PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

ON THE SUBGROUP SEPARABILITY OF GENERALIZED FREE PRODUCTS OF NILPOTENT GROUPS

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(Communicated by Warren J. Wong)

ABSTRACT. We prove that generalized free products of finitely generated nilpotent groups with cyclic amalgamation are subgroup separable

1. INTRODUCTION

Hall [7] showed that free groups are subgroup separable. This result was reproved by Burns [4] in a strengthened form. In [8], Mal′cev proved that polycyclic groups are subgroup separable. Subgroup separability is not only of interest group-theoretically but also is of interest for topological reasons. Thus Scott [10], using geometrical methods, showed that orientable surface groups are subgroup separable. This result was generalized by Brunner, Burns, and Solitar [3] who showed that the generalized free products of free groups with cyclic amalgamation are subgroup separable. More recently, Niblo, in his thesis [9], proved that the generalized free products of finitely generated Fuchsian groups with cyclic amalgamation are subgroup separable. Subgroup separability is related to the generalized word problem in the same way as residual finiteness is related to the word problem. Thus if G is a finitely presented subgroup separable group then G has a solvable generalized word problem. Topologically it is important because it is related to questions of embeddability of equivariant subspaces in their regular covering spaces. Scott raised the question whether all finitely generated 3-manifold groups are subgroup separable. This question was answered negatively by Burns, Karrass, and Solitar [5] where they showed that an infinite cyclic extension of a free group need not be subgroup separable. However the question whether all knot groups are subgroup separable remains open. Thurston [12, Problem 15] asked the question whether finitely generated

Received by the editors August 14, 1989 and, in revised form, November 19, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E06, 20E26, 20F18; Secondary 20F10.
Key words and phrases. Generalized free products, nilpotent groups, polycyclic groups, Hirsch length, residual finiteness, and subgroup separability.
The author gratefully acknowledges the partial support by the National Science and Engineering Research Council of Canada, Grant No. A-6064.

Kleinian groups are subgroup separable. He stated that it would be useful to know whether these groups are subgroup separable with respect to some special subgroups.

Unfortunately the classes of groups known to be subgroup separable are relatively few. In this paper, we prove that the generalized free products of finitely generated nilpotent groups with cyclic amalgamation are subgroup separable. The proof depends on a lemma of Brunner, Burns, and Solitar [3] which gives a sufficient condition for a generalized free product of groups with cyclic amalgamation to be subgroup separable. Their method made use of a modified concept of potency as introduced by Allenby and Tang [1] in their study of residual finiteness of generalized free products of groups.

2. Preliminaries

Definition 2.1. A group G is said to be *subgroup separable* if for every finitely generated subgroup H of G and every element $x \in G\backslash H$, there exists a subgroup L of finite index in G such that $H \subset L$ and $x \notin L$. Equivalently G is subgroup separable if for every finitely generated subgroup H of G and every element $x \in G\backslash H$, there exists a normal subgroup N of finite index in G such that $x \notin NH$.

We note that if we restrict $H = 1$ then the above definition is that of residual finiteness.

We need the following result of Brunner, Burns, and Solitar [3].

The BBS-Criterion. Let G be the generalized free product of the groups A and B with cyclic amalgamation. Then G is subgroup separable if A and B satisfy the following conditions:

For any $m+1$ nontrivial elements $x, g_1, \dots, g_m$ of A (B) and any n finitely generated subgroups $H_1, H_2, \dots, H_n$ of A (B), m, n finite, there exists a positive integer k such that for every positive integer l there is a normal subgroup N of A (B) of finite index with the following properties:

(i) $N \cap \langle x\rangle = \langle x^{kl}\rangle$;
(ii) for each H_j such that $H_j \cap \langle x\rangle = 1$, $NH_j \cap \langle x\rangle = \langle x^{kl}\rangle$;
(iii) for each H_j such that $H_j \cap \langle x\rangle \neq 1$, $NH_j \cap \langle x\rangle = H_j \cap \langle x\rangle$;
(iv) for each $g_i \notin \langle x\rangle$, $g_i \notin N\langle x\rangle$;
(v) for each pair (i, j) such that $H_j \cap \langle x\rangle g_i = \varnothing$, we have $NH_j \cap \langle x\rangle g_i = \varnothing$.

Definition 2.2. Let H be a subgroup of the group G. Then H is said to be *closed* in G if for all $x \in G$, $x^n \in H$ implies $x \in H$.

We also need the following result of Baumslag [2]:

Theorem 2.3. *Let G be a finitely generated torsion-free nilpotent group. Let H be a closed subgroup of G. Then $\bigcap_{i=1}^{\infty} HG^{p^i} = H$ for almost all primes p.*

Throughout this paper we shall use the following notations:

$G = A *_H B$ means G is the generalized free products of the groups A and B amalgamating the subgroup H.

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

If H is a subgroup of G then H^G is the normal closure of H in G.

$G^n = \langle x^n ; x \in G \rangle$.

If $N \lhd G$ and $\overline{G} = G/N$ then we use $\overline{x}$, $\overline{H}$ to denote the respective images of $x \in G$ and the subgroup H of G.

$Z_i(G)$ is the ith center of G.

3. Main result

Our purpose is to prove the following theorem:

Theorem. *Let A, B be finitely generated nilpotent groups. Then $G = A *_H B$ is subgroup separable if H is cyclic.*

We first make the following simple observation:

Lemma 3.1. *Let G be a finitely generated torsion-free nilpotent group. Let $x \in G$ and $X = \langle x \rangle$. Then $\langle x \rangle$ is closed in X^G.*

Proof. We use induction on the nilpotency class of G. If G is abelian or $x \in Z(G)$, then $X^G = \langle x \rangle$. Thus $\langle x \rangle$ is trivially closed in X^G. Suppose that G is nonabelian and $x \notin Z(G)$. Let $y \in X^G$ be such that $y^k = x^m$. Let $\overline{G} = G/Z(G)$. Then $\overline{y}^k = \overline{x}^m$. Therefore, by induction, we have $\overline{y} = \overline{x}^l$ for some integer l. This implies $y = x^l z$ for some $z \in Z(G)$. It follows that $y^k = (x^l z)^k = x^{lk} z^k = x^m$. Thus $z^{lk-m} = z^{-k}$. Since G is a torsion-free nilpotent group, $x \notin Z(G)$ implies that $\langle x \rangle \cap Z(G) = 1$. This means $z^{-k} = 1$, whence $z = 1$. Therefore $y = x^l \in \langle x \rangle$. Hence $\langle x \rangle$ is closed in X^G.

Corollary 3.2. *Let G be a finitely generated nilpotent group. Let $x \in G$ and $X = \langle x \rangle$. Then $\bigcap_{i=1}^{\infty} \langle x \rangle (X^G)^i = \langle x \rangle$.*

Proof. If x is of finite order then $(X^G)^i = 1$ for some integer i. Therefore we can assume G to be torsion-free. The corollary then follows from the above lemma and Theorem 2.3.

Lemma 3.3. *Let G be a finitely generated torsion-free nilpotent group. Let H be a subgroup of $Z(G)$ and $x \in G$ such that $\langle x \rangle \cap H = 1$. Then $\langle x \rangle$ is closed in $\langle x, H \rangle^G$.*

Proof. If $H = 1$ then, by Lemma 3.2, the lemma is true. We use induction on the Hirsch length of G. Let $A = \langle x, H \rangle^G$. If G is abelian then $A = \langle x \rangle \times H$. Thus $\langle x \rangle$ is closed in A. So we can assume G to be nonabelian and $x \notin Z(G)$. Since G is a finitely generated torsion-free nilpotent group, WLOG, we can assume H contains a maximal cyclic group $\langle z \rangle$ of G. Suppose there exists $y \in A$ such that $y^n = x^m$. Let $\overline{G} = G/\langle z \rangle$. Since $\langle z \rangle$ is a maximal cyclic group of G, $\overline{G}$ is torsion-free and of Hirsch length less than

G. Moreover, $\langle\overline{x}\rangle \cap \overline{H} = 1$. Also $\overline{A} = \langle\overline{x}, \overline{H}\rangle^{\overline{G}}$. Therefore, by induction, $\overline{y}^n = \overline{x}^m$ implies $\overline{y} \in \langle\overline{x}\rangle$. Thus, $y = x^l z^\alpha$. It follows that $x^m = y^n = x^{ln} z^{\alpha n}$, whence $x^{m-ln} = z^{\alpha n}$. This means $\alpha = 0$, whence $y = x^l \in \langle x\rangle$. This proves the lemma.

Corollary 3.4. *Let G be a finitely generated torsion-free nilpotent group. Let H be a subgroup of $Z(G)$ and $x \in G$ such that $\langle x\rangle \cap H = 1$. Then*

$$\bigcap_{i=1}^{\infty} \langle x\rangle(\langle x, H\rangle^G)^i = \langle x\rangle.$$

In order to apply the BBS-criterion we first prove the following propositions (cf. Propositions 1 and 2 [3]):

Proposition 3.5. *Let G be a finitely generated nilpotent group. Let H be a subgroup of G and let $x \in G$ be of infinite order such that $H \cap \langle x\rangle = 1$. Then there exists a postive integer r such that for all positive integers n there exists $N_n \lhd_f G$ such that $N_n H \cap \langle x\rangle = \langle x^{nr}\rangle$.*

Proof. We prove by induction on the Hirsch length l of G. The proposition is trivially true if $l = 0$.

Since the set of elements of finite order in G is a normal subgroup of finite order, WLOG we can assume G to be a finitely generated torsion-free group. Suppose $H \cap Z(G) \neq 1$. Let $1 \neq z \in H \cap Z(G)$ and $\overline{G} = G/\langle z\rangle$. Then $\overline{H} \cap \langle\overline{x}\rangle = 1$. Now $\overline{G}$ is of Hirsch length $l - 1$. Therefore, by induction, there exists $\overline{N}_n \lhd_f \overline{G}$ for $r > 0$ and for each n, such that $\overline{N}_n\overline{H} \cap \langle\overline{x}\rangle = \langle\overline{x}^{nr}\rangle$. Let N_n be the preimage of $\overline{N}_n$ in G. Clearly $N_n \lhd_f G$. Since $z \in H$ and $H \cap \langle x\rangle = 1$, we have $N_n H \cap \langle x\rangle = \langle x^{nr}\rangle$. Suppose next that $\langle H, x\rangle \cap Z(G) = 1$, and consider $\overline{G} = G/\langle z\rangle$, where $1 \neq z \in Z(G)$. Again, by induction, we can show that the required N_n exists. So we may suppose $\langle H, x\rangle \cap Z(G) \neq 1$ and $H \cap Z(G) = 1$. If $\langle x\rangle \cap Z(G) = \langle x^r\rangle$, then let $\overline{G} = G/\langle x^{nr}\rangle$. Clearly $\overline{H} \cap \langle\overline{x}\rangle = 1$ and the order of $\overline{x} = nr$. Since $\overline{G}$ is subgroup separable, it follows that for each positive integer n, there exists $\overline{N}_n \lhd_f \overline{G}$ such that $\overline{x}, \overline{x}^2, \ldots, \overline{x}^{nr-1} \notin \overline{N}_n\overline{H}$. Let N_n be the preimage of $\overline{N}_n$ in G. Then N_n is the required normal subgroup.

It remains to consider the case when $H \cap Z(G) = 1$, $\langle x\rangle \cap Z(G) = 1$, and $\langle H, x\rangle \cap Z(G) \neq 1$. Let $1 \neq z \in \langle H, x\rangle \cap Z(G)$. Let $\overline{G} = G/\langle z\rangle$. If $\langle\overline{H}\rangle \cap \langle\overline{x}\rangle = 1$ then, as before, the proposition can be proved by induction. Therefore, we assume $\overline{H} \cap \langle\overline{x}\rangle \neq 1$. Let $\overline{H} \cap \langle\overline{x}\rangle = \langle\overline{x}^t\rangle$. Then $\overline{x}^t = \overline{h}$ for some $h \in H$. This implies $h = x^t z^u$ for some integer u. Let h_1 be any element of H of the form $x^m z^k$. Then $\overline{h}_1 = \overline{x}^m \in \overline{H} \cap \langle\overline{x}\rangle$. It follows that $m = tl$, whence $h_1 = x^{tl} z^k$. Thus $h_1 h^{-l} = z^{k-ul} \in H \cap Z(G) = 1$. Hence $h_1 = h^l$. Therefore, the set $A = \{h \in H;\ h = x^m z^k\} = \langle x^t z^u\rangle$. Let $A_i = (A^G)^i$ with $A_1 = A^G$. If $A_1 \cap \langle z\rangle = \langle z^\alpha\rangle$ then $\langle z^{i\alpha}\rangle \subset A_i$. Since $\bigcap_{j=1}^{\infty}(A_j \cap \langle z\rangle) = 1$, it follows that, for each i, there exists a smallest positive integer k_i such that $A_{k_i} \cap \langle z\rangle \subset \langle z^{i\alpha}\rangle$.

We shall show that $\bigcap_{j=1}^{\infty}(HA_j \cap \langle x^t\rangle) = 1$. For each j, let $S_j = H \cap \langle x^t\rangle A_j$. Since $\langle x^t\rangle \cap \langle z^u\rangle = 1$, by Lemma 3.3, $\langle x^t\rangle$ is closed in $B = \langle x^t, z^u\rangle^G$. By Corollary 3.4, $\bigcap_{j=1}^{\infty}\langle x^t\rangle B^j = \langle x^t\rangle$. Thus $A_j \subset B^j$ implies $\bigcap_{i=1}^{\infty}\langle x^t\rangle A_j = \langle x^t\rangle$. Therefore $\bigcap_{j=1}^{\infty} S_j \subset \langle x^t\rangle$. Since $H \cap \langle x\rangle = 1$, it follows that $\bigcap_{j=1}^{\infty} S_j = 1$. Now $\bigcap_{j=1}^{\infty}(HA_j \cap \langle x^t\rangle) = \bigcap_{j=1}^{\infty}(S_jA_j \cap \langle x^t\rangle)$. Also $S_jA_j \cap \langle x^t\rangle$ is generated by $A_j \cap \langle x^t\rangle$ together with the set T_j of $x^{\beta_j} \in \langle x^t\rangle$ such that $x^{\beta_j}a_j \in S_j$ for some $a_j \in A_j$. Since $A_1 \cap \langle x^t\rangle = \langle x^{ct}\rangle$ for some integer c, it follows that $A_j \cap \langle x^t\rangle = \langle x^{jct}\rangle$. Let θ_j be the canonical homomorphism of G to G/A_j. Then $S_j\theta_j = H\theta_j \cap \langle (x\theta_j)^t\rangle = \langle (x\theta_j)^{d_jet}\rangle$ where e is given by $S_1\theta_1 = \langle (x\theta_1)^{et}\rangle$ and d_j is some integer with $d_1 = 1$. Therefore $T_j = \langle x^{d_jet}\rangle$. Moreover S_{j+1} and A_{j+1} are subgroups of S_j and A_j respectively. This implies $T_{j+1} \subset T_j$, whence $\langle x^{d_{j+1}et}\rangle \subset \langle x^{d_j et}\rangle$. Thus we can assume $d_j | d_{j+1}$. Let $d_j = n_jd_{j-1}$ for some integer n_j. Then $d_j = d_1n_2\cdots n_j$ it follows that

$$\bigcap_{j=1}^{\infty}(S_jA_j \cap \langle x^t\rangle) = \bigcap_{j=1}^{\infty}\langle x^{jct}, x^{(n_2\cdots n_j)et}\rangle. \tag{3.1}$$

Clearly for sufficiently large j, $\langle x^{jct}\rangle \subset \langle x^{(n_2\cdots n_i)et}\rangle = T_i$ for a given i. Moreover, $\bigcap_{j=1}^{\infty} S_j = 1$ implies that for sufficiently large j, $d_j > d_i$. Thus $\langle x^{(n_2\cdots n_j)et}\rangle \subsetneq T_i$. Therefore (3.1) implies $\bigcap_{j=1}^{\infty}(S_jA_j \cap \langle x^t\rangle) \subset \bigcap_{j=1}^{\infty} T_i = 1$, whence $\bigcap_{j=1}^{\infty}(HA_j \cap \langle x^t\rangle) = 1$. Thus for every integer d, there exists an integer m such that $HA_m \cap \langle x\rangle \subset \langle x^d\rangle$. In particular, for each n, there exists a smallest positive integer l_n such that $HA_{l_n} \cap \langle x\rangle \subset \langle x^{n\alpha t}\rangle$. Let $m_n = k_{nu}l_n$, recalling that k_{nu} is the smallest integer such that $A_{k_{kn}} \cap \langle z\rangle \subset \langle z^{nu\alpha}\rangle$. Then $HA_{m_n} \cap \langle x\rangle \subset \langle x^{n\alpha t}\rangle$ and $A_{m_n} \cap \langle z\rangle \subset \langle z^{n\alpha u}\rangle$. Let $U_n = A_{m_n}\langle z^{n\alpha u}\rangle$ and $\overline{G} = G/U_n$. Then $\overline{H} \cap \langle\overline{x}\rangle = \langle\overline{x}^{n\alpha t}\rangle$. Since $\overline{G}$ is subgroup separable, it follows that there exists $\overline{N}_n \lhd_f \overline{G}$ such that $\overline{x}, \overline{x}^2, \ldots, \overline{x}^{n\alpha t-1} \notin \overline{N}_n\overline{H}$. Thus $\overline{\overline{G}} = \overline{G}/\overline{N}_n$ is finite such that $\overline{\overline{H}} \cap \langle\overline{\overline{x}}\rangle = \langle\overline{\overline{x}}^{n\alpha t}\rangle$. Let N_n be the preimage of $\overline{\overline{N}}_n$ in G and let $r = \alpha t$. Then N_n is the required normal subgroup in G. This completes the proof.

Proposition 3.6. *Let G be a finitely generated nilpotent group. Let H be a finitely generated subgroup of G and $1 \neq x \in G$. If $g \in G$ is such that $H \cap \langle x\rangle g = \varnothing$ then there exists a subgroup L of finite index in G such that $H \subset L$ and $L \cap \langle x\rangle g = \varnothing$.*

Proof. By a theorem of Stebe [11], $g \notin \langle x\rangle H$ implies that there exists $N \lhd_f G$ such that $Ng \cap \langle x\rangle H = \varnothing$. Let $\overline{G} = G/N$. Then $\overline{g} \notin \langle\overline{x}\rangle\overline{H}$. This implies $\langle\overline{x}\rangle\overline{g} \cap \overline{H} = \varnothing$. Let $L = NH$. Clearly L is of finite index in G. If $y \in L \cap \langle x\rangle g$, then $\overline{y} \in \overline{L} \cap \langle\overline{x}\rangle\overline{g} = \overline{H} \cap \langle\overline{x}\rangle\overline{g}$ contradicting $\langle\overline{x}\rangle\overline{g} \cap \overline{H} = \varnothing$. Hence $L \cap \langle x\rangle g = \varnothing$ as required.

Applying Propositions 3.5 and 3.6 and using the same argument as in the Lemma of [3] we see that if A, B are finitely generated nilpotent groups then the conditions (i)–(v) of the BBS-criterion are satisfied. The theorem follows immediately.

4. Remark

Many residual properties of finitely generated nilpotent groups can be carried to polycyclic groups. In [6], Dyer proved that generalized free products of polycyclic groups are residually finite. Thus we propose the following problem:

Are generalized free products of polycyclic groups with cyclic amalgamation subgroup separable?

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