PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 2, February 2007, Pages 453–460
S 0002-9939(06)08485-1
Article electronically published on August 8, 2006

ON SUBSYMMETRIC BASES IN FRÉCHET SPACES

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. It is proved that any non-normable Fréchet space with a semi-symmetric absolute basis is isomorphic to the space ω of all scalar sequences. A similar result is shown for quasi-homogeneous absolute bases. It is also proved that any nuclear Fréchet space with a semi-subsymmetric basis is isomorphic to ω.

INTRODUCTION

Let E be a Fréchet space (i.e. a metrizable lcs over the field $\mathbb{K}$ of real or complex numbers) with a base (p_k) of continuous seminorms. A basis (e_n) in E, with the sequence (f_n) of coefficient functionals, is said to be *absolute* if

$$\forall\, k \in \mathbb{N}\ \forall\, x \in E : \sum_{n=1}^{\infty} |f_n(x)| p_k(e_n) < \infty.$$

By the Dynin-Mitiagin theorem any basis in a nuclear Fréchet space is absolute.

A basis (e_n) in E is *(semi-)symmetric* if any permutation $(e_{\pi(n)})$ of (e_n) is a basis in E (semi-)equivalent to (e_n). We say that a basis (e_n) in E is *(semi-)subsymmetric* if any permutation $(e_{\pi(n)})$ of (e_n) is a basis in E (semi-)equivalent to some subsequence (e_{k_n}) of (e_n). (Two sequences (x_n) in an lcs X and (y_n) in an lcs Y are *equivalent* if there exists an isomorphism T between their linear spans such that $Tx_n = y_n, n \in \mathbb{N}$, and *semi-equivalent* if for some $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ the sequences $(x_n), (\alpha_n y_n)$ are equivalent.)

Symmetric bases were studied in Banach spaces by Singer [8] and in locally convex spaces (lcs) by Garling [4], Ruckle [6], [7], Cac [3] and Terzioğlu [9].

Every basis in ω is symmetric because any two bases in ω are equivalent [2]. Terzioğlu [9] proved that any nuclear Fréchet space with a symmetric basis is isomorphic to ω. (His definition of a symmetric basis is equivalent to our definition which appears in [7].)

In this paper we show that any non-normable Fréchet space with a semi-symmetric absolute basis is isomorphic to ω (Theorem 2). We also prove that any Fréchet-Schwartz space with a semi-subsymmetric absolute basis is isomorphic to ω (Theorem 3). It follows that any nuclear Fréchet space with a semi-subsymmetric basis (in particular, with a subsymmetric basis) is isomorphic to ω (Corollary 4).

Received by the editors October 6, 2004 and, in revised form, September 6, 2005.
2000 *Mathematics Subject Classification.* Primary 46A35.
Key words and phrases. Symmetric, semi-symmetric, semi-subsymmetric, quasi-homogeneous bases.

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Next, we construct an uncountable family of mutually non-quasi-equivalent subsymmetric absolute bases in Fréchet spaces (Proposition 5). (Two sequences (x_n) in an lcs X and (y_n) in an lcs Y are *quasi-equivalent* if for some permutation π of $\mathbb{N}$ the sequences $(x_n), (y_{\pi(n)})$ are semi-equivalent.)

We say that a basis (e_n) in E is *(quasi-)homogeneous* if any subsequence (e_{k_n}) of (e_n) is (quasi-)equivalent to (e_n). (Homogeneous bases in Banach spaces are known as "subsymmetric bases" [8].)

We show that any non-normable Fréchet space with a quasi-homogeneous absolute basis is isomorphic to ω (Theorem 6). It follows that any nuclear Fréchet space with a quasi-homogeneous basis (in particular, with a homogeneous basis) is isomorphic to ω (Corollary 7).

Preliminaries

The linear span of a subset A of a linear space E is denoted by $\mathrm{lin}A$.

A non-decreasing sequence (p_k) of continuous seminorms on an lcs E is a *base* of continuous seminorms on E if for every continuous seminorm p on E there is $k \in \mathbb{N}$ with $p \leq p_k$. Any metrizable lcs E possesses a base (p_k) of continuous seminorms.

A Fréchet space F with a base (q_k) of continuous seminorms is isomorphic to ω if and only if for any $k \in \mathbb{N}$ the quotient space $(F/\ker q_k)$ is finite-dimensional [1].

A sequence (x_n) in an lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n : E \to \mathbb{K}$, $x \to \alpha_n$ $(n \in \mathbb{N})$ are continuous, then (x_n) is a *Schauder basis* in E.

Any basis in a Fréchet space is a Schauder basis.

Let (x_n) be a basis in a Fréchet space X and (y_n) a basis in a Fréchet space Y. Then the following conditions are equivalent:

(1) (x_n) is equivalent to (y_n);

(2) for any $(\alpha_n) \subset \mathbb{K}$ the series $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in X if and only if the series $\sum_{n=1}^{\infty} \alpha_n y_n$ is convergent in Y.

If a sequence (x_n) in a Fréchet space X is quasi-equivalent to a sequence (y_n) in a Fréchet space Y, then the closed linear spans of (x_n) and (y_n) are isomorphic.

Let E be a Fréchet space with a base (p_k) of continuous seminorms and let (e_n) be an absolute basis in E with the sequence (f_n) of coefficient functionals. Then we have the following:

(a) For any $M \subset \mathbb{N}$, $(e_n)_{n \in M}$ is an absolute basis in the closed linear span E_M of $(e_n)_{n \in M}$ and E is isomorphic to the product $E_M \times E_{\mathbb{N} \setminus M}$.

(b) The seminorms $p_k^*, k \in \mathbb{N}$, defined by

$$p_k^*(x) = \sum_{n=1}^{\infty} |f_n(x)| p_k(e_n), x \in E,$$

are continuous on E and (p_k^*) is a base of continuous seminorms on E.

(c) E is a nuclear space (respectively, a Schwartz space) if and only if for any $i \in \mathbb{N}$ there exists $j > i$ such that $\sum_{n=1}^{\infty} [p_i(e_n)/p_j(e_n)] < \infty$ (respectively, $[p_i(e_n)/p_j(e_n)] \to_n 0$) (we agree $[0/0] = 0$) [5].

Let $B = (b_{n,k})$ be an infinite matrix consisting of positive real numbers and satisfying the condition $\forall k, n \in \mathbb{N} : b_{n,k} \leq b_{n,k+1}$. The *Köthe space* associated with the matrix B is the Fréchet space

$$K(B) = \{(\alpha_n) \subset \mathbb{K} : \sum_{n=1}^{\infty} |\alpha_n| b_{n,k} < \infty \text{ for all } k \in \mathbb{N}\}$$

with the standard base (p_k) of continuous norms: $p_k((\alpha_n)) = \sum_{n=1}^{\infty} k|\alpha_n|b_{n,k}, k \in \mathbb{N}$. The sequence (e_n) of coordinate vectors is an absolute basis in $K(B)$.

Results

We will need the following lemma.

Lemma 1. *Assume that a sequence (x_n) in a metrizable lcs X is semi-equivalent to a sequence (y_n) in a metrizable lcs Y. Let (p_k) and (q_k) be bases of continuous seminorms on X and Y, respectively. Then there exist increasing functions $s, r : \mathbb{N} \to \mathbb{N}$ such that for all $i, j, n \in \mathbb{N}$ with $q_i(y_n)q_j(y_n) > 0$ we have*

$$\frac{q_i(y_n)}{q_{r(j)}(y_n)} \leq \frac{p_{s(i)}(x_n)}{p_{s(j)}(x_n)} \leq \frac{q_{r(i)}(y_n)}{q_j(y_n)}.$$

Proof. The sequence (x_n) is equivalent to $(\alpha_n y_n)$ for some $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$. Thus there exists an isomorphism $T : \text{lin}(x_n) \to \text{lin}(y_n)$ with $Tx_n = \alpha_n y_n, n \in \mathbb{N}$. By the continuity of T and T^{-1} we obtain

$$\forall k \in \mathbb{N}\ \exists s(k), r(k) \in \mathbb{N}\ \forall x \in \text{lin}(x_n) : q_k(Tx) \leq p_{s(k)}(x) \leq q_{r(k)}(Tx);$$

clearly, we can assume that $s(k) < s(k+1)$ and $r(k) < r(k+1)$ for any $k \in \mathbb{N}$. Then $\forall k, n \in \mathbb{N} : q_k(y_n) \leq p_{s(k)}(\alpha_n^{-1}x_n) \leq q_{r(k)}(y_n)$. Thus for all $k, n \in \mathbb{N}$ with $q_k(y_n) > 0$ we have $[q_k(y_n)/p_{s(k)}(x_n)] \leq |\alpha_n^{-1}| \leq [q_{r(k)}(y_n)/p_{s(k)}(x_n)]$. Hence for all $i, j, n \in \mathbb{N}$ with $q_i(y_n)q_j(y_n) > 0$ we get

$$\frac{q_i(y_n)}{p_{s(i)}(x_n)} \leq \frac{q_{r(j)}(y_n)}{p_{s(j)}(x_n)} \text{ and } \frac{q_j(y_n)}{p_{s(j)}(x_n)} \leq \frac{q_{r(i)}(y_n)}{p_{s(i)}(x_n)},$$

or equivalently

$$q_i(y_n)\, p_{s(j)}(x_n) \leq p_{s(i)}(x_n)\, q_{r(j)}(y_n) \text{ and } q_j(y_n)\, p_{s(i)}(x_n) \leq p_{s(j)}(x_n)\, q_{r(i)}(y_n).$$

This implies Lemma 1. □

Using this lemma, we shall show the following.

Theorem 2. *Any non-normable Fréchet space E with a semi-symmetric absolute basis (e_n) is isomorphic to ω. In particular, any non-normable Fréchet space with a symmetric absolute basis is isomorphic to ω.*

Proof. Let (p_k) be a base of continuous seminorms on E. Suppose, by contradiction, that E is not isomorphic to ω. Then, for some $k \in \mathbb{N}$, the space $(E/\ker p_k)$ is infinite-dimensional. Thus the set $M_1 = \{n \in \mathbb{N} : p_k(e_n) > 0\}$ is infinite. Put $M_2 = (\mathbb{N} \setminus M_1)$. Let E_i be the closed linear span of $\{e_n : n \in M_i\}$ for $i = 1, 2$. Clearly, $p_k^*|E_1$ is a continuous norm on E_1 and E is isomorphic to $E_1 \times E_2$. We shall prove that E has a continuous norm. It is obvious if M_2 is finite. Otherwise, there exists a permutation of $\mathbb{N}$ with $\pi(M_2) = M_1$. Since the basis (e_n) is semi-symmetric, the sequence $(e_{\pi(n)})$ is semi-equivalent to (e_n). Then $(e_{\pi(n)})_{n \in M_2}$ is semi-equivalent to $(e_n)_{n \in M_2}$. Thus E_1 is isomorphic to E_2. It follows that E possesses a continuous norm.

Since E is non-normable, then for $M = \{2n-1 : n \in \mathbb{N}\}$ or $M = \{2n : n \in \mathbb{N}\}$, the closed linear span F of $\{e_n : n \in M\}$ is non-normable. Without loss of generality we can assume that $p_k^*|F, k \in \mathbb{N}$, are pairwise non-equivalent norms on F. Then $\liminf_{n\in M}[p_k(e_n)/p_{k+1}(e_n)] = 0, k \in \mathbb{N}$. Let $\varphi : \mathbb{N}^3 \to M$ be an injection. We can construct an injection $\psi : \mathbb{N}^3 \to M$ such that

$$\frac{p_a(e_{\psi(w)})}{p_{a+1}(e_{\psi(w)})} < \frac{p_b(e_{\varphi(w)})}{p_c(e_{\varphi(w)})} \text{ for any } w = (a, b, c) \in \mathbb{N}^3.$$

(We put the elements of the set $\mathbb{N}^3$ in a sequence (t_m) and next we choose in turn $\psi(t_1), \psi(t_2), \dots$.)

Since the maps $\varphi, \psi : \mathbb{N}^3 \to M$ are injective and the sets $[\mathbb{N} \setminus \varphi(\mathbb{N}^3)], [\mathbb{N} \setminus \psi(\mathbb{N}^3)]$ are countable and infinite, there exists a permutation π of $\mathbb{N}$ with $\pi(\varphi(w)) = \psi(w)$ for all $w \in \mathbb{N}^3$. Then

$$\forall (a, b, c) \in \mathbb{N}^3 \ \exists n \in \mathbb{N} : \frac{p_a(e_{\pi(n)})}{p_{a+1}(e_{\pi(n)})} < \frac{p_b(e_n)}{p_c(e_n)}.$$

Since (e_n) is semi-symmetric, $(e_{\pi(n)})$ is semi-equivalent to (e_n) . By Lemma 1, there exist increasing functions $s, r : \mathbb{N} \to \mathbb{N}$ such that

$$\forall i, j, n \in \mathbb{N} : \frac{p_{r(i)}(e_{\pi(n)})}{p_j(e_{\pi(n)})} \geq \frac{p_{s(i)}(e_n)}{p_{s(j)}(e_n)}.$$

Hence, for $(a, b, c) = (r(1), s(1), s(r(1) + 1))$, we have

$$\forall n \in \mathbb{N} : \frac{p_a(e_{\pi(n)})}{p_{a+1}(e_{\pi(n)})} \geq \frac{p_b(e_n)}{p_c(e_n)};$$

a contradiction. Thus E is isomorphic to ω. □

For Fréchet-Schwartz spaces we shall show a stronger result.

Theorem 3. *Any Fréchet-Schwartz space E with a semi-subsymmetric absolute basis (e_n) is isomorphic to ω. In particular, any Fréchet-Schwartz space with a subsymmetric absolute basis is isomorphic to ω.*

Proof. Let (q_k) be a base of continuous seminorms on E. Suppose, by contradiction, that E is not isomorphic to ω. Then there is $i_1 \in \mathbb{N}$ with $\dim(E/\ker q_{i_1}) = \infty$. Let $i \geq i_1$. Put $N_i = \{n \in \mathbb{N} : q_i(e_n) > 0\}$. Clearly, the closed linear span E_i of $\{e_n : n \in N_i\}$ is an infinite-dimensional Fréchet-Schwartz space with an absolute basis $(e_n)_{n\in N_i}$ and $q_i^*|E_i$ is a continuous norm on E_i. Therefore $\lim_{n\in N_i}[q_i(e_n)/q_j(e_n)] = 0$ for some $j > i$. Thus we can construct inductively an increasing sequence $(i_k) \subset \mathbb{N}$ such that $\lim_{n\in N_{i_k}}[q_{i_k}(e_n)/q_{i_{k+1}}(e_n)] = 0$ for any $k \in \mathbb{N}$. Put $p_k = q_{i_k}$ and $M_k = N_{i_k}$ for all $k \in \mathbb{N}$; obviously $M_i \subset M_{i+1}$ for any $i \in \mathbb{N}$. Let $a_{i,j}(n) = [p_i(e_n)/p_j(e_n)]$ for $i, j \in \mathbb{N}$ with $i < j$ and $n \in M_i$. Then $\lim_{n\in M_i} a_{i,j}(n) = 0$ for all $i, j \in \mathbb{N}$ with $i < j$. Thus we can construct inductively an increasing sequence $(t_n) \subset M_1$ such that for any $n > 1$ we have

$$(1) \qquad \max_{1\leq p<q\leq n} a_{p,q}(t_n) < \min_{1\leq h<n} \min\{a_{h,n}(l) : l \in M_h, l \leq b_{h,n}\}$$

$$\text{where } b_{h,n} = \max_{h\leq d<w\leq n} \max\{f \in M_h : a_{d,w}(f) \geq a_{1,n}(t_{n-1})\}.$$

(If $h \leq d < w \leq n$, then the set $\{f \in M_h : a_{d,w}(f) \geq a_{1,n}(t_{n-1})\}$ is finite because $\lim_{f\in M_h} a_{d,w}(f) = 0$. Moreover, $\lim_{t\in M_1} \max_{1\leq p<q\leq n} a_{p,q}(t) = 0$ for any $n \in \mathbb{N}$.)

Let π be a permutation of $\mathbb{N}$ with $\pi(M_1) = M_1$ and $\pi(t_{3m}) = t_{3m}, \pi(t_{3m+1}) = t_{3m-1}$ for any $m \in \mathbb{N}$. Since the basis (e_n) is semi-subsymmetric, the sequence $(e_{\pi(n)})$ is semi-equivalent to (e_{k_n}) for some increasing sequence $(k_n) \subset \mathbb{N}$. Then $(e_{\pi(n)})_{n\in M_1}$ is semi-equivalent to $(e_{k_n})_{n\in M_1}$. Thus the linear span Y of $\{e_{k_n} : n \in M_1\}$ is isomorphic to the linear span of $\{e_n : n \in M_1\}$; so Y has a continuous norm. Therefore $\{k_n : n \in M_1\} \subset M_h$ for some $h \in \mathbb{N}$. Using Lemma 1, we infer that there exist increasing functions $s, r : \mathbb{N} \to \mathbb{N}$ such that

$$\frac{p_i(e_{k_n})}{p_{r(j)}(e_{k_n})} \le \frac{p_{s(i)}(e_{\pi(n)})}{p_{s(j)}(e_{\pi(n)})} \le \frac{p_{r(i)}(e_{k_n})}{p_j(e_{k_n})}$$

for all $i, j, n \in \mathbb{N}$ with $p_i(e_{k_n})p_j(e_{k_n}) > 0$. Hence for $(i, j) = (h, r(h) + 1)$ and $(u, v) = (h, r(r(h) + 1)), (p, q) = (s(h), s(r(h) + 1)), (d, w) = (r(h), r(h) + 1)$ we get $a_{u,v}(k_n) \le a_{p,q}(\pi(n)) \le a_{d,w}(k_n)$ for any $n \in M_1$, since $p_h(e_{k_n})p_{r(h)+1}(e_{k_n}) > 0$ for any $n \in M_1$.

Let $m \in \mathbb{N}$ with $3m \ge \max\{v, q, w\}$. Then $1 \le h < 3m$, $1 \le p < q \le 3m$, $h \le d < w \le 3m$, $k_{t_{3m}} \in M_h$, $k_{t_{3m+1}} \in M_h$, $a_{p,q}(t_{3m}) = a_{p,q}(\pi(t_{3m})) \ge a_{u,v}(k_{t_{3m}}) \ge a_{h,3m}(k_{t_{3m}})$ and $a_{d,w}(k_{t_{3m+1}}) \ge a_{p,q}(\pi(t_{3m+1})) = a_{p,q}(t_{3m-1}) \ge a_{1,3m}(t_{3m-1})$.

By (1) we obtain $k_{t_{3m}} > b_{h,3m}$ since $a_{p,q}(t_{3m}) \ge a_{h,3m}(k_{t_{3m}})$ and $k_{t_{3m}} \in M_h$. Moreover, $b_{h,3m} \ge k_{t_{3m+1}}$ because $k_{t_{3m+1}} \in M_h$ and $a_{d,w}(k_{t_{3m+1}}) \ge a_{1,3m}(t_{3m-1})$. Thus $k_{t_{3m}} > k_{t_{3m+1}}$. Hence $t_{3m} > t_{3m+1}$, so $3m > 3m + 1$; a contradiction. It follows that E is isomorphic to ω. □

Corollary 4. *Any nuclear Fréchet space with a semi-subsymmetric basis* (*in particular, with a subsymmetric basis*) *is isomorphic to* ω.

Now we show that there exist uncountably many mutually non-quasi-equivalent absolute subsymmetric bases in Fréchet spaces. Let $\mathcal{D}$ be the set of all sequences $a = (a_n) \subset (0, \infty)$ such that $\limsup_n \liminf_m |a_n - a_m| < \infty$. Let $a \in \mathcal{D}$. Put $B = (b_{n,k})$ where $b_{n,k} = k^{a_n}$ for all $n, k \in \mathbb{N}$. Denote by $D_\infty(a)$ the Köthe space $K(B)$. We shall prove the following.

Proposition 5. (a) *For any* $a \in \mathcal{D}$ *the coordinate basis in the Köthe space* $D_\infty(a)$ *is subsymmetric.*

(b) *There exists an uncountable family* $\mathcal{D}' \subset \mathcal{D}$ *such that the coordinate bases in* $D_\infty(a')$ *and* $D_\infty(a'')$ *are not quasi-equivalent for any* $a', a'' \in \mathcal{D}'$ *with* $a' \ne a''$.

Proof. (a) Let $a = (a_n) \in \mathcal{D}$ and let $(e_{\pi(n)})$ be a permutation of the coordinate basis (e_n) in $D_\infty(a)$. By the definition of $\mathcal{D}$, there exist $c \in \mathbb{N}$ and an increasing sequence $(k_n) \subset \mathbb{N}$ such that $|a_{\pi(n)} - a_{k_n}| \le (c - 1)$, $n \in \mathbb{N}$. Then

$$(a_{\pi(n)} + 1) \le c(a_{k_n} + 1) \le c^2(a_{\pi(n)} + 1), n \in \mathbb{N}.$$

Let $s(j) = j^c$ and $r(j) = j^{c^2}$ for each $j \in \mathbb{N}$ and $b_{n,k} = k^{a_n}$ for $n, k \in \mathbb{N}$. Then

$$\forall j, n \in \mathbb{N} : jb_{\pi(n),j} \le s(j)b_{k_n,s(j)} \le r(j)b_{\pi(n),r(j)}. \tag{2}$$

Consider a linear map $T : \mathrm{lin}(e_{\pi(n)}) \to \mathrm{lin}(e_{k_n})$, such that $Te_{\pi(n)} = e_{k_n}$ for any $n \in \mathbb{N}$. By (2) we get

$$\forall j \in \mathbb{N}\ \forall x \in \mathrm{lin}(e_{\pi(n)}) : p_j(x) \le p_{s(j)}(Tx) \le p_{r(j)}(x),$$

where (p_k) is the standard base of continuous norms on $D_\infty(a)$. Therefore T is an isomorphism. Thus $(e_{\pi(n)})$ is equivalent to (e_{k_n}), so (e_n) is subsymmetric.

(b) Let $\varphi : \mathbb{Q} \to \mathbb{N}$ be an injection. For $x \in \mathbb{R}$ let $(t_{x,n}) \subset (\mathbb{Q} \setminus \{x\})$ with $\lim t_{x,n} = x$ and $N_x = \{\varphi(t_{x,n}) : n \in \mathbb{N}\}$. Then $\{N_x : x \in \mathbb{R}\}$ is an uncountable family of infinite subsets of $\mathbb{N}$ such that the set $N_x \cap N_y$ is finite for all $x, y \in \mathbb{R}$ with $x \neq y$.

Let $(h_m) \subset (1, \infty)$ be an increasing sequence with $\lim_m h_{2m}^{-1} h_{2m+1} = \infty$. Put $I_m = (h_{2m-1}, h_{2m}), m \in \mathbb{N}$. For $x \in \mathbb{R}$ let $a_x = (a_{x,n})$ be a sequence with $\{a_{x,n} : n \in \mathbb{N}\} = \bigcup\{I_m \cap \mathbb{Q} : m \in N_x\}$. Clearly, $a_x \in \mathcal{D}$ for any $x \in \mathbb{R}$. Put $\mathcal{D}' = \{a_x : x \in \mathbb{R}\}$. We shall prove that the coordinate bases in $D_\infty(a_x)$ and $D_\infty(a_y)$ are not quasi-equivalent for any $x, y \in \mathbb{R}$ with $x \neq y$. Let $x, y \in \mathbb{R}$ with $x \neq y$.

Let π be a permutation of $\mathbb{N}$. Take an increasing sequence $(d_n) \subset (N_x \setminus N_y)$ and a sequence $(s_n) \subset \mathbb{N}$ such that $a_{x,\pi(s_n)} \in I_{d_n}, n \in \mathbb{N}$. Then $\lim_n a_{x,\pi(s_n)} = \infty$ and

$$(a_{x,\pi(s_n)}, a_{y,s_n}) \in \left[\left(\bigcup_{k=1}^{\infty} I_k \times \bigcup_{k=1}^{\infty} I_k\right) \setminus \bigcup_{k=1}^{\infty} I_k \times I_k\right], n \in \mathbb{N}.$$

Thus $a_{x,\pi(s_n)}, a_{y,s_n} \in (1, \infty)$ for $n \in \mathbb{N}$ and

$$\max\left\{\left(\frac{a_{x,\pi(s_n)}}{a_{y,s_n}}\right), \left(\frac{a_{y,s_n}}{a_{x,\pi(s_n)}}\right)\right\} \to_n \infty.$$

Hence

$$\max\left\{\left(\frac{a_{x,\pi(s_n)}+1}{a_{y,s_n}+1}\right),\ \left(\frac{a_{y,s_n}+1}{a_{x,\pi(s_n)}+1}\right)\right\} \to_n \infty$$

because

$$2\left(\frac{a_{x,\pi(s_n)}}{a_{y,s_n}}\right) > \left(\frac{a_{x,\pi(s_n)}+1}{a_{y,s_n}+1}\right) > \frac{1}{2}\left(\frac{a_{x,\pi(s_n)}}{a_{y,s_n}}\right), n \in \mathbb{N}.$$

Suppose that the permutation $(e_{\pi(n)})$ of the coordinate basis in $D_\infty(a_x)$ is semi-equivalent to the coordinate basis (e_n) in $D_\infty(a_y)$. Then by Lemma 1 there exist increasing functions $s, r : \mathbb{N} \to \mathbb{N}$ such that

$$\forall i, j, n \in \mathbb{N} : \frac{p_{y,i}(e_n)}{p_{y,r(j)}(e_n)} \leq \frac{p_{x,s(i)}(e_{\pi(n)})}{p_{x,s(j)}(e_{\pi(n)})} \leq \frac{p_{y,r(i)}(e_n)}{p_{y,j}(e_n)}$$

where $(p_{x,k})$ and $(p_{y,k})$ are the standard bases of continuous seminorms on $D_\infty(a_x)$ and $D_\infty(a_y)$, respectively. Thus

$$\forall i, j, n \in \mathbb{N} : \left[\frac{i^{a_{y,n}+1}}{r(j)^{a_{y,n}+1}}\right] \leq \left[\frac{s(i)^{a_{x,\pi(n)}+1}}{s(j)^{a_{x,\pi(n)}+1}}\right] \leq \left[\frac{r(i)^{a_{y,n}+1}}{j^{a_{y,n}+1}}\right].$$

Hence, for all $i, j, n \in \mathbb{N}$, we obtain

$$(a_{y,n}+1)\ln\left[\frac{i}{r(j)}\right] \leq (a_{x,\pi(n)}+1)\ln\left[\frac{s(i)}{s(j)}\right] \leq (a_{y,n}+1)\ln\left[\frac{r(i)}{j}\right].$$

Then for $(i, j) = (r(1)+1, 1)$ we get

$$\frac{\ln[i/r(j)]}{\ln[s(i)/s(j)]} \leq \left(\frac{a_{x,\pi(n)}+1}{a_{y,n}+1}\right) \leq \frac{\ln[r(i)/j]}{\ln[s(i)/s(j)]},\ n \in \mathbb{N}.$$

Thus

$$\max\left\{\left(\frac{a_{x,\pi(s_n)}+1}{a_{y,s_n}+1}\right), \left(\frac{a_{y,s_n}+1}{a_{x,\pi(s_n)}+1}\right)\right\} \not\to_n \infty;$$

a contradiction. This shows that the coordinate bases in $D_\infty(a_x)$ and $D_\infty(a_y)$ are not quasi-equivalent. □

Finally, we prove the following result.

Theorem 6. *Any non-normable Fréchet space E with a quasi-homogeneous absolute basis (e_n) is isomorphic to ω.*

Proof. Let (q_k) be a base of continuous seminorms on E. Suppose, by contradiction, that E is not isomorphic to ω. Then for some $k \in \mathbb{N}$, we have $\dim(E/\ker q_k) = \infty$, so the set $L = \{n \in \mathbb{N} : q_k(e_n) > 0\}$ is infinite. Denote by X the closed linear span of $(e_n)_{n\in L}$; clearly, $q_k^*|X$ is a continuous norm on X. Since the basis (e_n) is quasi-homogeneous, then $(e_n)_{n\in L}$ is quasi-equivalent to (e_n); so X is isomorphic to E. Thus E has a continuous norm. Without loss of generality we can assume that q_1 is a norm.

Let M be an infinite subset of $\mathbb{N}$. Since $(e_n)_{n\in M}$ is quasi-equivalent to (e_n), the closed linear span Y of $(e_n)_{n\in M}$ is isomorphic to E, so Y is non-normable. Hence

$$\forall\, k \in \mathbb{N}\ \exists\, l > k : \inf_{n\in M} \frac{q_k(e_n)}{q_l(e_n)} = 0.$$

Thus we can construct inductively an increasing sequence $(k_i) \subset \mathbb{N}$ and a decreasing sequence (M_i) of infinite subsets of $\mathbb{N}$ such that $\lim_{n\in M_i}[q_{k_i}(e_n)/q_{k_{i+1}}(e_n)] = 0$ for any $i \in \mathbb{N}$. Put $p_i = q_{k_i}, i \in \mathbb{N}$. Then there exists an increasing sequence $(t_n) \subset \mathbb{N}$ such that

$$\forall n \in \mathbb{N} : \min_{1\le v\le n} \frac{p_1(e_v)}{p_n(e_v)} > \max_{1\le i\le n} \frac{p_i(e_{t_n})}{p_{i+1}(e_{t_n})}. \tag{3}$$

Since (e_{t_n}) is quasi-equivalent to (e_n) , there is a permutation π of $\mathbb{N}$ such that $(e_{\pi(n)})$ is semi-equivalent to (e_{t_n}). By Lemma 1, there exist increasing functions $s, r : \mathbb{N} \to \mathbb{N}$ such that

$$\forall i, j, n \in \mathbb{N} : \frac{p_{s(i)}(e_{\pi(n)})}{p_{s(j)}(e_{\pi(n)})} \le \frac{p_{r(i)}(e_{t_n})}{p_j(e_{t_n})}.$$

In particular, for $(a, b, c) = (r(1), s(1), s(r(1) + 1))$ we have

$$\forall n \in \mathbb{N} : \frac{p_b(e_{\pi(n)})}{p_c(e_{\pi(n)})} \le \frac{p_a(e_{t_n})}{p_{a+1}(e_{t_n})}.$$

Using (3), we get $\pi(n) > n$ for any $n \ge c$ because

$$\frac{p_1(e_{\pi(n)})}{p_n(e_{\pi(n)})} \le \frac{p_b(e_{\pi(n)})}{p_c(e_{\pi(n)})}$$

and $a \le n$ for any $n \ge c$. Hence $\pi(\{i \in \mathbb{N} : i \ge c\}) \subset \{i \in \mathbb{N} : i > c\}$, so $\pi(\{i \in \mathbb{N} : i < c\}) \supset \{i \in \mathbb{N} : i \le c\}$; this is impossible.

Thus E is isomorphic to ω. □

Corollary 7. *Any nuclear Fréchet space with a quasi-homogeneous basis (in particular, with a homogeneous one) is isomorphic to ω.*

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