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AN ABSOLUTELY EXTREMAL FLOW WITH A NONABSOLUTELY EXTREMAL FACTOR

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ABSTRACT. It is shown that a homomorphic image of an absolutely extremal flow is not necessarily absolutely extremal.

1. INTRODUCTION

The notions of affine embedding and absolute extremality were defined in [1]. It was shown there that the property of being absolutely extremal is not preserved under products. We show here that a homomorphic image of an absolutely extremal point is not necessarily absolutely extremal. (This answers a problem suggested in [1].) For our example of an absolutely extremal flow with a nonabsolutely extremal factor we will use a construction similar to that in [3], and results concerning absolute extremality proved in [1 and 2]. For the convenience of reading we state here the relevant results:

In all that follows (X, T) is a minimal metric flow. $\mathscr{P}(X)$ denotes the space of probability measures on X endowed with the weak* topology. Whenever $\psi: (X, T) \to (Q, T)$ is an affine embedding (that is, Q is a compact convex subset of a locally convex linear topological space, $T: Q \to Q$ is an affine homeomorphism, ψ is a continuous 1-1 equivariant map, and $Q = \overline{\text{co}}\, \psi X)$ we will denote by β the barycenter map from $\mathscr{P}(X)$ onto Q (i.e., $\beta\lambda = \int_X \psi x \, d\lambda(x)$, $\lambda \in \mathscr{P}(X))$. A point $x_0 \in X$ is said to be absolutely extremal if for every affine embedding $\psi: X \to Q$, $\psi(x_0)$ is an extreme point of Q. (X, T) is an absolutely extremal flow if every point of X is absolutely extremal.

$L = L(X) = \{(x_1, x_2) \in X \times X | \Delta$ is the unique minimal set in the orbit closure of $(x_1, x_2)\}$, $P(X) = \{(x_1, x_2) \in X \times X | x_1$ and x_2 are proximal$\}$, and if $\varphi: (X, T) \to (Y, T)$ is a flow homomorphism then $X \underset{\varphi}{\times} X = \{(x_1, x_2) \in X \times X | \varphi x_1 = \varphi x_2\}$.

By Corollary 2.2 and the proof of Theorem 2.3 in [1],

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(1.1) If for $x_0 \in X$ and $\mu \in \mathscr{P}(X)$ $\beta\mu = \psi x_0$ in some affine embedding $\psi: X \to Q$, then there exists an F_σ subset A of X with $\mu(A) = 1$, $\overline{x} \in X$, and a sequence of integers $\{n_i\}$ s.t. $T^{n_i}x \to \overline{x}$ for every $x \in A \cup \{x_0\}$.

This implies Theorem 2.3 of [1].

(1.2) Every distal point in (X, T) is absolutely extremal.

We will need also Proposition 1.1 of [2].

(1.3) If $x_0 \in X$ is one of three doubly asymptotic points then x_0 is not absolutely extremal.

And finally, Lemma 3.1 in [1].

(1.4) If in some affine embedding $\psi: X \to Q$ $\beta\mu = \psi x$ $(\mu \in \mathscr{P}(X), x \in X)$ and $y \in X$ is an atom of μ then $(x, y) \in L$.

Considering these last two results one might conjecture that a converse of (1.4) and a generalization of (1.3) holds, i.e. that whenever x_0, y_0, z_0 are three points in X each pair of which is in L, x_0 (and/or y_0, z_0) is not absolutely extremal. We conclude with an example showing this conjecture to be false.

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2. An absolutely extremal flow with a nonabsolutely extremal factor

Throughout this discussion I will denote $[0, 1)$ with addition modulo 1, $\alpha \in I$ will be a fixed irrational number, and R_α the rotation by α on I.

Given a decreasing sequence $(t_k)_{k=0}^\infty$ of numbers in $(0, 1]$ s.t. $t_0 = 1$, $t_k \downarrow 0$ and $t_k \neq \cos 2\pi n\alpha$ for every $k \geq 1$ and $n \in \mathbf{Z}$, let $f: [-1, 1]\setminus(\{t_k\}_{k=1}^\infty \cup \{0\}) \to \{-1, 0, 1\}$ be defined as follows:

$$f(t) = \begin{cases} 1, & \text{if } t = 1 \text{ or } t_k < t < t_{k-1} \text{ where } k \geq 1 \text{ is odd.} \\ 0, & \text{if } t_k < t < t_{k-1} \text{ where } k \geq 2 \text{ is even.} \\ -1, & \text{if } -1 \leq t < 0. \end{cases} \tag{2.1}$$

Denote

$$f_{\xi, n} = f(\cos 2\pi(\xi + n\alpha))$$

(whenever this is well defined), and define $\tilde{x} \in \{-1, 0, 1\}^{\mathbf{Z}}$ by

$$\tilde{x}(n) = f(\cos 2\pi n\alpha) = f_{0, n}.$$

Let T be the shift on $\Omega = \{-1, 0, 1\}^{\mathbf{Z}}$ and $X = \overline{O(\tilde{x})}$ the orbit closure of $\tilde{x}$ in Ω. If the t_k's are chosen so that for each $\xi \in I$ $f(\cos 2\pi(\xi + n\alpha))$ is defined for "enough" n's then for every $x \in X$ there exists a *unique* $\xi \in I$ s.t.,

$$x(n) = f(\cos 2\pi(\xi + n\alpha)) = f_{\xi, n}. \tag{2.2}$$

For every $n \in \mathbf{Z}$ for which $f_{\xi, n}$ is well defined. (Note that if $f_{\xi, n} = f_{\eta, n}$ then $\operatorname{sgn} \cos 2\pi(\xi + n\alpha) = \operatorname{sgn} \cos 2\pi(\eta + n\alpha)$, so if such equality holds for, say, all $n \leq n_0$ for some integer n_0, then $\xi = \eta$.)

From (2.2) it easily follows that X is an almost 1-1 extension of (I, R_α) and thus a minimal flow. Let $\pi: X \to I$ denote the flow homomorphism assigning to each $x \in X$ the unique $\xi \in I$ satisfying (2.2).

Our first example is of a flow X constructed as above which is not absolutely extremal but has an extremal extension. For this example set $t_0 = 1$ and $\xi_0 = 0$. Then, when $t_j \in (0, 1]$ and $\xi_j \in I$ have already been chosen for $0 \le j < k$, $t_j = \cos 2\pi\xi_j$, choose $0 < t_k < t_{k-1}/2$ s.t. $t_k \neq \cos 2\pi(\xi_j + n\alpha)$ for every $0 \le j < k$ and $n \in \mathbf{Z}$ and let $\xi_k \in I$ satisfy $t_k = \cos 2\pi\xi_k$. (Then for every $\xi \in I$ $f_{\xi,n}$ is well defined for every $n \in \mathbf{Z}$ except for maybe one.)

It is easy to verify that in the above construction the set of ξ's in I s.t. $\pi^{-1}(\xi)$ is a singleton is $I \setminus \{\xi_j - m\alpha, (1/4) - m\alpha, (3/4) - m\alpha | m \in \mathbf{Z}, j \ge 1\}$. If $\xi = \xi_j - m\alpha$ for some $j \ge 1$ and $m \in \mathbf{Z}$, then $\pi^{-1}(\xi)$ consists of exactly two points which differ only on the m th coordinate which is 0 in one of these points and 1 in the other. If $\xi = (1/4) - m\alpha$ or $\xi = (3/4) - m\alpha$ for some $m \in \mathbf{Z}$ then $\pi^{-1}(\xi)$ consists of exactly three points which differ only on the m th coordinate. Thus if $x \in X$ and $\pi x = (1/4) - m\alpha$ or $\pi x = (3/4) - m\alpha$ for some $m \in \mathbf{Z}$, then x is one of three doubly asymptotic points and therefore is not absolutely extremal by (1.3). That is, the flow X is not absolutely extremal.

We now define an almost periodic extension Y of X which is absolutely extremal.

Let $Y = X \times I$ and let $T: Y \to Y$ be a homeomorphism defined by

$$T(x, a) = (Tx, a + g(x))$$

for every $x \in X$, $a \in I$, where $g: X \to I$ is a continuous function. (Y, T) is an almost periodic extension of (X, T). By defining g properly we get that Y has the property

(2.3) if x_0, x_1 and x_{-1} are three different points in X s.t. $\pi x_0 = \pi x_1 = \pi x_{-1}$ and for some $k \in \mathbf{Z}$ $x_0(k) = 0, x_1(k) = 1$ and $x_{-1}(k) = -1$, then for every $a \in I (x_0, a)$ and (x_1, a) are doubly asymptotic and proximal to (x_{-1}, b) for every $b \in I$ (and to no other point).

Denote by $\varphi: Y \to X$ the projection on the first coordinate. Y is a distal extension of the minimal flow X and hence is a disjoint union of minimal sets. Each of these minimal sets is projected by φ onto X and hence must include at least one point of each fiber over X. By (2.3) (and using the same notations as in (2.3)), all the points in the fiber over x_{-1} are proximal to the same point (x_0, a) and thus all lie in the same minimal set as (x_0, a). Therefore there is a unique minimal set in Y, i.e. (Y, T) is minimal. Now, if in some affine embedding of Y $\psi: Y \to Q$, ψy_0 is not an extreme point, then there exists $\mu \in \mathscr{P}(Y)$, $\mu \neq \delta_{y_0}$ s.t. $\beta\mu = \psi y_0$. Without loss of generality we may assume that y_0 is an atom of μ. By (1.1) there exists an F_σ subset A of Y s.t. $\mu(A) = 1$ and all the points in A are simultaneously proximal. Since any two points in Y which lie in the same fiber over X are distal, such A includes at most one point of each fiber. Since the proximality of (x, a) and (x', a') in Y implies that x and x' are proximal in X, and in X each point has only finite number of points proximal to it, such A is finite (and in particular,

$A = \operatorname{supp}\mu)$. Therefore, if $\pi\psi y_0 \notin \{(1/4) - m\alpha, (3/4) - m\alpha | m \in \mathbf{Z}\}$ then there are at most two points in $\operatorname{supp}\mu$. But then if $\mu \neq \delta_{y_0}$, $\beta\mu \neq \psi y_0$.

If $\pi\psi y_0 = (1/4) - m\alpha$ or $\pi\psi y_0 = (3/4) - m\alpha$ for some $m \in \mathbf{Z}$ then (using again the same notations as in (2.3)) $y_0 = (x, a)$ for some $a \in I$, where $x = x_0$ or x_1 or x_{-1}.

Consider for example the case when $y_0 = (x_0, a)$. By (2.3) for every $b \in I$ there exist $b' \neq b$ and a sequence (n_i) of integers s.t. $T^{n_i}((x_0, a), (x_{-1}, b')) \to (y', y')$ for some $y' \in Y$. Passing to a subsequence if necessary, we may assume that $T^{n_i}((x_{-1}, b), (x_{-1}, b')) \to (y'', y')$. Since Y is an almost periodic extension of X and y' and y'' are two different points lying in the same fiber over X, y' and y'' are distal. Since $T^{n_i}((x_0, a), (x_{-1}, b)) \to (y', y'')$, the pair $((x_0, a), (x_{-1}, b))$ is not in $L(Y)$. Thus by (1.4) (and since $\operatorname{supp}\mu$ is finite), for every $b \in I$ $(x_{-1}, b) \notin \operatorname{supp}\mu$. Hence in this case also $\operatorname{supp}\mu$ consists of at most two points $((x_0, a)$ and $(x_1, a))$ and thus if $\mu \neq \delta_{y_0} = \delta_{(x_0, a)}$ then $\beta\mu \neq \psi y_0$. Similarly it can be shown that if $y_0 = (x, a)$ where $x = x_1$ or x_{-1} then y_0 is absolutely extremal.

To complete the example we need only construct a continuous function $g: X \to I$ so that (2.3) is satisfied. Let then $<$ be the order on $I\backslash\{0\}$ induced by the usual order on $(0, 1)$. For every $n \geq 0$ choose $s_n \in I$ s.t. $0 < s_n \leq (1/4)$, $s_n \downarrow 0$, and $s_n = (1/4) + m_n\alpha$ where $m_0 = 0$ and for every $n \geq 0$ $m_n < m_{n+1}$. Choose for every $n \geq 0$ $r_n \in I$ s.t. $(3/4) \leq r_n < 1$, $r_n \uparrow 1$, and $r_n = (3/4) + l_n\alpha$ where $l_0 = 0$ and for every $n \geq 0$, $l_n < l_{n+1}$. Let

$$C = \overline{\{\pi^{-1}(t) | (1/4) < t < (3/4)\}}$$

and for every $n \geq 0$ let

$$A_n = \overline{\{\pi^{-1}(t) | s_{n+1} < t < s_n\}},$$
$$B_n = \overline{\{\pi^{-1}(t) | r_n < t < r_{n+1}\}}.$$

Each of these sets is a clopen set in X, they are pairwise disjoint and the union of all of them is $X\backslash\{\pi^{-1}(0)\} = X\backslash\{\tilde{x}\}$.

Define $g: X \to I$ as follows:

On $C \cup \{\tilde{x}\}$ $g = 0$. On each A_n $g = \rho_n \circ \pi$ where ρ_n is an increasing map taking the interval $\pi A_n = [s_{n+1}, s_n]$ onto an interval of the form $[0, 1/2^j]$. On B_n $g = \delta_n \circ \pi$ where δ_n is an increasing map from $\pi B_n = [r_n, r_{n+1}]$ onto $[0, 1/2^j]$. The j's are chosen so that for $n = 0, 1$ $gA_n = gB_n = [0, 1/2]$, then for the next 4 n's $gA_n = gB_n = [0, 1/4]$, etc.. Obviously, g is continuous on

$$X\backslash\{\tilde{x}\} = \left[\bigcup_{n\geq 0}(A_n \cup B_n)\right] \cup C,$$

and it is easily verified that g is also continuous at $\tilde{x}$.

Now, for every $n \in \mathbf{Z}$,

$$T^n(x, a) = (T^n x, a + g_n(x))$$

where

$$g_0(x) = 0, \qquad g_n(x) = \sum_{j=0}^{n-1} g(T^j x) \quad \text{for } n \geq 1,$$

and

$$g_n(x) = \sum_{j=-n}^{-1} -g(T^j x) \quad \text{for } n \leq -1.$$

Two points (x, a) and (x', a') in Y are proximal iff

(i) x and x' are proximal (which in our case means $\pi x = \pi x'$), and
(ii) $a - a'$ is a limit point of $(g_n(x) - g_n(x'))_{n \in \mathbf{Z}}$.

There can be an n for which $g_n(x) - g_n(x') \neq 0$ only if $\pi x = \pi x'$ is in the orbit closure of $1/4$ or $3/4$ in I, and then only if on the mth coordinate where x and x' differ, $x(m) = -1$ and $x'(m) = 0$ or 1. For example, suppose $\pi x = \pi x' = 1/4$, $x(o) = -1$ and $x'(o) = 0$ or 1. Then $g(T^j x) \neq g(T^j x')$ only for j's in the sequence $(m_n)_{n=0}^{\infty}$, and $(g(T^{m_n} x) - g(T^{m_n} x'))_{n=0}^{\infty}$ is the sequence $1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \ldots$. Thus $(g_n(x) - g_n(x'))_{n \in \mathbf{Z}}$ has all the diadic numbers in I, and hence the whole of I, as limit points. All the other cases are similar.

3. An absolutely extremal flow with three points pairwise in L

Let α, I and f be defined as in §2, only this time choose the t_k's differently: Set $t_0 = 1$. Now let $n_1 > 0$ be an integer s.t. $n_1\alpha + (1/4) \in (0, 1/4)$, and set $t_1 = \cos 2\pi(n_1\alpha + 1/4)$. Let $n_2 > n_1$ be an integer s.t. $\cos 2\pi(n_2\alpha + 3/4) < (1/2)t_1$, and set $t_2 = \cos 2\pi(n_2\alpha + 3/4)$. Assuming n_j and t_j has been chosen for $1 \leq j < k$ let $n_k > n_{k-1}$ be an integer s.t. $\cos 2\pi(n_k\alpha + 1/4) < (1/2)t_{k-1}$, if k is odd, and set $t_k = \cos 2\pi(n_k\alpha + 1/4)$. If k is even choose $n_k > n_{k-1}$ s.t. $\cos 2\pi(n_k\alpha + 3/4) < (1/2)t_{k-1}$ and set $t_k = \cos 2\pi(n_k\alpha + 3/4)$.

As in the construction of X in §2, let $\tilde{z} \in \{-1, 0, 1\}^{\mathbf{Z}}$ be defined by

$$\tilde{z}(n) = f_{0,n} = f(\cos 2\pi n\alpha).$$

Let Z be the orbit closure of $\tilde{z}$ in $\Omega = \{-1, 0, 1\}^{\mathbf{Z}}$ under the shift T. Then for every $z \in Z$ there exists a unique $\xi \in I$ s.t.,

$$z(n) = f_{\xi,n}. \tag{3.1}$$

For every $n \in \mathbf{Z}$, or for every $n \in \mathbf{Z}$ except for a sequence of the form $(m + n_{2j-1})_{j=1}^{\infty}$, or for every $n \in \mathbf{Z}$ except for a sequence of the form $(m + n_{2j})_{j=1}^{\infty}$.

Let $\pi: Z \to I$ be the flow homomorphism assigning to each $z \in Z$ the unique $\xi \in I$ satisfying (3.1). (Z, T) is thus a minimal, almost 1-1 extension of (I, R_α) and therefore $L(Z) = P(Z) = Z \underset{\pi}{\times} Z$.

If $\xi \notin \{(1/4) - m\alpha, (3/4) - m\alpha | m \in \mathbf{Z}\}$ then $\pi^{-1}(\xi)$ is a singleton. If $\xi = (1/4) - m\alpha$ then $\pi^{-1}(\xi)$ consists of exactly three points x_0, x_1, x_{-1}

which differ on the coordinates $(m+n_{2j-1})_{j=1}^{\infty}$, where

$$x_0(m)=0, \qquad x_1(m)=1, \qquad x_{-1}(m)=-1 \quad \text{and for every } j\geq 1,$$
$$x_0(m+n_{2j-1})=x_1(m+n_{2j-1})=1, \qquad x_{-1}(m+n_{2j-1})=0.$$

Similarly, if $\xi=3/4-m\alpha$ then $\pi^{-1}(\xi)$ consists of exactly three points x_0, x_1, x_{-1} which differ only on the coordinates $(m+n_{2j})_{j\geq 1}$, where

$$x_0(m)=0, \qquad x_1(m)=1, \qquad x_{-1}(m)=1 \quad \text{and for every } j\geq 1,$$
$$x_0(m+n_{2j})=x_1(m+n_{2j})=0, \qquad x_{-1}(m+n_{2j})=1.$$

In both cases, x_0 and x_1 are doubly asymptotic while x_0 and x_{-1} are not doubly asymptotic (though the pair (x_0, x_{-1}) is in L). In such case x_0 cannot be an absolutely extremal point, as the following argument shows:

Claim. If X is a minimal metric flow and $x_0, y_0, z_0 \in X$ are three different points s.t. $(x, x_0)\in P$ iff $x=x_0, y_0$ or z_0, x_0 and y_0 are doubly asymptotic and x_0 and z_0 are not doubly asymptotic, then x_0 is absolutely extremal.

Proof. Assume $T^{n_i}x_0 \to x$ and $T^{n_i}z_0 \to z$, $x\neq z$. If there exists an affine embedding of X $\psi\colon X\to Q$ s.t. for some measure $\delta_{x_0}\neq\mu\in\mathscr{P}(X)$, $\beta\mu=\psi x_0$, then without loss of generality we may assume that $\operatorname{supp}\mu\subseteq\{y_0, z_0\}$, i.e. $\mu=\alpha\delta_{y_0}+(1-\alpha)\delta_{z_0}$, $0\leq\alpha\leq 1$. Thus

$$\psi x_0=\alpha\psi y_0+(1-\alpha)\psi z_0$$
$$\begin{array}{ccccc} T^{n_i}\psi x_0 & = & \alpha T^{n_i}\psi y_0 & + & (1-\alpha)T^{n_i}\psi z_0 \\ \downarrow & & \downarrow & & \downarrow \\ \psi x & = & \alpha\psi x & + & (1-\alpha)\psi z \end{array}$$

which contradicts $x\neq z$.

Note that in our specific example by a similar argument x_1 and x_{-1} are also absolutely extremal and thus (Z, T) is an absolutely extremal flow.

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