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# ON THE GEOMETRIC INTERPRETATION OF SEMINORMALITY

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ABSTRACT. We give a new characterization of the property of seminormality for 1-dimensional local rings; and with the help of this result we develop extensions of the theorems of Salmon and Bombieri which interpret this property in terms of singularities of algebraic curves.

**1. Introduction.** Let $R$ be a reduced, 1-dimensional noetherian ring with normalization (in its total quotient ring) a finite $R$-module. It is known that $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X])$ is an isomorphism if, and only if, the conductor of the normalization, viewed as an ideal of the larger ring, is its own radical **[B-M**, 7.12], [**E**, 4.3]. These conditions are equivalent to what is now called *seminormality* [**T**, 3.6]. If $R$ is the local ring of a point of a scheme, then the point is called seminormal provided that $R$ is seminormal. Assume that $p$ is a singular point of an algebraic curve over an algebraically closed field. Salmon [**S**] proves that in the plane curve case, $p$ is seminormal if, and only if, it is an ordinary double point. Bombieri [**B**] generalizes this result by dropping the planar restriction and proving that $p$ is seminormal if, and only if, analytically the curve locally at $p$ consists of smooth branches with linearly independent tangents, i.e., the curve has "normal crossings" at $p$. Now in [**S**] it is claimed that the theorem is true over arbitrary ground field provided that "ordinary double point" is interpreted to mean that the tangent cone splits over the algebraic closure into two distinct lines. This is not quite true. In [**B**] it is stated, but not really proved, that the seminormality of $p$ is equivalent to "$p$ is an ordinary $n$-fold point, where $n$ is the dimension of the Zariski tangent space at $p$". We shall give a new characterization of seminormality of a 1-dimensional local ring which shows that this property may be described purely algebraically in terms of the associated graded ring, or purely geometrically in terms of the projectivized tangent cone. This result has as corollary the theorem stated in [**B**] and the correct version of the theorem claimed in [**S**]. We conclude with a brief exposition of how the "normal crossings" result of [**B**] extends to the abstract case.

*Standing notation.* $R$ is a reduced, noetherian, 1-dimensional local ring, with normalization $D$ a finite $R$-module. $M$, $k$, $e(R)$ and $\mathrm{emdim}(R)$ denote the

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maximal ideal, residue field, multiplicity and embedding dimension of $R$. Let $J$ denote the Jacobson radical of $D$ and $K = D/J$. For any semilocal ring $A$, $G(A)$ denotes the associated graded ring (with respect to its Jacobson radical).

*Observe that $R$ is seminormal if, and only if, $M = J$.*

**2. The algebraic-geometric viewpoint.** The first theorem is the result referred to in §1.

THEOREM 1. *The following statements are equivalent.*

(1) *$R$ is seminormal.*

(2) *$G(R)$ is reduced and $e(R) =$ emdim$(R)$.*

(3) Proj$(G(R))$ *is reduced and $e(R) =$ emdim$(R)$.*

Before proceeding with the proof we state and discuss the following geometric corollary.

COROLLARY 1. *Let $p$ be a closed point of an algebraic (or algebroid) curve at which the Zariski tangent space has dimension $n$. Then:*

(1) (*cf.* [S]) *$p$ is seminormal if, and only if, it is an $n$-fold point at which the projectivized tangent cone is reduced.*

(2) (cf. [B]) *For algebraically closed ground field, $p$ is seminormal if, and only if, it is an ordinary $n$-fold point (i.e., a point of multiplicity $n$ with $n$ distinct tangents).*

(1) is clearly a special case of (1)$\leftrightarrow$(3) of the theorem, and (2) is valid because over algebraically closed ground field "ordinary point" and "projectivized tangent cone reduced" are equivalent. Something weaker than algebraic closure suffices: it is enough that all points on the normalization of the curve lying over $p$ be rational over the residue field at $p$. Henceforth if our local ring $R$ has this property, we say that $R$ has *$k$-rational normalization*. In connection with (1) and [S] we consider an example. Suppose that char$(k) =$ 2 and that $b^2 = a \in k$, with $b \in \bar{k} - k$. We consider the plane curve over $k$ defined by the equation $Y^2 = aX^2 + X^3$. This curve is absolutely reduced and irreducible (and therefore a variety in anyone's sense of the term) since over $\bar{k}$ the equation becomes $(Y + bX)^2 = X^3$. Let $R$ be the local ring of the origin on this curve. Clearly $e(R) =$ emdim$(R) = 2$, and $G(R) = k[X, Y]/Y^2 + aX^2$ is reduced; hence $R$ is seminormal. But $Y^2 + aX^2 = (Y + bX)^2$; i.e., the affine tangent cone does not split over $\bar{k}$, as claimed in [S], into two distinct lines. But this is in some sense the unique counterexample. Observe that, in general, $G(R)$ will remain reduced over $\bar{k}$ if, and only if, $K$ is a separable $k$-algebra; and this is the case whenever $e(R) <$ char$(k)$.

Now for the proof of Theorem 1 we require the following technical characterization of seminormality.

LEMMA. *$R$ is seminormal if, and only if, the following conditions hold.* (i) $e(R) =$ emdim$(R)$; (ii) $MJ \cap M = M^2$.

PROOF. Since $\dim(R) = 1$, $e(R) = k\text{-}\dim(D/MD)$. But $J$ and $D$ are isomorphic $R$-modules since $J = xD$ for some regular element $x$ of $J$. Hence $e(R) = k\text{-}\dim(J/MJ)$. Suppose that $R$ is seminormal; i.e., $M = J$. Then $e(R) = k\text{-}\dim(J/MJ) = k\text{-}\dim(M/M^2) = \operatorname{emdim}(R)$, and $MJ \cap M = M^2$. To prove the converse observe that the inclusion $M \to J$ induces a $k$-linear map $M/M^2 \to J/MJ$ with kernel $MJ \cap M/M^2$. By (ii) this map is injective; and by (i) $k\text{-}\dim(M/M^2) = k\text{-}\dim(J/MJ)$. So (i) and (ii) imply that the map is surjective. Thus $J = M + MJ$; whence $M = J$ (Nakayama's lemma).

PROOF OF THEOREM 1. (1) $\to$ (2). Observe that the $k$-linear maps $M^n/M^{n+1} \to J^n/J^{n+1}$ induce a graded $k$-algebra homomorphism $G(R) \to G(D)$. In any event $G(D)$ is reduced: $G(D) \cong K[T]$, $T$ an indeterminate. If $R$ is seminormal, then $e(R) = \operatorname{emdim}(R)$ (Lemma), and $G(R)$ is reduced because the map $G(R) \to G(D)$ is injective when $M = J$. (2) $\to$ (1). Suppose that $R$ is not seminormal. Then by the Lemma, there is $x \in MJ \cap M$, $x \notin M^2$. Now $x^n \in M^nJ^n$ $(n > 0)$, and for sufficiently large $n$, $J^n \subset M$ because $J$ is the radical of the conductor. For such $n$, $x^n \in M^{n+1}$; i.e., the leading form of $x$ in $G(R)$ is nilpotent. (2) $\to$ (3) is clear. (3) $\to$ (2). Since the points of $\operatorname{Proj}(G(R))$ correspond to the minimal prime ideals of $G(R)$, the fact that $\operatorname{Proj}(G(R))$ is reduced implies that the minimal primary components of the $O$-ideal in $G(R)$ are prime. Hence in order to prove that $G(R)$ is reduced, it suffices to show that the irrelevant maximal ideal of $G(R)$ is not an associated prime of the $O$-ideal. One can prove this directly; but instead we quote a more general result of J. Sallÿ [Sy] valid for Cohen-Macaulay local rings $R$ of arbitrary dimension: $G(R)$ is Cohen-Macaulay provided that $\operatorname{emdim}(R) = e(R) + \dim(R) - 1$.

REMARK. One can prove (3) $\to$ (1) by using the "$M = J$" criterion and the methods of §1 of Lipman's paper [L].

We conclude this section by giving a more explicit description of $G(R)$. For $T$ an indeterminate let $G = \{f \in K[T] \| f(0) \in k\}$.

COROLLARY 2. *$R$ is seminormal if, and only if, $G(R) \cong G$ (isomorphism of graded $k$-algebras).*

PROOF. Let $t \in G(D)$ be the leading form of $x$, where $J = xD$. Then $t \mapsto T$ induces a graded $K$-algebra isomorphism of $G(D)$ with $K[T]$. If $M = J$, then this isomorphism identifies $G(R)$ with $G$. Note that if $G(R) \cong G$, then $k\text{-}\dim(M^n/M^{n+1}) = k\text{-}\dim(K)$ $(n > 0)$. Since in any event we have $e(R) = k\text{-}\dim(M^n/M^{n+1})$ for all large $n$, it follows that $e(R) = \operatorname{emdim}(R)$ if $G(R) \cong G$. And $G$ is clearly reduced.

REMARK. (Thanks to the referee for bringing to light this supplement to Corollary 2.) Observe first that $G$–say by the conductor criterion–is seminormal with normalization $K[T]$. It follows that $G$ is the seminormalization of each of its graded $k$-subalgebras having normalization $K[T]$. (See [T] for "seminormalization.") Next note that $G(R)$ is reduced if, and only if, the homomorphism $G(R) \to G(D)$ is injective. (Proof: essentially identical with the "reduced" part of the proof of (1)$\leftrightarrow$(2) of Theorem 1.) It follows that when $G(R)$ is reduced, the identification of $G(D)$ with $K[T]$ identifies $G(R)$ with a $k$-subalgebra of $G$ having normalization $K[T]$. These comments, together with Corollary 2, prove: *R is seminormal if, and only if, G(R) is reduced and seminormal*. Consequently: *D is obtained from R by glueing over M if, and only if, G(D) is obtained from G(R) by glueing over the homogeneous maximal ideal.* (See [T] for "glueing.")

Let $\{M_i | 1 \leqslant i \leqslant n\}$ be the set of maximal ideals of $D$, and $K_i = D/M_i$. For each $i$ let $k\text{-dim}(K_i) = d(i)$ and let $\{a_{ij} | 1 \leqslant j \leqslant d(i)\}$ be a $k$-basis of $K_i$. Since $K$ is the $k$-algebra direct sum of $\{K_i\}$, the set $\{a_{ij}\}$ is canonically identified with a subset of $K$. Let $X = \{X_{ij} | 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant d(i)\}$ be a set of indeterminates, and define a $k$-algebra homomorphism $k[X] \to K[T]$ by $X_{ij} \mapsto a_{ij}T$. One easily verifies that the image is $G$ and the kernel is of the form $Sk[X]$, where $S = S_1 \cup \cdots \cup S_n \cup \{X_{ij}X_{uv} | i \neq u\}$ and $S_i$ is a set of homogeneous generators of the kernel of the $k$-algebra homomorphism $k[X_{i1}, \ldots, X_{id(i)}] \to K[T]$. Hence:

Corollary 3. *R is seminormal if, and only if,* $G(R) \cong k[X]/Sk[X]$ *(graded k-algebra isomorphism). And, in particular, when R has k-rational normalization, R is seminormal if, and only if,*

$$G(R) \cong k[X, \ldots, X_n]/\{X_iX_j | i \neq j\}k[X_1, \ldots, X_n].$$

**3. The algebroid-geometric viewpoint.** We retain in force the notation introduced at the end of §2. Let $\hat{G} = \{f \in K[[T]] \| f(0) \in k\}$. One easily verifies: $\hat{G}$ is local with normalization $K[[T]]$; $\hat{G}$ is seminormal; $X_{ij} \mapsto a_{ij}T$ defines a continuous $k$-algebra homomorphism with image $\hat{G}$ and kernel $Sk[[X]]$. Assume for the moment that $R$ is complete and equicharacteristic. Then: $k$ is contained in $R$; $K$ is contained in $D$; and we have a continuous $K$-algebra isomorphism $D \cong K[[T]]$, which identifies $R$ with $\hat{G}$ if, and only if, $R$ is seminormal. Consequently, for complete equicharacteristic $R$, we have that $R$ is seminormal if, and only if, $R \cong k[[X]]/Sk[[X]]$ (continuous $k$-algebra isomorphism). All this is noted in **[B]**, albeit for the case in which $R$ has $k$-rational normalization.

Now let "^" denote completion. Let $P_i$ denote the minimal prime ideal of $\hat{R}$ corresponding to $M_i$; i.e., $P_i = Q_i \cap \hat{R}$, where $Q_i$ is the unique minimal prime ideal of $\hat{D}$ contained in $\hat{M}_i$. Recall that the set of (algebroid, or analytic) *branches* of $R$ is $\{\hat{R}/P_i\}$. (Properly speaking we should say that $\{\text{Spec}(\hat{R}/P_i)\}$ is the set of branches of Spec($R$).) Since the Zariski tangent space $Z(R)$ of $R$ is the $k$-linear dual of $M/M^2 = \hat{M}/\hat{M}^2$, we see that $Z(R) = Z(\hat{R})$. Furthermore, since $Z(\hat{R}/P)$ is the dual of $\hat{M}/P + \hat{M}^2$ = the annihilator in $Z(\hat{R})$ of the subspace $P + \hat{M}^2/\hat{M}^2$, the Zariski tangent spaces of the branches are canonically embedded in $Z(R)$, and we may therefore meaningfully speak of their linear dependence or independence.

Observe that by the "$M = J$" criterion, $\hat{R}$ is seminormal if, and only if $R$, is seminormal. With this fact and the above explicit description of complete equicharacteristic $R$, one easily shows that the following theorem is valid in the equicharacteristic case.

THEOREM 2. *The following are equivalent.*

(1) *$R$ is seminormal.*

(2) *The branches of $R$ are seminormal and the Zariski tangent space of $R$ is the direct sum of the Zariski tangent spaces of the branches.*

(3) *The branches of $R$ are seminormal and $\hat{M}$ has a minimal basis $\{x_{ij} | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant d(i)\}$, with $\{x_{uj} | u \neq i\}$ a basis for $P_i$ (for each $i$).*

*Claim.* Theorem 2 is valid without the equicharacteristic assumption. (3) $\to$ (1) is valid since from (3) one can deduce the explicit description of $G(R)$ given in Corollary 3. (1) $\to$ (2) $\to$ (3) can be proved in the spirit of the equicharacteristic case, but the technical details must necessarily be somewhat different owing to the absence of a coefficient field. We omit these details which are tedious, but quite straightforward, manipulations of standard techniques of elementary linear and local algebra.

Finally we single out the special case wherein $R$ has $k$-rational normalization. As in the classical case, "normal crossings" means "smooth branches with linearly independent tangents".

COROLLARY 4. *Suppose that $R$ has $k$-rational normalization. Then $R$ is seminormal if, and only if, $R$ has normal crossings.*

REMARKS. Recent results of F. Orecchia [**O**, 1.1, 2.3, 2.9] imply the validity of (1) $\leftrightarrow$ (2) of Theorem 2. Theorem 1 remains valid if the standing hypothesis "reduced with finite normalization" is weakened to "Cohen-Macaulay". But this is not a true generalization: under the weaker hypothesis (1), (2) and (3) each imply the stronger hypothesis. Theorem 2 does not remain valid under the weaker hypothesis; e.g., $R = k[[X, Y]]/XY^2k[[X, Y]]$ satisfies (2) and (3) but not (1). If "branches" means $\{\hat{R}/P_i\}$, $\{P_i\}$ the set of *primary* components of $O$, then Theorem 2 is valid under the weaker hypothesis. But the criticism of the above generalization of Theorem 1 also applies here.

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