

## SOME UNCOMPLEMENTED UNIFORM ALGEBRAS

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ABSTRACT. The uniform algebras specified by the Wermer embedding theorem are not linearly homeomorphic to a complemented subspace of any Banach space  $C(Y)$ .

Let  $A$  be a uniform algebra on a compact Hausdorff space  $X$ . Assume there is a character in the maximal ideal space  $M(A)$ , contained in a nontrivial Gleason part  $P$  and admitting a unique representing measure  $m$  on  $X$ .

THEOREM.  $A$  is never linearly homeomorphic to a complemented subspace of a space  $C(Y)$ ,  $Y$  a compact Hausdorff space.

REMARKS. Glicksberg [4] asked whether a closed nonselfadjoint subalgebra of  $C(X)$  must be uncomplemented in  $C(X)$ , and answered this in the affirmative when  $X$  is a compact group and  $A$  is translation invariant. Rudin [9] had shown that the disc algebra is uncomplemented in  $C(T)$  ( $T$  the unit circle) and that  $H^\infty(dt)$  ( $dt$  Haar measure on  $T$ ) is uncomplemented in  $L^\infty(dt)$ . Pełczyński [6] and Rosenthal [8] proved a theorem like the one in this paper for the disc algebra (Rosenthal's result is stated for algebras of generalized analytic functions). We will use their basic idea. Finally Pełczyński [7] proved that  $A$  is uncomplemented in  $C(X)$  if the annihilator of  $A$  is separable. This applies, in particular, to uniform algebras, such as  $R(X)$ , with at most a countable number of nontrivial Gleason parts and no completely singular annihilating measure in the sense of the F. and M. Riesz theorem [3, VI. 2.3 and VI. 3.2]. Our theorem applies, in particular, to Dirichlet algebras, logmodular algebras, the polydisc algebras  $A(D^n)$ , and algebras of generalized analytic functions (cf. [3, p. 169]), and is somewhat disjoint from the result of Pełczyński.

PROOF. An injective Banach space is one for which all closed isomorphic embeddings into Banach spaces are complemented in these spaces. It is known [5] that complemented subspaces of injective spaces

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are injective and that  $C(X)^{**}$  (the second dual of  $C(X)$ ) is an injective space. Thus, it is enough to prove that the second dual  $A^{**}$  of  $A$  contains a complemented subspace linearly homeomorphic to the space  $H^\infty(dt)$ .

We prove first that  $A^{**}$  contains a complemented subspace isomorphic to  $H^\infty(m)$ , the  $w^*$ -closure of our algebra  $A$  in the Banach space  $L^\infty(m)$ . By the Riesz representation theorem, the Lebesgue decomposition theorem and the Radon-Nikodym theorem, we have (up to isometric isomorphism),  $C(X)^* = L^1(m) \oplus S$ ,  $S$  being the closed subspace of all measures singular to  $m$ . By the F. and M. Riesz theorem,  $A^\perp = A^\circ \oplus Z$ , where  $A^\circ$  is the part of the annihilator contained in  $L^1(m)$  (this subspace can be identified (cf. [1, 4.2.12]), but that is irrelevant here), and  $Z$  consists of those annihilating measures singular to  $m$ . Now ( $\sim$  stands for "linearly homeomorphic to"),

$$A^* = C(X)^*/A^\perp \sim (L^1(m)/A^\circ) \times (S/Z) \quad \text{and} \\ A^{**} \sim (L^1(m)/A^\circ)^* \times (S/Z)^*.$$

We have used here the fact that, for Banach spaces and closed subspaces, if  $A = B \oplus C$  and  $A' = B' \oplus C'$ , with  $A', B', C'$  subspaces of  $A, B, C$ , resp., we have  $A/A' \sim (B/B') \times (C/C')$  (consider the natural map). Observe also, in this general context, that  $B/B'$  is isomorphic to a complemented subspace of  $A/A'$  (the image of  $B/B'$  in  $A/A'$  under the natural map).

We claim that  $(L^1(m)/A^\circ)^* = H^\infty(m)$ . Indeed, identifying  $A$  with a subspace of  $L^\infty(m)$ ,  $A^\circ$  is precisely the (backward) polar of  $A$ , and  $(L^1(m)/A^\circ)^* = A^{\circ\circ}$ , the double polar of  $A$ . By the double polar theorem, this must be  $H^\infty(m)$ .

Finally we prove that  $H^\infty(dt)$  (which we identify with  $H^\infty(D)$ , the algebra of all bounded analytic functions on the open disc  $D$ ) is (isometrically) isomorphic to a complemented subspace of  $H^\infty(m)$ . The hypotheses of the theorem are precisely those of the Wermer embedding theorem [3, VI. 7.2 and V. 7.1] so, by that result, the Gleason part  $P$  is an analytic disc and  $H^\infty(P)$ , the algebra of all bounded analytic functions on  $P$  is (isometrically) isomorphic to  $H^\infty(D) = H^\infty(dt)$ . By a theorem of Gamelin ([2, Theorem 8.5], cf. also, [3, V. Example 6]),  $H^\infty(P)$  is (isometrically) isomorphic to a complemented subspace of  $H^\infty(m)$ , a topological supplement being the ideal of those functions in  $H^\infty(m)$  vanishing on  $P$ . This completes the proof.

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