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THE ENDS OF PRODUCT MANIFOLDS

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ABSTRACT. In this paper we provide simple point set properties which characterize the m-spheres S^m, open m-cells E^m, closed cells I^m, and annuli $A^m = [0, 1)\times S^{m-1}$. It is important to notice that the Poincaré conjecture is not used in dimension 3 or 4.

In this paper we provide simple point set properties which characterize m-spheres S^m and open m-cells E^m among m-manifolds. Accepting these two main theorems, we establish corollaries which characterize closed cells I^m and annuli $A^m=[0, 1)\times S^{m-1}$. Then we prove the two main theorems. It is important to note that the Poincaré conjecture is not used in dimension 3 or 4.

In the following M will mean an m-manifold, for $m\geqq 2$, and P is a point in M. We will always be asking for geometric consequences of supposing that a certain subset of M (such as $M-P$) is a product $A\times B$ of topological spaces where neither A nor B is a single point. Recall that such factors A, B are necessarily generalized manifolds in the sense of Wilder ([**1**], [**2**]). We will repeatedly use the fact that A or B is therefore a manifold if its dimension is $\leqq 2$ [**1**].

1. The proofs of the first two theorems are deferred until the next section.

THEOREM 1. *$M=E^m$ is the only connected noncompact m-manifold such that $M-P$ is a product space.*

THEOREM 2. *$M=S^m$ is the only connected compact m-manifold such that $M-P$ is a product space.*

COROLLARY 3. *It is not necessary in Theorem 2 to assume that M is connected.*

PROOF. Let N be the component of M with P in N. By Theorem 2, $N=S^m$. Now let $M-P=X\times Y$ and X_0, Y_0 be the components of X and Y

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for which $N-P=X_0\times Y_0$. Then X_0 and Y_0 are contractible (hence noncompact). If X_1 is another component of X, then $X_1\times Y_0$ is a noncompact component of M. This contradiction means X is connected. Similarly Y, and hence M, is connected. Thus $M=N=S^m$.

Without the connectivity, Theorem 1 fails, even for $m=2$. There are two distinct 2-manifolds M and M_1 such that $M-P=M_1-P_1=(E^1\cup S^1)\times E^1$ is a product of E^1 by a disjoint union.

We now turn our attention to manifolds M with boundary bM and interior Int(M). P will always be a point in the interior of M and Q a point in bM.

THEOREM 4. *These conditions are equivalent for a connected m-manifold with* $bM=S^{m-1}$:

(a) $M=I^m$.

(b) Int M *is a product space and* M *is compact.*

(c) M *is a compact product space.*

(d) Int $M-P$ *is a product space.*

(e) $M-Q$ *is a product space and* M *is compact.*

PROOF. (a) clearly implies the rest of the list. Each of (e) and (c) implies (b). We will show (b) implies (a). Let $N=M/bM$ be the quotient space. Then $N-(bM/bM)$ is Int M, which is a product space. Since N is compact, $N=S^m$ by Theorem 2. Now Int $M=E^m$ which implies (a) by Schoenflies [**3**]. That (d) implies (a) follows from Theorem 1.

In condition (d) compactness is a consequence rather than a hypothesis. According to Theorem 7, below, this is often true of condition (e) as well.

THEOREM 5. *These conditions are equivalent for a connected m-manifold* M *with* $bM=S^{m-1}$:

(a) $M=[0,1)\times S^{m-1}=A^m$.

(b) Int M *is a product space and* M *is not compact.*

(c) M *is a noncompact product space.*

PROOF. Clearly (a) implies (c) and (c) implies (b). We shall show that (b) implies (a) by forming the manifold $N=M/bM$. Since $N-(bM/bM)=$ Int M is a product space, $N=E^m$ by Theorem 1. The theorem follows.

The next theorem fully describes closed cells and annuli entirely in point set terms.

THEOREM 6. *Let* M *be a connected m-manifold with compact boundary* bM. *If each of* M *and* $bM-Q$ *is a product space, then* M *is either* I^m *or* $[0,1)\times S^{m-1}$.

PROOF. By Corollary 3, bM is S^{m-1}. If M is compact, $M=I^m$ by Theorem 4(c). If not, then $M=A^m$ by Theorem 5(c).

For most m, we have two other characterizations of I^m. Whenever X is a generalized manifold, let bX denote its boundary.

THEOREM 7. *If* $M-Q=X\times Y$ *is a product space with* $bX=\varnothing$, *then*
(a) *the double* $2M$ *is* S^m, *so* M *is compact*;
(b) bM *is a homotopy* $(m-1)$*-sphere*;
(c) $\text{Int } M=E^m$;
(d) $M=I^m$, *unless* $m=4$ *or* 5; *and*
(e) $M-Q=(bM-Q)\times[0, 1)$.

PROOF. $2M-Q=2(X\times Y)=X\times 2Y$, because $bX=\varnothing$ and because Q is in bM. Thus, either $2M=S^m$ by Theorem 2, or $2M=E^m$ by Theorem 1. In the latter case we use the proof of Theorem 1 to conclude further that X is a compact (generalized) homotopy $(m-1)$-sphere and that $2Y=E^1$. Now, $Y=[0, 1)$ and $M-Q=X\times[0, 1)$. Since $bM-Q=X$ and since X is compact and $bM-Q$ is not, we have reached a contradiction. Thus, $2M=S^m$ as stated.

If dim $Y>1$, then $bM-Q=b(M-Q)=b(X\times Y)=X\times bY$ is a product. By Theorem 2, $bM=S^{m-1}$. By Theorem 4(b), M is I^m, completing the theorem.

Thus, we let dim $Y=1$. Since $E^m=S^m-Q=2M-Q=X\times 2Y$, we have $2Y=E^1$ and $Y=[0,1)$ and X is contractible. Further, $bM-Q=b(M-Q)=X$. This means bM is a homotopy sphere. Unless $m=4$ or 5, $bM=S^{m-1}$ and $M=I^m$ by Theorem 4(b). Finally, $E^m=S^m-Q=2M-Q=X\times 2Y=X\times E^1=\text{Int } M$, completing the theorem.

THEOREM 8. *If* bM *is compact and* Int $M-P=X\times Y$ *is a product space, then*
(a) $\text{Int } M=E^m$, *so* M *is compact*;
(b) bM *is a homotopy sphere*;
(c) $2M=S^m$; *and*
(d) $M=I^m$, *unless* $m=4$ *or* 5.

PROOF. (a) follows from Theorem 1 and implies (b). (c) holds because $2M$ is the union of two m-cells. (d) follows from the Poincaré conjecture in the correct dimension.

2. In order to prove the two main theorems, we begin by studying the ends of open generalized manifolds. See Siebenmann [**4**] for a discussion of ends and Wilder [**1**, Chapter 8] for a definition of generalized manifolds.

Let M be a j-connected generalized manifold and let M have an isolated end E which is k-connected (i.e. E has small k-connected neighborhoods). Although generally no relationship exists between j and k, it is instructive to examine the following class of doughnut-like manifolds.

Let $N=S^{m+1}\times I^{n+2}$. Now, Int $N=S^{m+1}\times E^{n+2}$ is m-connected and has one isolated end whose groups are those of $bN=S^{m+1}\times S^{n+1}$. Thus, $j=m$, and $k=\min\{m, n\}$. It follows from Corollary 11 below that, just as in this example, it is always true that $j\geqq k$ for such products. Proofs of the next three propositions appear in the author's dissertation [5]. Let $M=A\times B$ be an open generalized m-manifold.

LEMMA 9. *If neither A nor B is compact, then M has exactly one end.*

Let $M=A\times B$ have at least one isolated end E.

LEMMA 10. *If E has a k-connected neighborhood and if A is not compact, then B is k-connected, even if B is not compact.*

The import here is that B is a retract of every such neighborhood.

COROLLARY 11. *If M is k-connected at E, and if A is not compact, then B is k-connected. Thus, if neither A nor B is compact, then each of M, A and B is k-connected. That is, $j\geqq k$.*

LEMMA 12. *If $K\subset E^1\times B$ is compact, then there is another copy of K in $E^1\times B$ disjoint from K.*

PROOF. Since K is compact, there is an r in E^1 such that $K\subset(-r, r)\times B$. The homomorphism h given by $h(x, b)=(x+3r, b)$ is a translation which moves K disjoint from itself.

THEOREM 13. *If M is an m-manifold with P in* Int M, *and if $M-P=A\times B$ is a product space, then M is $(m-2)$-connected. Moreover, if B is compact, then $M-P=E^1\times B$.*

PROOF. By general position (in a coordinate chart about P), the homotopy groups of M and $M-P$ agree up to dimension $m-2$. If neither A nor B is compact, then we are done by Corollary 11. Let B be compact. Using Poincaré duality on B, $H_iB=Z_2$ for $i=\dim B$. From Corollary 11, B is $(m-2)$-connected; that is, $H_nB=0$ for $n\leqq m-2$. Thus $\dim B=m-1$ and $\dim A=1$. This means $A=E^1$ and $M-P=E^1\times B$. Now, $M-P$ is $(m-2)$-connected, because B is.

Now we are equipped to prove the two main theorems. Let P be in M.

PROOF OF THEOREM 1. By Lemma 9, if neither A nor B is compact, then $M-P$ has just one end. Since this makes M compact, we may instead assume that B is compact.

By Theorem 13, $M-P=E^1\times B$. This makes M the open cone CB on B. Thus $M=E^m$ by Brown's open star theorem [6].

PROOF OF THEOREM 2. By Theorem 13, M is $(m-2)$-connected. Using the Poincaré theorem, we find $M=S^m$ for $m\geqq 5$. Thus, let $m\leqq 4$. From

Theorem 13, we find that if B is compact, $M-P=E^1\times B$ has two ends. Since this denies that M is compact, we may assume that neither A nor B is compact. It follows that each of $M-P$, A, and B is contractible. If $m=2$, $A=B=E^1$ so $M=S^2$. If $m=3$, $A=E^1$, $B=E^2$ or vice-versa, so $M=S^3$. Let $m=4$. If dim $B=2$, then $A=B=E^2$, so $m=S^4$. We have left only the case $m=4$ and dim $B=3$. In this case $A=E^1$ and $M-P=E^1\times B$. We decompose $M=e^4\cup K$ as a disjoint union of an open 4-cell e^4 and a compact residual set K with P not in K. Using Lemma 12 to translate K in $E^1\times B$, we find that M is a union of two open cells, and, hence, $M=S^4$; see [7].

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