IVERSEN'S FORMULA FOR THE SECOND CHERN CLASSES OF REGULAR SURFACES IN ANY CHARACTERISTIC

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Dedicated to the 100*th anniversary of D. K. Faddeev's birth*

ABSTRACT. The formula mentioned in the title is proved.

INTRODUCTION

Let S, T be complete nonsingular surfaces over an algebraically closed field k of any characteristic, and let $h : T \to S$ be a finite separable morphism of degree n. We establish a formula that expresses the Euler characteristic (understood as the degree of the second Chern class $\int c_{2,T}$) of T via the Euler characteristic of S and some local terms associated with components of the branch divisor $B_h = h^* R_h$ of h and with certain points on B_h (here R_h is the ramification divisor).

Let $B_h = \sum_i b_i B_i$, where the B_i are prime divisors on S. Then

$$\chi_T - n\chi_S = \sum_i b_i \chi_{B_i} + \sum_Q \lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}}).$$

Here Q runs over the closed points of T, and $\lambda_f(A'/A)$ is a certain invariant defined explicitly for an extension of complete 2-dimensional regular local rings A'/A and for a (sufficiently general) element f of the maximal ideal of A; this element must be a local equation of a curve at $h(Q)$ in any fixed sufficiently good pencil of curves on S. This invariant is defined in terms of the different of all $A'/\mathfrak{q}$ over $A/(\mathfrak{q} \cap A)$, the invariants of singularity of arcs corresponding to $A'/\mathfrak{q}$ and $A/(\mathfrak{q} \cap A)$, and the invariants of intersection of the latter arc with the branch divisor, where $\mathfrak{q}$ runs over the prime divisors of f in A'. (For the precise statement, see the definitions in §§1, 3, 4 and Theorem 7.4.)

The term $\lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}})$ does not vanish only for a finite number of points Q, all of them lying on the ramification divisor of f.

What is also important, this term depends on the infinitesimal (rather than merely local) behavior of h, i.e., on the properties of extensions of *completed* local rings, and this reduces the further analysis to some questions related only to complete regular local rings.

The formula under discussion is a 2-dimensional analog of the Riemann–Hurwitz formula. In characteristic 0 it was established by Iversen in [Iv].

Remark 0.0.1. I was not able to avoid the dependence on f in the definition of the term λ that describes ramification in codimension 2. However, I expect that λ_f is independent of f (and, therefore, the formula is in its final form) in case there is no ferocious ramification. (This condition means that all morphisms of curves induced by the given finite morphism of surfaces are separable.)

In the good (nonferocious) case, we can show directly that, at all "nonexceptional" points, λ_f does not depend on the choice of a pencil of curves if the pencils are "sufficiently general". Then the exceptional points can be managed presumably by a local-global argument such as that in [L].

§1. Definitions, notation, and preliminary facts

For an arbitrary domain A, we denote by $\tilde{A}$ the integral closure of A; $\delta(A) = l_A(\tilde{A}/A)$; $\nu(A)$ is the number of maximal prime ideals in $\tilde{A}$; $\mathfrak{q}_A$ is the conductor of A, i.e., $\{c \in \tilde{A} \mid c\tilde{A} \subset A\}$; $Q(A)$ is the field of fractions.

If C is a reduced irreducible curve and P is a closed point on it, we denote $\nu_P(C) = \nu(\mathcal{O}_{C,P})$.

If A is a 1-dimensional domain, $a \in A$, and $a \neq 0$, we denote $\mathrm{ord}_A\, a = l_A(A/aA)$.

If A is a 1-dimensional local domain, $\omega \in \Omega_A$, and v is the valuation in $\tilde{A}$, we denote $v(\omega) = v(g)$, where $\omega = g\, dt$, and t is any prime element of $\tilde{A}$.

If A is a local ring, we denote by $\mathfrak{m}_A$ the maximal ideal of A and by $\widehat{A}$ the completion of A.

S_i denotes the set of i-dimensional points of a scheme S.

$k(S)$ denotes the field of rational functions on an integral scheme S.

Let C be a divisor on a complete regular surface S over a perfect field k. Its arithmetic genus is defined as

$$p_a(C) = 1 + \frac{1}{2}(C + K_S.C).$$

Lemma 1.1. *Let A be a 1-dimensional local domain such that $\tilde{A}$ is finite over A. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be all maximal ideals of $\tilde{A}$. Then for any $a \in A$, $a \neq 0$, we have*

$$\mathrm{ord}_A\, a = \sum_{i=1}^{n} \mathrm{ord}_{\tilde{A}_{\mathfrak{m}_i}}\, a \cdot [\tilde{A}_{\mathfrak{m}_i}/\mathfrak{m}_i \tilde{A}_{\mathfrak{m}_i} : A/\mathfrak{m}_A].$$

Proof. See [F, Example A.3.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 1.1.1. *Under the assumptions of Lemma 1.1, suppose that A is a k-algebra, where k is a field. Then*

$$\dim_k(A/a) = \sum_{i=1}^{n} \dim_k(\widetilde{A_{\mathfrak{m}_i}}/a).$$

Wild different. Let B/A be a finite separable extension of complete discrete valuation rings. The order of the different $\mathfrak{D}_{B/A}$ can be written in the form

$$v_B(\mathfrak{D}_{B/A}) = e_{B/A} - 1 + d(B|A),$$

where $d(B|A) \geq 0$, and $d(B|A) = 0$ if and only if the extension is tame. The term $d(B|A)$ is called the *wild different* of B/A; we agree that $d(B|A) = \infty$ if B/A is nonseparable.

Let $\mathcal{O}$ be a complete 2-dimensional regular local ring with a coefficient subfield k. Suppose $b \in \mathcal{O}$ and $a \in \mathfrak{m} \setminus \{0\}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. In this situation we introduce the quantity $d(a,b) \in \mathbb{N} \cup \{+\infty\}$.

If b is an irreducible element, we denote by $\overline{a}$ the image of a in $B = \widetilde{\mathcal{O}/b}$. Then

$$d(a,b) = d(B|k[[\overline{a}]])$$

if $\overline{a} \neq 0$, and $d(a,b) = +\infty$ otherwise. In the general case, if $b = \varepsilon \prod_i p_i^{r_i}$ is a canonical factorization, we put

$$d(a,b) = \sum_i r_i d(a, p_i).$$

Let A be a complete discrete valuation ring with a coefficient subfield k and valuation v.

Lemma 1.2. *For any $f \in \mathfrak{m}_A$, $f \neq 0$, we have $v(df) = v(f) - 1 + d(A|k[[f]])$.*

Proof. Choose an arbitrary prime element $\pi \in A$. Then $A = A_0[\pi]$, where $A_0 = k[[f]]$; $f = F(\pi)$ for some $F \in Xk[[X]]$. Let $F_0 \in A_0[X]$ be the characteristic polynomial of π. Then $f - F(\pi) = 0$ implies $F(X) - f = F_0(X)\rho(X)$ for some $\rho \in A_0[[X]]$, and $v_{A_0}(f) = v_{A_0}(F_0(0)) = 1$ implies $v_{A_0}(\rho(0)) = 0$. We obtain

$$F'(X) = F_0'(X)\rho + F_0\rho',$$

and $F'(\pi) = F_0'(\pi)\rho(\pi)$. Therefore,

$$v(df) = v(F'(\pi)d\pi) = v(F'(\pi)) = v(F_0'(\pi)) = v(\mathfrak{D}_{A/A_0})$$

by [Se, Chapter III, Proposition 11, Corollary 2]. $\qquad\square$

Finite determinacy. Let k be an algebraically closed field, let $\mathfrak{m}$ be the maximal ideal of $k[[X_0, \ldots, X_n]]$, and let $f \in \mathfrak{m}$. The *Tyurina ideal* of f is defined as follows:

$$j(f) = \left(f, \frac{\partial f}{\partial X_0}, \ldots, \frac{\partial f}{\partial X_n} \right).$$

Proposition 1.3. *Suppose $f \in \mathfrak{m}$ and $\mathfrak{m}^l \subset j(f)$. Then for any $g \equiv f \mod \mathfrak{m}^{2l+1}$ we have $k[[X,Y]]/(f) \cong k[[X,Y]]/(g)$.*

Proof. This is exactly [GK, Lemma 2.6]. $\qquad\square$

Generalization of the Weierstrass preparation lemma. Here k is an arbitrary perfect field.

Lemma 1.4. *Let $f \in k[[X,Y]]$ be irreducible. Then, after a possible exchange of X and Y, we have $f = uf_0$, where $u \in k[[X,Y]]^*$, and*

$$f_0 = Y^n + a_1 Y^{n-1} + \cdots + a_{n-1}Y + a_n$$

is a separable polynomial in Y, where $a_i \in Xk[[X]]$.

Proof. This is exactly [Iv, Lemma 2.4]. $\qquad\square$

Corollary 1.4.1. *Let $\mathcal{O}$ be a complete 2-dimensional local ring with coefficient subfield k, and let π be an irreducible element of $\mathcal{O}$. Then $d\pi \notin \pi\Omega_{\mathcal{O}/k}$.*

Proof. Since $\Omega_{\mathcal{O}/k}$ is a free $\mathcal{O}$-module with a basis ds, dt, an equivalent statement is that either $\pi \nmid \frac{\partial \pi}{\partial s}$, or $\pi \nmid \frac{\partial \pi}{\partial t}$. But this follows from Lemma 1.4. $\qquad\square$

Ramification divisor and branch divisor. For a finite separable morphism of two-dimensional schemes $h : T \to S$ of degree n, we consider the corresponding *ramification divisor*

$$R_h = \sum_{\eta \in T_1} l_{\mathcal{O}_{T,\eta}}(\Omega_{T/S,\eta}) \cdot D_\eta,$$

where D_η is a prime divisor with the generic point η, and the *branch divisor*

$$B_h = f_* R_h.$$

§2. Analytic adjunction formula

Let A be a 1-dimensional complete local domain with an algebraically closed coefficient subfield k. Assume that $\operatorname{emb\,dim} A \leq 2$. In other words, A is isomorphic to $k[[X,Y]]/(f)$, where f is an irreducible element of the maximal ideal of $k[[X,Y]]$.

Proposition 2.1. *There exists an irreducible polynomial $f \in Xk[X,Y] + Yk[X,Y]$ such that A is isomorphic to $k[[X,Y]]/(f)$.*

Proof. By Corollary 1.4.1, the Tyurina ideal $j(f)$ is an ideal of definition in $k[[X,Y]]$, and we are done by Proposition 1.3. $\qquad\square$

Proposition 2.2. *We have*
$$\dim_k(\tilde{A}/\mathfrak{q}_A) = 2\delta(A).$$

Proof. Let $A_0 = (k[X,Y]/(f))_{(X,Y)}$, where f is as in Proposition 2.1. Then A is the completion of A_0, $\dim_k(\tilde{A}/\mathfrak{q}_A) = \dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0})$, and $\delta(A) = \delta(A_0)$. It remains to show that $\dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0}) = 2\delta(A_0)$, but this is [Sa, Theorem 5]. $\qquad\square$

Let s,t be generators of the maximal ideal of A such that $f \nmid \frac{\partial f}{\partial t}$, let v be the valuation in $\tilde{A}$, and let $\overline{x}$ be the class of $x \in k[[X,Y]]$ in A.

Proposition 2.3. *We have $v(d\overline{s}) < \infty$, and*
$$v(\mathfrak{q}_A) + v(d\overline{s}) = v\Big(\overline{\frac{\partial f}{\partial t}}\Big).$$

Proof. Indeed, $\dim_k(\tilde{A}/\mathfrak{q}_A) = \dim_k(\widetilde{A_0}/\mathfrak{q}_{A_0})$ as in the preceding proof; we may assume that $s,t \in A_0$. It remains to apply [Sa, Theorem 3bis]. $\qquad\square$

Theorem 2.4. *Let s,t be as above. Then*
$$2\delta(A) + v(d\overline{s}) = v\Big(\overline{\frac{\partial f}{\partial t}}\Big).$$

Proof. This follows from Propositions 2.2 and 2.3. $\qquad\square$

§3. Tame and wild singularity

In this section k is an algebraically closed field, $\mathcal{O}$ is a 2-dimensional complete regular local ring with the coefficient field k, and K is the fraction field of $\mathcal{O}$. If $f,g \in \mathcal{O}$ are such that (f,g) is an ideal of definition, we denote $(f.g) = \dim_k \mathcal{O}/(f,g)$; it is easy to verify the bimultiplicativity of $(f.g)$.

Let $\pi_1, \ldots, \pi_r$ be pairwise nonassociated prime elements of $\mathcal{O}$, and let $f = \pi_1 \ldots \pi_r$. We introduce the *tame singularity of f*:
$$\operatorname{sing}_\mathcal{O}^t f = 2\sum_i \delta(\mathcal{O}/\pi_i) - r + \sum_{i \neq j}(\pi_i.\pi_j) + 1.$$

Next, we introduce the *wild singularity* $\operatorname{sing}_\mathcal{O}^w f$ of f. Assume that there exist regular local parameters s,t of $\mathcal{O}$ such that

(i) $d(f,s)$, $d(s,f)$, $d(f,\frac{\partial f}{\partial t})$, $d(s,\frac{\partial f}{\partial t})$ are all finite, and

(ii) $\pi_i \nmid \frac{\partial \pi_i}{\partial t}$ for any i; $s \nmid \frac{\partial f}{\partial t}$.
Then
$$\operatorname{sing}_\mathcal{O}^w f = -d(f,s) + d(s,f) + d\big(f, \tfrac{\partial f}{\partial t}\big) - d\big(s, \tfrac{\partial f}{\partial t}\big).$$

(In Corollary 3.3.1 we shall see that this value is independent of the choice of s, t.) If no required s, t exist, we put $\operatorname{sing}_{\mathcal{O}}^{w} f = \infty$.

Our plan is to express the Milnor number $\mu(f) = \left(\frac{\partial f}{\partial s} \cdot \frac{\partial f}{\partial t} \right)$ as the sum of tame and wild singularities. It is easily seen that this value is independent of the choice of s and t.

Remark 3.0.1. In the prime characteristic case, the Milnor number is not an invariant of $\mathcal{O}/(f)$, i.e., $\varepsilon \in \mathcal{O}^{*}$ does not imply $\mu(\varepsilon f) = \mu(f)$ (see [GK, Subsection 2.1]). However, it is easy to show that $\varepsilon \in k[[f]]^{*}$ implies $\mu(\varepsilon f) = \mu(f)$.

Lemma 3.1. *The local parameters s, t in $\mathcal{O}$ can be chosen so that*

1) $\pi_i \nmid \frac{\partial \pi_i}{\partial t}$ *for every i;*

2) $s \nmid \frac{\partial f}{\partial t}$;

3) *if* $\operatorname{char} k = p > 0$, *then the conditions* $s' \equiv s \mod (s,t)^2$ *and* $s' \in k[[s, t^p]]$ *imply* $s' \nmid \frac{\partial f}{\partial t}$.

Proof. Let s_0 and t_0 be arbitrary regular local parameters in $\mathcal{O}$. Put

$$s = s_0 + \alpha t_0 + \beta t_0^2,$$
$$t = t_0,$$

where α, β are some elements of k. It is easily seen that

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial s_0},$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} - \alpha \frac{\partial}{\partial s_0} - 2\beta t \frac{\partial}{\partial s_0}.$$

By Corollary 1.4.1, we have $\pi_i \nmid d\pi_i$ for any i, i.e., $\pi_i \nmid \frac{\partial \pi_i}{\partial s_0}$ or $\pi_i \nmid \frac{\partial \pi_i}{\partial t_0}$. It follows that the set M of all pairs (α, β) satisfying condition 1) is nonempty (because k is infinite) and open on the plane.

After a possible intermediate change of variables, we may assume that $(0, 0) \in M$. The relation $\pi_i \nmid \frac{\partial \pi_i}{\partial t_0}$ implies that $\pi_i \nmid \frac{\partial f}{\partial t_0}$, whence $df \neq 0$. Let f_N denote the form in f of minimal degree such that $df_N \neq 0$. Observe that either of the conditions $s|\frac{\partial f}{\partial t}$ and $s'|\frac{\partial f}{\partial t}$ implies

$$(1) \qquad\qquad\qquad s \Big| \frac{\partial f_N}{\partial t} + \frac{\partial f_{N+1}}{\partial t}$$

in $\mathcal{O}/(s_0, t_0)^{N+1}$; in this ring we put

$$\frac{\partial f_N}{\partial t} + \frac{\partial f_{N+1}}{\partial t} = s \cdot \left(\sum_{i=0}^{N-2} g_i s_0^{N-2-i} t_0^i + \sum_{i=0}^{N-1} h_i s_0^{N-1-i} t_0^i \right).$$

After the substitution $s = s_0 + \alpha t_0 + \beta t_0^2$, we obtain

$$\frac{\partial f}{\partial t} = \sum_{i=0}^{N-1} (g_i + \alpha g_{i-1}) s_0^{N-1-i} t_0^i + \sum_{i=0}^{N} (h_i + \alpha h_{i-1} + \beta g_{i-2}) s_0^{N-i} t_0^i.$$

On the other hand, put

$$f_N = \sum_{i=0}^{N} a_i s_0^{N-i} t_0^i;$$

$$f_{N+1} = \sum_{i=0}^{N+1} b_i s_0^{N+1-i} t_0^i.$$

In the same factor ring we compute

$$\frac{\partial f}{\partial t} = \sum_{i=0}^{N-1} ((i+1)a_{i+1} - \alpha(N-i)a_i)s_0^{N-1-i} t_0^i$$

$$- \sum_{i=0}^{N} ((N-i-1)b_{i+1} + \alpha(N-i+1)b_i + 2\beta(N-i+1)a_{i-1})s_0^{N-i} t_0^i.$$

Note that the g_i and h_i (but not a_i and b_i) depend on α and β.

Comparing forms of degree $N-1$, we obtain the following system of relations:

$$g_0 = a_1 - \alpha \cdot N a_0,$$
$$g_1 + \alpha g_0 = 2a_2 - \alpha \cdot (N-1)a_1,$$
$$g_2 + \alpha g_1 = 3a_3 - \alpha \cdot (N-2)a_2,$$
$$\cdots$$
$$g_{N-2} + \alpha g_{N-3} = (N-1)a_{N-1} - \alpha \cdot 2a_{N-2},$$
$$\alpha g_{N-2} = N a_N - \alpha \cdot a_{N-1},$$

whence

(2) $$N(a_n - \alpha a_{N-1} + \alpha^2 a_{N-2} - \cdots \pm \alpha^N a_0) = 0.$$

Suppose that the statement to be proved is false; this means in particular that (1) is satisfied for all α such that $(\alpha, 0) \in M$, i.e., for infinitely many values of α. Then (2) is valid for infinitely many α's, and we see that $N = 0$ in k. This concludes the proof in the case where char $k = 0$, whereas for char $k = p > 0$ we obtain $p|N$, whence

(3) $$g_i = (i+1)a_{i+1}, \qquad i = 0, 1, \ldots, N-2.$$

Next, comparing forms of degree N, we obtain the system of relations

(4) $$h_i + \alpha h_{i-1} + \beta g_{i-2} = (N-i-1)b_{i+1} + \alpha(N-i+1)b_i + 2\beta(N-i+1)a_{i-1},$$

$i = 0, \ldots, N$ (here we agree that $g_{-2} = g_{-1} = 0$, etc.). Assuming $\beta = 0$, we obtain

$$b_{N+1} - \alpha b_N + \alpha^2 b_{N-1} - \cdots \pm \alpha^{N+1} b_0 = 0$$

for infinitely many values of α, whence $b_i = 0$ for any i.

Finally, let β take any nonzero value such that $(\alpha, \beta) \in M$ for infinitely many values of α. Relations (3) and (4) imply that

$$h_i + \alpha h_{i-1} = (-3)\beta(i-1)a_{i-1},$$

$i = 0, \ldots, N$. Therefore,

$$\sum_{i=0}^{N-1} (-1)^i (N-1-i)\beta a_{N-1-i}\alpha^i = 0.$$

Since the value of a polynomial vanishes for infinitely many values of α, all the coefficients must be zero, i.e., $a_n = 0$ if $p \nmid n$. We obtain $f_N \in k[[s_0^p, t_0^p]]$, in contradiction with $df_N \neq 0$. □

Lemma 3.2. *Suppose s, t are regular local parameters, q is an irreducible element of $\mathcal{O}$, and $q \mid \frac{\partial q}{\partial t}$. Then $q = q_1 \varepsilon$, where $q_1 \in k[[s]][t]$, $\varepsilon \in \mathcal{O}^*$, and $\frac{\partial q_1}{\partial t} = 0$.*

Proof. This follows immediately from the Weierstrass preparation lemma. $\qquad\square$

Corollary 3.2.1. *Let s, t be regular local parameters, and let $\mathfrak{q}$ be a prime ideal of height 1 in $\mathcal{O}$ such that $d\overline{s} = 0$ in $\Omega_{\mathcal{O}/\mathfrak{q}}$, where $\overline{s}$ is the class of s in $\mathcal{O}/\mathfrak{q}$. Then the ideal $\mathfrak{q}$ has a generator q such that $\frac{\partial q}{\partial t} = 0$ and $q \equiv s \mod (s, t)^2$.*

Proof. Let q_0 be an arbitrary generator of $\mathfrak{q}$; then $dq_0 = \frac{\partial q_0}{\partial s} ds + \frac{\partial q_0}{\partial t} dt$ implies $\overline{\left(\frac{\partial q_0}{\partial t}\right)} \cdot d\overline{t} = 0$, whence $\overline{\left(\frac{\partial q_0}{\partial t}\right)} = 0$. $\qquad\square$

Proposition 3.3. *Let s, t be any regular local parameters of $\mathcal{O}$ satisfying the conditions of Lemma 3.1. Assume that $\mu(f) < \infty$. Then $d(f, s)$, $d(s, f)$, $d\left(f, \frac{\partial f}{\partial t}\right)$, and $d\left(s, \frac{\partial f}{\partial t}\right)$ are all finite, and*

$$\mu(f) = \mathrm{sing}_{\mathcal{O}}^t f - d(f, s) + d(s, f) + d\left(f, \frac{\partial f}{\partial t}\right) - d\left(s, \frac{\partial f}{\partial t}\right).$$

Proof. First, we show that $d\left(f, \frac{\partial f}{\partial t}\right) < \infty$, $d\left(s, \frac{\partial f}{\partial t}\right) < \infty$, and

$$(5) \qquad \mu(f) = \left(f \cdot \frac{\partial f}{\partial t}\right) + d\left(f, \frac{\partial f}{\partial t}\right) - \left(s \cdot \frac{\partial f}{\partial t}\right) - d\left(s, \frac{\partial f}{\partial t}\right).$$

It suffices to check that $d(f, q) < \infty$, $d(s, q) < \infty$, and

$$(6) \qquad (f.q) = \left(\frac{\partial f}{\partial s} \cdot q\right) - d(f, q) + (s.q) + d(s, q),$$

where q is any irreducible divisor of $\frac{\partial f}{\partial t}$. It is a commonplace that all other terms on both sides of the relation are also finite.

By Corollary 1.1.1, for any $a \in \mathcal{O}$ such that $q \nmid a$ we have $(a.q) = v(\overline{a})$, where $\overline{a}$ is the class of a in $\mathcal{O}/q$, and v is the valuation in $\widetilde{\mathcal{O}/q}$. Since $df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$, we obtain $d\overline{f} = \overline{\frac{\partial f}{\partial s}} d\overline{s}$ in $\Omega_{\mathcal{O}/q}$.

Suppose that $d\overline{s} = 0$ in $\mathcal{O}/q$. Only the case where char $k = p > 0$ is nontrivial. By Corollary 3.2.1, a generator s' of the ideal (q) can be chosen so that $s' \equiv s \mod (s, t)^2$, $s' \in k[[s, t^p]]$. Then by Lemma 3.1 we obtain $s' \nmid \frac{\partial f}{\partial t}$, in contradiction with $q \mid \frac{\partial f}{\partial t}$.

Thus, $v(d\overline{s}) < \infty$ and $v(d\overline{f}) < \infty$. Applying Lemma 1.2 twice, we see that $d(f, q)$ and $d(s, q)$ are finite, and

$$v(\overline{f}) = v(d\overline{f}) + 1 - d(f, q)$$

$$= v\left(\overline{\frac{\partial f}{\partial s}}\right) + v(d\overline{s}) + 1 - d(f, q)$$

$$= v\left(\overline{\frac{\partial f}{\partial s}}\right) + v(\overline{s}) - d(f, q) + d(s, q),$$

which proves (6) and (5).

Similarly, considering $\Omega_{\mathcal{O}/s}$ instead of $\Omega_{\mathcal{O}/q}$, we obtain $d(f, s) < \infty$ and

$$(7) \qquad (f.s) - 1 = \left(\frac{\partial f}{\partial t} \cdot s\right) - d(f, s).$$

We combine (5) and (7):

$$\mu(f) = \left(f \cdot \frac{\partial f}{\partial t}\right) - ((f.s) - 1) + d\left(f, \frac{\partial f}{\partial t}\right) - d\left(s, \frac{\partial f}{\partial t}\right) - d(f, s).$$

It remains to show that $d(s, f) < \infty$, and

$$\left(f \cdot \frac{\partial f}{\partial t}\right) - ((f.s) - 1) = d(s, f) + \operatorname{sing}_{\mathcal{O}}^t f.$$

It suffices to check the relation

$$\left(\pi_i \cdot \frac{\partial \pi_i}{\partial t}\right) - (\pi_i.s) = 2\delta(\mathcal{O}/\pi_i) - 1 + d(s, \pi_i)$$

for every i. By Corollary 1.1.1, this is equivalent to

$$(8) \qquad v\left(\overline{\frac{\partial \pi_i}{\partial t}}\right) - v(\bar{s}) + 1 - d(s, \pi_i) = 2\delta(\mathcal{O}/\pi_i),$$

where v is the valuation in $\widetilde{\mathcal{O}/\pi_i}$. By Lemma 1.2, $d\bar{s} \neq 0$ implies $d(s, \pi_i) < \infty$, and (8) can be rewritten as

$$v\left(\overline{\frac{\partial \pi_i}{\partial t}}\right) - v(d\bar{s}) = 2\delta(\mathcal{O}/\pi_i).$$

However, this identity is none other than the analytic adjunction formula of Theorem 2.4 for the ring $\mathcal{O}/\pi_i$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.3.1. $\operatorname{sing}_{\mathcal{O}}^w f$ *is independent of the choice of* s, t.

Examples. Let x, y be any system of regular local parameters in $\mathcal{O}$.
  1. Let $f = x^l - y^m$, $p \nmid l$, $p \nmid m$. Then $\operatorname{sing}_{\mathcal{O}}^w f = 0$.
  2. Let $f = y^p + y^M - x^2$, $p \nmid M$. Taking $s = x, t = y$, we compute

$$d(f, s) = d(s, f) = d\left(s, \frac{\partial f}{\partial t}\right) = 0,$$

and

$$d\left(f, \frac{\partial f}{\partial t}\right) = M - p,$$

whence $\operatorname{sing}_{\mathcal{O}}^w f = M - p$.
  3. Let $f = y^p - x^2$. Then $d(f, x) = \infty$, whence for any choice of regular local parameters s, t we have $d(f, \frac{\partial f}{\partial t}) = \infty$, and $\operatorname{sing}_{\mathcal{O}}^w f = \infty$.

§4. Extensions of 2-dimensional complete regular local rings

In this section $\mathcal{O}'/\mathcal{O}$ is a finite separable extension of complete 2-dimensional regular local rings of some degree n, both having an algebraically closed coefficient subfield k.

Proposition 4.1. *Let* π *be a prime element of* $\mathcal{O}$ *that does not divide a local equation* β *of the branch divisor of* $\mathcal{O}'/\mathcal{O}$, *and let* $\pi = \pi_1 \ldots \pi_r$, *where* $\pi_1, \ldots, \pi_r$ *are irreducible elements of* $\mathcal{O}'$. *Then*

$$\operatorname{sing}_{\mathcal{O}'}^t \pi - n \operatorname{sing}_{\mathcal{O}}^t \pi = (\beta.\pi) - (n - 1) - \sum_{i=1}^r d((\mathcal{O}'/\pi_i)|(\mathcal{O}/\pi)).$$

Proof. We choose regular local parameters s, t in $\mathcal{O}$ so that $\pi \nmid \frac{\partial \pi}{\partial t}$. This can be done by Lemma 3.1. Fixing j, we choose regular local parameters u, v in $\mathcal{O}'$ so that $\pi_j \nmid \frac{\partial \pi_j}{\partial v}$. Denote $A_j = \widetilde{\mathcal{O}'/\pi_j}$.

Now we do some computations in $\Omega_{Q(A_j)/k}$; the elements of $\mathcal{O}'$ (respectively, of $\Omega_{\mathcal{O}'/k}$) will be denoted by the same letters as their images in A_j (respectively, in $\Omega_{Q(A_j)/k}$). First, we have

$$\frac{\partial \pi_j}{\partial u} du + \frac{\partial \pi_j}{\partial v} dv = d\pi_j = 0.$$

Next, the derivations $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ of $Q(\mathcal{O})$ can be uniquely extended up to continuous derivations of $Q(\mathcal{O}')$, also denoted by $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$. We have

$$\frac{\partial}{\partial u} = \frac{\partial s}{\partial u} \cdot \frac{\partial}{\partial s} + \frac{\partial t}{\partial u} \cdot \frac{\partial}{\partial t},$$

because these derivations coincide on the elements s and t, and $Q(\mathcal{O}')$ is separable over $Q(\mathcal{O}) = Q(k[[s, t]])$. Therefore, the identity

$$ds = \left(\frac{\partial s}{\partial u} \frac{\partial \pi_j}{\partial v} du + \frac{\partial s}{\partial v} \frac{\partial \pi_j}{\partial v} dv \right) \left(\frac{\partial \pi_j}{\partial v} \right)^{-1}$$

$$= \left(\frac{\partial s}{\partial u} \frac{\partial \pi_j}{\partial v} du - \frac{\partial s}{\partial v} \frac{\partial \pi_j}{\partial u} du \right) \left(\frac{\partial \pi_j}{\partial v} \right)^{-1}$$

$$= \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial \pi_j}{\partial u} & \frac{\partial \pi_j}{\partial v} \end{vmatrix} \left(\frac{\partial \pi_j}{\partial v} \right)^{-1} du$$

in $\Omega_{Q(A_j)/k}$ implies

$$\left(\frac{\partial \pi}{\partial t} \right)^{-1} ds = \left(\frac{\partial \pi}{\partial t} \right)^{-1} \begin{vmatrix} \frac{\partial s}{\partial s} & \frac{\partial s}{\partial t} \\ \frac{\partial \pi_j}{\partial s} & \frac{\partial \pi_j}{\partial t} \end{vmatrix} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \left(\frac{\partial \pi_j}{\partial v} \right)^{-1} du$$

(9)
$$= \left(\frac{\partial \pi}{\partial t} \right)^{-1} \frac{\partial \pi_j}{\partial t} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \left(\frac{\partial \pi_j}{\partial v} \right)^{-1} du$$

$$= \prod_{i \neq j} \pi_i^{-1} \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \left(\frac{\partial \pi_j}{\partial v} \right)^{-1} du.$$

We shall apply the analytic adjunction formula of Theorem 2.4. Let w_j and w be the valuations in A_j and $A = \widetilde{\mathcal{O}/\pi}$, respectively, and let $e_j = e(A_j/A)$, $d_j = d(A_j|A)$. We denote by ρ_j and ρ any local parameters in A_j and A, respectively. We write the left-hand side and the right-hand side of (9) in the form $Ld\rho_j = L_0 d\rho$ and $Rd\rho_j$, respectively, and compute the valuations of L and R.

From the exact sequence

$$\Omega_{A/k} \otimes_A A_j \to \Omega_{A_j/k} \to \Omega_{A_j/A} \to 0$$

it is clear that $d\rho = ad\rho_j$, where a is a generator of the different of A_j/A, that is, $w_j(a) = e_j - 1 + d_j$. We have $w(L_0) = -w\left(\frac{\partial \pi}{\partial t} \right) + w(ds)$. From this it follows that

$w_j(L) = \left(-w\left(\frac{\partial \pi}{\partial t}\right) + w(ds)\right)e_j + e_j - 1 + d_j.$ Next,

$$w_j(R) = -\sum_{i \neq j} w_j(\pi_i) + w_j\left(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}\right) - w_j\left(\frac{\partial \pi_j}{\partial v}\right) + w_j(du).$$

Thus, we have

$$\left(-w\left(\frac{\partial \pi}{\partial t}\right) + w(ds)\right)e_j + e_j - 1 + d_j = -\sum_{i \neq j} w_j(\pi_i) + w_j\left(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}\right) - w_j\left(\frac{\partial \pi_j}{\partial v}\right) + w_j(du).$$

We apply the analytic adjunction formula to both sides, and apply Corollary 1.1.1 on the right-hand side:

$$(-2\delta(\mathcal{O}/\pi) + 1)e_j + d_j = 1 - \sum_{i \neq j}(\pi_i.\pi_j) + (\beta'.\pi_j) - 2\delta(\mathcal{O}'/\pi_j),$$

where β' is an equation of the ramification divisor in $\mathcal{O}'/\mathcal{O}$. Finally, we sum over j:

$$-n(\operatorname{sing}_{\mathcal{O}}^t \pi - 1) + \sum_j d_j = -(\operatorname{sing}_{\mathcal{O}'}^t \pi - 1) + (\beta.\pi). \qquad \square$$

Definition of λ_f. For the branch divisor of $\mathcal{O}'/\mathcal{O}$ (viewed as a closed subscheme in $\operatorname{Spec}\mathcal{O}$), denote by β_i the equations of its prime components and by b_i their multiplicities. We take an element $f \in \mathcal{O}$ that is a product of pairwise nonassociated prime elements in $\mathcal{O}$ such that $\operatorname{sing}_{\mathcal{O}}^w f < \infty$ and none of the β_i divides f. Let $f = \pi_1' \ldots \pi_r'$ be a factorization of f in $\mathcal{O}'$. Every π_i' divides exactly one prime divisor π_i of f in $\mathcal{O}$; we denote by A_i' and A_i the integral closures of $\mathcal{O}'/\pi_i'$ and $\mathcal{O}/\pi_i$, respectively. We define

$$\lambda_f(\mathcal{O}'/\mathcal{O}) = \sum_i b_i(1 - d(f, \beta_i)) - (n-1) - \sum_{i=1}^r d(A_i'|A_i) + \operatorname{sing}_{\mathcal{O}'}^w f - n \operatorname{sing}_{\mathcal{O}}^w f.$$

Conjecture 4.2. *Assume that there is no ferocious ramification in $\mathcal{O}'/\mathcal{O}$. Let $f, f' \in \mathcal{O}$ be elements such that $\lambda_f(\mathcal{O}'/\mathcal{O})$ and $\lambda_{f'}(\mathcal{O}'/\mathcal{O})$ are defined. Then $\lambda_f(\mathcal{O}'/\mathcal{O}) = \lambda_{f'}(\mathcal{O}'/\mathcal{O})$.*

Example. An ample series of examples can be constructed if for the role of $\mathcal{O}'$ we take the integral closure of $\mathcal{O}$ in the extension of the fraction field of $\mathcal{O}$ given by the equation $x^p - x = t^{-pm+1}u^{-pn}$, where t, u are fixed regular local parameters in $\mathcal{O}$, and m, n are nonnegative integers. In is easily seen that such an $\mathcal{O}'$ is always regular.

In this paper we only consider an example of a nonexceptional point on a component of the branch divisor without ferocious ramification. For this, take $m > 0$ and $n = 0$. (The notion of an exceptional point is due to Deligne [D]; see also Brylinski [Br]. Roughly speaking, these are the points on the ramification divisor where the codimension 2 ramification invariants take their nongeneric values.)

Let $y = t^m x$. Then $y^p - t^{(p-1)m}y = t$, whence $\mathcal{O}' \supset \mathcal{O}[y] = k[[y, u]]$. Since $\mathcal{O}[y]$ is regular, we have $\mathcal{O}[y] = \mathcal{O}'$, and

$$t = y^p - y^{p+(p-1)(pm-1)} + O(y^{p+2(p-1)(pm-1)}).$$

The branch divisor B consists of one component $t = 0$, and $b_1 = (p-1)pm$.

Let $f = t - u^i$, $i > 0$, $(i, p) = 1$. Then the expansion of f in $k[[y, u]]$ is

$$f = F(y, u) = y^p - y^{p+(p-1)(pm-1)} - u^i + O(y^{p+2(p-1)(pm-1)}).$$

We have

$$\text{sing}_{\mathcal{O}'}^{w} f = d(f,y) - d(y,f) - d\big(f, \frac{\partial f}{\partial u}\big) + d\big(y, \frac{\partial f}{\partial u}\big)$$
$$= 0 - 0 - (i-1)d\big(k[[y]]|k[[F(y,0)]]\big) + 0$$
$$= (i-1)(p-1)(pm-1).$$

It follows that

$$\lambda_f(\mathcal{O}'/\mathcal{O}) = (p-1)pm - (p-1) - d\big(k[[\widetilde{[y,u]}]]/f|k[[u]]\big) + (i-1)(p-1)(pm-1)$$
$$= (p-1)(pm-1) - i(p-1)(pm-1) + (i-1)(p-1)(pm-1) = 0.$$

§5. Severi's formula

In this section, S is a regular geometrically irreducible complete surface over a perfect field k; K is the field of functions on S.

Lemma 5.1. *Let s,t be regular local parameters at $P \in S$. Then for any point P' in some neighborhood of P, the functions $s - s(P')$ and $t - t(P')$ are regular local parameters at P'.*

Proof. Let U be a neighborhood of P such that s and t are regular functions on U. Consider the morphism $f : U \to \mathbb{A}_k^2$ determined by the pair of functions s,t. Obviously, f is unramified at P, whence f is also unramified at all points of some neighborhood of P. □

Let ω be a nonzero rational 1-differential on S, i.e., $\omega \in \Omega_{K/k}$, and $\omega \neq 0$. We define a divisor (ω) and a 0-cycle $\langle \omega \rangle$ on S.

Let $P \in S$, and let s,t be regular local parameters at P. Then $\omega = f_P \cdot (a_P \, ds + b_P \, dt)$, where a_P and b_P are coprime in $\mathcal{O}_{S,P}$, and $f_P \in K$. Then (ω) is defined as the divisor on S such that in a neighborhood of any closed point P it coincides with the divisor of the function f_P; this can be done in view of Lemma 5.1. Finally, by definition,

$$\langle \omega \rangle = \sum_{P \in S_0} \dim_k \mathcal{O}_{S,P}/(a_P, b_P) \cdot P.$$

Theorem 5.2. *In the group $A_0(S)$ we have*

$$c_{2,S} = \langle \omega \rangle + (\omega) \cdot K_S - (\omega) \cdot (\omega).$$

Proof. See [Y, Corollary to Theorem 2'], or [K]. See also the proof involving higher adèles in [P]. □

§6. Computation of the second Chern class with the help of a pencil of curves

Let S be as in the preceding section; k is assumed to be algebraically closed. Here a *pencil of curves on S* is treated as a dominant rational map of S in $\mathbb{P}_k^1$. In other words, this is a surjective morphism

$$\mathcal{C} : S \setminus B \to \mathbb{P}_k^1$$

that cannot be extended to any point of B; B is a closed subset of S, the so-called set of base points of a given pencil of curves. Theorem 3 in [Sh, Chapter II, §3] implies that $\dim B = 0$.

We shall consider only pencils $\mathcal{C}$ of curves satisfying the following condition:

($*$) the fiber of $\mathcal{C}$ over any point $s \in \mathbb{P}_k^1$ is a reduced subscheme.

The closure of $\mathcal{C}^{-1}(s) \subset S \setminus B$ in S will be denoted by $\mathcal{C}_s$. The theorem on dimension of fibers of a morphism (see, e.g., [H, Chapter II, Example 3.22]) implies that $\mathcal{C}_s$ is of pure dimension 1. The schemes $\mathcal{C}_s$ are called *curves in* $\mathcal{C}$.

It is easily seen that all curves in $\mathcal{C}$ belong to one and the same divisor class on S. Therefore, for an arbitrary divisor D on S the intersection multiplicity $(\mathcal{C}_s.D)$ is independent of s; it will be denoted simply by $(\mathcal{C}.D)$.

We require also that the following condition be fulfilled:

($**$) for any base point b all curves in $\mathcal{C}$ pass through b and any two of them meet at b transversally.

If $s = \mathcal{C}(P)$, the curve $\mathcal{C}_s$ will be denoted also by $\mathcal{C}_P$.

For $P \in S \setminus B$, we put $\operatorname{sing}_P^w \mathcal{C} = \operatorname{sing}_{\widehat{\mathcal{O}_{S,P}}}^w (t_s \circ \mathcal{C})$ and $\operatorname{sing}_P^t \mathcal{C} = \operatorname{sing}_{\widehat{\mathcal{O}_{S,P}}}^t (t_s \circ \mathcal{C})$, where $s = \mathcal{C}(P)$ and t_s is a local parameter at s. Clearly, $\operatorname{sing}_P^t \mathcal{C}$ is independent of t_s. Remark 3.0.1 shows that $\operatorname{sing}_P^w \mathcal{C}$ is also independent of t_s. Our third requirement is as follows.

($***$) $\operatorname{sing}_P^w \mathcal{C} < \infty$ for all $P \in S \setminus B$.

We say that a pencil $\mathcal{C}$ on S is *separable* if it satisfies conditions ($*$), ($**$), and ($***$).

Proposition 6.1. *Let S be a regular projective surface, and let $C_1, \ldots, C_n$ be prime divisors. Then there exists a pencil of curves $\mathcal{C}$ on S with the set of base points B such that*:
  – *for any i we have $B \cap C_i = \varnothing$;*
  – *for any i, $\mathcal{C}$ induces a separable morphism $C_i \to \mathbb{P}_k^1$;*
  – *$\mathcal{C}$ is separable.*

Proof. The proposition is obvious in the case of $S = \mathbb{P}_k^2$; the general case can easily be derived by applying the following two lemmas.   □

Lemma 6.2. *Under the conditions of Proposition 6.1, S admits a finite separable morphism g onto $\mathbb{P}_k^2$ such that $C_1, \ldots, C_n$ are not components of its ramification divisor R_g.*

Proof. Assume that $S \subset \mathbb{P}^N$. We choose arbitrary closed points $P_i \in C_i$, $i = 1, \ldots, n$, and let H_i be the projective closure of the tangent plane to S at P_i embedded into $\mathbb{P}^N$. Choose a projective subspace $W \cong \mathbb{P}^{N-3}$ in $\mathbb{P}^N$ that meets neither S nor any of H_i. Then the projection with the center W determines a finite morphism of S onto $\mathbb{P}^2$ unramified at any of the P_i.   □

Lemma 6.3. *Let $h : T \to S$ be a finite separable morphism of surfaces, and let $\mathcal{C}$ be a separable pencil on S such that the curves in $\mathcal{C}$ have no common components with the branch divisor B_h. Then the pencil $\mathcal{D} = \mathcal{C} \circ h$ on T is also separable.*

Proof. Let b be any base point of $\mathcal{D}$. Since h is unramified at b, regular local parameters at $h(b)$ generate the maximal ideal of $\mathcal{O}_{T,b}$. In other words, if two curves are regular at $h(b)$ and meet transversally at this point, the same is true for their preimages with respect to the point b. This proves that $\mathcal{D}$ satisfies ($**$). Condition ($*$) can be verified by a similar argument applied to any regular point on a given curve in $\mathcal{D}$ such that h is unramified at that point.

To verify condition ($***$), it suffices to note that the Jacobi matrix of h is of rank 2 at any point of T.   □

For a separable pencil $\mathcal{C}$ on S, we introduce the 0-cycle

$$\operatorname{sing} \mathcal{C} = \sum_{P \in S \setminus B} (\operatorname{sing}_P^t \mathcal{C} + \operatorname{sing}_P^w \mathcal{C}) P.$$

Since all curves in $\mathcal{C}$ belong to the same divisor class, their arithmetic genera are the same; this common value will be denoted by $p_a(\mathcal{C})$.

Proposition 6.4. *Let $\mathcal{C}$ be a separable pencil of curves on S with the set of base points B. Then*

$$\int c_{2,S} = \int \text{sing}\, \mathcal{C} - \#B - 4(p_a(\mathcal{C}) - 1).$$

Proof. We pick any rational function on the projective line such that its divisor of zeros as well as its divisor of poles has degree 1. (If we choose the point at infinity suitably, any such function is of the form $\lambda\frac{X-\alpha}{X-\beta}$ for some $\alpha, \beta, \lambda \in k$.) Let f denote the inverse image of this function in $k(S)$. The idea of the proof is to apply Theorem 5.2 to df.

The above explicit description of the original function implies that any other choice of such a function would change f to $\frac{af+b}{cf+d}$, where $a, b, c, d \in k$, $ad - bc \neq 0$. It follows that $\langle df \rangle = \sum_{P \in S_0} n_P P$ is independent of this choice.

Now we compute n_P for $P \notin B$. Let D be any curve in $\mathcal{C}$ that is distinct from $\mathcal{C}_P$. If the divisor of f is $\mathcal{C}_P - D$, then f is a local equation of $\mathcal{C}_P$ at P. If s and t are local parameters at P, then $df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$. We have

$$n_P = \left(\frac{\partial f}{\partial s} . \frac{\partial f}{\partial t} \right) = \mu_{\widehat{\mathcal{O}_{S,P}}}(f) = \text{sing}_P^t\, \mathcal{C} + \text{sing}_P^w\, \mathcal{C};$$

i.e., the 0-cycles $\langle df \rangle$ and $\text{sing}\, \mathcal{C}$ have equal coefficients at P.

Now, let $P \in B$. For any choice of f, its divisor is $C - D$, where C and D are curves in $\mathcal{C}$ that meet transversally at P. Then f can be written as g/q, where g and q are local equations of C and D at P. We have

$$df = \frac{g\, dq - q\, dg}{g^2},$$

whence $n_P = 1$.

Thus, we obtain

(10) $$\langle df \rangle = \text{sing}\, \mathcal{C} + \sum_{P \in B} P.$$

Next, we compute (df). The divisor of f is $C - D$, where C and D are some curves in $\mathcal{C}$; we shall prove that

(11) $$(df) = -2D.$$

Take any $P \notin B$, and let s and t be regular local parameters. If $P \in C$, then, as we saw in the calculation of $\langle df \rangle$, here $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ are coprime, i.e., $P \notin \text{Supp}(df)$. If $P \in D$, then $df = -f^2 \cdot d(f^{-1})$ and, as above, $P \notin \text{Supp}(d(f^{-1}))$. Finally, if $P \notin C \cup D$, then P belongs to the zero divisors of $f - f(P)$, and this case can be reduced to the case of $P \in C$ by replacing f with $f - f(P)$.

Substituting (10) and (11) in the formula of Theorem 5.2, we obtain the following identity in $A_0(S)$:

$$c_{2,S} = \text{sing}\, \mathcal{C} + \sum_{P \in B} B - 2(C.K_S) - 4(C.C),$$

where C is an arbitrary curve in $\mathcal{C}$. Clearly, condition $(**)$ in the definition of a separable pencil implies that $(C.C) = \sum_{P \in B} B$. Therefore,

$$c_{2,S} = \text{sing}\, \mathcal{C} - \sum_{P \in B} B - 2(C.K_S) - 2(C.C).$$

Now it suffices to calculate the degrees and to apply the definition of the arithmetic genus. □

We prove another property of pencils.

Proposition 6.5 (generalized Plücker equation). *Let $\mathcal{C}$ be a pencil of curves on S with the set of base points B, let D be a reduced irreducible curve on S that is not a component of any curve in $\mathcal{C}$, and let $D \cap B = \varnothing$. Assume that the restriction φ of the morphism $\mathcal{C}$ to D is a separable morphism of D onto a projective line. Then*

$$\sum_{P \in D} \left((\mathcal{C}_P.D)_P - \nu_P(D) + d_P \right) = 2(\mathcal{C}.D) + 2p_g(D) - 2,$$

where d_P is the sum of the wild differents of φ at all points of the normalization of D over P.

Proof. We denote by $\lambda : \tilde{D} \to D$ the normalization morphism. Let $P \in D$. For $P' \in \lambda^{-1}(D)$, let $e_{P'}$ denote the ramification index of $\varphi \circ \lambda$ at P'. Corollary 1.1.1 implies

$$\sum_{P' \in \lambda^{-1}(P)} e_{P'} = (\mathcal{C}_P.D)_P.$$

It is easily seen that $\deg(\varphi \circ \lambda) = (\mathcal{C}.D)$, and it remains to apply the Riemann–Hurwitz formula to the morphism $\varphi \circ \lambda$. □

§7. Morphisms of surfaces

In this section we consider a finite separable morphism $h : T \to S$ of degree n.

It is known (see, e.g., [Ii]) that in $A_1(T)$ we have

(12) $$K_T = h^* K_S + R_f.$$

A local equation of the ramification divisor can be determined as follows.

Lemma 7.1. *Let u, v be regular local parameters at $Q \in T$, and let s, t be regular local parameters at $h(Q)$. Then $\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$ is a local equation of R_f at Q.*

Proof. In the exact sequence

$$\Omega_{\mathcal{O}_{S,h(Q)}/k} \otimes_{\mathcal{O}_{S,h(Q)}} \mathcal{O}_{T,Q} \to \Omega_{\mathcal{O}_{T,Q}/k} \to \Omega_{\mathcal{O}_{T,Q}/\mathcal{O}_{S,h(Q)}} \to 0,$$

the first arrow is a homomorphism of two free $\mathcal{O}_{T,Q}$-modules with bases ds, dt and du, dv (respectively); ds and dt are mapped to $\frac{\partial s}{\partial u} du + \frac{\partial s}{\partial v} dv$ and $\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv$. Localizing with respect to all prime ideals of $\mathcal{O}_{T,Q}$ of height 1, we see that for any prime divisor containing Q with generic point η we have

$$l_{\mathcal{O}_{T,\eta}}(\Omega_{T/S,\eta}) = v_\eta \left(\begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} \right),$$

where v_η is the valuation of $k(T)$ associated with η. □

We fix a pencil $\mathcal{C}$ as in Proposition 6.1 and denote by $\mathcal{D}$ the pencil of curves $\mathcal{C} \circ h$ on T (the components of B_h are taken for the role of $C_1, \ldots, C_n$), and by B and B' the sets of base points of $\mathcal{C}$ and $\mathcal{D}$, respectively.

It is clear that $\#B' = n \cdot \#B$. Therefore, by Proposition 6.4,

(13) $$\chi_T - n\chi_S = \int \operatorname{sing} \mathcal{D} - n \int \operatorname{sing} \mathcal{C} - 2((2p_a(\mathcal{D}) - 2) - n(2p_a(\mathcal{C}) - 2)).$$

Lemma 7.2 (Zeuthen)**.** *Let C be a divisor on S, and let $D = h^*C$. Then*

$$(2p_a(D) - 2) - n(2p_a(C) - 2) = (C.B_h).$$

Proof. Using (12) and the projection formula, we obtain

$$
\begin{aligned}
2p_a(D) - 2 &= (D.D) + (K_T.D) \\
&= n(C.C) + (h^*K_S.D) + (R_h.D) \\
&= n(C.C) + n(K_S.C) + (B_h.C) \\
&= n(2p_a(C) - 2) + (B_h.C).
\end{aligned}
$$
□

By definition, we have

$$
\int \operatorname{sing} \mathcal{D} - n \int \operatorname{sing} \mathcal{C}
$$
$$
= \sum_{P \in S \setminus B} \Big(\sum_{Q \in h^{-1}(P)} \operatorname{sing}_Q^t \mathcal{D}_Q - n \operatorname{sing}_P^t \mathcal{C}_P + \sum_{Q \in h^{-1}(P)} \operatorname{sing}_Q^w \mathcal{D}_Q - n \operatorname{sing}_P^w \mathcal{C}_P \Big).
$$

For an effective divisor C on S without multiple components and without common components with B_h, and for a point Q on T, we introduce the notation

$$
d_Q(C) = \sum_{\pi, \pi'} d\big((\widehat{\mathcal{O}_{T,Q}}/\pi') \big| (\widehat{\mathcal{O}_{S,h(Q)}}/\pi) \big),
$$

where π runs over the nonassociated prime divisors in $\widehat{\mathcal{O}_{S,h(Q)}}$ of a local equation of C at $h(Q)$, and π' runs over the nonassociated prime divisors of π in $\widehat{\mathcal{O}_{T,Q}}$.

Proposition 7.3. *Let C be an effective divisor on S without multiple components and without common components with B_h, let P be any point of $\operatorname{Supp} C$, and let f be a local equation of $C_{h(Q)}$ at $h(Q)$. Then*

$$
\sum_{Q \mapsto P} \operatorname{sing}_{\widehat{\mathcal{O}_{T,Q}}}^t (f \circ h) - n \operatorname{sing}_{\widehat{\mathcal{O}_{S,P}}}^t f = (C.B_h)_P - (n - \#h^{-1}(P)) - \sum_{Q \mapsto P} d_Q(C).
$$

Proof. We can immediately reduce the proposition to a similar statement in which S is the spectrum of a complete 2-dimensional regular local ring. Next, it reduces to the case where T is connected, i.e., is also the spectrum of a complete 2-dimensional regular local ring.

Let $C = \sum C_i$, where the C_i are prime divisors. For $i \neq j$ we have

$$
n(C_i.C_j)_P = \sum_{Q \in h^{-1}(P)} (h^*C_i.h^*C_j)_Q.
$$

Therefore, the proposition reduces immediately to the case where C is a prime divisor, and this case is none other than Proposition 4.1. □

Theorem 7.4. *Let $B_h = \sum b_i B_i$ be the branch divisor of h. Let $\mathcal{C}$ be any separable pencil on S such that none of the B_i is a component of any curve in $\mathcal{C}$. Then*

$$(14) \qquad \chi_T - n\chi_S = \sum_i b_i(2p_g(B_i) - 2) + \sum_Q \lambda_f(\widehat{\mathcal{O}_{T,Q}}/\widehat{\mathcal{O}_{S,h(Q)}}),$$

where f is a local equation of $\mathcal{C}_{h(Q)}$ at $h(Q)$.

Proof. Proposition 7.3 implies

$$\int \text{sing}\, \mathcal{D} - n \int \text{sing}\, \mathcal{C} = \sum_{P \in S \setminus B} \left((\mathcal{C}_P.B_h)_P - (n - \#h^{-1}(P)) - \sum_{Q \mapsto P} d_Q(\mathcal{C}_P) \right.$$
$$\left. + \sum_{Q \mapsto P} \text{sing}_Q^w \mathcal{D}_Q - n \, \text{sing}_P^w \mathcal{C}_P \right) P.$$

Together with (13) and Lemma 7.2, this yields

$$\chi_T - n\chi_S = -2(\mathcal{C}.B_h) + \sum_{P \in S \setminus B} \left((\mathcal{C}_P.B_h)_P - (n - \#h^{-1}(P)) - \sum_{Q \mapsto P} d_Q(\mathcal{C}_P) \right.$$
$$\left. + \sum_{Q \mapsto P} \text{sing}_Q^w \mathcal{D}_Q - n \, \text{sing}_P^w \mathcal{C}_P \right).$$

Therefore,

$$\chi_T - n\chi_S = -2 \sum_i b_i(\mathcal{C}.B_i) + \sum_i \sum_{P \in S} b_i((\mathcal{C}_P.B_i)_P - \nu_P(B_i))$$
$$+ \sum_P \left(\sum_i b_i \nu_P(B_i) - (n - \#h^{-1}(P)) - \sum_{Q \mapsto P} d_Q(\mathcal{C}_P) \right.$$
$$\left. + \sum_{Q \mapsto P} \text{sing}_Q^w \mathcal{D}_Q - n \, \text{sing}_P^w \mathcal{C}_P \right).$$

Note that for any i the morphism $\varphi_i : B_i \to \mathbb{P}_k^1$ determined by $\mathcal{C}$ is separable. Indeed, if $P \in B_i$ is any point of B_i regular on B_h and such that $\mathcal{C}_P$ meets B_h at P transversally, then a local parameter at $\varphi_i(P)$ on $\mathbb{P}_k^1$ is mapped to a local parameter at P on B_i. Thus, we can apply Proposition 6.5, obtaining

$$\chi_T - n\chi_S = \sum_i b_i(2p_g(B_i) - 2)$$
$$+ \sum_P \left(\sum_i \left(-b_i \sum_{P'} d(\widehat{\mathcal{O}_{\widetilde{B_i},P'}} | \widehat{\mathcal{O}_{\mathbb{P}_k^1,\varphi_i(P)}}) + b_i \nu_P(B_i) \right) \right.$$
$$\left. - (n - \#h^{-1}(P)) - \sum_{Q \mapsto P} d_Q(\mathcal{C}_P) + \sum_{Q \mapsto P} \text{sing}_Q^w \mathcal{D}_Q - n \, \text{sing}_P^w \mathcal{C}_P \right),$$

where P' runs over the points over P of the normalization $\widetilde{B_i}$ of B_i. Obviously,

$$n - \#h^{-1}(P) = \sum_{Q \mapsto P} (n_Q - 1),$$

where n_Q is the degree of $\widehat{\mathcal{O}_{T,Q}}$ over $\widehat{\mathcal{O}_{S,P}}$. It remains to rewrite b_i as a sum over $Q \mapsto P$ of similar infinitesimal terms. This completes the proof of Theorem 7.4.   □

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