

SOME INEQUALITIES FOR THE MULTIPLICATOR OF A FINITE GROUP

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ABSTRACT. The paper is devoted to the derivation of certain upper and lower bounds for the multiplier of a finite group. The lower bounds enable us to give a sufficient condition for a finite group to have nontrivial multiplier.

1. Introduction. In this note, some inequalities for the multiplier, $M(G)$, of a finite group G are given. The main results can also be obtained by applying the spectral sequence due to R. C. Lyndon [5]. However, I feel it is worth while proving them directly using a result due to I. Schur [7] embodied in 1.1 below.

The definition of the multiplier of a finite group may be found in [3, Kap V, §23], as may the basic facts about multipliers.

Notation. The notation used is as follows: the exponent of a finite group X is denoted by $e(X)$; if x and y are two elements of some group then $x^{-1}y^{-1}xy$ is denoted by $[x, y]$ and $y^{-1}xy$ is denoted by x^y ; if X and Y are two subgroups of a group G then $[X, Y]$ is the subgroup of G generated by all $[x, y]$ with x in X and y in Y ; finally, the lower central series of a group G is denoted by $G = \gamma_1(G) \supseteq \gamma_2(G) = G' \supseteq \gamma_3(G) \supseteq \cdots$, where for $j \geq 1$, $\gamma_{j+1}(G) = [\gamma_j(G), G]$.

All other notation, where not explained, will be standard.

1.1. Let G be a finite group and $G = F/R$ a presentation for G as a factor-group of the free group F . Then

$$M(G) \cong (F' \cap R)/[F, R].$$

2. Some lemmas. The results of this section are two lemmas. The proof of the first lemma is much the same as that of Lemma 2.1 of [4] and so will be omitted. The lemmas are used to give the results of §§3 and 4. In applying Lemma 2.1 and its corollary, the reader should recall that, for two finite groups A and B ,

$$A \otimes B \cong (A/A') \otimes (B/B').$$

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LEMMA 2.1. *Let G be a finite group and B a normal subgroup. Set $A=G/B$. Let $G=F/R$ be a presentation for G as a factor-group of the free group F and suppose $B=S/R$ so that $A \cong F/S$. Then $[F, S]/[F, R][F, S, F]S'$ is isomorphic with a factor-group of $A \otimes B$.*

COROLLARY 2.2 (SEE [4]). *Further to the notation and assumptions of Lemma 2.1, let B be a central subgroup of G . Then $[F, S]/[F, R]S'$ is an epimorphic image of $A \otimes B$.*

LEMMA 2.3. *Let G be a finite group with a normal subgroup K and set $H=G/K$. Then there exists a finite group L with a normal subgroup M such that*

- (i) $G' \cap K \cong L/M$,
- (ii) $M \cong M(G)$,
- (iii) $M(H)$ is an epimorphic image of L .

PROOF. Let $G=F/R$ be a presentation for G as a factor-group of the free group F and suppose $K=T/R$ so that $H \cong F/T$. Then

$$G' \cap K = (F'R \cap T)/R = (F' \cap T)R/R \\ \cong (F' \cap T)/(F' \cap R) \cong ((F' \cap T)/[F, R])/((F' \cap R)/[F, R])$$

so that (i) and (ii) follow using 1.1.

Next,

$$M(H) \cong (F' \cap T)/[F, T] \cong ((F' \cap T)/[F, R])/([F, T]/[F, R])$$

so that (iii) holds.

Note that the kernel of the epimorphism described in Lemma 2.3(iii) is $[F, T]/[F, R]$. We shall use this fact in §4.

3. Lower bounds. The main result of this section is:

THEOREM 3.1. *Let G be a finite group and K any normal subgroup. Set $H=G/K$. Then*

- (i) $|M(H)|$ divides $|M(G)| |G' \cap K|$,
- (ii) $e(M(H))$ divides $e(M(G))e(G' \cap K)$,
- (iii) $d(M(H)) \leq d(M(G)) + d(G' \cap K)$.

PROOF. In the notation of Lemma 2.3 we have $|L|=|M(G)| |G' \cap K|$, $e(L)$ divides $e(M(G))e(G' \cap K)$ and $d(L) \leq d(M(G)) + d(G' \cap K)$.

The results now follow by Lemma 2.3(iii).

COROLLARY 3.2. *Let G be a finite d -generator group of order p^n . Then*

$$p^{d(d-1)/2} \leq |M(G)| |G'| \leq p^{n(n-1)/2}.$$

PROOF. Suppose we take $K=\Phi(G)$, the Frattini subgroup of G , in part (i) of Theorem 3.1. Then H is elementary abelian of order p^d so that

$|M(H)| = p^{d(d-1)/2}$ (see, for example, [3]) and the left-hand inequality holds.

The right-hand inequality may be found in [4].

Corollary 3.2 gives some measure of the spread of $|M(G)|/|G'|$. In particular, it shows that the smaller the multiplier of G the larger the derived group of G , and conversely.

Suppose now that G is a 3-generator p -group with trivial multiplier. Then Corollary 3.2 shows that the order of G' is at least as large as p^3 so that $|G| \geq p^6$. This means that 3-generator p -groups of orders no more than p^5 have nontrivial multipliers. This fact can, of course, be obtained from tables of groups of orders dividing p^5 .

It is often useful to know when a group has nontrivial multiplier. Theorem 3.1 gives some sufficient conditions for a group to have nontrivial multiplier and perhaps the most interesting is:

COROLLARY 3.3. *Let G be a 3-generator finite p -group with $d(G')$ no more than 2. Then $M(G)$ is nontrivial.*

PROOF. If we take $K = \Phi(G)$ in Theorem 3.1(iii) then H is elementary abelian of order p^3 so that

$$d(M(H)) = 3 \leq d(M(G)) + d(G') \leq d(M(G)) + 2.$$

The results now follow.

It is well known (see for example [3, p. 642]) that p -groups needing at least four generators have nontrivial multipliers so that Corollary 3.3 extends the list of p -groups with nontrivial multipliers.

Again, it follows immediately from Corollary 3.3 that 3-generator p -groups of order no more than p^5 have nontrivial multipliers.

COROLLARY 3.4. *Let G be a t -generator finite p -group. Then in any presentation for G with t generators and ρ relations, we must have*

$$\rho \geq \frac{1}{2}t(t+1) - d(G').$$

PROOF. If a finite group X can be presented with n generators and m relations, then it is well known (see [3, p. 642]) that $d(M(X)) \leq m - n$.

Suppose then that $\rho < \frac{1}{2}t(t+1) - d(G')$ for some finite t -generator p -group G . Then

$$\begin{aligned} d(M(G)) &\leq \rho - t < \frac{1}{2}t(t+1) - d(G') - t \\ &= \frac{1}{2}t(t-1) - d(G'). \end{aligned}$$

On the other hand, Theorem 3.1(iii), with $K = \Phi(G)$, shows that

$$d(M(G)) \geq \frac{1}{2}t(t-1) - d(G')$$

so that we have a contradiction.

4. Upper bounds. In this section, some upper bounds are given in terms of normal subgroups and factor-groups.

THEOREM 4.1. *Let G be a finite group and B a central subgroup. Set $A = G/B$. Then*

- (i) $|M(G)| |G' \cap B|$ divides $|M(A)| |M(B)| |A \otimes B|$,
- (ii) $d(M(G)) \leq d(M(A)) + d(M(B)) + d(A \otimes B)$,
- (iii) $e(M(G))$ divides $e(M(A))e(M(B))e(A \otimes B)$.

Before proving Theorem 4.1, we make the following:

DEFINITION 4.2. *Let X be a finite group. We say that X has (special) rank $r(X)$ if every subgroup of X may be generated by $r(X)$ elements and there is at least one subgroup that cannot be generated by fewer than $r(X)$ elements.*

PROOF OF THEOREM 4.1. *Let $G = F/R$ be a presentation for G as a factor-group of the free group F and let $B = S/R$ so that $A \cong F/S$. Then $[F, S] \leq R$.*

- (i) By Lemma 2.3,

$$|M(G)| |G' \cap B| = |L| = |M(A)| |[F, S]/[F, R]].$$

Now, $([F, S]/[F, R])/([F, R]S'/[F, R])$ is isomorphic with $[F, S]/[F, R]S'$. Hence,

$$|M(G)| |G' \cap B| = |M(A)| |[F, R]S'/[F, R]| |[F, S]/[F, R]S'|.$$

But

$$\begin{aligned} [F, R]S'/[F, R] &\cong S'/(S' \cap [F, R]) \\ &\cong (S'/[S, R])/((S' \cap [F, R])/[S, R]) \end{aligned}$$

and, since $S' \leq [F, S] \leq R$,

$$S'/[S, R] = (S' \cap R)/[S, R] \cong M(B).$$

Hence (i) follows by Corollary 2.2.

- (ii) By Lemma 2.3 again

$$\begin{aligned} d(M(G)) &\leq r(L) \leq r(M(A)) + r([F, S]/[F, R]) \\ &= d(M(A)) + d([F, S]/[F, R]), \end{aligned}$$

since $M(A)$ and $[F, S]/[F, R]$ are finite abelian groups (for $[F, S]' \leq R' \leq [F, R]$). The result now follows as for (i).

- (iii) This again follows as for (i).

Note that if B is cyclic, Theorem 4.1(i) generalises a result of J. A. Green [2] obtained by using the Lyndon spectral sequence.

COROLLARY 4.3. *Let G be a finite t -generator group of order p^n . Then there exists an integer $h = h(G)$ with $t-1 \leq h \leq n-1$ such that*

$$d(M(G)) \leq ht - \frac{1}{2}t(t-1).$$

PROOF. If G is cyclic, we obtain the result by choosing $h=0$. Suppose then that G is noncyclic and choose B_1 in $Z(G)$ to be cyclic of order $e(Z(G))$. Then by Theorem 4.1(ii),

$$d(M(G)) \leq d(M(G/B_1)) + d((G/B_1) \otimes B_1) = d(M(G/B_1)) + d(G/B_1).$$

For $j \geq 2$, if G/B_{j-1} is noncyclic, choose B_j/B_{j-1} in $Z(G/B_{j-1})$ to be cyclic of order $e(Z(G/B_{j-1}))$.

Suppose G/B_k is noncyclic for all k . By finiteness, there exists an integer l such that $B_l = G$. Hence, $G/B_{l-1} = B_l/B_{l-1}$ and we have a contradiction.

Hence if h is the minimum number of steps required to obtain cyclicity we have

$$d(M(G)) \leq \sum_{j=1}^h d(G/B_j).$$

Since $e(Z(G/B_j)) \geq p$ for all $j \geq 0$ (where $B_0 = 1$), it is clear that $h \leq n - 1$. Also, $t \leq h + 1$ so that

$$d(M(G)) \leq \frac{1}{2}t(t - 1) + \sum_{j=1}^{h-t+1} d(G/B_j) \leq \frac{1}{2}t(t - 1) + (h - t + 1)t.$$

If G is the direct product of t finite cyclic p -groups then it is clear that $h = t - 1$ so that the bound of Corollary 4.3 is attained by all finite abelian p -groups.

Corollary 4.3 generalises a result implicit in [6], but the two results coincide when G is an elementary abelian p -group.

Let $G = F/R$ be a presentation for the finite p -group G as a factor-group of the free group F . Let $\Gamma_{j+1} = \gamma_{j+1}(F)$ for all j . Since $G' = F'R/R$ we have, by 1.1, that

$$M(G/G') \cong (F' \cap F'R)/[F, F'R] = F'/[F, F'R].$$

With this notation we have:

THEOREM 4.4. *Let G be a finite p -group of nilpotency class c and let Q_j denote the quotient group $G/\gamma_j(G)$ for $2 \leq j \leq c$. Then*

$$(i) \quad |G'| |M(G)| \leq |M(G/G')| \prod_{j=1}^{c-1} |Q_{j+1} \otimes \gamma_{j+1}(G)|,$$

$$(ii) \quad d(M(G)) \leq d(M(G/G')) + \sum_{j=1}^{c-1} d(Q_{j+1} \otimes \gamma_{j+1}(G)),$$

$$(iii) \quad e(M(G)) \leq e(M(G/G')) \prod_{j=1}^{c-1} e(Q_{j+1} \otimes \gamma_{j+1}(G)).$$

PROOF. (i) In the above notation,

$$\begin{aligned}
 |G'| |M(G)| &= |F'/[F, R]| = |M(G/G')| |[F, F'R]/[F, R]| \\
 &= |M(G/G')| |[F, \Gamma_{j+2}R]/[F, R]| \prod_{k=1}^j |[F, \Gamma_{k+1}R]/[F, \Gamma_{k+2}R]|
 \end{aligned}$$

for all $j \geq 1$. Now, $1 = \gamma_{c+1}(G) = \Gamma_{c+1}R/R$ so that $\Gamma_{c+1} \leq R$ and $[F, \Gamma_{c+1}R] = [F, R]$. Next, $\gamma_j(G) = \Gamma_jR/R$ for all $j \geq 2$. Thus

$$[F, R](\Gamma_jR)' [F, \Gamma_jR, F] = [F, R]\Gamma_{j+2} = [F, \Gamma_{j+1}R]$$

and (i) follows by Lemma 2.1.

(ii) We have,

$$r(F'/[F, R]) \leq r(M(G/G')) + r([F, \Gamma_2R]/[F, R])$$

so that

$$d(M(G)) \leq d(M(G/G')) + \sum_{j=1}^{c-1} r([F, \Gamma_{j+1}R]/[F, \Gamma_{j+2}R]),$$

and (ii) again follows by Lemma 2.1.

(iii) This follows as for (i) and (ii).

Note that part (i) of Theorem 4.4 gives a different result to that of [1], as can be seen by considering the groups $G(\alpha, \beta, \gamma)$, $\gamma \geq \alpha > \beta$, defined by $G(\alpha, \beta, \gamma) = \langle a, b, c, d, e, f \rangle$ with defining relations:

$$\begin{aligned}
 a^{p^\gamma} &= b^{p^\alpha} = c^{p^\beta} = d^{p^\alpha} = e^{p^\beta} = f^{p^\beta} = 1; \\
 a^d &= a, a^e = a, a^f = a, d^e = d, d^f = d, e^f = e; \\
 a^b &= ad, d^b = d, e^b = e, f^b = f; \\
 a^c &= ae, b^c = bf, d^c = d, e^c = e, f^c = f.
 \end{aligned}$$

Part (i) of Theorem 4.4 gives $|M(G)| \leq p^{2(\alpha+2\beta)+3\beta}$ and [1] gives

$$|M(G)| \leq p^{3(\alpha+2\beta)},$$

where $G = G(\alpha, \beta, \gamma)$.

COROLLARY 4.5. *If G is a finite p-group of special rank r and nilpotency class c then*

$$d(M(G)) \leq \frac{1}{2}r((2c - 1)r - 1).$$

PROOF. If $d(G) = t$ then it follows from Theorem 4.4(ii) that

$$d(M(G)) \leq \frac{1}{2}t(t - 1) + t \sum_{j=1}^{c-1} d(\gamma_{j+1}(G)).$$

The result now follows.

Note that the bound of Corollary 4.5 is attained by all abelian p -groups. As a final application of Theorem 4.4 we have:

COROLLARY 4.6. *Further to the notation and assumptions of Theorem 4.4 let $e_j = \min\{e(Q_{j+1}), e(\gamma_{j+1}(G))\}$ for $1 \leq j \leq c-1$. Then $e(M(G)) \leq e(G|G') \prod_{j=1}^{c-1} e_j$; in particular, if G has exponent p^e then $e(M(G)) \leq p^{ec}$.*

PROOF. If A is a finite abelian p -group then it is easy to see that $e(M(A))$ is no more than $e(A)$. Also, from the properties of tensor products (see [3]), it follows that $e(Q_{j+1} \otimes \gamma_{j+1}(G)) \leq e_j$. The result follows.

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