A RELATION BETWEEN RIESZ AND RIEMANN SUMMABILITY

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The Cesàro and the Riemann methods of summation are both intimately connected with the theories of Fourier and trigonometric series. It is therefore natural that the relation between these methods should have been investigated in some detail. For instance Verblunsky [2] has proved that, when k is a positive integer, summability $(C, k-\delta)$ implies summability $(R, k+1)$; and Kuttner [1] has proved that, for $k=1, 2$, summability (R, k) implies summability $(C, k+\delta)$.

Riesz's typical means generalise Cesàro summability and there is a corresponding generalisation for Riemann summability. Since both these methods are appropriate for dealing with almost periodic functions, we wish to establish a connection between them.

The notation we shall use is the following.

Riesz's typical means. If $k>0$, $0=\lambda_0<\lambda_1<\lambda_2< \cdots$ and

$$\sum_{\lambda_n<\omega}\left(1-\frac{\lambda_n}{\omega}\right)^k u_n \to s$$

as $\omega\to\infty$, then $\sum u_n$ is said to be summable (R, λ_n, k) to s.

Riemann summability. If k is a positive rational number with odd denominator, $0<\lambda_1<\lambda_2< \cdots$ and

$$u_0+\sum_{n=1}^{\infty}\left(\frac{\sin \lambda_n h}{\lambda_n h}\right)^k u_n \to s$$

as $h\to 0$, then $\sum u_n$ is said to be summable (R, k, λ_n) to s.

The object of this note is to prove the following result.

THEOREM. *Suppose* $\lambda_0=0$, $0<p\leqq\lambda_{n+1}-\lambda_n\leqq q$ *for all* n *and* $\sum_{n=0}^{\infty} u_n$ *is summable* $(R, \lambda_n, 1)$ *to* t. *Then* $\sum_{n=0}^{\infty} u_n$ *is also summable* (R, k, λ_n) *to* t *when* $k>2$ (*and* k *is a rational number with odd denominator*).

Before proceeding with the proof we might mention that, although our result is new even when $\{\lambda_n\}=\{n\}$, from the point of view of applications it would be more desirable to proceed from a nonintegral Riesz mean to Riemann summability of an integral order. However this problem appears to be a very formidable one because of the difficulty of dealing with nonintegral Riesz means.

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PROOF OF THE THEOREM. For $n \geqq 1$ we put

$$t_n = \sum_{r=0}^{n} \left(1 - \frac{\lambda_r}{\lambda_n}\right) u_r$$

so that the hypothesis of the theorem implies that $t_n \to t$ as $n \to \infty$. We have

$$\lambda_n t_n = \lambda_n s_n - \sum_{r=0}^{n} \lambda_r u_r,$$

where $s_n = u_0 + u_1 + \cdots + u_n$, and so

$$\lambda_{n+1} t_{n+1} - \lambda_n t_n = \lambda_{n+1} s_{n+1} - \lambda_n s_n - \lambda_{n+1} u_{n+1} = (\lambda_{n+1} - \lambda_n) s_n.$$

It follows that, for $n \geqq 2$,

$$u_n = \frac{\lambda_{n+1} t_{n+1} - \lambda_n t_n}{\lambda_{n+1} - \lambda_n} - \frac{\lambda_n t_n - \lambda_{n-1} t_{n-1}}{\lambda_n - \lambda_{n-1}}; \tag{1}$$

and, since $\lambda_0 = 0$, (1) is easily seen to hold for $n = 1$ also, whatever value is assigned to t_0.

Taking a suitable $k > 2$ we define τ_h, for $h > 0$, by

$$\tau_h = u_0 + \sum_{n=1}^{\infty} \left(\frac{\sin \lambda_n h}{\lambda_n h}\right)^k u_n,$$

the series being convergent since $\lambda_{n+1} - \lambda_n \geqq p$ and so $u_n = o(\lambda_n)$. We put

$$c(x) = \begin{cases} 1, & \text{for } x = 0, \\ \left(\dfrac{\sin x}{x}\right)^k, & \text{for } x \neq 0. \end{cases}$$

Then, by (1) and since also $u_0 = t_1$,

$$\tau_h = c(\lambda_0 h) t_1 + \sum_{n=1}^{\infty} c(\lambda_n h) \left(\frac{\lambda_{n+1} t_{n+1} - \lambda_n t_n}{\lambda_{n+1} - \lambda_n} - \frac{\lambda_n t_n - \lambda_{n-1} t_{n-1}}{\lambda_n - \lambda_{n-1}}\right).$$

Now t_n is bounded, $c(\lambda_n h) = O(1/n^k)$ and $\lambda_n/(\lambda_{r+1} - \lambda_r) = O(n)$. Hence the series for τ_h may be written

$$\begin{aligned} \tau_h &= \sum_{n=1}^{\infty} \left(\frac{c(\lambda_{n+1} h) - c(\lambda_n h)}{\lambda_{n+1} - \lambda_n} - \frac{c(\lambda_n h) - c(\lambda_{n-1} h)}{\lambda_n - \lambda_{n-1}}\right) \lambda_n t_n \\ &= \sum_{n=1}^{\infty} \alpha_{h,n} t_n. \end{aligned}$$

We wish to prove that this transformation is regular, i.e. that $\tau_h \to t$ as $h \to 0$. This is so if (and only if) the three Toeplitz conditions are satisfied, namely

(i) $\alpha_{h,n} \to 0$ as $h \to 0$, for all n,

(ii) $\sum_{n=1}^{\infty} |\alpha_{h,n}|$ is uniformly bounded for all $h > 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_{h,n} \to 1$ as $h \to 0$.

The first and the third condition are easily shown to be satisfied. In the first place the continuity of $c(x)$ shows that $\alpha_{h,n} \to 0$ as $h \to 0$. Also

$$\sum_{n=1}^{\infty} \alpha_{h,n} = 1$$

for all $h > 0$. This is either proved directly or by taking $u_0 = 1$, $u_n = 0$ for $n \geqq 1$ so that $t_n = 1$ for all n and $\tau_h = 1$ for all $h > 0$.

We still have to prove (ii). We have

$$\begin{aligned} \alpha_{h,n} &= \left(\frac{c(\lambda_{n+1}h) - c(\lambda_n h)}{\lambda_{n+1}h - \lambda_n h} - \frac{c(\lambda_n h) - c(\lambda_{n-1}h)}{\lambda_n h - \lambda_{n-1}h} \right) \lambda_n h \\ &= \{ c'(\eta_n) - c'(\xi_n) \} \lambda_n h \\ &= (\eta_n - \xi_n) c''(x_n) \lambda_n h, \end{aligned}$$

where $\lambda_{n-1}h < \xi_n < \lambda_n h < \eta_n < \lambda_{n+1}h$ and $\xi_n < x_n < \eta_n$. Now

$$\begin{aligned} c''(x) = k(k-1) &\left(\frac{\sin x}{x} \right)^{k-2} \left(\frac{x \cos x - \sin x}{x^2} \right)^2 \\ &+ k \left(\frac{\sin x}{x} \right)^{k-1} \left(\frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3} \right) \end{aligned}$$

and so

$$|c''(x)| < \begin{cases} A & \text{if } 0 < x < 1, \\ A/x^k & \text{if } x \geqq 1, \end{cases}$$

where A is a constant.

Let N be the integer such that

$$\lambda_N h < 1 \leqq \lambda_{N+1} h.$$

For $n \leqq N$,

$$|\alpha_{h,n}| < (\eta_n - \xi_n) A \lambda_n h < 2qhA\lambda_n h < 2Aq^2 nh^2$$

and so

$$\sum_{n=1}^{N} |\alpha_{h,n}| < 2Aq^2h^2 \sum_{n=1}^{N} n = Aq^2h^2N(N+1) < Bh^2N^2$$
$$\leqq Bh^2(\lambda_N/p)^2 < B/p^2 = K,$$

say.

For $n > N$,

$$|\alpha_{h,n}| < (\eta_n - \xi_n) \frac{A}{x_n^k} \lambda_n h < \frac{2Aq^2nh^2}{(\lambda_{n-1}h)^k} \leqq \frac{2Aq^2}{\left(\frac{1}{2}p\right)^k h^{k-2}n^{k-1}}$$

and so

$$\sum_{n=N+1}^{\infty} |\alpha_{h,n}| < \frac{2Aq^2}{\left(\frac{1}{2}p\right)^k h^{k-2}} \sum_{n=N+1}^{\infty} \frac{1}{n^{k-1}} < \frac{C}{h^{k-2}(N+1)^{k-2}}$$
$$< \frac{Cq^{k-2}}{(h\lambda_{N+1})^{k-2}} \leqq Cq^{k-2} = L,$$

say. Hence, for all $h > 0$,

$$\sum_{n=1}^{\infty} |\alpha_{h,n}| < K + L$$

and the theorem is therefore proved.

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References

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