [11] for the properties of $Q_{ms}(R)$. Note in particular that $Q_{ms}(R) \subseteq Q_{mr}(R)$.

Lemma 7. *Let R be a prime ring. If any two nilpotent elements of R commute, then $Q_{ms}(R)$ has no nontrivial idempotents.*

Proof. Let e be a nonzero idempotent in $Q_{ms}(R)$. Then by [11, Proposition 2.1] and [2, Theorem 2.1.1 (v)] there exist a dense right ideal D_1 of R and a dense left ideal D_2 of R such that $0 \neq eD_1 \subseteq R$ and $0 \neq D_2e \subseteq R$. Observe that, for any $x, y \in D_1D_2$, the elements $ex(1-e)$ and $(1-e)ye$ in R are nilpotent. Thus we have $ex(1-e)ye = (1-e)yex(1-e)$ for all $x, y \in D_1D_2$. Multiplying both sides of this equation by e, we get $eD_1D_2(1-e)D_1D_2e = 0$ and so $D_2(1-e)D_1 = 0$ by the primeness of R. Hence $e = 1$ by [2, Theorem 2.1.1 (v)]. □

We conclude this paper with some remarks.

Remark 8. One might expect unital simple rings with nontrivial zero-divisors to contain nontrivial idempotents. This is not true, even in the case of noetherian rings. There is an example due to Zalesskii-Neroslavskii of a simple noetherian ring having 1 and nontrivial zero-divisors but no nontrivial idempotents. (See [14, §38, The Zalesskii-Neroslavskii Example] for details.)

Remark 9. If Herstein's question (c) has a negative answer, then the maximal right quotient ring $Q_{mr}(R)$ of the simple ring R in the possible counterexample is not Dedekind-finite by Theorem 4. That is, there exist $u, v \in Q_{mr}(R)$ such that $uv = 1$ while $vu \neq 1$. Set $e_i = v^{i-1}u^{i-1} - v^iu^i$ for $i = 1, 2, 3, \cdots$; then the elements $e_1, e_2, e_3, \cdots$ constitute a set of infinitely many nonzero orthogonal idempotents [9]. Thus such a simple ring R would have the following two properties:

1. $Q_{ms}(R)$ has no nontrivial idempotents (by Lemma 7).
2. $Q_{mr}(R)$ has infinitely many nonzero orthogonal idempotents.

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