INDEPENDENT RECURSIVE AXIOMATIZABILITY IN ARITHMETIC

J. P. JONES

1. **Introduction.** According to results of Tarski [7, p. 524] and Reznikoff [6], every theory may be axiomatized independently. However, there exist axiomatizable theories for which there are no recursively enumerable sets of independent axioms. The first example of such a theory was constructed by Kreisel [2].

Concerning formalized arithmetic, the well-known theories R and Q of [8] are easily seen to be independently recursively axiomatizable. Also, Peano arithmetic P and each of its axiomatizable extensions is independently recursively axiomatizable. (This result follows from an old theorem of Mostowski [4], according to which the extensions of P are *reflexive* theories. Montague and Tarski have shown [3] that axiomatizable reflexive theories are independently recursively axiomatizable.)

M. Poúr-El has proven [5] the existence of an axiomatizable nonindependently recursively axiomatizable extension of Q which is compatible with P (although not necessarily a subtheory of elementary arithmetic).

It is therefore natural to ask whether all axiomatizable subtheories of elementary arithmetic are independently recursively axiomatizable. By proving the following theorem we will show that this is not the case.

THEOREM 1. *There exist axiomatizable theories T_1 and T_2 such that*

$$R \subset T_2 \subset Q \subset T_1 \subset P$$

and neither T_1 nor T_2 is independently recursively axiomatizable.

Kreisel's method [2] for constructing a nonindependently axiomatizable theory was to add a new predicate $P(x)$ to the syntax of Q and the following axioms:

$$(\forall x)[(\exists y)K(y, x) \rightarrow P(x)] \quad \text{and} \quad P(\Delta_n) \qquad \text{for all } n$$

where the formula $(\exists y)K(y, x)$ represents a hypersimple set in Q.

The idea of our construction is to replace the predicate $P(x)$ with a formula of arithmetic, $A(x)$. Roughly speaking, such a formula should have the property that its numeric substitution instances,

Received by the editors February 13, 1969.

$A(\Delta_0)$, $A(\Delta_1)$, $\cdots$, are independent over some finitely axiomatizable theory in which the recursively enumerable sets may be represented.

Thus our construction is a refinement of that of Kreisel. We shall also make use of the following important result of Poúr-El [**5**]. Her Theorem sets forth the precise relationship between hypersimplicity and independent axiomatizability.

THEOREM 2. *Let* $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ *be any r.e. set of axioms for a theory* T. *Then* T *is nonindependently recursively axiomatizable if and only if the set*

$$\{i : \{\alpha_0, \alpha_1, \cdots, \alpha_{i-1}\} \vdash \alpha_i\}$$

is hypersimple.

2. **Definitions and notation.** All theories discussed in this paper, with the exception of Kreisel's, are formalized within a *fixed* first order predicate logic with identity, (specifically that of [**8**, p. 51]). This formalism has the following nonlogical constants: the individual constant symbol $\mathbf{0}$, the unary operation symbol $\boldsymbol{S}$ (successor), and the two binary operation symbols $\boldsymbol{+}$ and $\boldsymbol{\cdot}$. As in [**8**], the formula $x \boldsymbol{\leqq} y$ is an abbreviation for the formula $(\exists v)(v \boldsymbol{+} x \boldsymbol{=} y)$. In this formalism, the *numerals*, Δ_n, are defined as follows: $\Delta_0 = \mathbf{0}$ and $\Delta_{n+1} = \boldsymbol{S}(\Delta_n)$ for $0 \leqq n$.

For our purposes, a *theory* is simply any set of first order sentences on this formalism which contains the predicate calculus as a subset and is closed under logical deduction. Thus we identify a theory with the set consisting of its provable (valid) sentences. A theory is completely determined by specifying its (nonlogical) axioms. If S and T are theories with the *same syntax* and $S \subseteq T$ then we say that S is a *subtheory* of T and we write $S \subseteq T$. If $S \subseteq T$ and $S \neq T$ then we say S is a *proper subtheory* of T and we write $S \subset T$.

Let β be a sentence and let X be a set of sentences. If β is *provable* in a theory T, we write $\vdash_T \beta$ or simply $\beta \in T$. If β is *deducible* from the set X within the theory T, we write $X \vdash_T \beta$. If β is deducible from X within the predicate calculus, we write $X \vdash \beta$. We usually omit brackets and simply present the elements of X in a list.

DEFINITION 1. A consistent set X of sentences is said to be *independent* if for each $\beta \in X$

$$X - \{\beta\} \nvdash \beta.$$

DEFINITION 2. A theory T is said to be *independently recursively axiomatizable* if there exists an independent recursive set of (nonlogical) axioms for T.

REMARK. Craig's Theorem [1] asserts the equivalence of the notion of *recursive* axiomatizability with the notion of *recursive enumerable* axiomatizability. The notion of independent *recursive* axiomatizability and the notion of independent *recursive enumerable* axiomatizability are also equivalent. For if $\{\alpha_1, \alpha_2, \alpha_3, \cdots\}$ is any r.e. set of independent sentences, then the set consisting of the sentences α_1, $\alpha_2 \wedge \alpha_2$, $\alpha_3 \wedge \alpha_3 \wedge \alpha_3$, $\cdots$, is evidently independent and recursive (by virtue of increasing length).

DEFINITION 3. A relation $R(x_1, x_2, \cdots, x_k)$ is said to be *definable* in a theory T if there exists a formula $A(x_1, x_2, \cdots, x_k)$ with k free variables such that

if $R(n_1, n_2, \cdots, n_k)$ then $\vdash_T A(\Delta_{n_1}, \Delta_{n_2}, \cdots, \Delta_{n_k})$ and
if not $R(n_1, n_2, \cdots, n_k)$ then $\vdash_T \sim A(\Delta_{n_1}, \Delta_{n_2}, \cdots, \Delta_{n_k})$.

DEFINITION 4. A relation $R(x_1, x_2, \cdots, x_k)$ is said to be *representable* in a theory T if there exists a formula $A(x_1, x_2, \cdots, x_k)$ with k free variables such that

$$R(n_1, n_2, \cdots, n_k) \quad \text{if and only if} \quad \vdash_T A(\Delta_{n_1}, \Delta_{n_2}, \cdots, \Delta_{n_k}).$$

DEFINITION 5. A set A of natural numbers is said to be *hyperimmune* if it is infinite and if no recursive function f has the property that for each n, $f(n) \geqq$ the nth element of A in increasing order. An r.e. set whose complement is hyperimmune is said to be *hypersimple.*

For reference we list the axioms for the three theories R, Q and P of [8]. We recall that $R \subset Q \subset P$.

AXIOMS OF R (SCHEMA).

1. $\Delta_n + \Delta_m = \Delta_{n+m}$,
2. $\Delta_n \cdot \Delta_m = \Delta_{n \cdot m}$,
3. $\Delta_n \neq \Delta_m$, if $n \neq m$,
4. $(\forall x)(x \leqq \Delta_n \rightarrow x = \Delta_0 \vee x = \Delta_1 \vee \cdots \vee x = \Delta_n)$,
5. $(\forall x)(x \leqq \Delta_n \vee \Delta_n \leqq x)$.

AXIOMS OF Q.

1. $(\forall x)(\forall y)(S(x) = S(y) \rightarrow x = y)$.
2. $(\forall y)(S(y) \neq 0)$,
3. $(\forall x)(x \neq 0 \rightarrow (\exists y)(x = S(y)))$,
4. $(\forall x)(x + 0 = x)$,
5. $(\forall x)(\forall y)(x + S(y) = S(x + y))$,
6. $(\forall x)(x \cdot 0 = 0)$,
7. $(\forall x)(\forall y)(x \cdot S(y) = x \cdot y + x)$.

AXIOMS OF P (PEANO'S ARITHMETIC).

1. *The seven axioms of Q, and*

2. *all sentences which are particular instances of the following induction scheme*:

$$\Phi(O) \wedge (\forall u)[\Phi(u) \rightarrow \Phi(S(u))] \rightarrow (\forall u)\Phi(u).$$

3. **Preliminaries.** Let H be a fixed hypersimple set of natural numbers which, for convenience, does not contain 0. Let $L(y, x)$ be a recursive binary relation with the property that, for every natural number n,

$$n \in H \text{ if and only if there exists an } m \text{ such that } L(m, n).$$

(L could be taken to be the graph of a recursive function which enumerates H.) According to [8, p. 56], every recursive set is definable in R. Thus we may choose a fixed formula $K(y, x)$ with two free variables which defines the relation $L(y, x)$ in R. The formula $K(y, x)$ will be used in the axiomatization of T_1 and T_2.

4. **Construction of T_1.** Let $A_1(x)$ and $F_1(x)$ be abbreviations for the following two formulas:

$$A_1(x) = (\forall z)(S(z) = z \rightarrow x \cdot z = z), \qquad F_1(x) = S(x) \neq x.$$

$A_1(x)$ will play the role played by $P(x)$ in Kreisel's construction.

Axioms of T_1.
σ_1. *The conjunction of the seven axioms of* Q,
Σ_1. $(\forall x)[F_1(x) \wedge (\exists y)F_1(y) \wedge K(y, x) \rightarrow A_1(x)]$,
$A_1(\Delta_n)$ *for all* n.

Now the sentence $(\forall x)A_1(x)$ is evidently provable in P (by induction). Consequently we have $Q \subseteq T_1 \subseteq P$.

The following lemma establishes the independence of the sentences $A_1(\Delta_n)$ in a quite strong sense. Here we make heavy use of the fact that Q has nonstandard models.

Lemma 1. *For each set M of natural numbers there exists a model $\mathfrak{M}_1(M)$ of Q with the following properties*:
(i) *The sentence $A_1(\Delta_m)$ is true in the model if and only if $m \in M$.*
(ii) *The sentence Σ_1 is true in the model if and only if $H \subseteq M$.*

Proof. The domain of the model is to be the set

$$\{0, 1, 2, \cdots, \infty_0, \infty_1\}.$$

The operations S, $+$, and $\cdot$ are defined in the usual way upon the natural numbers. We define $n + \infty_i = \infty_i + n = \infty_i$, $\infty_i + \infty_j = \infty_i \cdot \infty_j = \infty_i$, and $S(\infty_i) = \infty_i$. Also, for a natural number n,

$$\infty_i \cdot n = \infty_i \quad \text{if } 0 < n, \qquad n \cdot \infty_i = \infty_i \quad \text{if } n \in M,$$
$$= 0 \quad \text{if } 0 = n; \qquad = \infty_{1-i} \quad \text{if } n \notin M.$$

It is routine to check that $\mathfrak{M}_1(M)$, so defined, is in fact a model of Q and that property (i) obtains. To see that property (ii) holds, observe that the interpretation of $F_1(y)$ in this model is that "y is a natural number." Now the formula $K(y,x)$ defines the relation $L(y, x)$ in the theory Q and $\mathfrak{M}_1(M)$ is a model of Q. Therefore $(\exists y)F_1(y) \wedge K(y, \Delta_m)$ is true in the model if and only if $m \in H$. So by property (i), Σ_1 is true in the model if and only if $H \subseteq M$. This completes the proof of Lemma 1.

We show that T_1 is nonindependently recursively axiomatizable. Consider the following axiomatization α_n of T_1:

$$\alpha_0 = \sigma_1 \wedge \Sigma_1 \wedge A(\Delta_0), \qquad \alpha_n = A(\Delta_n) \quad \text{for } n = 1, 2, 3, \cdots.$$

We use Theorem 2 of §1. It suffices to prove that $H = I$ where H is the hypersimple set of §3 and $I = \{i: \{\alpha_0, \alpha_1, \cdots, \alpha_{i-1}\} \vdash \alpha_i\}$.

From the way in which the α_n was defined, we have that $i \in I$ if and only if $0 < i$ and

(iii) $\Sigma_1, A_1(\Delta_0), A_1(\Delta_1), \cdots, A_1(\Delta_{i-1}) \vdash_Q A_1(\Delta_i)$.

To see that $H \subseteq I$, suppose $i \in H$. Then $0 < i$ and there exists an n such that $L(n, i)$ and hence $\vdash_Q K(\Delta_n, \Delta_i)$. Consequently $\Sigma_1 \vdash_Q A_1(\Delta_i)$ and so $i \in I$.

To see that $I \subseteq H$, suppose $i \in I$. Then $0 < i$ and (iii) holds. Let $M = H \cup \{0, 1, \cdots, i-1\}$ and consider the model $\mathfrak{M}_1(M)$ of Lemma 1. By properties (i) and (ii) of Lemma 1 the antecedents, $\Sigma_1, A(\Delta_0), A(\Delta_1), \cdots, A(\Delta_{i-1})$, are true in this model. Consequently, $A(\Delta_i)$ is true in the model because, according to (iii), it is deducible from sentences true in the model. So by property (i) of Lemma 1 we have $i \in M$. So $i \in H$ and we have shown that $I = H$.

5. **Construction of T_2.** Let $A_2(x)$ and $F_2(x)$ be abbreviations for the following two formulas:

$$A_2(x) = (\forall z)(x \cdot S(z) = x \cdot z + x), \quad F_2(x) = (x = 0 \vee (\exists u)(x = S(u))).$$

We wish to construct a nonindependently axiomatizable theory T_2 such that $R \subset T_2 \subset Q$.

However, the theory R is not finitely axiomatizable [**8**, p. 55] and this particular property of Q was crucial in our construction of T_1. For this reason, we construct first a finitely axiomatizable extension Q' of R such that $R \subset Q' \subset Q$. We then construct T_2 so that $R \subset Q' \subset T_2 \subset Q$.

AXIOMS OF Q'.

1. $(\forall x)(\forall y)(F_2(x)\wedge F_2(y)\rightarrow(S(x)=S(y)\rightarrow x=y))$,
2. $(\forall x)(S(x)\neq 0)$,
3. $(\forall x)(x+0=x)$,
4. $(\forall x)(\forall y)(F_2(x)\wedge F_2(y)\rightarrow(x+S(y)=S(x+y)))$,
5. $(\forall x)(F_2(x)\rightarrow x\cdot 0=0)$,
6. $(\forall x)(\forall y)(F_2(x)\wedge F_2(y)\rightarrow(x\cdot S(y)=x\cdot y+x))$,
7. $(\forall x)(\forall y)(\sim F_2(x)\rightarrow y\leqq x)$,
8. $(\forall x)(\forall y)(F_2(x)\wedge F_2(y)\leftrightarrow F_2(x+y))$,
9. $(\forall x)((F_2(x)\wedge x\neq 0)\rightarrow(\exists z)(F_2(z)\wedge x=S(z)))$.

AXIOMS OF T_2.

σ_2. *The conjunction of the nine axioms of* Q',

Σ_2. $(\forall x)[F_2(x)\wedge(\exists y)F_2(y)\wedge K(y, x)\rightarrow A_2(x)]$,

$A_2(\Delta_n)$ *for all* n.

We claim that $R\subseteq Q'\subseteq T_2\subseteq Q$. Evidently $Q'\subseteq T_2$. $T_2\subseteq Q$ because the sentences $(\forall x)(F_2(x))$ and $(\forall x)A_2(x)$ are *axioms* of Q. To check that $R\subseteq Q'$ is straightforward but tedious. We omit the details. As before we make use of a nonstandard model.

LEMMA 2. *For each set* M *of natural numbers there exists a model,* $\mathfrak{M}_2(M)$ *of* Q' *with the following properties*:

(i) *The sentence* $A_2(\Delta_m)$ *is true in the model if and only if* $m\in M$.

(ii) *The sentence* Σ_2 *is true in the model if and only if* $H\subseteq M$.

PROOF. The domain of the model is to be the set $\{0, 1, 2, \cdots, \infty\}$. The operations S, $+$ and $\cdot$ are defined in the usual way upon the natural numbers. We define $\infty+\infty=\infty$, $\infty\cdot\infty=n+\infty=\infty+n=\infty$. However, $S(\infty)=2$. We multiply as follows:

$$\begin{aligned}\infty\cdot n = n\cdot\infty &= n \qquad &\text{if } n\in M,\\ &= n+1 \qquad &\text{if } n\notin M.\end{aligned}$$

It is easy to check that $\mathfrak{M}_2(M)$ is a model of Q' and that property (i) holds. As for property (ii), the argument given in the proof of Lemma 1. §4 will establish this, if, in this argument, we replace Q by Q' and change subscripts from 1 to 2. This completes the proof of Lemma 2.

Similarly, the argument to the effect that T_2 is nonindependently recursively axiomatizable is identical with the argument given in §4 for T_1. This completes the construction of T_2.

References

1. W. Craig, *On axiomatizability within a system*, J. Symbolic Logic **18** (1953), 30–32.

2. G. Kreisel, *Independent recursive axiomatization*, J. Symbolic Logic **22** (1957), 109.

3. R. Montague and A. Tarski, *Independent recursive axiomatizability*, Mimeographed notes, Summer Institute for Symbolic Logic, Cornell University, Ithaca, N. Y., 1957, p. 270.

4. A. Mostowski, *On models of axiomatic systems*, Fund. Math. **39** (1952), 133–158.

5. M. B. Poúr-El, *Hypersimplicity as a necessary and sufficient condition for non-independent axiomatization*, Z. Math. Logik Grundlagen Math. **14** (1968), 449–456.

6. I. Reznikoff, *Tout ensemble de formules de la logique classique est équivalent à un ensemble independent*, C. R. Acad. Sci. Paris **260** (1965), 2385–2388.

7. A. Tarski, *Grundzüge des systemenkalkuls*. I, Fund. Math. **25** (1935), 503–526.

8. A. Tarski, A. Mostowski and R. Robinson, *Undecidable theories*, North-Holland, Amsterdam, 1953, ix+98 pp.

University of Calgary