

CONTINUOUS CLOSURE, AXES CLOSURE, AND NATURAL CLOSURE

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ABSTRACT. Let R be a reduced affine \mathbb{C} -algebra with corresponding affine algebraic set X . Let $\mathcal{C}(X)$ be the ring of continuous (Euclidean topology) \mathbb{C} -valued functions on X . Brenner defined the *continuous closure* I^{cont} of an ideal I as $IC(X) \cap R$. He also introduced an algebraic notion of *axes closure* I^{ax} that always contains I^{cont} , and asked whether they coincide. We extend the notion of axes closure to general Noetherian rings, defining $f \in I^{\text{ax}}$ if its image is in IS for every homomorphism $R \rightarrow S$, where S is a one-dimensional complete seminormal local ring. We also introduce the *natural closure* I^{\natural} of I . One of many characterizations is $I^{\natural} = I + \{f \in R : \exists n > 0 \text{ with } f^n \in I^{n+1}\}$. We show that $I^{\natural} \subseteq I^{\text{ax}}$ and that when continuous closure is defined, $I^{\natural} \subseteq I^{\text{cont}} \subseteq I^{\text{ax}}$. Under mild hypotheses on the ring, we show that $I^{\natural} = I^{\text{ax}}$ when I is primary to a maximal ideal and that if I has no embedded primes, then $I = I^{\natural}$ if and only if $I = I^{\text{ax}}$, so that I^{cont} agrees as well. We deduce that in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, if $f = 0$ at all points where all of the $\frac{\partial f}{\partial x_i}$ are 0, then $f \in (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})R$. We characterize I^{cont} for monomial ideals in polynomial rings over \mathbb{C} , but we show that the inequalities $I^{\natural} \subseteq I^{\text{cont}}$ and $I^{\text{cont}} \subseteq I^{\text{ax}}$ can be strict for monomial ideals even in dimension 3. Thus, I^{cont} and I^{ax} need not agree, although we prove they are equal in $\mathbb{C}[x_1, x_2]$.

CONTENTS

1. Introduction	3316
2. Properties of continuous closure	3318
3. Seminormal rings	3322
4. Axes closure and one-dimensional seminormal rings	3327
5. Special and inner integral closure, and natural closure	3332
6. The ideal generated by the partial derivatives	3339
7. Naturally closed primary ideals are axes closed	3340
8. Multiplying by invertible ideals and rings of dimension 2	3347
9. A negative example and a fiber criterion for exclusion from the continuous closure	3349
10. Mixed natural closure and continuous closure for monomial ideals in polynomial rings	3351
11. A bigger axes closure	3358
Acknowledgment	3361
References	3361

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1. INTRODUCTION

Holger Brenner [Bre06] introduced a new closure operation on ideals in finitely generated \mathbb{C} -algebras called *continuous closure*, and asked whether it is the same as an algebraic notion called *axes closure* that he introduced. He proved this for ideals in a polynomial ring that are primary to a maximal ideal and generated by monomials. We shall relate this closure to some variant notions of integral closure, the *special part of the integral closure* over a local ring, introduced in [Eps10], and the *inner integral closure*, a notion explored here that exists without an explicit name in the literature, and also to a notion we introduce called *natural closure*. We shall prove that if I is an unmixed ideal in any affine \mathbb{C} -algebra, then I is continuously closed if and only if it is axes closed. See Theorem 7.8, Corollary 7.14, and Corollary 7.15. We also provide further conditions under which continuous closure equals axes closure or natural closure.

In consequence we can prove, for example, that if f is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$ that vanishes wherever its partial derivatives all vanish, then there are continuous functions g_j from $\mathbb{C}^n \rightarrow \mathbb{C}$ such that

$$f = \sum_{j=1}^n g_j \frac{\partial f}{\partial x_j}.$$

See Theorem 6.1.

On the other hand, we show that continuous closure is sometimes strictly smaller than axes closure. Indeed, in §9, we give an example (followed by a method of generating such examples) of a monomial ideal in a polynomial ring over \mathbb{C} which is continuously closed but not axes closed.

After hearing the second-named author give a talk on the results of this paper, Kollár [Kol12] studied continuous closure in the context of coherent sheaves on schemes over \mathbb{C} and gave an algebraic characterization that permits the notion of continuous closure to be defined in a larger context. In a further paper [FK13], continuous closure is studied over topological fields other than \mathbb{C} , particularly for the field of real numbers.

Let R be a finitely generated \mathbb{C} -algebra. Map a polynomial ring $\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow R$ onto R as \mathbb{C} -algebras. Let $\mathfrak{A} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be the kernel ideal, and let X be the set of points in \mathbb{C}^n where all elements of \mathfrak{A} vanish. X may be identified with the set of maximal ideals of R . Then X has a Euclidean topology, and the topological space X is independent of the presentation of R . We let $\mathcal{C}(Y)$ denote the ring of complex-valued continuous functions on any space Y . Polynomial functions on \mathbb{C}^n , when restricted to X , yield a ring $\mathcal{C}[X]$ which is isomorphic to R_{red} , the original ring modulo the ideal N of nilpotents. (Nothing will be lost in the sequel if we restrict our attention to reduced rings R (i.e., rings without nonzero nilpotents).) Thus, we have a \mathbb{C} -homomorphism $R \rightarrow \mathcal{C}(X)$ which is injective when R is reduced. The *continuous closure* of $I \subseteq R$, denoted I^{cont} , is the contraction of $IC(X)$ to R . That is, if $I = (f_1, \dots, f_m)R$, then $f \in I^{\text{cont}}$ precisely when there are continuous functions $g_i : X \rightarrow \mathbb{C}$ such that

$$f|_X = g_1 f_1|_X + \dots + g_m f_m|_X,$$

where $h|_X$ indicates the image of $h \in R$ in $\mathcal{C}(X)$. Henceforth, we focus on the case where R is reduced, and omit $|_X$ from the notation. However, we can state many

of the results without this hypothesis: one can typically pass at once in the proofs to the case where the ring is reduced.

In this paper we study this closure and several other closures that are related, obtaining satisfying answers to many questions that were open even for polynomial rings.

Let L be an algebraically closed field. We are especially concerned with the case where $L = \mathbb{C}$ is the complex numbers. A finitely generated L -algebra R is called a *ring of axes* over L if it is one-dimensional, reduced, and either smooth, with just one irreducible component, or else is such that the corresponding algebraic set is the union of n smooth irreducible curves, and there is a unique singular point, which is the intersection of any two of the components, such that the completion of the local ring at that point is isomorphic with $L[[x_1, \dots, x_n]]/(x_i x_j \mid 1 \leq i < j \leq n)$.

We now restrict ourselves to the case of the complex numbers. In [Bre06] Brenner obtains a structure theorem for the ideals of such a completed local ring that enables him to prove that in a ring of axes over \mathbb{C} , for every ideal I , $I = I^{\text{cont}}$. The *axes* closure I^{ax} of an ideal I of R is defined to be the set of elements r such that for every \mathbb{C} -homomorphism $R \rightarrow S$, where S is a ring of axes, one has $r \in IS$. (Here as in the sequel, we use the standard abuse of notation $r \in IS$ to mean that $\varphi(r) \in \varphi(I)S$, where $\varphi : R \rightarrow S$ is the map from R to S .) The results of [Bre06] imply that $I^{\text{cont}} \subseteq I^{\text{ax}}$ in general and that they agree for ideals of polynomial rings that are primary to maximal ideals and are generated by monomials. As mentioned above, we prove here that continuous closure coincides with axes closure for all ideals of affine \mathbb{C} -algebras that are primary to a maximal ideal, and in many other cases. We also show that an unmixed ideal (one that has no embedded primes) is axes closed if and only if it is continuously closed, and that there exist continuously closed ideals which are not axes closed, which answers a question raised by Brenner.

In §3 we prove that an element r of an affine \mathbb{C} -algebra is in the axes closure of $I \subseteq R$ if and only if $x \in IS$ for every homomorphism of R to an excellent (respectively, complete) Noetherian one-dimensional seminormal ring S . We use the latter definition to extend the notion of axes closure to all Noetherian rings. See Corollaries 4.2 and 4.4 and Definition 4.3.

Here is a brief sketch of the contents of the paper.

In §2, we discuss some important properties of continuous closure that we will need. Some of this material is reviewed from [Bre06], but in some cases we need sharper or more general results. §3 is devoted to seminormal rings and their connections to continuous and axes closures. In §4, we extend the definition of axes closure to general Noetherian rings, characterizing it by maps to excellent one-dimensional seminormal rings, and we show that this agrees with the original definition in Brenner's setting. In §5 we discuss the concepts of special and inner integral closure, and introduce the notion of natural closure. We also introduce the notion of I -relevant ideals, which are used to characterize when an ideal is naturally closed, and which play a key role in proving the results of §7. We show that the natural closure is contained in the axes closure and, wherever it is defined, the continuous closure. This "traps" continuous closure between two algebraically defined closures. This is the main tool used in §7 to prove results when axes closure and continuous closure agree.

One of the main results of §6 has already been stated in the second paragraph of this Introduction.

§7 is mostly devoted to a number of important cases where natural closure and axes closure agree, and contains several of our main results. When these two agree and continuous closure is defined, it agrees as well. This yields the central result that an unmixed ideal in an affine \mathbb{C} -algebra is continuously closed if and only if it is axes closed. We also give a characterization of seminormal rings in terms of axes closed ideals.

In §8 we show that continuous and axes closure agree in the locally factorial two-dimensional case. In §9, we develop a “fiber criterion” to exclude certain elements from the continuous closure of an ideal. This allows us to construct examples of continuously closed ideals that are not axes closed (even a monomial ideal in a three-dimensional polynomial ring). We apply this criterion in §10 to show that for monomial ideals in polynomial rings over \mathbb{C} , continuous closure always equals mixed natural closure, which is defined in that section. Finally, we introduce in §11 a closure operation $^{\text{AX}}$ that is similar to $^{\text{ax}}$ and agrees with it in equal characteristic 0, but is based on weakly normal rings instead of seminormal ones, and thus is sometimes bigger in positive characteristic. The two notions and their relative usefulness are discussed.

We conclude this Introduction by reminding the reader of the definition of a term we have already used several times.

Definition 1.1. A closure operation $\#$ on (the ideals of) a ring R is an inclusion preserving function from ideals to ideals such that if the value on I is denoted $I^\#$, then for all ideals $I \subseteq R$, $I \subseteq I^\# = (I^\#)^\#$.

We refer the reader to [Eps12] for a detailed treatment of closure operations and their properties.

2. PROPERTIES OF CONTINUOUS CLOSURE

Given a homomorphism $R \rightarrow S$ of finitely generated \mathbb{C} -algebras we get an induced map in the other direction of the corresponding algebraic sets, $X \leftarrow Y$, which is continuous in the Euclidean topologies (it is defined coordinatewise by restricted polynomial functions), and so there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{C}(X) & \longrightarrow & \mathcal{C}(Y) \\
 \uparrow & & \uparrow \\
 \mathbb{C}[X] & \longrightarrow & \mathbb{C}[Y] \\
 \uparrow & & \uparrow \\
 R & \longrightarrow & S
 \end{array}$$

where the top and middle horizontal arrows are induced by the map $Y \rightarrow X$.

Hence (cf. [Bre06]) we have the following.

Proposition 2.1 (Persistence of continuous closure). *If $h : R \rightarrow S$ is a homomorphism of finitely generated \mathbb{C} -algebras, I is an ideal of R , and $f \in I^{\text{cont}}$, then $h(f) \in (IS)^{\text{cont}}$.*

If I is an ideal of a ring R and \mathcal{F} is a subset of R , let $I :_R \mathcal{F} = \{r \in R \mid \text{for all } f \in \mathcal{F}, fr \in I\}$. If \mathcal{F} consists of a single element f , this coincides with $I :_R f = I :_R fR$. Note that $I :_R \mathcal{F} = \bigcap_{f \in \mathcal{F}} (I :_R f)$ and that if J is the ideal generated by \mathcal{F} , then $I :_R \mathcal{F} = I :_R J$.

Proposition 2.2. *Let I be an ideal of an affine \mathbb{C} -algebra R , and let $\mathcal{F} \subseteq R$. If I is a continuously closed ideal, so is $I :_R \mathcal{F}$ for every set $\mathcal{F} \subseteq R$, and so is the contraction of IR_W to R for every multiplicative system W .*

Proof. The second statement follows from the first, because the contraction of IR_W to R is the union of the ideals $I :_R w$ for $w \in W$, and since this set is directed, one can choose $w \in W$ so that the contraction is the same as $I :_R w$. Moreover, the statement for \mathcal{F} reduces to the case of a single element f , since an intersection of continuously closed ideals is evidently continuously closed.

Now suppose that $r \in R$ is a linear combination with continuous coefficients g_1, \dots, g_h of elements f_1, \dots, f_h of $I :_R f$. Then $fr = \sum_{i=1}^h g_i(ff_i)$ where every $ff_i \in I$, and so $fr \in I^{\text{cont}} = I$, and $r \in I :_R f$, as required. □

Corollary 2.3. *If R is an affine \mathbb{C} -algebra and I is a continuously closed ideal of R , then so is every primary component of I for a minimal prime P of I .*

Proof. The minimal primary component corresponding to P is the contraction of IR_W to R , with $W = R - P$. □

For any ring homomorphism $R \rightarrow S$, if $I, J \subseteq R$ the product of the contractions of IS and JS is obviously contained in the contraction of $(IJ)S = (IS)(JS)$. Applying this to the map $\mathbb{C}[X] \rightarrow \mathcal{C}(X)$, we have the following.

Proposition 2.4. *If R is an affine \mathbb{C} -algebra and I, J are ideals of R , then $I^{\text{cont}}J^{\text{cont}} \subseteq (IJ)^{\text{cont}}$* □

The following result is proved in the standard-graded case in [Bre06].

Theorem 2.5. *Let R be a finitely generated \mathbb{N} -graded \mathbb{C} -algebra with $R_0 = \mathbb{C}$ and let $F_1, \dots, F_h \in R$ be homogeneous elements of positive degrees d_1, \dots, d_h . Suppose F is homogeneous of degree d , where $d > d_i, 1 \leq i \leq h$. Let $I = (F_1, \dots, F_h)R$. Suppose that every element of positive degree has a power in I . Then $F \in I^{\text{cont}}$.*

Proof. We may assume that R is reduced. We map a graded polynomial ring

$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow R,$$

so that

$$R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{A},$$

where \mathfrak{A} is the kernel and the map preserves degree. Let each x_j have degree e_j . Define an action of \mathbb{C} on \mathbb{C}^n by this rule: if $z = (z_1, \dots, z_n)$, then let

$$tz := (t^{e_1}z_1, \dots, t^{e_n}z_n),$$

and let

$$\|z\| := \sqrt{\sum_{j=1}^n |z_j|^{2/e_j}}.$$

Then if H is homogeneous of degree δ in the polynomial ring, $H(tz) = t^\delta H(z)$. Moreover, $\|tz\| = |t|\|z\|$. The action then stabilizes $X = V(\mathfrak{A})$. Let $\mathbf{0}$ be the origin in \mathbb{C}^n . Then $\mathbf{0} \in X$, and the F_j vanish simultaneously only at $\mathbf{0}$. Hence, $\sum_i |F_i|^2$ vanishes only at $\mathbf{0}$, and we have

$$1 = \sum_j \frac{\overline{F_j}}{\sum_i |F_i|^2} F_j$$

on $X - \{0\}$. Multiplying by F yields $F = \sum_j g_j F_j$, where the g_j are continuous on $X - \{0\}$. Let $t = \|z\|$. For $z \neq 0$ in X , let $y = y(z) = t^{-1}z$. Then

$$F(z) = F(ty) = t^d F(y) = t^d \sum_j g_j(y) F_j(y) = \sum_j t^{d-d_j} g_j(y) F_j(ty).$$

Define

$$h_j(z) = \|z\|^{d-d_j} g_j(y).$$

Then h_j is continuous on $X - \{0\}$, and its limit as $z \rightarrow 0$ is 0 because $\|z\|^{d-d_j} \rightarrow 0$, while g_j is bounded on the set $\{y \in X : \|y\| = 1\}$, which is where $z/\|z\|$ varies, since this set is closed and bounded and thus compact in the Euclidean topology. Since F vanishes at the origin, we are done. \square

Discussion 2.6. We can extend the notion of continuous closure to the local ring $R_{\mathfrak{m}}$ of an affine \mathbb{C} -algebra R at a maximal ideal \mathfrak{m} as follows. Let I be an ideal of $R_{\mathfrak{m}}$. Let X be the affine algebraic set $\text{Max Spec}(R)$ (in the Euclidean topology), let $x \in X$ correspond to \mathfrak{m} , and let S denote the ring of germs of continuous \mathbb{C} -valued functions on X at x . Then define I^{cont} as the contraction of IS to $R_{\mathfrak{m}}$.

When I is an ideal of R , we write $I^{\text{cont},x}$ for $(IR_{\mathfrak{m}})^{\text{cont}}$.

Proposition 2.7. *Let R be an affine \mathbb{C} -algebra and let X be the corresponding algebraic set. Let I be an ideal of R , and let $f \in R$. Then $f \in I^{\text{cont}}$ if and only if for all $x \in X$, $f/1 \in (IR_{\mathfrak{m}})^{\text{cont}}$, where \mathfrak{m} is the maximal ideal of R corresponding to x .*

Proof. Let f_1, \dots, f_h generate I . It is clear that if $f = \sum_{i=1}^h g_i f_i$ with the g_i continuous on X , the equation persists as to when we take germs at $x \in X$. For the converse, suppose that $f \in R$ has image in $(IR_{\mathfrak{m}})^{\text{cont}}$ for all \mathfrak{m} . Then for every $x \in X$, x has a neighborhood U_x in the Euclidean topology on X such that

$$f|_{U_x} = \sum_{i=1}^h g_i^x f_i|_{U_x}$$

on U_x , where the g_i^x are continuous functions on U_x . By making the neighborhoods U_x smaller we may also assume that the g_i^x are bounded on U_x . The open cover $\{U_x \mid x \in X\}$ has a locally finite refinement by open sets V_λ such that there are continuous $[0, 1]$ -valued functions b_λ on X with the property that b_λ vanishes off V_λ and such that

$$1 = \sum_{\lambda} b_\lambda,$$

i.e., the b_λ give a partition of unity. Each V_λ is contained in some U_x , and so there are continuous \mathbb{C} -valued functions g_i^λ on each V_λ , bounded on V_λ (obtained by restricting suitable g_i^x), such that

$$f|_{V_\lambda} = \sum_{i=1}^h g_i^\lambda f_i|_{V_\lambda}.$$

Then

$$f = \sum_{i=1}^h \left(\sum_{\lambda} g_i^\lambda b_\lambda \right) f_i$$

and every $\sum_{\lambda} g_i^\lambda b_\lambda$ is a continuous function on X when defined to be 0 off V_λ . \square

Corollary 2.8. *Let R be a finitely generated \mathbb{C} -algebra, with X the associated affine variety. Let $I \subseteq R$ be an ideal, and let $f \in R$. Let $\{X_j\}_{j \in \Lambda}$ be an affine open cover of X , with $R_j = \Gamma(X_j, \mathcal{O}_X)$. Then $f \in I^{\text{cont}}$ if and only if $f \in (IR_j)^{\text{cont}}$ for all $j \in \Lambda$.*

Discussion 2.9 (Ideal closures and gradings). At this point, we are only aiming to prove Proposition 2.10 below, but we will eventually want to prove similar results for other closure operations where the issue is more difficult. Let R be a \mathbb{Z}^h -graded ring, where $h > 0$ is an integer. Note that this case includes \mathbb{N}^h -gradings and, of course, \mathbb{N} -gradings. If $\alpha = (\alpha_1, \dots, \alpha_h)$ is a k -tuple of units of R_0 , where the subscript is the zero element in \mathbb{Z}^h , there is a degree-preserving automorphism θ_α of R that multiplies forms of degree (k_1, \dots, k_h) by $\alpha_1^{k_1} \cdots \alpha_h^{k_h}$. Suppose that $\#$ is a closure operation on ideals of R such that the closure of an ideal that is stable under these automorphisms is again stable under these automorphisms. Suppose that R_0 contains an infinite field or, more generally, that for every integer $N > 0$ R_0 contains N units $\alpha_1, \dots, \alpha_N$ such that the elements $\alpha_i - \alpha_j$ for $i \neq j$ are also invertible. Then whenever I is homogeneous, its closure $I^\#$ is also homogeneous. By induction on h one can reduce to the case where $h = 1$. The result then follows from the invertibility of Vandermonde matrices (α_i^{j-1}) , where the α_i are distinct units whose nonzero differences are units. If $f \in I^\#$ is an element whose nonzero homogeneous components occur in degrees $d, \dots, d + N - 1$, it suffices to have N units whose distinct differences are also units in R_0 to conclude that the homogeneous components of f are in $I^\#$. See Discussion (4.1) in [HH94b]. If R_0 does not have sufficiently many units to carry through the argument, one can seek a family of \mathbb{Z}^h -graded R -algebras S_N containing R such that $R \subseteq S_N$ preserves degrees and such that for all homogeneous ideals $I \subseteq R$ and all $N > 0$, $I^\# \subseteq (IS_N)^\#$, while $(IS_N)^\# \cap R = I^\#$. Then, if $f \in I^\#$, we have that $f \in (IS_N)^\#$ for all N , and for N sufficiently large this will imply that all homogeneous components of f are in $(IS_N)^\#$ and, hence, in $(IS_N)^\# \cap R = I^\#$. In particular, this method will succeed if one can choose S_N to be the localization of $R[t_1, \dots, t_N]$ at the element g_N that is the product of all the t_i and all the $t_i - t_j$ for $i \neq j$, which may be thought of as $R_0[t_1, \dots, t_N]_{g_N} \otimes_{R_0} R$, and the grading is taken so that the degree zero part is $R_0[t_1, \dots, t_N]_{g_N}$. Note that these S_N are smooth and faithfully flat over R . This discussion applies also to the case when $I^\#$ is an ideal defined ring-theoretically in terms of I , even when $\#$ is not a closure operation, such as inner integral closure, which is treated in §5.

If R is a finitely generated \mathbb{Z}^h -graded \mathbb{C} -algebra and if $\alpha_1, \dots, \alpha_h$ are nonzero elements of \mathbb{C} , the automorphisms θ_α are \mathbb{C} -automorphisms. Hence, if I is homogeneous, I^{cont} is stable under these automorphisms by the persistence of continuous closure (Proposition 2.1). Therefore, simply because \mathbb{C} is an infinite field in R_0 , we have the following.

Proposition 2.10. *Let R be a finitely generated \mathbb{Z}^h -graded \mathbb{C} -algebra, and suppose that \mathbb{C} is contained in R_0 , where the subscript indicates the zero element in \mathbb{Z}^h . Let I be a homogeneous ideal of R with respect to this grading. Then I^{cont} is also a homogeneous ideal of R with respect to this grading. \square*

3. SEMINORMAL RINGS

In this section we review certain facts about seminormal rings and prove that a reduced affine \mathbb{C} -algebra is seminormal if and only if every ideal generated by a nonzerodivisor is axes closed (equivalently, continuously closed).

Recall [Swa80] that a ring R is *seminormal* if it is reduced, and whenever f is an element of the total quotient ring of R such that $f^2, f^3 \in R$, we have that $f \in R$.¹

Given a reduced Noetherian ring R with total ring of fractions \mathcal{T} , there is a unique smallest seminormal extension R^{sn} of R within \mathcal{T} , called the *seminormalization* of R .

For rings containing a field of characteristic 0 the property of being seminormal is equivalent to the property of being weakly normal. See Vitulli's recent survey article [Vit11] for a treatment of both notions. (We will revisit the concept of weak normality in §11.) We collect several facts about seminormality that we will need in the proposition below. A reference for each part is given with the statement, except for (6), which is immediate from the definition of seminormal, and (7), which follows at once from (5) and (6) because the (strict) Henselization is a directed union of localized étale extensions.

Proposition 3.1. *Suppose R, S are reduced Noetherian rings. Let R' be the integral closure of R in its total ring of fractions.*

- (1) R^{sn} is the set of all $b \in R'$ such that for any $\mathfrak{p} \in \text{Spec } R$, $b/1 \in R_{\mathfrak{p}} + \text{Jac}(R'_{\mathfrak{p}})$, where Jac denotes the Jacobson radical and $R'_{\mathfrak{p}}$ is the localization of R' at the multiplicative set $R \setminus \mathfrak{p}$. [Tra70]
- (2) If $g : R \rightarrow S$ is faithfully flat and S is seminormal, then R is seminormal. [GT80, Corollary 1.7]
- (3) If R is seminormal and W is a multiplicative set, then $W^{-1}R$ is seminormal. [GT80, Corollary 2.2]
- (4) Suppose the integral closure of R in its total quotient ring is module-finite over R . The following are equivalent: [GT80, Corollary 2.7]
 - (a) R is seminormal.
 - (b) $R_{\mathfrak{m}}$ is seminormal for all $\mathfrak{m} \in \text{Max Spec } R$.
 - (c) $R_{\mathfrak{p}}$ is seminormal for all $\mathfrak{p} \in \text{Spec } R$.
 - (d) $R_{\mathfrak{p}}$ is seminormal for all $\mathfrak{p} \in \text{Spec } R$ such that $\text{depth } R_{\mathfrak{p}} = 1$.
- (5) Suppose $g : R \rightarrow S$ is flat with all fibers geometrically reduced and generic fibers normal (e.g., a normal homomorphism). If R is seminormal, then so is S . [GT80, Proposition 5.1] In particular, if S is smooth over R , which includes the case where S is étale over R , and R is seminormal, then S is seminormal.
- (6) A directed union of seminormal rings is seminormal.
- (7) If R is local and seminormal, then the Henselization of R and the strict Henselization of R are seminormal.
- (8) Suppose R is excellent and local. R is seminormal $\iff \hat{R}$ is seminormal. [GT80, Corollary 5.3]
- (9) Let X be an indeterminate over R . R is seminormal $\iff R[[X]]$ is seminormal. [GT80, Proposition 5.5]

¹The concept was introduced by Traverso [Tra70] for a more restricted class of rings. Swan showed that under Traverso's assumptions, Swan's definition was equivalent to Traverso's.

- (10) Let R be a reduced affine \mathbb{C} -algebra. Let R' be the integral closure of R in its total ring of quotients. Let X and Y be the varieties associated to R , R' , respectively. If $\pi : Y \rightarrow X$ is the map induced from the inclusion $R \hookrightarrow R'$, then the seminormalization of R consists of all regular functions f on Y such that $f(y) = f(z)$ whenever $y, z \in Y$ are such that $\pi(y) = \pi(z)$. [LV81, special case of Theorem 2.2]
- (11) Let R be a reduced affine \mathbb{C} -algebra, and let S be the seminormalization of R . Then the map of affine algebraic sets corresponding to the inclusion $R \subseteq S$ is a homeomorphism in both the Zariski and Euclidean topologies. [AB69, Theorem 1]

We note that passing to the seminormalization of a ring does not affect continuous closure in the following sense.

Proposition 3.2. *Let R be a reduced affine \mathbb{C} -algebra, and let S be the seminormalization of R . Let I be an ideal of R . Then $(IS)^{\text{cont}} \cap R = I^{\text{cont}}$.*

Proof. By Proposition 3.1(11), the affine algebraic sets associated with S and R are homeomorphic. Call both of them X . The ideals I and IS have the same generators $f_1, \dots, f_n \in R$. The condition that f be a continuous linear combination of these elements is independent of whether we think of the problem over R or over S . \square

The following is a characterization of complete local one-dimensional seminormal Noetherian rings. It is based on Traverso's "glueing" construction.

Theorem 3.3. *Let k be a field, let L_1, \dots, L_n be finite algebraic extension fields of k , and let (V_i, \mathfrak{m}_i) be discrete valuation rings such that $V_i/\mathfrak{m}_i = L_i$. Let S be the subring of $\prod_{i=1}^n V_i$ consisting of all n -tuples (v_1, \dots, v_n) such that there exists $\alpha \in k$ such that $v_i \equiv \alpha \pmod{\mathfrak{m}_i}$ for all i . Then S is one-dimensional, local, and seminormal.*

Conversely, let (R, \mathfrak{m}, k) be a complete one-dimensional seminormal Noetherian local ring. Then there exist such extension fields L_i and such DVRs V_i (which, moreover, are complete) such that R is isomorphic to the ring S described above.

Proof. Consider any so-described S , and let $W := \prod_{i=1}^n V_i$. Note that $\mathfrak{m} = \prod_{i=1}^n \mathfrak{m}_i \subseteq S$ consists of all nonunits of S , and so is the unique maximal ideal of S . Let $u \in \mathfrak{m}$ be an element of this product that is nonzero in every coordinate. Then u is a nonzerodivisor in S , and $uW \subseteq \mathfrak{m} \subseteq S$. It follows that W is the normalization of S . It is then clear that S is one-dimensional. Since W is spanned over S by elements that map to a basis for $\prod_{i=1}^n L_i$ over k , W is module-finite over S , and S is Noetherian by Eakin's theorem; see [Eak68] or [Nag68]. Finally, we check seminormality. Let $0 \neq v = (v_1, \dots, v_n) \in W$ be such that $v^2, v^3 \in S$. Then there exist $\alpha, \beta \in k - \{0\}$ such that $v_i^2 \equiv \alpha \pmod{\mathfrak{m}_i}$ and $v_i^3 \equiv \beta \pmod{\mathfrak{m}_i}$ for all i . Consider the element $\gamma := \beta/\alpha \in k$. It follows easily that $v_i \equiv \gamma \pmod{\mathfrak{m}_i}$ for all i , whence $v \in S$. Thus, S is seminormal.

Now let (R, \mathfrak{m}, k) be a complete one-dimensional seminormal Noetherian local ring. Let R' be the normalization of R . Note that $R' = \prod_{i=1}^n V_i$, where $(V_i, \mathfrak{m}_i, L_i)$ are discrete valuation rings, complete since R is complete. In particular, if $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal primes of R , then $V_i = (R/\mathfrak{p}_i)'$. Moreover, since R' is module-finite over R , it follows that each L_i is module-finite (i.e., finite algebraic) over k . Let S be as described in the statement of the theorem for these particular k , L_i , and V_i . Clearly R embeds as a subring of S , since for any $r \in R$, the map $R \rightarrow R'$ sends

$r \mapsto (\bar{r}_1, \dots, \bar{r}_n)$ (where \bar{r}_i is the residue class of $r \pmod{\mathfrak{p}_i}$), and the residue class of each \bar{r}_i modulo \mathfrak{m}_i is clearly the same as the residue class of $r \pmod{\mathfrak{m}}$, which is, of course, in k . So all we need to show is that the induced injective map from R to S is surjective.

Let $v = (v_1, \dots, v_n) \in S$. Then there is some $\alpha \in k$ such that $v_i \equiv \alpha \pmod{\mathfrak{m}_i}$ for all i . Take any $r \in R$ such that $r \equiv \alpha \pmod{\mathfrak{m}}$. Let $w := v - r = (v_1 - \bar{r}_1, \dots, v_n - \bar{r}_n)$. By construction, $w \in \prod_{i=1}^n \mathfrak{m}_i = \text{Jac}(R')$. But by part (1) of Proposition 3.1, we have $\mathfrak{m} = \text{Jac}(R')$, since R is seminormal. Hence, $w \in \mathfrak{m}$, whence $v = r + w \in R + \mathfrak{m} = R$, as was to be shown. \square

Note. In Traverso’s terminology, S is the *glueing of R' over \mathfrak{m}* . That is, S is the pullback of the following diagram of ring homomorphisms:

$$\begin{array}{ccc} & \prod_{i=1}^n V_i & \\ & \downarrow & \\ k & \longrightarrow & \prod_{i=1}^n L_i \end{array}$$

Note. The above theorem may also be deduced from the machinery developed in [Yos82], although it does not appear explicitly.

Let L be an algebraically closed field. The notion of a ring of axes over L is defined in the Introduction. We shall also use the term *affine ring of axes* over L to emphasize the distinction from other notions described below. By a *complete* (or *formal*) *ring of axes* over L , we mean a ring of the form

$$L[[x_1, \dots, x_n]] / (x_i x_j : 1 \leq i < j \leq n),$$

where the x_i are formal power series indeterminates. Such rings are known to be seminormal.² When the x_i are indeterminates over an algebraically closed field L , we shall refer to

$$L[x_1, \dots, x_n] / (x_i x_j : 1 \leq i < j \leq n)$$

as a *polynomial axes ring*. Both complete axes rings and polynomial axes rings are seminormal. In fact, we have, we have parts (a) and (b) of the following proposition from [Bom73], while parts (c) and (d) follow easily from parts (a) and (b). (See also [Gib89] and [GW77] for connections with the notion of F -purity.)

Proposition 3.4. *Let L be an algebraically closed field.*

- (a) *A complete axes ring over L is seminormal.*
- (b) *Every complete local one-dimensional seminormal ring of equal characteristic with algebraically closed residue class field L is isomorphic with a complete ring of axes over L .*
- (c) *Every affine ring of axes over L is seminormal.*
- (d) *A one-dimensional affine L -algebra R is seminormal if and only if there are finitely many étale L -algebra maps $\theta_i : R \rightarrow A_i$, where the A_i are affine rings of axes over L , and every maximal ideal of R lies under a maximal ideal of some A_i .*

²Indeed, one can see this as a special case of Theorem 3.3.

Proof. As already mentioned, parts (a) and (b) are proved in [Bom73]. Part (c) is then immediate from parts (4)(b) and (8) of Proposition 3.1, the definition of affine ring of axes over L , and part (a) above. For part (d), we first prove “if”. Note that if $\tilde{\mathfrak{m}}$ in $A = A_i$ lies over \mathfrak{m} in R , then $R_{\mathfrak{m}} \rightarrow A_{\tilde{\mathfrak{m}}}$ is faithfully flat. Since A is seminormal by part (c), so is $A_{\tilde{\mathfrak{m}}}$, and the result follows from Proposition 3.1 part (2). To prove “only if” it suffices to construct a cover by sets $\text{Spec}(A_i)$ that may be infinite, since we may use the quasicompactness of $\text{Spec}(R)$ to pass to a finite subcover. Therefore, it suffices to construct an étale extension A that is a ring of axes for each maximal ideal \mathfrak{m} of R so that $\mathfrak{m}A \neq A$. If $R_{\mathfrak{m}}$ is regular we may simply take A to be R_f for a suitable element f . For the finitely many choices of \mathfrak{m} such that $R_{\mathfrak{m}}$ is singular, we know the completion of $R_{\mathfrak{m}}$ is a formal ring of axes. This implies that there is an étale extension A of R that is an affine ring of axes with $\mathfrak{m}A \neq A$. To see why this is true, let $S = R_{\mathfrak{m}}$. Let B denote the normalization of S , which is semilocal and regular. The normalization C' of the completion C of $R_{\mathfrak{m}}$ is the product of the rings $L[[x_i]]$. Because S is an excellent domain, $C' \cong C \otimes_S B$. Let T denote the Henselization of S . Then $B \otimes_S T$ is regular and module-finite over the Henselian local ring T , and so is a finite product of discrete valuation domains. The completion of these gives the various $L[[x_i]]$. T is a direct limit of étale extensions of R in which there is a unique maximal ideal $\tilde{\mathfrak{m}}$ lying over \mathfrak{m} . For a sufficiently large such extension A , $B \otimes_R A$ will contain all of the idempotents of $B \otimes_S T$, and will be regular. One may localize A at one element not in $\tilde{\mathfrak{m}}$ so that it is a ring of axes with a unique singularity at $\tilde{\mathfrak{m}}$. \square

Discussion 3.5. Let R be a finitely generated \mathbb{C} -algebra and let \mathfrak{m} be a maximal ideal of R . Map a polynomial ring $T = \mathbb{C}[x_1, \dots, x_n]$ onto R so that the x_i map to generators of \mathfrak{m} . Then $\mathbb{C}[x_1, \dots, x_n] \subseteq \mathbb{C}\{\{x_1, \dots, x_n\}\}$, the ring of convergent power series in x_1, \dots, x_n , and $S = \mathbb{C}\{\{x_1, \dots, x_n\}\} \otimes_T R_{\mathfrak{m}}$ is the *analytic completion* of $R_{\mathfrak{m}}$. This ring is a local, excellent, Henselian, faithfully flat extension of $R_{\mathfrak{m}}$, and we have $R_{\mathfrak{m}} \subseteq S \subseteq \widehat{R}_{\mathfrak{m}}$ with the second inclusion faithfully flat as well. We shall refer to $\mathbb{C}\{\{x_1, \dots, x_n\}\}/(x_i x_j : i \neq j)$ as an *analytic axes ring*.

Proposition 3.6. *Let R be a finitely generated \mathbb{C} -algebra of dimension one. Then R is seminormal if and only if for every maximal ideal \mathfrak{m} of R such that $R_{\mathfrak{m}}$ is not regular, the analytic completion of $R_{\mathfrak{m}}$ is an analytic axes ring.*

Proof. We know that R is seminormal if and only if each $\widehat{R}_{\mathfrak{m}}$ is, and this is automatic if $R_{\mathfrak{m}}$ is regular (which includes any isolated points). It is therefore sufficient to show that the analytic completion (A, \mathfrak{m}_A) of $R_{\mathfrak{m}}$ is an analytic ring of axes if $\widehat{R}_{\mathfrak{m}} = \widehat{A}$ is a formal ring of axes. Since A is one-dimensional, excellent, and Henselian, its minimal primes P_1, \dots, P_n correspond bijectively to those of \widehat{A} via extension and contraction. Let $Q_i = \bigcap_{j \neq i} P_j$. Then $Q_i \widehat{A}$ is a principal ideal generated by an element not in $(\mathfrak{m}_A \widehat{A})^2$, $Q_i Q_j = 0$ for $i \neq j$, and $\sum_i Q_i \widehat{A} = \mathfrak{m}_A \widehat{A}$, from which it follows that Q_i is a principal ideal generated by an element not in \mathfrak{m}_A^2 , that $Q_i Q_j = 0$ for $i \neq j$, and that $\sum_i Q_i = \mathfrak{m}_A$. Let x_i generate Q_i . Then the x_i are a minimal set of generators of \mathfrak{m}_A , $x_i x_j = 0$ for $i \neq j$, and since $P_i \widehat{A} = \sum_{j \neq i} Q_j \widehat{A}$, we have that P_i is generated by $\{x_j : j \neq i\}$ for each i . The map $\mathbb{C}[X_1, \dots, X_n] \rightarrow A$ (sending $X_i \mapsto x_i$ for all i) induces a \mathbb{C} -homomorphism θ of $B = \mathbb{C}\{\{X_1, \dots, X_n\}\}/(X_i X_j : i \neq j)$ to A such that the map of completions is an isomorphism. Thus, this map is injective. If \mathcal{P}_i is generated by the X_j for $j \neq i$,

then $B/P_i \rightarrow A/P_i$ is a map from $\mathbb{C}\{\{X_i\}\} \rightarrow A/P_i \cong \mathbb{C}\{\{x_i\}\} \subseteq A$ which induces an isomorphism of completions, and so must be an isomorphism. Since A is the sum of the subrings $\mathbb{C}\{\{x_i\}\}$, θ is surjective. \square

Both in this section and the next we shall need to use Artin approximation to descend a map from an affine K -algebra to a complete ring of axes over an algebraically closed extension field L of K to a map to an étale extension of a polynomial ring of axes, which will be seminormal. Specifically, we have the following.

Theorem 3.7 (Descent via Artin approximation). *Let K be an algebraically closed field, let L be an algebraically closed extension field, let R be an affine K -algebra, let I be an ideal of R , and let $f \in R$. Let $R \rightarrow S$ be a K -algebra homomorphism to a complete ring of axes S over L such that the image of f is not in IS . Then there is a K -algebra homomorphism $R \rightarrow S_0$, where S_0 is an étale extension of a polynomial ring of axes over K such that the image of f is not in IS_0 . Moreover, if the image of f is not a zerodivisor in S , the map $R \rightarrow S_0$ may be chosen to satisfy the additional condition that the image of f in S_0 is not a zerodivisor.*

Proof. S is a complete ring of axes with algebraically closed residue class field L , and such a ring is the completion of a polynomial axes ring $B = L[x_1, \dots, x_n]/(x_i x_j : i < j)$ localized at the maximal ideal \mathfrak{m} generated by the x_j . Call the localized ring B_0 . Moreover, we can choose N so large that the image of f is not in $IB + \mathfrak{m}^N B$. Think of R as $K[y_1, \dots, y_s]/(g_1, \dots, g_h)$. Then the images z_j of the y_1, \dots, y_s in T give solutions of the equations $g_j = 0$ in T , and we may use Artin approximation, i.e., the main result of [Art69], to find a solution z'_1, \dots, z'_s of these equations in the Henselization B_0^h of B_0 congruent to the z_j modulo $\mathfrak{m}^N B$. We can map $R \rightarrow B_0^h$ as a K -algebra so that the images of the y_j map to the z'_j and we still have that $f \notin (I + \mathfrak{m}^N)B_0^h$. In particular, $f \notin IB_0^h$. Since B_0^h is a direct limit of finitely generated étale extensions of B , we have a finitely generated étale extension C of B and a K -algebra map $R \rightarrow C$ such that $f \notin IC$.

Hence, $C = B[w_1, \dots, w_m]/(g_1, \dots, g_m)$ is such that the image of the Jacobian determinant $\det(\partial g_i / \partial w_j)$ is a unit of C .

Now let A denote a varying but finitely generated K -subalgebra of L sufficiently large to contain all the coefficients of the g_j . Let

$$B_A = A[x_1, \dots, x_n]/(x_i x_j : i < j)$$

and let

$$C_A = B_A[w_1, \dots, w_m]/(g_1, \dots, g_m).$$

Then C is the direct limit of the rings C_A , and so the Jacobian determinant is invertible in C_A for all sufficiently large A . We may therefore choose A so large that C_A is étale over B_A and the map $R \rightarrow C$ factors as $R \rightarrow C_A \rightarrow C$. If D is any A -algebra we write B_D and C_D for $D \otimes_A B_A$ and $D \otimes_A C_A$, respectively. Let \mathcal{K} be the fraction field of A . Then $C = C_L \cong L \otimes_{\mathcal{K}} C_{\mathcal{K}}$ is faithfully flat over $C_{\mathcal{K}}$. Hence, $f \notin IC_{\mathcal{K}}$.

We have the exact sequences

$$\begin{aligned} 0 &\rightarrow IC_A \rightarrow C_A \rightarrow C_A/IC_A \rightarrow 0, \\ 0 &\rightarrow fC_A + IC_A \rightarrow C_A \rightarrow C_A/(fC_A + IC_A) \rightarrow 0, \text{ and} \\ 0 &\rightarrow IC_A \rightarrow (fC_A + IC_A) \rightarrow W_A \rightarrow 0, \end{aligned}$$

where W_A is a cyclic C_A -module spanned by the image of f . By the lemma of generic freeness (cf. [Mat86, Theorem 24.1], [HR74, Lemma 8.1]), we can localize at one nonzero element $a \in A$ so that all of the modules in these sequences become A_a -free. We change notation and continue to write objects with the subscript A : A has been replaced by A_a . The A -free module W_A is not 0, since this is true even after we apply $\mathcal{K} \otimes_A _$. Let μ be any maximal ideal of A . Then $A/\mu = K$, and we use the subscript K to indicate these various algebras and modules after tensoring with $K = A/\mu$ over A . We have a map $R \rightarrow C_K$. Because $W_K \neq 0$, we have that the image of f is not in $IC_K \subseteq C_K$. But C_K is a finitely generated étale extension of B_K , which is a polynomial ring of axes over K .

We now consider the modifications needed to preserve the condition that the image of f be a nonzerodivisor. Hence, the image of f in $L[[x_1, \dots, x_n]]/(x_i x_j)$ is not in any of the ideals P_i generated by all of the x_j for $j \neq i$. Then $f \notin P_i + \mathfrak{m}^N$ for sufficiently large N . Thus, we may choose N so large that when we apply Artin approximation, the image of f in the Henselization B_0^h is not in any of the ideals $P_i \cap B_0^h$. When we replace the Henselization by a finitely generated étale extension C of B , multiplication by the image of f may have a kernel, but the kernel will be killed by localization at one element of C not in the contraction of the maximal ideal of B_0^h . Thus, we may assume that the image of f is not a zerodivisor in C . For the last step in the descent, when we pass from C_A to $C_{A/\mu}$, we need to preserve the exactness of the sequence $0 \rightarrow C_A \xrightarrow{\phi} C_A \rightarrow C_A/\phi C_A \rightarrow 0$, where ϕ is the image of f , when we apply $(A/\mu) \otimes_A _$. We can do this by localizing at one element of $A - \{0\}$ so that all of the terms of the sequence become A -free. \square

4. AXES CLOSURE AND ONE-DIMENSIONAL SEMINORMAL RINGS

We want to extend the notion of axes closure to a larger class of rings. We first note the following theorem.

Theorem 4.1. *Let R be a Noetherian ring, let I be an ideal of R , and let $f \in R$. Then statements (1), (2), and (3) below are equivalent.*

Moreover, if R is a finitely generated algebra over a field K of characteristic 0 and L is the algebraic closure of K , then the first six of the statements below are equivalent. If we also assume that $K = L = \mathbb{C}$, then all seven of the statements below are equivalent.

- (1) *For every map from R to an excellent one-dimensional seminormal ring S , the image of f is in IS .*
- (2) *For every map from R to an excellent local one-dimensional seminormal ring S , the image of f is in IS .*
- (3) *For every map from R to a complete local one-dimensional seminormal ring S , the image of f is in IS .*
- (4) *For every map from R to a complete local one-dimensional seminormal ring S with algebraically closed residue field, $f \in IS$.*
- (5) *For every K -algebra map from R to a complete axes ring S with residue field L , $f \in IS$.*
- (6) *For every K -algebra map from R to a finitely generated étale extension S of a polynomial axes ring over L , $f \in IS$.*
- (7) *For every \mathbb{C} -algebra map θ from R to an analytic ring of axes (S, \mathfrak{n}) over \mathbb{C} such that $\theta^{-1}(\mathfrak{n})$ is maximal in R , $f \in IS$.*

Proof. (1) \implies (2) \implies (3) \implies (4) \implies (5) is obvious. If R is only assumed Noetherian we can prove (3) \implies (1) as follows. Suppose that there is a map $R \rightarrow S$, where S is one-dimensional and seminormal, such that $f \notin IS$. Then this can be preserved when we localize at some maximal ideal of S and then complete. The completion of $S_{\mathfrak{m}}$ is still seminormal. In the rest of the proof we assume the additional hypothesis on R .

To see that (5) \implies (6): If (6) fails, we also have $f \notin IS_{\mathfrak{m}}$ for a local ring $S_{\mathfrak{m}}$ of S and also for the completion T of $S_{\mathfrak{m}}$. S is seminormal, and hence so is T , which will have algebraically closed residue class field L . T is a complete axes ring; see Proposition 3.4.

We next show that (6) \implies (1). Assume (6) and suppose that (1) fails for S . Then it also fails for some localization of S at a maximal ideal and for the completion of that ring. Hence, we may assume that S is complete local. By part (7) of Proposition 3.1, we may replace S by its strict Henselization, which will have algebraically closed residue class field since we are in equal characteristic 0, and we may then complete again. We may therefore assume that S is a complete ring of axes with algebraically closed residue class field \mathcal{L} . Extend K to a coefficient field for S , which we also denote \mathcal{L} . Then we have $K \subseteq L \subseteq \mathcal{L} \subseteq S$. We may now replace R by $L \otimes_K R$: we still have a map from this ring to S , using the embedding $L \hookrightarrow \mathcal{L}$. The result now follows from Theorem 3.7.

Finally, it is clear that (1) \implies (7), and it will suffice to prove that (7) \implies (6). Suppose that we have a map $R \rightarrow A$, where A is étale over a polynomial ring of axes such that $f \notin IA$. Then A is a one-dimensional seminormal ring, and we can preserve that $f \notin IA$ by localizing at some maximal ideal μ of A . The inverse image will be a maximal ideal of R , since R and A are affine \mathbb{C} -algebras. Let (S, \mathfrak{n}) be the analytic completion of A_{μ} . Note that $\widehat{S} \cong \widehat{A_{\mu}}$. Then $A, A_{\mu}, \widehat{A_{\mu}}$, and S are seminormal by parts (5), (3), (2), and (8) of Proposition 3.1, and S is an analytic axes ring by Proposition 3.6. Since S is faithfully flat over A_{μ} and $f \notin IA_{\mu}$, it follows that $f \notin IS$, which completes the proof. \square

Corollary 4.2. *If R is an affine \mathbb{C} -algebra, the equivalent conditions (1) through (6) of Theorem 4.1 hold if and only if $f \in I^{\text{ax}}$.*

Proof. If (1) through (6) hold, it is clear that $f \in I^{\text{ax}}$, since rings of axes are one-dimensional excellent seminormal rings. Suppose that $f \in I^{\text{ax}}$. It suffices to verify condition (6). Since $L = K = \mathbb{C}$, by Proposition 3.1(5) it suffices to show that for every map to a one-dimensional affine seminormal ring S over R , the image of f is in IS . If not, we can choose a maximal ideal \mathfrak{m} of S such that $f \notin IS_{\mathfrak{m}}$. By part (d) of Proposition 3.4 there is an étale map $S \rightarrow A$ such that A is an affine ring of axes over \mathbb{C} and has a maximal ideal $\tilde{\mathfrak{m}}$ lying over \mathfrak{m} . Since the image of f is not in $IS_{\mathfrak{m}}$ and $S_{\mathfrak{m}} \rightarrow A_{\tilde{\mathfrak{m}}}$ is faithfully flat, we have that the image of f is not in $IA_{\tilde{\mathfrak{m}}}$ and, hence, not in IA . \square

Definition 4.3. Theorem 4.1 shows that conditions (1), (2), and (3) are equivalent for any Noetherian ring R . We therefore define $f \in R$ to be in the *axes closure* I^{ax} of I in the general case if these three equivalent conditions hold. Once we have made this definition, we have at once the following corollary.

Corollary 4.4. *Let R be a finitely generated K -algebra, where K is a field of characteristic 0, $I \subseteq R$ is an ideal, and $f \in R$. Then $f \in I^{\text{ax}}$ if and only if the equivalent conditions (1) through (6) of Theorem 4.1 hold.*

The following remark is obvious from the definition, but it is, nonetheless, quite important.

Proposition 4.5. *Let R be an excellent one-dimensional seminormal ring. Then every ideal of R is axes closed.* □

Proposition 4.6. *Let R be a Noetherian ring.*

- (1) *The axes closure of an ideal is contained in its integral closure.*
- (2) *If R is a normal domain, every principal ideal is axes closed.*

Proof. One may test integral closure by mapping to Noetherian valuation domains, and these may be replaced by their completions, which are excellent. The second statement follows, since principal ideals are integrally closed in a normal domain. □

Lemma 4.7. *If $I, J \subseteq R$, then $I^{\text{ax}}J^{\text{ax}} \subseteq (IJ)^{\text{ax}}$. In particular, if $r \in R$, $r(I^{\text{ax}}) \subseteq (rI)^{\text{ax}}$.*

Proof. Let $\theta : R \rightarrow B$ denote any homomorphism to an excellent one-dimensional seminormal ring. The first statement follows since $I^{\text{ax}}J^{\text{ax}}B = (I^{\text{ax}}B)(J^{\text{ax}}B) = (IB)(JB) = (IJ)B$, and the second statement follows from the case where $J = rR$. □

In §7 we prove that an excellent ring is seminormal if and only if every principal ideal generated by a non-zero-divisor is axes closed; see Theorem 7.17.

Example 4.8. We give an example of a reduced, finitely generated \mathbb{C} -algebra of pure dimension two, with precisely two minimal primes, which is seminormal, although one of its quotients by a minimal prime is not seminormal. Its normalization is the product of two polynomial rings in two variables. It has a principal ideal generated by a zero-divisor that is not axes closed. This example is similar to [GT80, Example 2.11].

Let $S = \mathbb{C}[u, v] \times \mathbb{C}[x, y]$. Let R be the subring of S generated over \mathbb{C} by $q = (u^2, x)$, $r = (u^3, y)$, $s = (v, 0)$, and $t = (uv, 0)$. Then $w = q^3 - r^2 = (0, x^3 - y^2) \in R$, and $z = s + w = (v, x^3 - y^2)$ is a nonzerodivisor in S and, hence, in R . Thus, $e = (1, 0) \in S$ is in the total quotient ring of R , since it is integral over R and $ze = (v, 0) \in R$, $(u, 0)$ is integral over R since its square is qe , which is integral over R , and it is in the total quotient ring of R , since $z(u, 0) \in R$. It follows that the integral closure of R is S .

Then R consists of all pairs of the form $(P(u^2, u^3) + vH(u, v), P(x, y))$, where $H(u, v)$ is an arbitrary polynomial in u, v and P is an arbitrary polynomial in x, y . Alternatively, R consists of all pairs $(Q(u, v), P(x, y))$ such that $Q(u, v) \equiv P(u^2, u^3) \pmod{v\mathbb{C}[u, v]}$. From the latter description we see that R is seminormal, for if $Q(u, v)^2 \equiv P(u^2, u^3)^2 \pmod{v\mathbb{C}[u, v]}$ and $Q(u, v)^3 \equiv P(u^2, u^3)^3 \pmod{v\mathbb{C}[u, v]}$, then $Q(u, v) \equiv P(u^2, u^3) \pmod{v\mathbb{C}[u, v]}$.

The two minimal primes of S contract to incomparable primes $P = (\mathbb{C}[u, v] \times 0) \cap R$ and $Q = (0 \times \mathbb{C}[x, y]) \cap R$. Clearly, $P \cap Q = 0$. Note that $(v, 0), (uv, 0) \in P - Q$ and $(0, x^3 - y^2) \in Q - P$. Hence, P and Q constitute all the minimal primes of R . We have that $R/Q \cong \mathbb{C}[u^2, u^3, uv, v]$, which is not seminormal, while $R/P \cong \mathbb{C}[x, y]$.

The maximal spectrum of R , in the Euclidean topology, is the union of two complex planes, $\text{Max Spec}(\mathbb{C}[u^2, u^3, uv, v])$, which may be identified topologically with $\text{Max Spec}(\mathbb{C}[u, v]) = \mathbb{C}^2$, and $\text{Max Spec}(\mathbb{C}[x, y])$. These meet along a topological

line which may be identified with $\mathcal{V}(v)$ in the first plane and with $\mathcal{V}(x^3 - y^2)$ in the second plane.

Then u represents a continuous function on \mathbb{C}^2 that may be restricted to the intersection and so viewed as a function on the closed set $\mathcal{V}(x^3 - y^2)$ in the second plane. Hence, by the Tietze extension theorem there is a continuous function θ on the second plane that extends the restriction of u . The pair (u, θ) represents a continuous \mathbb{C} -valued function on $\text{Spec}(R)$, and we have that $(uv, 0) = (u, \theta)(v, 0)$. It follows that $(uv, 0)$ is in the continuous closure of $(v, 0)R$ in R and, hence, in the axes closure. But it is not in the ideal.

In §7 we prove that in a reduced affine \mathbb{C} -algebra, axes closure, and continuous closure agree for principal ideals generated by a nonzerodivisor. We do not know whether they agree for principal ideals generated by a zerodivisor.

The following result, together with Proposition 2.7 and Corollary 2.8, gives one sense in which the issue of whether axes closure and continuous closure agree in affine \mathbb{C} -algebras is local.

Proposition 4.9. *Let R be a Noetherian ring, let I be an ideal of R , and let $f \in R$.*

- (a) *If $\varphi : R \rightarrow S$ is any homomorphism and $f \in I^{\text{ax}}$, then $\varphi(f) \in (IS)^{\text{ax}}$. (In other words, axes closure is persistent.) Hence, the contraction of an axes closed ideal is axes closed.*
- (b) *$f \in I^{\text{ax}}$ if and only if for each $P \in \text{Spec } R$, one has $f \in (IR_P)^{\text{ax}}$.*
- (c) *$f \in I^{\text{ax}}$ if and only if this holds in an affine open neighborhood of each prime ideal of R . In particular, I is axes closed if and only if this is true for an affine open neighborhood of each prime of R .*

Proof. The first statement in (a) is immediate from the definition, and the second statement follows at once from the first statement.

For part (b): Part (a) guarantees that if $f \in I^{\text{ax}}$, this remains true when we localize. It will suffice to show that if $f \in (IR_P)^{\text{ax}}$ for all P , then $f \in I^{\text{ax}}$. If not, we can map to a one-dimensional local seminormal ring (S, Q) such that $f \notin IS$. But then f is not in $(IR_P)^{\text{ax}}$, where P is the contraction of Q .

Part (c) follows at once. □

Proposition 4.10. *Let R be a Noetherian ring, and let I be an ideal. Let $f \in R$, let J be an ideal of R , and let W be a nonempty multiplicative system in R . If I is axes closed, then so are $(I :_R f)$, $(I :_R J)$, $\bigcup_{n=1}^{\infty} (I :_R J^n)$, and the contraction $\mathfrak{A} = \{r \in R : \text{for some } w \in W, wr \in I\}$ of $IW^{-1}R$ to R .*

Proof. Suppose $g \notin (I :_R f)$. Then $fg \notin I$, and we can choose a homomorphism $h : R \rightarrow A$, where A is a one-dimensional excellent seminormal ring such that $h(fg) \notin IA$. But then $h(g) \notin (IA :_A h(f)) \supseteq (I :_R f)A$, which shows that $g \notin (I :_R f)^{\text{ax}}$. This establishes the first statement.

Since $I :_R J = \bigcap_{j \in J} (I :_R j)$, the second statement follows. The ideals $(I :_R J^N)$ form an ascending chain, and so the union is equal to one of them. This proves the third statement. Finally, $\mathfrak{A} = \bigcup_{w \in W} (I :_R w)$. The union is directed, since $(I :_R v) \cup (I :_R w) \subseteq (I :_R vw)$. The family of ideals $\{(I :_R v) \mid v \in W\}$ therefore has a maximal element, which must be maximum, and the union consequently has the form $(I :_R w)$. □

Proposition 4.11. *In any Noetherian ring R , every axes closed ideal is an intersection of primary axes closed ideals. If R is an affine \mathbb{C} -algebra, these may be taken to be primary to maximal ideals.*

Proof. Let $f \notin I$, where I is axes closed. Then we can choose a map $h : R \rightarrow A$, where A is one-dimensional, excellent, and seminormal, such that $h(f) \notin IA$. In any Noetherian ring, every ideal is an intersection of ideals J that are primary to maximal ideals. We can choose such a primary ideal J of A so that $h(f) \notin J$, and then $h^{-1}(J)$ will be primary and axes closed with $f \notin h^{-1}(J)$. If R is an affine \mathbb{C} -algebra, we may take A to be an affine ring of axes over \mathbb{C} . In this case, the inverse image of the radical of J , which is a maximal ideal of A , is a maximal ideal m of R , and the inverse image of J is primary to m . \square

Proposition 4.12. *Let \mathfrak{m} be a maximal ideal of a Noetherian ring R , and let I be primary to \mathfrak{m} and axes closed. Then the extensions of I to $S = R_{\mathfrak{m}}$ and to $S = \widehat{R}_{\mathfrak{m}}$ are axes closed. This is also true if S is the Henselization of $R_{\mathfrak{m}}$ or, when R is a finitely generated algebra over \mathbb{C} , the analytic completion of $R_{\mathfrak{m}}$.*

Proof. Consider the case where $S = \widehat{R}_{\mathfrak{m}}$. Let $f \in S$ be an element of $S - IS$. Then, since $R/I \rightarrow S/IS$ is an isomorphism, we may choose $g \in R$ such that $g \equiv f \pmod{IS}$. By Definition 4.3 and Theorem 4.1, we can choose a map from $R \rightarrow A$, where A is a complete local seminormal ring of dimension 1 such that $g \notin IA$. Since I maps into the maximal ideal \mathfrak{m}_A of A (or else IA would be the unit ideal), we have that \mathfrak{m} maps into \mathfrak{m}_A . But then the map extends continuously (in the \mathfrak{m} -adic and \mathfrak{m}_A -adic topologies) to a map $S = \widehat{R}_{\mathfrak{m}} \rightarrow A$. Since the image of g is not in IA and $f \equiv g \pmod{IS}$, we have that the image of $f \notin IA$. Thus, IS is axes closed. In the other cases, if f were in the axes closure of IS , it would be in the axes closure of $I\widehat{R}_{\mathfrak{m}}$ as well, and we know this is $I\widehat{R}_{\mathfrak{m}}$. But in every instance $S \rightarrow \widehat{R}_{\mathfrak{m}}$ is faithfully flat, so that the contraction of $I\widehat{R}_{\mathfrak{m}}$ to S is IS . \square

The following result is a weak result on the compatibility of axes closure with smooth base change. It suffices to prove that axes closures of homogeneous ideals are homogeneous. See also Theorem 7.11, which is a much more difficult result on compatibility of axes closure with smooth base change.

Proposition 4.13. *Let R be Noetherian and let S be faithfully flat, essentially of finite type, and smooth over R . Then for every ideal $I \subseteq R$, $(IS)^{\text{ax}}$ contracts to I^{ax} in R .*

Proof. Suppose that f is in the contraction of $(IS)^{\text{ax}}$ but not in I^{ax} . Then we can choose a homomorphism $R \rightarrow A$, where (A, \mathfrak{m}_A) is a complete local one-dimensional seminormal ring such that the image of f is not in IA . Then $S_A = S \otimes_R A$ is faithfully flat, essentially of finite type, and smooth over A . By part (5) of Proposition 3.1, S_A is seminormal, and it is excellent. If we localize at a minimal prime Q of $\mathfrak{m}_A S_A$ in S_A , we obtain a one-dimensional, seminormal, local, faithfully flat, excellent extension B of A . Since the image of f is not in IA , we have that the image of f is not in IB . But since $R \rightarrow B$ factors as $R \rightarrow S \rightarrow S_A \rightarrow B$, this contradicts the assumption that f is in $(IS)^{\text{ax}}$. \square

Proposition 4.14. *Let R be Noetherian, and suppose that R is \mathbb{Z}^h -graded, where $h \geq 1$. Let I be a homogeneous ideal of R with respect to this grading. Then I^{ax} is also homogeneous with respect to this grading.*

Proof. We want to apply Discussion 2.9. The only difficulty is that R may not have sufficiently many units. But each of the rings S_N constructed in Discussion 2.9 is finitely presented, faithfully flat, and smooth over R , and so the result follows from that discussion and Proposition 4.13 just above. \square

5. SPECIAL AND INNER INTEGRAL CLOSURE, AND NATURAL CLOSURE

Let R be a Noetherian ring, $I \subseteq R$, and $r \in R$. The following conditions are well known to be equivalent:

- (1) There is a monic polynomial $f(x) \in R[x]$ of some degree d such that the coefficient of x^{d-j} is in I^j , $1 \leq j \leq d$, and such that $f(r) = 0$.
- (2) For every map $R \rightarrow V$, where V is a DVR, $rV \subseteq IV$. In other words, r has order at least as large as IV under the valuation associated with V .

The elements satisfying these equivalent conditions form an ideal I^- called the *integral closure* of I .

The *special part of the integral closure* of I [Eps10] is defined when (R, \mathfrak{m}) is local and $I \subseteq \mathfrak{m}$. It consists of all r which satisfy a monic polynomial as in (1) such that the coefficient of x^{d-j} is in $\mathfrak{m}I^j$ for $1 \leq j \leq d$. The special part of the integral closure is an ideal containing $\mathfrak{m}I$ and contained in I^- , but it typically does not contain I .

More generally, for any ideal J of a Noetherian ring R we can define the J -special integral closure $I^{-J^{\text{sp}}}$ of I to consist of all elements r in R that satisfy a J -special polynomial over I . This means that the polynomial is monic of degree $d \geq 1$ and the coefficient of x^{d-j} is in JI^j for $1 \leq j \leq d$. We shall soon see that the condition depends only on $\text{Rad}(J)$ and not on J itself. Our main interest is in the cases where $J = I$ or $J = \mathfrak{m}$.

Note that while the integral closure of a Noetherian domain need not be Noetherian, it is still true that it is a Krull ring: principal ideals have only finitely many minimal primes, the localization at such a minimal prime is a DVR, and one has primary decomposition for principal ideals. Compare [Nag62, pp. 115-117 and Theorem (33.10) on p. 118].

If $I = 0$ or $J = 0$, then $I^{-J^{\text{sp}}}$ is the ideal of nilpotent elements of R and is (0) if R is a domain.

Theorem 5.1. *Let R be a Noetherian domain, let I and J be nonzero ideals of R , and let $r \in R$. The following conditions are equivalent:*

- (1) r is in the J -special integral closure of I .
- (2) There exists an integer $n \geq 1$ such that $r^n \in (JI^n)^-$. In this case, all multiples tn of n have the same property; in fact, $r^{tn} \in (J^t I^{tn})^-$.
- (3) There exists an integer $n \geq 1$ such that for all maps $R \hookrightarrow V$ where V is a DVR, then

$$\text{ord}_V(r) \geq \text{ord}_V(J)/n + \text{ord}_V(I).$$

- (4) For all maps $R \hookrightarrow V$ where V is a DVR, $\text{ord}_V(r) \geq \text{ord}_V(I)$, and the inequality is strict if $\text{ord}_V(J) > 0$.

- (5) Write $I = (f_1, \dots, f_h)R$ and $J = (g_1, \dots, g_k)R$ with the elements f_i, g_j all nonzero. Then for any choice of $1 \leq i \leq h, 1 \leq j \leq k$, and \mathfrak{p} a minimal prime of f_iS or g_jS where S is the normalization of $R[I/f_i][J/g_j]$, if we let $V := S_{\mathfrak{p}}$, then $\text{ord}_V(r) \geq \text{ord}_V(I)$, with the inequality strict if $\text{ord}_V(J) > 0$.

If $J = I$, we may instead use the minimal primes of the f_jS in the normalizations S of the rings $R[I/f_i]$.

Proof. The second statement in (2) is clear. (2) \implies (1) since the equation showing integral dependence for r^n on JI^n may be viewed as an equation that r satisfies, and this provides the J -special polynomial over I . We prove (1) \implies (3). Suppose (1) holds with a J -special polynomial over I of degree n . Then for some $j, 1 \leq j \leq n$,

$$\text{ord}_V(r^n) \geq \text{ord}_V(JI^j) + \text{ord}_V(r^{n-j}),$$

and so

$$j \text{ord}_V(r) \geq \text{ord}_V(J) + j \text{ord}_V(I)$$

and

$$\text{ord}_V(r) \geq \text{ord}_V(J)/j + \text{ord}_V(I).$$

Hence, we have

$$\text{ord}_V(r) \geq \text{ord}_V(J)/n + \text{ord}_V(I),$$

no matter what j is.

(3) \implies (4) is clear, and (4) \implies (5) is clear.

Therefore, the proof will be complete if we can show that (5) \implies (2). Choose a value of n so large that condition (3) holds with this value of n for all of the finitely many valuation rings described in (5). Now consider any injection $R \hookrightarrow V$ where V is a DVR. Then IV is generated by the image of some f_i , and JV is generated by some g_j . For these two elements, the map $R \hookrightarrow V$ factors as $R \hookrightarrow R[I/f_i][J/g_j] \hookrightarrow V$ and, hence, $R \hookrightarrow S_{ij} \hookrightarrow V$, where S_{ij} is the normalization of $R[I/f_i][J/g_j]$. We claim that in $S_{ij}, r^n \in JI^n S_{ij} = g_j f_i^n S_{ij}$. Since S_{ij} is a Krull domain, it suffices to see this after localizing at each of the minimal primes of $g_j f_i^n S_{ij}$. Since each of these is a minimal prime of g_j or f_i , this produces one of the discrete valuation rings W for which we have assumed that

$$\text{ord}_W(r) \geq \text{ord}_W(J)/n + \text{ord}_W(I).$$

Multiplying by n gives the result we need. Since $r^n \in JI^n S_{ij}$, this continues to hold in V . But then $r^n \in (JI^n)^-$, as required.

When $J = I$, both are generated by the image of some f_i after expanding to V , and so the map $R \hookrightarrow V$ factors as $R \hookrightarrow R[I/f_i] \hookrightarrow V$, and the rest of the proof is the same. □

Corollary 5.2. *For any Noetherian ring R with ideals I and $J, r \in R$ is in the J -special integral closure of I if and only if for some $n, r^n \in (JI^n)^-$. This may be tested modulo every minimal prime P of R .*

The J -special integral closure of I is an ideal, depends only on $\text{Rad}(J)$, and lies between $\text{Rad}(J)I^-$ and $\text{Rad}(J) \cap I^-$.

Proof. Say the minimal primes of R are P_1, \dots, P_h . If $r^{n_i} \in (JI^{n_i})^- \text{ mod } P_i$ for $1 \leq i \leq h$, and N is the product of the n_i , by Theorem 5.1(2) we have that $r^N \in (JI^N)^- \text{ mod } P_i$ for all i , and this implies $r^N \in (JI^N)^-$ in R . If we have J -special equations over I satisfied for $r \text{ mod } P_i$, the value of the product

of these on r is nilpotent, and so a power of the product will be 0. Thus, the equivalence reduces to the domain case, where we already know it.

In the domain case, $I^{-J^{sp}}$ is an ideal; this is true if either I or J is 0. If not, we can use (3) to characterize the J -special integral closure. If r_1 and r_2 satisfy condition (3) with integers n_1 and n_2 , their sum satisfies condition (3) with $n = \max\{n_1, n_2\}$, while closure under multiplication is obvious. In the general case, the J -special integral closure is the intersection of the images of what one gets modulo various minimal primes.

The second statement in (2) shows that J -special integral closure is contained in J^t -special integral closure, and the opposite inclusion is obvious. It is clear that the special part of the integral closure contains $J I^-$, and we may replace J by $\text{Rad}(J)$. It is also obvious that $I^{-J^{sp}}$ is contained in both $\text{Rad}(J)$ and I^- . \square

Corollary 5.3. *Let (R, \mathfrak{m}) be a Noetherian local domain. An element $r \in R$ is in the special part of the integral closure of I iff for all $R \hookrightarrow V$ with V a DVR centered on \mathfrak{m} , if $IV \neq 0$, then $\text{ord}_V(r) > \text{ord}_V(I)$.*

Proof. Since $\text{ord}_V(\mathfrak{m})$ will be positive, the condition is necessary. Because there are only finitely many valuation rings needed in the test in part (5) of Theorem 5.1, we can always choose a value of n that will work for all of these. \square

The *inner integral closure* of I is defined as the I -special integral closure of I . Instead of $I^{-I^{sp}}$ we write $I_{>1}$. This construction is developed to some extent in [HS06, Section 10.5] (from which we obtain our notation) and in [GV11, Section 4]; some of our results in this section may overlap with the results in these two references. We emphasize that this *is not a closure operation*. It does not usually contain I , but it is contained in the integral closure of I . It may be thought of as the “inner” part of the integral closure. Note that by part (c) of Proposition 5.5 below, the inner integral closure of I is the same as the inner integral closure of I^- , and by part (d) of Proposition 5.5, the ideal $I_{>1}$ is itself integrally closed. The notation $I_{>1}$ agrees with the notation given in [HS06, Definition 10.5.3 on p. 206], where an infinite family of related notions ($I_{>\alpha}$ for all positive rational numbers α) is considered, but no name is given for $I_{>1}$.

Theorem 5.4. *Let R be a Noetherian ring, let I be an ideal of R , and let $r \in R$. The following conditions are equivalent and characterize when r is in the inner integral closure of I :*

- (1) *The element r satisfies a monic polynomial $f(x) \in R[x]$ of some degree d such that the coefficient of x^{d-j} is in I^{j+1} , $1 \leq j \leq d$.*
- (2) *There is an integer n such that r^n is integral over I^{n+1} .*
- (3) *There is an integer n such that $r^n \in I^{n+1}$.*
- (4) *For every prime ideal P containing I , $r/1$ is in the special part of the integral closure of IR_P .*
- (5) *For every map $R \rightarrow V$, where V is a DVR, such that IV is not 0 and not V , $\text{ord}_V(r) > \text{ord}_V(I)$.*
- (6) *The condition in (5) holds for all V such that $R \rightarrow V$ kills a minimal prime \mathfrak{p} of R .*
- (7) *If R is a domain, and $I = (f_1, \dots, f_h)$ where the f_i are nonzero, the condition in (5) holds for all V arising as the localization at a minimal prime of one of the f_i in the normalization of one of the rings $R[I/f_i]$.*

(8) *If R is an excellent domain and every prime ideal is an intersection of maximal ideals³ (e.g., if R is a finitely generated \mathbb{C} -algebra), the condition in (5) holds for discrete valuations centered on a maximal ideal m of R .*

Proof. We already have (1) \implies (2) \implies (5) \implies (6) \implies (7). We can see that (2) \iff (3) as follows. It suffices to show \implies ; the converse is obvious. But if r^n is integral over I^{n+1} , then there exists h such that for all N , $(I^{n+1} + r^n)^{N+h} = (I^{n+1})^{N+1}(I^{n+1} + r^n)^{h-1}$, so that $r^{n(N+h)} \in I^{(n+1)(N+1)}$ for all N . But if $N > nh - n - 1$ we have that $(n + 1)(N + 1) > n(N + h)$, as required.

Because the condition in (7) (and in part (5) of the preceding theorem) involves only finitely many DVRs, we can choose n that works for these, and then the preceding theorem yields (1) and (2). Condition (4) and the preceding corollary yield condition (5), since $IV \neq V$ in (5) implies that V is centered on a prime containing I .

It remains only to show the equivalency of condition (8), which is obviously necessary. When R is excellent, the normalization S_i of $R[I/f_i]$ is finitely generated over R . Let V be one of the localizations of S_i at a minimal prime of f_j . Let v generate the maximal ideal of V . We can choose a localization T_0 of S_i at one element such that v generates a prime ideal in T_0 . We can write r and f_i as units of V times powers of v , say $r = uv^\nu$, $f_i = u'v^\mu$, where $\nu > \mu$. We may localize T_0 further, but at finitely many elements, so that it contains u and u' . Call the resulting ring T . Then v is prime in T , and the domain T/vT is excellent. It follows that there is a maximal ideal \mathfrak{m} of T/vT such that $(T/vT)_{\mathfrak{m}}$ is regular. The pullback \mathcal{M} of \mathfrak{m} to T will be a maximal ideal of T containing v such that $T_{\mathcal{M}}$ is regular and the image of v is a regular parameter. The contraction of \mathcal{M} to R is a maximal ideal because R is a Hilbert ring. The valuation given by order with respect to powers of \mathcal{M} will give the same result for the orders of r and I as V . \square

Proposition 5.5. *Let I be an ideal of the Noetherian ring R .*

- (a) *If W is a multiplicative system of R , $(I_W)_{>1} = (I_{>1})_W$.*
- (b) *If S is integral over R , $(IS)_{>1} \cap R = I_{>1}$.*
- (c) *If $I \subseteq J$, then $I_{>1} \subseteq J_{>1}$. If $I \subseteq J \subseteq I^-$, then $I_{>1} = J_{>1}$.*
- (d) *The ideal $I_{>1}$ is integrally closed.*

Proof. For part (a), if $r^n \in I^{n+1}R_W$, then for some $w \in W$ we have $wr^n \in I^{n+1}$, and then $(wr)^n \in I^{n+1}$, which shows that $wr \in I_{>1}$. The other implication is trivial.

For (b), if $r \in R$ is such that r^n is in the integral closure of $I^{n+1}S$, then r^n is in the integral closure of I^{n+1} . The other containment is obvious.

The first statement in (c) is obvious from condition (2) or (3) of Theorem 5.4. The second statement is clear provided that we can show that an element r in the inner integral closure of I^- is in the inner integral closure of I . But $(I^-)^{n+1} \subseteq (I^{n+1})^-$, and so if r^n is in the former it is in the latter.

Part (d) follows because condition (5) of Theorem 5.4 shows that $I_{>1}$ is an intersection of valuation ideals. This fact also follows from [HS06, Proposition 10.5.2, part (4)] coupled with the final remark in the paragraph following [HS06, Definition 10.5.3]. \square

Theorem 5.6. *Let R be a finitely generated \mathbb{C} -algebra. Then $I_{>1} \subseteq I^{\text{cont}}$.*

³I.e. R is a Hilbert ring.

Proof. Consider a polynomial for r as in condition (1) of Theorem 5.4, with indeterminate variable x . Suppose that the coefficient of x^{d-j} is a sum of t_j terms, each the product of $j + 1$ elements of I . We replace all $t_j(j + 1)$ elements of I involved by variables. This gives an equation $F = x^d + P_1x^{d-1} + \cdots + P_d = 0$, where P_j is homogeneous of degree $j + 1$ in $t_j(j + 1)$ variables, and all the variables from distinct P_i, P_j are mutually disjoint. Consider the ring S defined by adjoining x and all the other variables to \mathbb{C} and killing the polynomial F . In this ring, let J be the ideal generated by all variables other than x . The radical of J contains x as well. We have a homomorphism $S \rightarrow R$ which takes x to r and such that $JS \subseteq I$. Thus, it suffices to show that $x \in J^{\text{cont}}$. But S has an \mathbb{N} -grading: we give the variables occurring in P_j degree $\frac{j}{j+1}(d + 1)!$, $1 \leq j \leq d$, and we give x degree $(d + 1)!$. Since x has a higher degree than any generator of J , the result follows from Theorem 2.5. \square

It follows that $I_{>1} \subseteq I^{\text{ax}}$ in an affine \mathbb{C} -algebra. In fact this holds more generally. To see this, first we give the following useful lemma.

Lemma 5.7. *Let (R, \mathfrak{m}, k) be a one-dimensional complete Noetherian seminormal local ring. Then for any ideal J of R , we have $\mathfrak{m} \cdot J^- \subseteq J$.*

Proof. First, note that it is enough to prove the lemma for a *primary* ideal. Let $J = Q_1 \cap \cdots \cap Q_t$ be a primary decomposition, and suppose the lemma holds for primary ideals. Then $\mathfrak{m}J^- \subseteq \mathfrak{m}Q_1^- \cap \cdots \cap \mathfrak{m}Q_t^- \subseteq Q_1 \cap \cdots \cap Q_t = J$.

Let L_i and V_i be as in the description given in Theorem 3.3. For each i , let t_i be a generator of the maximal ideal of V_i . We have $\text{Spec } R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n, \mathfrak{m}\}$, where \mathfrak{p}_i is the kernel of the map $R \rightarrow V_i$.

If J is \mathfrak{p}_i -primary for some i , it follows from the minimality of \mathfrak{p}_i and the fact that R is reduced that $J = \mathfrak{p}_i$. Then $\mathfrak{m}J^- = \mathfrak{m}\mathfrak{p}_i^- = \mathfrak{m}\mathfrak{p}_i \subseteq \mathfrak{p}_i = J$.

Thus, we may assume J is \mathfrak{m} -primary. Then for each i , there is some integer $1 \leq e_i < \infty$ such that $JV_i = t_i^{e_i}V_i$. Hence,

$$J^- = \bigoplus_{i=1}^n t_i^{e_i} V_i.$$

Pick any i between 1 and n . Then from the structure of R , it follows that there is some element of the form $c := ut_i^{e_i} + \sum_{j \neq i} v_j t_j^{f_j} \in J$, where u is a unit of V_i , the $v_j \in V_j$, and the $f_j \geq e_j$. Then for a typical element $vt_i^{e_i+1} \in t_i^{e_i+1}V_i$ (where $v \in V_i$), we have $u^{-1}vt_i \in R$ so that $vt_i^{e_i+1} = (u^{-1}vt_i)c_i \in J$. Hence, $\mathfrak{m}J^- = \bigoplus_{i=1}^n t_i^{e_i+1}V_i \subseteq J$. \square

Next, we have the following theorem, which follows from Theorem 5.6 when R is a finitely generated \mathbb{C} -algebra, but in fact holds in a more general setting as we see below.

Theorem 5.8. *Let R be an excellent Noetherian ring, and let I be an ideal. Then $I_{>1} \subseteq I^{\text{ax}}$.*

Proof. Since the persistence property holds for both inner integral closure and axes closure, we may assume that (R, \mathfrak{m}, k) is a complete local one-dimensional seminormal ring, in which case what we want to show is that $I_{>1} \subseteq I$. Pick Noetherian valuation rings $(V_i, (t_i), L_i)$, $1 \leq i \leq n$, as in Theorem 3.3. If $I = R$ there is nothing to prove, so we may assume $I \subseteq \mathfrak{m} = \bigoplus_{i=1}^n t_i V_i$. For each i , either $IV_i = 0$ or

$IV_i = t_i^{e_i} V_i$ for some $1 \leq e_i < \infty$. If we use the convention $t_i^\infty = 0$, then we have $I^- = \bigoplus_{i=1}^n t_i^{e_i} V_i$, where $1 \leq e_i \leq \infty$ for each i .

By part (5) of Theorem 5.4, we have $I_{>1} \subseteq \bigoplus_{i=1}^n t_i^{e_i+1} V_i = \mathfrak{m}I^-$. But by Lemma 5.7, we have $\mathfrak{m}I^- \subseteq I$, which completes the proof. \square

We next observe that $I_{>1} = (I^-)_{>1}$, and so the operation that sends I to $I + I_{>1}$ is a closure operation in the sense of Definition 1.1.

Definition 5.9. For an ideal I , we let $I^\natural := I + I_{>1}$, the *natural closure* of I . If $I = I^\natural$ we say that I is *naturally closed*. Evidently, if $I \subseteq I_1 \subseteq I^-$ and $I = I^\natural$, then $I_1 = I_1^\natural$.

By a *valuation* of R we simply mean a map $R \rightarrow V$ where (V, \mathfrak{n}) is a discrete valuation ring. It may have a kernel, even if R is a domain. If $IV \neq 0$ has order $k > 0$, call the valuation ideal \mathfrak{A} arising from contraction of \mathfrak{n}^{k+1} the *I-relevant valuation ideal* of $R \rightarrow V$. If $IV = 0$, we call the contraction of 0, i.e., $\text{Ker}(R \rightarrow V)$, the *I-relevant valuation ideal* of $R \rightarrow V$.

If R is a domain and I is a nonzero ideal of R , then for every nonzero element $f \in I$ we may take the normalization S of the $R[I/f]$ and consider the discrete valuation rings arising as localizations of S at a minimal prime of fS . There are only finitely many such valuation rings, and if R is excellent, they are small. Following standard terminology, we refer to these rings as the *Rees valuation rings* of I .

In the case where R is not a domain, we make the following conventions about the Rees valuations of I . For every minimal prime P of R not containing I , we include the Rees valuations of $I(R/P)$ among the Rees valuations for I . If P contains I , we shall think of the fraction field $\kappa(P)$ of R/P as a “degenerate” valuation ring, and we include the maps $R \rightarrow \kappa(P)$ as Rees valuations. The order of every element not in P is 0, while the elements of P may be viewed as having order $+\infty$.

Theorem 5.10. *Let R be Noetherian and I an ideal of R .*

- (a) *There are finitely many valuations $R \rightarrow (V_i, \mathfrak{n}_i)$ such that $I_{>1}$ is the intersection of the I-relevant valuation ideals of these valuations. The V_i may be chosen to be the Rees valuations.*
- (b) *$I = I^\natural$ if and only if there are finitely many valuations $R \rightarrow (V_i, \mathfrak{n}_i)$ with I-relevant valuation ideals \mathfrak{A}_i such that $\bigcap_i \mathfrak{A}_i \subseteq I$. These valuations may be chosen to be the Rees valuations of I .*

Proof. For part (a), we know that $r \in I_{>1}$ if and only if that holds modulo each minimal prime of I . Thus, it suffices to show that there are valuation ideals as specified for every R/\mathfrak{p} and IR/\mathfrak{p} . We may thus reduce to the domain case. If $I \subseteq \mathfrak{p}$ we take the relevant valuation ideal to be 0. We may localize at any height one prime of the normalization of R/\mathfrak{p} and do this. Otherwise we use the valuation ideals coming from the rings $R[I/f]$.

The “only if” part of (b) is obvious. For the “if” part, suppose that we have $\bigcap_i \mathfrak{A}_i \subseteq I$. To show that $I = I^\natural$, it suffices to show that $I_{>1} \subseteq I$. Let $u \in I$. Then $u \in \mathfrak{A}_i$ for all i , and the result is obvious. \square

Proposition 5.11. *Let I be an ideal of R and let W be a multiplicative system in R . Then $(I_W)^\natural = (I^\natural)_W$.*

Proof. $(I_W)^\natural = I_W + (I_W)_{>1} = I_W + (I_{>1})_W$, by Proposition 5.5(a), but $I_W + (I_{>1})_W = (I + I_{>1})_W = (I^\natural)_W$. \square

Theorem 5.12. *Let $R \rightarrow S$ be a flat homomorphism of excellent Noetherian rings whose fibers are geometrically regular. Let I be ideal of R . Then $(IS)_{>1} = I_{>1}S$ and $(IS)^\natural = (I^\natural)S$. In particular, if (R, \mathfrak{m}, k) is local, this holds if S is \widehat{R} or $S = R(t)$, the localization of the polynomial ring $R[t]$ at $\mathfrak{m}R[t]$.*

Proof. The statement for natural closure is immediate from the statement for inner integral closure. Evidently, $I_{>1}S \subseteq (IS)_{>1}$. To see the opposite inclusion, it will suffice to show that for every I -relevant ideal \mathfrak{B} for a Rees valuation of $I(R/\mathfrak{p})$, where \mathfrak{p} is a minimal prime of R , $\mathfrak{B}S$ is an IS -relevant ideal of a valuation on S . For $I_{>1}$ is the intersection of the ideals \mathfrak{B} , and the intersection of the ideals $\mathfrak{B}S$ will be $(I_{>1})S$ and will also contain $(IS)_{>1}$.

To prove this, we may replace $R \rightarrow S$ by $R/\mathfrak{p} \rightarrow S/\mathfrak{p}S$, which is still flat with geometrically regular fibers. Thus, we may assume that R is a domain. Let (V, \mathfrak{n}) be a Rees valuation of I . Then V is essentially of finite type over R , since R is excellent, and so $T = S \otimes_R V$ is essentially of finite type over S and is Noetherian and flat over V with geometrically regular fibers. It follows that $T/\mathfrak{n}T$ is reduced. Let Q_1, \dots, Q_s be the minimal primes of $\mathfrak{n}T$. Then every T_{Q_i} is a valuation ring, and $Q_i T_{Q_i} = \mathfrak{n}T_{Q_i}$. Let h be the order of IV so that $IV = \mathfrak{n}^h$. Then $(IS)T_{Q_i} = (IV)T_{Q_i} = \mathfrak{n}^h T_{Q_i} = Q_i^h T_{Q_i}$, for each $i = 1, \dots, s$. Hence, the contraction \mathfrak{A}_i of Q_i^{h+1} to S is an IS -relevant ideal for the valuation ring T_{Q_i} . Consequently, $(IS)_{>1} \subseteq \mathfrak{A}_i$.

We next show that $\bigcap_i \mathfrak{A}_i = \mathfrak{B}S$. Since \mathfrak{n}^{h+1} maps into $Q_i^{h+1}T_{Q_i}$, \supseteq is clear. We need to show that $\bigcap_i \mathfrak{A}_i \subseteq \mathfrak{B}S$. First observe that since $(R/\mathfrak{B}) \hookrightarrow V/\mathfrak{n}^{h+1}$ is injective and S is R -flat, we may apply $S \otimes_R _$ to obtain that

$$S/\mathfrak{B}S \hookrightarrow (V/\mathfrak{n}^{h+1}) \otimes_R S \cong T/\mathfrak{n}^{h+1}T$$

is injective. Thus, $\mathfrak{n}^{h+1}T$ lies over $\mathfrak{B}S$. Since V is a discrete valuation ring and $V \rightarrow T$ is flat with regular fibers, T is regular. (In fact, we only need that T is normal.) Moreover, \mathfrak{n} is principal, and so $\mathfrak{n}^{h+1}T$ is a principal ideal generated by a nonzerodivisor. It follows that its primary decomposition is simply $\bigcap_{i=1}^s Q_i^{(h+1)}$ (where the notation denotes symbolic powers), since $Q_i T_{Q_i} = \mathfrak{n}T_{Q_i}$. Hence, $\mathfrak{B}S$, which is the contraction of $\mathfrak{n}^{h+1}T$, is also the contraction of $\bigcap_{i=1}^s Q_i^{(h+1)}$. This is the same as the intersection of the contractions of the ideals $Q_i^{(h+1)}$. But $Q_i^{(h+1)}$ is simply the contraction of $Q_i^{h+1}T_{Q_i}$ to T , and it follows that the contraction of $Q_i^{(h+1)}$ is \mathfrak{A}_i . Thus, $\bigcap_{i=1}^s \mathfrak{A}_i = \mathfrak{B}S$, as claimed.

Finally, since $(IS)_{>1} \subseteq \mathfrak{A}_i$ for every i , we have that $(IS)_{>1} \subseteq \mathfrak{B}S$ for each of the \mathfrak{B} . Since there are only finitely many \mathfrak{B} , finite intersection commutes with flat base change, and the intersection of the \mathfrak{B} is $I_{>1}$, we then have $(IS)_{>1} \subseteq I_{>1}S$, proving the other needed inclusion. \square

It follows from the proof that the hypothesis that the fibers be geometrically regular may be weakened. It suffices that the fibers are geometrically reduced and whenever $R \rightarrow V$ is a map to a discrete valuation ring, $V \otimes_R S$ is normal.

In the case of a finitely presented smooth R -algebra S , we do not need any hypothesis of excellence on S .

Proposition 5.13. *Let R be a ring and let S be a finitely presented R -algebra that is smooth over R . Then for every ideal I of R , $(IS)_{>1} = I_{>1}S$ and $(IS)^\natural = I^\natural S$.*

Proof. The second conclusion follows from the first, and, for the first, it suffices to prove that $(IS)_{>1} \subseteq I_{>1}S$. First note that there is a subring R_0 of R finitely generated over the prime ring Λ in R and a finitely presented smooth R_0 -algebra S_0 such that $S \cong R \otimes_{R_0} S_0$, and that S is the union of the rings $S_1 = R_1 \otimes_{R_0} S_0$ as R_1 runs through all subrings of R finitely generated over R_0 . These in turn are finitely generated over Λ and, therefore, excellent. If $f \in (IS)_{>1}$, then we can choose such an R_1 so that $f \in (I_1 S_1)_{>1}$, where I_1 is an ideal of R_1 generated by elements in I . But then R_1 and S_1 satisfy the hypotheses of Theorem 5.12, and so we can conclude that $f \in (I_1)_{>1} S_1 \subseteq I_{>1} S$, as required. \square

Proposition 5.14. *Let R be a \mathbb{Z}^h -graded ring, let $h > 0$, and let I be a homogeneous ideal with respect to this grading. Then $I_{>1}$ and I^\natural are also homogeneous with respect to this grading.*

Proof. It suffices to prove this for $I_{>1}$. Although $_{>1}$ is not a closure operation, Discussion 2.9 applies: it is defined ring-theoretically, and will be stable under the automorphisms θ_α . The only issue in the proof is that R may not have sufficiently many units. However, the rings S_N introduced in Discussion 2.9 are finitely presented, smooth, and faithfully flat over R , and the result now follows from Discussion 2.9, Proposition 5.13, and the fact that since S_N is faithfully flat over R , $I_{>1} S_N \cap R = I_{>1}$. \square

6. THE IDEAL GENERATED BY THE PARTIAL DERIVATIVES

Theorem 6.1. *Let S be a localization of $R = \mathbb{C}[x_1, \dots, x_n]$ at one element. Let $f \in R$, and let $\mathcal{J} = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$.*

- (a) *For any nonconstant $f \in R$ there exists $g \in R$ such that $(1 - gf)f$ is in the inner integral closure of \mathcal{J} and, hence, the continuous closure.*
- (b) *If $f \in \text{Rad}(\mathcal{J}S)$, then f is in the inner integral closure of $\mathcal{J}S$ and hence in the continuous closure of $\mathcal{J}S$.*

Proof. We first prove (b). Consider a valuation $S \rightarrow V$ centered on a maximal ideal \mathcal{M} that contains $\mathcal{J}S$. Then we have $\alpha : S_{\mathcal{M}} \rightarrow V$, and f is also in \mathcal{M} . We may assume that \mathcal{M} corresponds to the origin, and so passing to completions, α induces $\theta : \mathbb{C}[[x_1, \dots, x_n]] \rightarrow L[[t]]$ for some field L with $\mathbb{C} \subseteq L$. Then $\theta(f)$ is in the maximal ideal of $L[[t]]$ and is nonzero. By the chain rule, its derivative is in $\mathcal{J}L[[t]]$. The derivative has order exactly one less than that of $\theta(f)$. This shows that the order of f is strictly larger than the order of \mathcal{J} .

To prove (a), let $\mathfrak{B} = \text{Rad}(\mathcal{J}) : f$. Then no maximal ideal can contain $\mathfrak{B} + fR$. When we localize at such a maximal ideal, the chain rule shows that for all valuations centered on it, the order of f is larger than the order of \mathcal{J} , and so $f \in \mathcal{J}^- \subseteq \text{Rad}(\mathcal{J})$, which contradicts the fact that the maximal ideal contains $\text{Rad}(\mathcal{J}) : f$. Thus, we can choose $h \in \mathfrak{B}$ such that that $h + fg = 1$, and so $1 - fg \in \mathfrak{B}$, and $f(1 - fg) \in \text{Rad}(\mathcal{J})$. We can apply part (b) to the ring R_h where $h = 1 - fg$. Hence, h has a power, also of the form $1 - fg$, which multiplies f into the inner integral closure of \mathcal{J} . \square

7. NATURALLY CLOSED PRIMARY IDEALS ARE AXES CLOSED

In this section we prove that if I is a primary ideal of an excellent Noetherian ring, then $I = I^{\text{ax}}$ iff $I = I^{\natural}$. This shows that for ideals primary to maximal ideals, natural closure and axes closure agree. Moreover, in the case of an affine \mathbb{C} -algebra, since $I^{\natural} \subseteq I^{\text{cont}} \subseteq I^{\text{ax}}$, we have the corresponding result for continuous closure as well. It then follows that for an unmixed ideal I of an affine \mathbb{C} -algebra, $I = I^{\text{cont}}$ if and only if $I = I^{\text{ax}}$.

We first extend the I -relevant terminology from Definition 5.9 as follows.

Definition 7.1. Let D be a Dedekind domain with a nonzero principal ideal tD where t is prime. Given a map $R \rightarrow D$ we say that \mathfrak{A} is the I -relevant associated with (D, tD) if either $ID = 0$ and $\mathfrak{A} = \text{Ker}(R \rightarrow D)$ or $ID = t^k D$ and \mathfrak{A} is the contraction of $t^{k+1}D$.

This is the same as saying that \mathfrak{A} is the I -relevant ideal of $R \rightarrow D_Q$, where $Q = tD$.

We need the following result, which is contained in [EH79, Corollary 1, p. 158].

Theorem 7.2. *Let R be a regular Noetherian ring, let P be a prime ideal of R , and let \mathcal{M} be a family of maximal ideals of R containing P whose intersection is P and such that for all $\mathfrak{m} \in \mathcal{M}$, $R_{\mathfrak{m}}/PR_{\mathfrak{m}}$ is regular. Let n be a positive integer. Then $\bigcap_{\mathfrak{m} \in \mathcal{M}} \mathfrak{m}^n = P^{(n)}$, the n th symbolic power of P .*

Corollary 7.3. *Let R , P , and \mathcal{M} be as in Theorem 7.2. Moreover, suppose that the prime ideal P is principal with generator π . Let n_1, \dots, n_k be positive integers. Then the set of maximal ideals \mathfrak{m} in \mathcal{M} such that $\pi^{n_i} \in \mathfrak{m}^{n_i} - \mathfrak{m}^{n_i+1}$ also has intersection P . In particular, this set is nonempty.*

Proof. By a straightforward induction on k , we reduce to the case where $k = 1$. We write n for n_1 . Let $f \in R - P$. It suffices to show that there exists $\mathfrak{m} \in \mathcal{M}$ such that $f \notin \mathfrak{m}$ and $\pi^n \notin \mathfrak{m}^{n+1}$. Let $g \in R - P$. Then there is an element of \mathcal{M} that does not contain g . It follows that the intersection of the set \mathcal{N} of maximal ideals in \mathcal{M} that do not contain f is also P . By Theorem 7.2, $\bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^{n+1} = P^{(n+1)} = \pi^{n+1}R$. Therefore, we can choose $\mathfrak{m} \in \mathcal{N}$ such that $\pi^n \notin \mathfrak{m}^{n+1}$. □

Key Lemma 7.4. *Let (R, \mathfrak{m}, K) be an excellent local domain of dimension $d \geq 2$ with infinite residue class field and let I be an \mathfrak{m} -primary ideal. Let f be a nonzero element of I . Let S be the normalization of $R[I/f]$, and let $R \rightarrow V = S_P$ be a Rees valuation of I , where P is a minimal prime of fS . Let \mathfrak{n} denote the maximal ideal of V . Let IV have order $h - 1 \geq 1$, and let \mathfrak{B} be the contraction of the proper nonzero ideal \mathfrak{n}^h of V to R . Let $g_1, \dots, g_h \in R - \{0\}$, and let $n_i = \text{ord}_V(g_i)$. Then there exists an algebra T finitely generated over S by adjunction of fractions, a prime ideal Q of T of height $d - 1$ such that T/Q is a Dedekind domain, and a principal height one prime $\pi T/Q$ of T/Q such that $IT/Q = \pi^{h-1}T/Q$, $\mathfrak{B}T/Q = \pi^h(T/Q)$, and the image of g_i in T/Q is a unit times π^{n_i} . In particular, by including a given nonzero element r of R among the g_i , we may choose Q so that r has nonzero image in T/Q .*

Proof. By the dimension formula, S/P is an affine K -algebra of dimension $d - 1$. Since S_P is regular, we can localize at one fraction of $S - P$ to produce S_1 that is regular, and we can localize at one element of $S - P$ to produce a localization S_2 such that S_2/PS_2 is regular as well. We may also localize so that $P = yS_2$ is

principal. We replace S by S_2 and drop the subscript. By Corollary 7.3 we can choose a maximal ideal \mathfrak{m} of S_2 containing P so that the orders of a finite set of generators of I , and of \mathfrak{B} , as well as of g are all the same with respect to the m -adic valuation on $S_{\mathfrak{m}}$ as they are in V . We may extend y to a set of elements $y_1 = y, y_2, \dots, y_d$ in \mathfrak{m} that generate $\mathfrak{m}S_{\mathfrak{m}}$. We may localize at one element of $S - \mathfrak{m}$ and then assume that these elements generate \mathfrak{m} . Consider the leading forms of g_i and the various generators of I and \mathfrak{B} in $\text{gr}_{\mathfrak{m}}S$. After a linear change of generators over K (which is infinite) that fixes y and replaces each y_i for $i > 1$ by $y_i + c_i y_i$, we may assume that all of these leading forms have a term that is a scalar times a power of y : the exponent on y is the order. The ideal $S_{\mathfrak{m}}/(y_2, \dots, y_d)S_{\mathfrak{m}}$ is a regular domain in which the image of y generates a prime ideal. This will remain true after we localize at one element of $S - \mathfrak{m}$. This localization is T , and we may take $Q = (y_2, \dots, y_d)T$. \square

Corollary 7.5. *Let R be an excellent ring, and let I be an \mathfrak{m} -primary ideal of R , where \mathfrak{m} is maximal and R/\mathfrak{m} is infinite. $R \rightarrow V$ be a Rees valuation with I -relevant ideal \mathfrak{A} . Let r be a nonzerodivisor in R . Then there exist finitely many surjections $R \rightarrow D_i$ such that D_i is a one-dimensional Dedekind domain that is finitely generated over the residue class field $K = R/\mathfrak{m}$ and nonzero primes $t_i D_i$ such that \mathfrak{A} is the intersection of the I -relevant ideals \mathfrak{B}_i of these maps. Moreover, the surjections may be chosen in such a way that the image of r in every D_i is nonzero.*

Proof. If the Rees valuation has kernel \mathfrak{p} we may work with R/\mathfrak{p} , IR/\mathfrak{p} , and $\mathfrak{A}/\mathfrak{p}$. The image of r in R/\mathfrak{p} is not 0. Thus, we may assume that R is a domain and that r is a nonzero element of R . Consider all finite intersections of ideals \mathfrak{B} containing \mathfrak{A} of the type described (including the condition that the image of r be nonzero in every D_i). Since R/\mathfrak{A} has DCC, one of these is minimum. If it is not \mathfrak{A} , choose g in it that is not in \mathfrak{A} . By the Key Lemma 7.4, we can construct \mathfrak{B} so that the image of r in the corresponding Dedekind domain is not 0 and so that it contains \mathfrak{A} but not g , a contradiction. \square

Corollary 7.6. *Let I be an \mathfrak{m} -primary naturally closed ideal of an excellent ring R , where \mathfrak{m} is maximal and R/\mathfrak{m} is infinite. Let $r \in R$ be a nonzerodivisor. Then there is a radical ideal $J \subseteq I$ such that R/J has pure dimension one, the image of r is not a zerodivisor in R/J , and $I(R/J) = I/J$ is naturally closed. Moreover, I/J is primary to \mathfrak{m}/J in R/J .*

Proof. By Theorem 5.10(a), we may pick ideals \mathfrak{A}_i that are I -relevant from Rees valuation rings and whose intersection is contained in I . For each \mathfrak{A}_i pick \mathfrak{B}_{ij} as in Corollary 7.5 whose intersection is within \mathfrak{A}_i , and let \mathfrak{q}_{ij} be the kernel of the map onto a Dedekind domain of dimension one that is used in the construction of \mathfrak{B}_{ij} . This may be done so that r is not in any of \mathfrak{q}_{ij} . Take $J = \bigcap_{i,j} \mathfrak{q}_{ij}$. Then by Theorem 5.10(b), R/J has the required property. \square

Lemma 7.7. *Let R be a one-dimensional excellent Noetherian reduced ring, and let S be the seminormalization of R . Let I be an ideal of R whose minimal primes are all height one maximal ideals and such that I is naturally closed. Then $IS \cap R = I$.*

Proof. Suppose that $f \in (IS \cap R) - I$. By replacing f by a multiple we may assume that $I :_R Rf$ is a maximal ideal \mathfrak{m} of R . If we replace R by $R_{\mathfrak{m}}$ and S by $S_{\mathfrak{m}}$ we still

have that $f \notin IR_{\mathfrak{m}}$, while natural closure commutes with localization. It follows that we may assume that (R, \mathfrak{m}, K) is local of dimension one. Consider the local extension rings R_1 of R with $R \subseteq R_1 \subseteq S$ and choose R_1 maximal in this family such that f is not in the natural closure of IR_1 , which will still be primary to the maximal ideal of R_1 . Thus, we may replace R by R_1 , and I by the natural closure of IR_1 , and we still have a counterexample. If $R_1 = S$ we are done. Hence, there is some element $u \in S - R$ such that $u^2, u^3 \in R$. Note that $R[u] = R + Ru$. By replacing u by a multiple we may assume that the annihilator of the image of u in $(R + Ru)/R$ is a prime ideal of R . Since u is in the total quotient ring of R , it is multiplied into R by a nonzerodivisor, and it follows that we may assume that the annihilator is a height one maximal ideal, which must be \mathfrak{m} . Since $u^3 = (u^2)u \in R$, we must have $u^2 \in \mathfrak{m}$, and it follows likewise that $u^3 \in \mathfrak{m}$.

Hence, $R[u] = R + Ru$ is local with maximal ideal $\mathfrak{m} + Ru$. Let $J = IR[u] = I + Iu$. We shall show that $Iu \subseteq I_{>1} \subseteq I$, since $I = I^{\natural}$. First note that since $u^2 \in \mathfrak{m}$, some power of u is in I , say $u^k \in I$. Then $(Iu)^k = I^k u^k \subseteq I^{k+1}$, as required. Since every element of Iu is in R , this shows that $Iu \subseteq I_{>1}$. Hence, $J = I$. Now suppose $f \in J + J_{>1}$, where the calculation of $J_{>1}$ is in $R[u]$. Then $f = v + w$, where $v \in J = I$. Then $f - v \in I_{>1}R[u] \cap R$. By Proposition 5.5(b), $f - v \in I_{>1}$, and so $f \in I^{\natural}$, a contradiction. \square

Theorem 7.8. *Let R be an excellent Noetherian ring. Let I be an ideal primary to a prime ideal P such that I is naturally closed. Then I is axes closed. Moreover, if $r \in R$ is not a zerodivisor and $f \notin I^{\text{ax}}$, then there is a map $R \rightarrow A$, where A is a one-dimensional excellent seminormal ring, which may further be assumed to be complete local such that $f \notin IA$ and the image of r is a nonzerodivisor in A .*

Proof. Let $f \in I^{\text{ax}} - I^{\natural}$. Since natural closure commutes with localization by Proposition 5.11, we may replace R by R_P : f is not in IR_P since I is primary to P . By the persistence of axes closure, $f \in (IR_P)^{\text{ax}}$. We have therefore constructed a new counterexample in which P is maximal. We revert to our original notation and call the ring R , but we assume that $P = \mathfrak{m}$ is a maximal ideal. Second, we may replace R by $R(t)$, by Theorem 5.12. Hence, we may assume without loss of generality that the residue field K is infinite. The element r is still a nonzerodivisor.

By Corollary 7.6, we can preserve the fact that f is in the axes closure but not the natural closure of I while passing to a reduced local ring of pure dimension one that is a homomorphic image of R , and such that the image of r is not a zerodivisor in this ring. Thus, we need only consider the issue in dimension one. Let S be the seminormalization of R . Then $IS \cap R = I$ by Lemma 7.7, and so $f \notin IS$, which shows that $f \notin I^{\text{ax}}$ by Definition 4.3. Note also that r remains a nonzerodivisor in S . Finally, we may replace S by a suitable completed localization. \square

We also note the following.

Theorem 7.9. *Let R be an excellent Noetherian ring. Then the following conditions on an ideal I are equivalent:*

- (1) *I is axes closed.*
- (2) *I is an intersection of primary naturally closed ideals.*

Proof. By Proposition 4.11, we already know that I is axes closed if and only if it is an intersection of primary axes closed ideals, and for primary ideals, being naturally closed coincides with being axes closed by Theorem 7.8. \square

Discussion 7.10. Although we have not been able to determine whether axes closure commutes with localization, we can show that it commutes with smooth base change in certain cases. Before stating our results, we recall the notion of *intersection flatness* from [HH94a], where it is introduced just before the statement of (7.18). An R -module S (typically, S will be an R -algebra here) is *intersection-flat* or \cap -flat if S is flat, and for every finitely generated R -module M and every collection of submodules $\{M_\lambda\}_{\lambda \in \Lambda}$ of M , the obvious injection

$$S \otimes_R \left(\bigcap_{\lambda} M_\lambda \right) \hookrightarrow \bigcap_{\lambda} (S \otimes_R M_\lambda)$$

is an isomorphism. The \cap -flat modules include R and are closed under arbitrary direct sum and passing to direct summands, and therefore include the projective R -modules. In [HH94a] it is observed that if R is complete local and $R \rightarrow S$ is flat local, then S is \cap -flat over R , using Chevalley's theorem, and that $R[[x]]$, where R is Noetherian and x denotes a finite string of variables, is \cap -flat over R .

If S is an \cap -flat R -algebra that is faithfully flat, where R is Noetherian, and W is a multiplicative system consisting of elements of S that are nonzerodivisors on S/PS for every prime ideal P of R (which implies that no element of W is in mS for m maximal in R), then by Lemma (5.10) of [AHH93], $W^{-1}S$ is \cap -flat. In particular, the localization of a polynomial ring over R at a multiplicative system of polynomials each of which has the property that its coefficients generate the unit ideal is \cap -flat over R .

Theorem 7.11. *Let $R \rightarrow S$ be a flat homomorphism of excellent Noetherian rings with geometrically regular fibers.*

- (a) *If I is an unmixed ideal of R that is axes closed, then IS is axes closed.*
- (b) *If S is \cap -flat, then for every ideal I of R , $(IS)^{\text{ax}} = I^{\text{ax}}S$. In particular, this holds when S is a localization of a polynomial ring in finitely many variables over an excellent ring R at a multiplicative system consisting of polynomials each of which has the property that its coefficients generate the unit ideal in R .*

Proof. For part (a), note that the primary components of I are axes closed, and since finite intersection commutes with flat base change it will suffice to prove the result when I is primary, say to P . Then IS is naturally closed in S by Theorem 5.12. If IS is unmixed, its primary components will also be naturally closed and so axes closed, and it will follow that IS is axes closed. But R/I has a finite filtration by torsion-free R/P -modules. It follows that the associated primes of S/IS are the same as those of S/PS . Since R/P is a domain and $R/P \rightarrow S/PS$ is flat with geometrically regular fibers, S/PS is reduced.

For part (b), note that by Theorems 7.9 and 7.8, I^{ax} is an intersection of axes closed primary (hence unmixed) ideals J_λ . Then $I^{\text{ax}}S = \bigcap_{\lambda} J_\lambda S$, and so it suffices to show that every $J_\lambda S$ is axes closed, which follows from part (a). \square

Theorem 7.12. *Let R be an excellent Noetherian ring, let I be an ideal of R , and let $r \in R$ be a nonzerodivisor. In testing whether $f \in I^{\text{ax}}$ in R , it suffices to consider maps $h : R \rightarrow A$ where A is one-dimensional, seminormal, and $h(r)$ is not a zerodivisor in A . Moreover, A may be chosen to be complete local.*

Proof. If $f \notin I^{\text{ax}}$, by Proposition 4.11, we may choose a primary axes closed ideal J such that $I \subseteq J$ and $f \notin J$. The result then follows from Theorem 7.8. \square

We do not know whether axes closure commutes with localization. This would be true if an axes closed ideal remained axes closed after localization. The following result sheds light on the problem.

Theorem 7.13. *Let R be an excellent Noetherian ring, and let I be an axes closed ideal. The following conditions are equivalent:*

- (1) *I is a finite intersection of primary naturally closed ideals (and so has an irredundant primary decomposition in which the primary components are all naturally closed ideals).*
- (2) *I is a finite intersection of primary axes closed ideals (and so has an irredundant primary decomposition in which the primary components are all axes closed ideals).*
- (3) *For every associated prime ideal P of I , IR_P is axes closed in R_P .*
- (4) *For every multiplicative system W in R , $IW^{-1}R$ is axes closed in $W^{-1}R$.*

Proof. Note in (1) and (2) that if one can express I as a finite intersection of naturally closed or axes closed primary ideals, one may obtain an irredundant primary decomposition as usual by intersecting those primary to the same prime and omitting terms that are not needed.

The first two conditions are equivalent, since a primary ideal is naturally closed if and only if it is axes closed in an excellent ring by Theorem 7.8. We have that (1) \implies (4), because primary naturally closed ideals remain primary naturally closed ideals or become the unit ideal when one localizes. Since (4) \implies (3) is evident, it suffices to show that (3) \implies (2). Assume (3). We use induction on the number of associated primes of I . If there is only one associated prime, I is primary to it, and the result is clear.

We next reduce to the local case. Let P be an associated prime of I . Then IR_P is axes closed in R_P , by hypothesis, and if we can prove the result for R_P and IR_P , we may write IR_P as a finite intersection of axes closed primary ideals in R_P . The contractions of these ideals to R will be finitely many axes closed primary ideals. If we intersect all of these as P varies, we obtain I , for if $f \notin I$, then f has a multiple not in I such that the annihilator of $(I + Rf)/I$ is an associated prime of I . Thus, $f/1 \notin IR_P$, and so it fails to be in at least one of the primary components \mathfrak{A} of IR_P , and f will not be in the contraction of \mathfrak{A} to R .

Hence, it suffices to prove the result when $R = (R, P)$ is local and P is an associated prime of I . Let J denote the ideal $\bigcup_{t=1}^{\infty} (I : P^t)$, the saturation of I with respect to P . Then J is axes closed, by Proposition 4.10. If we localize at any prime of R other than P , I and J have the same extension. Thus, I and J have the same associated primes, except that P is an associated prime of I and not J , and so J has fewer associated primes than I . Thus, J remains axes closed when we localize at any of its associated primes. By the induction hypothesis, J is a finite intersection of primary axes closed ideals. We know that J/I is killed by a power of P , and so has finite length. Let \mathcal{S} be the set of ideals contained in J that are finite intersections of primary axes closed ideals that contain I . Note that we have shown $J \in \mathcal{S}$. \mathcal{S} is a directed family by \supseteq , since it is closed under finite intersection. Since J/I has finite length, we can choose $J_0 \in \mathcal{S}$ such that the length of J_0/I is minimum. We can complete the proof by showing that $J_0 = I$. But if $f \in J_0 - I$, by Proposition 4.11 we can choose a primary axes closed ideal Q that contains I and not f . Then $Q \cap J_0 \in \mathcal{S}$, and Q/I is strictly contained in J_0/I and, therefore, of smaller length, a contradiction. Thus, $J_0 = I$, as required. \square

Corollary 7.14. *For a primary ideal I in an affine \mathbb{C} -algebra, $I = I^{\text{ax}}$, $I = I^{\text{cont}}$, and $I = I^{\natural}$ are equivalent. Moreover, in an affine \mathbb{C} -algebra, an unmixed ideal is continuously closed if and only if it is axes closed.*

Proof. We know that the continuous closure lies between the natural closure and the axes closure. Hence, the first statement follows at once from Theorem 7.8. If I is unmixed and continuously closed, each primary component is a primary component for a minimal prime of I and is continuously closed as well by Corollary 2.3. Hence, each primary component is axes closed, and an intersection of axes closed ideals is axes closed. \square

Corollary 7.15. *Let I be an ideal of an excellent ring R such that R/I is zero-dimensional. Then the natural closure of I is the same as the axes closure of I . Moreover, if R is an affine \mathbb{C} -algebra, the continuous closure of I is the same as well.*

Proof. Since R/I^{\natural} is zero-dimensional, we may replace I with I^{\natural} . Let \mathfrak{m} be a prime (hence maximal) ideal that contains I . Since $I = I^{\natural}$, we have $IR_{\mathfrak{m}} = (IR_{\mathfrak{m}})^{\natural}$ by Proposition 5.11. But since $IR_{\mathfrak{m}}$ is $\mathfrak{m}R_{\mathfrak{m}}$ -primary by the dimension assumption, it follows from Theorem 7.8 that $IR_{\mathfrak{m}} = (IR_{\mathfrak{m}})^{\text{ax}}$. By Proposition 4.9(a), then, we have that each $IR_{\mathfrak{m}} \cap R$ is axes closed in R . Since I is the intersection of these ideals, it follows that I itself is axes closed. \square

Proposition 7.16. *If R is a finitely generated \mathbb{C} -algebra, \mathfrak{m} is a maximal ideal and I is an \mathfrak{m} -primary ideal, then I^{cont} is the contraction of $(IR_{\mathfrak{m}})^{\text{cont}}$ to R . This means that if $f \in R$, then $f \in I^{\text{cont}}$ if and only if $f/1 \in (IR_{\mathfrak{m}})^{\text{cont}}$, i.e., if and only if the germ of f at the origin is in the extension of I to the ring of germs of continuous \mathbb{C} -valued functions at the origin.*

Proof. It is clear that $J = I^{\text{cont}}$ is contained in the contraction of $(IR_{\mathfrak{m}})^{\text{cont}}$ to R . To complete the argument, it will suffice to show that J , which is an \mathfrak{m} -primary continuously closed ideal, is contracted from the ring T of germs of continuous \mathbb{C} -valued functions at $x \in X = \text{MaxSpec}(R)$ (with the Euclidean topology), where x corresponds to \mathfrak{m} . Because $J = J^{\text{cont}}$, we know that $J = J^{\text{ax}}$ by Corollary 7.15 just above. Let $f \in R - J$. Then by part (7) of Theorem 4.1 we can choose $\theta : R \rightarrow (A, \mathfrak{n})$ such that $f \notin JA$, where θ is a \mathbb{C} -homomorphism, A is an analytic ring of axes over \mathbb{C} , and $\theta^{-1}(\mathfrak{n})$ is a maximal ideal. But since $JA \neq A$, we have $JA \subseteq \mathfrak{n}$, whence $\mathfrak{n} \supseteq \sqrt{JA} \supseteq \sqrt{JA} = \mathfrak{m}A$, so that $\mathfrak{m} \subseteq \theta^{-1}(\mathfrak{n})$. Thus $\mathfrak{m} = \theta^{-1}(\mathfrak{n})$. By Lemma 3.5 of [Bre06], JA is contracted from T . Hence, $f \notin JT$. Thus, J is contracted from T , as required. \square

Theorem 7.17. *Let R be a reduced excellent ring. R is seminormal if and only if every principal ideal generated by a nonzerodivisor is axes closed.*

Proof. We first prove “if”. Suppose that g is an element of the total quotient ring of R such that $g^2, g^3 \in R$. We must show that $g \in R$. Since g is in the total quotient ring of R , we can choose f , a nonzerodivisor in R , such that $gf \in R$. We claim that gf is in the axes closure of f . We use the test for being in the axes closure provided by Theorem 7.12. Suppose that we have any ring map $\theta : R \rightarrow S$ such that S is an excellent seminormal one-dimensional ring and $u := \theta(f)$ is a nonzerodivisor. Let

$w := \theta(fg)$. Consider the element $v = w/u$ of the total quotient ring of S . We have

$$u^2v^2 = (uw)^2 = w^2 = \theta(fg)^2 = \theta(f^2)\theta(g^2) = u^2\theta(g^2).$$

Since u^2 is a nonzerodivisor in the total quotient ring of S , it follows that $\theta(g^2) = v^2$, and in particular that $v^2 \in S$. Similar computations show that $\theta(g^3) = v^3$, so that $v^3 \in S$ as well. Since S is seminormal, it follows that $v \in S$. So we have

$$\theta(fg) = w = uv \in uS = \theta(f)S.$$

Then by choice of θ and Theorem 7.12, it follows that $fg \in (fR)^{\text{ax}} = fR$. That is, there is some $r \in R$ such that $fg = fr$. Since f is a nonzerodivisor, it follows that $g = r \in R$, so that R is seminormal.

Now assume instead that R is excellent seminormal. We may assume that $\dim R \geq 2$, since otherwise every ideal of R is axes closed by Proposition 4.5. Suppose some element $g \in R$ is in the axes closure of fR , where f is a nonzerodivisor. Let S be the integral closure of R . Since S is a product of finitely many normal domains (the normalizations of the quotients of R by its various minimal primes), every principal ideal of S is integrally closed, hence axes closed, so $g \in fS$. That is, $g = hf$ for some $h \in S$. If we can show that h is in the seminormalization of R in S , which is R , then we are done. To this end we use the criterion from Proposition 3.1(1). For the remainder of the proof we may change notation: we replace R by R_P (where P is an arbitrary prime ideal of R), and so assume that (R, P, K) is local, and we replace S by S_P , which is the integral closure of R_P . Then S is semilocal, and we denote the maximal ideals of S by Q_1, \dots, Q_n . We want to show that h is in $R + Q_1 \cap \dots \cap Q_n$. Let c_i denote the image of h in L_i , $1 \leq i \leq n$, where $L_i = S/Q_i$. Note that we may identify K with a subfield of L_i for every i . Note that L_i is a finite algebraic extension of K for every i . We shall show that all of the c_i are in K and that they are all equal. If $r \in R$ represents their common value, then $h - r$ is in all of the Q_i , which yields the desired conclusion.

For each Q_i , choose a prime ideal q_i of S contained in Q_i and maximal with respect to not containing f . Then Q_i/q_i is a minimal prime of $f(S/q_i)$, and so Q_i/q_i has height one. Let $p_i = Q_i \cap R$. Then R/p_i is a local domain, and since $R/p_i \hookrightarrow S/q_i$ is module-finite, we must have that $\dim(R/p_i) = 1$, by the dimension formula [Mat86, Theorem 15.6]. The image of f is a nonzerodivisor in both R/p_i and $(S/q_i)_{Q_i}$, and so is a nonzerodivisor in both of their completions. We have an induced map of the completions $C_i \rightarrow D_i$, which are one-dimensional reduced complete local rings. Choose a minimal prime of D_i . Its contraction to C_i will not contain the image of f , and so is also a minimal prime. We get an induced map of quotient domains $\overline{C}_i \hookrightarrow \overline{D}_i$. In each, the image of f is nonzero. Let V_i be the normalization of \overline{D}_i ; then V_i is a complete local discrete valuation domain whose residue class field is an extension of L_i and contains K .

Let W_i denote the seminormalization of \overline{C}_i . Since V_i is normal, the normalization of \overline{C}_i may be constructed as a subring of V_i , and we may view $W_i \subseteq V_i$. Then W_i is a one-dimensional seminormal ring, and we have a map $R \twoheadrightarrow R/p_i \hookrightarrow C_i \twoheadrightarrow \overline{C}_i \hookrightarrow W_i \subseteq V_i$. Let \mathcal{W}_i denote the subring of V_i consisting of all elements with image in K modulo the maximal ideal of V_i . Then $\overline{C}_i \subseteq \mathcal{W}_i$, and whenever $a \in V_i$ is such that $a^2, a^3 \in \mathcal{W}_i$, one has that $a \in \mathcal{W}_i$ as well. Thus, $W_i \subseteq \mathcal{W}_i$.

We shall show $c_i \in K$, using that the image of g is in fW_i . We have a commutative diagram

$$\begin{CD} S @>\beta>> V_i \\ @VVV @VVV \\ R @>\alpha>> W_i \end{CD}$$

where the vertical maps are inclusions. Then $\alpha(g) = \beta(g) = \beta(fh) = \beta(f)\beta(h) = \alpha(f)\beta(h)$. Since $\alpha(g) \in \alpha(f)W_i$ (which holds because g is in the axes closure of fR) and since $\alpha(f)$ is a nonzerodivisor on V_i , it follows that $\beta(h) \in W_i$. This implies that $\beta(h) \in \mathcal{W}_i$, and so its residue c_i is in K .

Finally, suppose that we have c_i and c_j in K for $i \neq j$. We shall show $c_i = c_j$. Let V_i and V_j be as above. Each is a complete discrete valuation domain whose residue class field is an algebraic extension of L_i and, hence, of K . Enlarge V_i to a complete discrete valuation domain \mathcal{V}_i whose residue class field is the algebraic closure of K . Thus, there is a K -isomorphism θ between the residue class fields of \mathcal{V}_i and \mathcal{V}_j . Let F be the field in question. Let $\mathcal{A} = \mathcal{V}_i \times_{\theta} \mathcal{V}_j$ be the pullback of the surjection $\mathcal{V}_i \times \mathcal{V}_j \rightarrow F \times F$ along the diagonal embedding $F \hookrightarrow F \times F$. By Theorem 3.3, \mathcal{A} is an excellent local one-dimensional seminormal ring. We have maps $\eta_i : S \rightarrow V_i \subseteq \mathcal{V}_i$ and (η_i, η_j) , and therefore maps $S \rightarrow \mathcal{V}_i \times \mathcal{V}_j$. It is clear that the image of R lies in \mathcal{A} , since θ is a K -isomorphism. Hence, we have a commutative diagram

$$\begin{CD} S @>\beta>> \mathcal{V}_i \times \mathcal{V}_j \\ @VVV @VVV \\ R @>\alpha>> \mathcal{A} \end{CD}$$

where the vertical maps are inclusions. Once again, $\alpha(g) = \beta(g) = \beta(fh) = \beta(f)\beta(h) = \alpha(f)\beta(h)$, and since $\alpha(g) \in \alpha(f)\mathcal{A}$ and $\alpha(f)$ is a nonzerodivisor on \mathcal{A} (and hence also on $\mathcal{V}_i \times \mathcal{V}_j$, which is the normalization of \mathcal{A}), we must have $\beta(h) \in \mathcal{A}$. This implies that the residues of h correspond under θ , and since these are in K and θ is a K -isomorphism, they must be equal. □

Corollary 7.18. *Let R be a reduced affine \mathbb{C} -algebra, and let f be a nonzerodivisor of R . Then $(fR)^{\text{cont}} = (fR)^{\text{ax}} = fS \cap R$, where S is the seminormalization of R . In particular, if S is seminormal and f is a nonzerodivisor, fS is both continuously closed and axes closed.*

Proof. By Theorem 7.17, fS is axes closed and, consequently, continuously closed as well. Hence, $(fR)^{\text{ax}} \subseteq (fS)^{\text{ax}} \cap R = fS \cap R$. By Proposition 3.2, $(fR)^{\text{cont}} = (fS)^{\text{cont}} \cap R$, and since fS is axes closed, $(fS)^{\text{cont}} = fS$, so that $(fR)^{\text{cont}} = fS \cap R$. This shows that $(fR)^{\text{ax}} \subseteq (fR)^{\text{cont}}$, and since we always have the opposite inclusion, it follows that all three of $(fR)^{\text{cont}}$, $(fR)^{\text{ax}}$, and $fS \cap R$ are equal. □

8. MULTIPLYING BY INVERTIBLE IDEALS AND RINGS OF DIMENSION 2

In this section we prove that continuous closure and axes closure agree in locally factorial affine \mathbb{C} -algebras of dimension 2. In particular, this holds for the polynomial ring in two variables over \mathbb{C} . In §9 we give an example which shows that they do not agree in the polynomial ring in three variables over \mathbb{C} . In order to prove the main result of this section, we need some preliminary results.

Lemma 8.1. *Let $\#$ be a closure operation (see Definition 1.1) on R . Suppose that for any two ideals I, J or R , $I\#J\# \subseteq (IJ)\#$. Suppose further that for any nonzerodivisor $r \in R$ and any ideal J such that $J = J\#$, we have $rJ = (rJ)\#$. Let I be an ideal of R that is locally free of rank one. Then for every ideal J of R , $I(J\#) = (IJ)\#$. In particular, if I is locally free of rank one, then $I = I\#$.*

Proof. Evidently, $I(J\#) \subseteq I\#J\# \subseteq (IJ)\#$, and so it suffices to show that $(IJ)\# \subseteq I(J\#)$. Since the latter contains IJ , it suffices to show that the latter is closed. We may replace J by $J\#$, and so assume that $J = J\#$, and we want to prove that IJ is closed. Since I is projective of rank one, it is an invertible ideal, and we may choose an ideal I' such that $I'I = rR$, where r is a nonzerodivisor. Then $I'(IJ)\# \subseteq (I'IJ)\# = (rJ)\# = r(J\#) = (I'I)J\# = I'(I(J\#))$. Multiplying by I then yields that $r((IJ)\#) = r(I(J\#))$. Since r is a nonzerodivisor, it follows that $(IJ)\# = I(J\#)$. The final statement is the case where $J = R$. \square

Theorem 8.2. *Let X be a closed affine algebraic set over \mathbb{C} , and let $R = \mathbb{C}[X]$. Let I be an ideal that is locally free of rank one, and let J be any ideal. If R is seminormal, then $(IJ)^{\text{cont}} = I(J^{\text{cont}})$.*

Proof. By Lemma 8.1, Proposition 2.4, and Corollary 7.18, we may assume that $I = rR$, where r is a nonzerodivisor in R . Also by Proposition 2.4, $r(J^{\text{cont}}) \subseteq (rJ)^{\text{cont}}$. Now suppose that $u \in (rJ)^{\text{cont}} \subseteq (rR)^{\text{cont}}$. Since R is seminormal, we know from Corollary 7.18 that rR is continuously closed, and so we may write $u = rf$ for some $f \in R$. Then $rf = u = \sum_{i=1}^n g_i r f_i$ (where f_1, \dots, f_n is a generating set for the ideal J) for some $g_1, \dots, g_n \in \mathcal{C}(X)$. Since r is not a zerodivisor in $\mathcal{C}(X)$, we have $f = \sum_{i=1}^n g_i f_i$ as well. \square

Theorem 8.3. *Let R be a seminormal excellent Noetherian ring. Let I be an ideal of R that is locally free of rank one and let J be any ideal of R . Then $(IJ)^{\text{ax}} = I(J^{\text{ax}})$.*

Proof. By Lemma 8.1, Lemma 4.7, and Theorem 7.17, we reduce at once to the case where $I = rR$ is generated by a nonzerodivisor r . Since $r(J^{\text{ax}}) \subseteq (rJ)^{\text{ax}}$, it suffices to prove the other inclusion. Since rR is axes closed by Theorem 7.17, it suffices to show that if $rf \in (rJ)^{\text{ax}}$, then $f \in J^{\text{ax}}$. If we have a counterexample, then by Theorem 7.12, there is a map $h : R \rightarrow A$, where A is a one-dimensional excellent seminormal ring, $h(r)$ is a nonzerodivisor in A , and $h(f)$ is not in $h(J)A$. Then $h(rf) = h(r)h(f) \notin h(r)h(J)A = h(rJ)A$, and so $rf \notin (rJ)^{\text{ax}}$, a contradiction. \square

Discussion 8.4. Let R be a locally factorial domain. When R is factorial, every nonzero ideal \mathfrak{A} is uniquely the product of a principal ideal (which may be R) and an ideal of height at least two (which may also be R : the height of the unit ideal is $+\infty$). The principal ideal is generated by the greatest common divisor of any given set of generators of \mathfrak{A} , which is unique up to a unit multiplier, and is the same as the greatest common divisor of all elements of \mathfrak{A} . When R is only *locally* factorial, we may say instead that every nonzero ideal \mathfrak{A} factors uniquely as the product of an ideal that is locally free of rank one and an ideal of height at least two. One can perform the factorization uniquely in every local ring of R , since the local rings are factorial. But one can actually carry this out on a cover by open affines: it is clear that the factorizations on two affines will be the same on the overlaps, since the factorization is unique in every local ring of R . To get the factorization on a

neighborhood of a prime Q , consider the height one primes P_1, \dots, P_k of R that contain \mathfrak{A} and are contained in Q . Each P_i becomes principal when expanded to R_Q . Localize R at one element $f \notin Q$ so that each $P_i R_f$ is principal, say $\pi_i R_f$, and so that the only height one primes of R_f that contain $\mathfrak{A} R_f$ are the $P_i R_f$. Suppose that $\mathfrak{A} R_{P_i} = \pi_i^{a_i} R_{P_i}$. Then $\mathfrak{A} R_f$ factors as rJ where $r = \pi_1^{a_1} \cdots \pi_k^{a_k}$, since comparing primary decompositions shows that $\mathfrak{A} \subseteq rR_f$. The factor J is not contained in any height one prime of R_f , and so this is the desired factorization.

If, moreover, R has dimension at most two, then when we factor \mathfrak{A} in this way, the second factor J is either the unit ideal or is contained only in maximal ideals of height two, and is unmixed in the sense of having no embedded primes.

Theorem 8.5. *Let R be a domain of dimension two that is a locally factorial affine \mathbb{C} -algebra. Then axes closure and continuous closure agree for R .*

Proof. The result is immediate from the preceding discussion, Theorem 8.2, Theorem 8.3, and Corollary 7.15. □

9. A NEGATIVE EXAMPLE AND A FIBER CRITERION
FOR EXCLUSION FROM THE CONTINUOUS CLOSURE

We begin with an inclusion lemma for axes closure.

Lemma 9.1. *Let R be an excellent Noetherian ring. Let I be an ideal and let I^- be the integral closure of I . Let $P \in \text{Spec } R$ and let $J := (P \cdot I^-) \cap (I : P)$. Then $J \subseteq I^{\text{ax}}$.*

Proof. Let $f : R \rightarrow (A, \mathfrak{m})$ be a ring homomorphism, where A is a complete local one-dimensional seminormal ring. If $f(P) \not\subseteq \mathfrak{m}$, then $JA \subseteq (I : P)A = IA$. Thus, we may assume that $f(P) \subseteq \mathfrak{m}$. In that case, the image of J is contained in $\mathfrak{m}(IA)^-$. But by Lemma 5.7, $\mathfrak{m}(IA)^- \subseteq IA$. Hence, $JA \subseteq IA$, as required. □

Example 9.2. In the polynomial ring $\mathbb{C}[u, v, x]$, the element uvx is in the axes closure of $I = (u^2, v^2, uvx^2)$ but not the continuous closure.

The first statement follows from Lemma 9.1, applied to the ideal I and the prime ideal $P = (u, v, x)$, since $uv \in I^-$.

Now suppose that uvx is in the continuous closure of I , say

$$uvx = fu^2 + gv^2 + huvx^2,$$

where f, g, h are continuous functions of u, v, x in that order. Let $a = h(0, 0, 0)$. Choose a constant $c \neq 0$ such that $ch(0, 0, c) \neq 1$; this is possible since $xh(0, 0, x) \rightarrow 0 \cdot a = 0$ as $x \rightarrow 0$. Substitute $x = c$ in the displayed equation. Then

$$cuv = f_0u^2 + g_0v^2 + h_0c^2uv,$$

where the subscript indicates the function of u, v obtained by substituting $x = c$ in f, g, h , respectively. The new equation involves only u, v . The function $c - h_0c^2$ has value $c(1 - ch(0, 0, c)) \neq 0$ at the origin in the u, v -plane, and so has a continuous inverse s on a neighborhood U of the origin. Then

$$uv = Fu^2 + Gv^2,$$

where the coefficients $F := sf_0$ and $G := sg_0$ are continuous functions defined on U . But this yields a contradiction. To see this, let $A := \{(u, v) \mid u = v\} \cap U$ and $B := \{(u, v) \mid u = -v\} \cap U$. On the set $A \setminus \mathbf{0}$, we have $F + G = 1$, so that by

continuity, $F(0,0) + G(0,0) = 1$. But on $B \setminus \mathbf{0}$, we have $F + G = -1$, so that $F(0,0) + G(0,0) = -1$ (again by continuity), whence $1 = -1$, an absurdity.

Generalizing the counterexample. If $R = \mathbb{C}[X]$ for an affine algebraic set X and $x \in X$, recall from Discussion 2.6 that we write $I^{\text{cont},x}$ for the set of elements of the local ring R_x of $\mathbb{C}[X]$ at x that are continuous linear combinations of elements of I on some Euclidean neighborhood of x in X . This is the contraction to R_x of the extension of I to the ring of germs of continuous (in the Euclidean topology) functions on X at x .

Suppose that $B \rightarrow R$ is a \mathbb{C} -homomorphism of finitely generated \mathbb{C} -algebras such that B, R are reduced. Let $Y = \text{Max Spec}(B)$ and $X = \text{Max Spec}(R)$. Thus, we have a map $\pi : X \rightarrow Y$. If $y \in Y$, let \overline{R}^y denote the coordinate ring of the reduced fiber over y , i.e., if \mathfrak{m}_y is the maximal ideal of B corresponding to y , then $\overline{R}^y = (R/\mathfrak{m}_y R)_{\text{red}}$. $\text{Max Spec}(\overline{R}^y)$ may be identified with the fiber $X_y = \pi^{-1}(y)$, and \overline{R}^y with the ring of regular functions on $\pi^{-1}(y)$. We have a surjection $R \rightarrow \overline{R}^y$ for every y , which may be thought of as a restriction of regular functions from X to X_y . If $g \in R$, we write g^y for the image of g in \overline{R}^y .

Theorem 9.3 (Fiber criterion for exclusion from continuous closure). *Let B, R be as in the paragraph above, and let notation be as in that paragraph. Suppose that $f, g \in R$ and $I, J \subseteq R$ are ideals. Suppose that:*

- (1) $f \notin J^{\text{cont}}$ in R .
- (2) $\{x \in X : g^{\pi(x)} \notin (I\overline{R}^{\pi(x)})^{\text{cont},x}\}$ is dense in X in the Euclidean topology.

Then $gf \notin (I + gJ)^{\text{cont}}$ in R .

Before giving the proof, we show how the example from the beginning of this section can be analyzed using this criterion. Let $B = \mathbb{C}[x] \subseteq \mathbb{C}[x, u, v] = R$. Let $f = x, g = uv, I = (u^2, v^2)R$, and $J = x^2R$. Note that $f \notin J^{\text{cont}}$ in R . The fibers are simply the rings obtained by specializing x to a complex constant, and all of them may be identified with $\mathbb{C}[u, v] \subseteq R$. In this case, $uv \notin (u^2, v^2)^{\text{cont},x}$ in all fibers $\mathbb{C}[u, v]$ for all x . To see this, observe that (u^2, v^2) is primary to the maximal ideal (u, v) of $\mathbb{C}[u, v]$. It is clear that $uv \notin (u^2, v^2)^{\natural}$, which is the same as $(u^2, v^2)^{\text{cont}} = (u^2, v^2)^{\text{ax}}$ by Corollary 7.15, and we may apply Proposition 7.16 to conclude that $uv \notin (u^2, v^2)^{\text{cont},(0,0)}$. Hence, $xuv \notin (u^2, v^2, x^2uv)^{\text{cont}}$ in R .

Proof of the fiber criterion. Let u_1, \dots, u_h generate I and let v_1, \dots, v_k generate J . Suppose that

$$fg = \sum_{i=1}^h \alpha_i u_i + \sum_{j=1}^k \beta_j g v_j,$$

wheres the α_i and β_j are continuous. Then $\gamma = f - \sum_{j=1}^k \beta_j v_j$ is a continuous function on X that does not vanish identically, since $f \notin J^{\text{cont}}$. Then $U = \gamma^{-1}(\mathbb{C} - \{0\})$ is a nonempty subset of X that is open in the Euclidean topology. Hence, it must meet $\{x \in X : g^{\pi(x)} \notin (I\overline{R}^{\pi(x)})^{\text{cont},x}\}$. Thus, we may choose $x \in X$ such that $\gamma(x) \neq 0$ (since $x \in U$) and $g^{\pi(x)} \notin (I\overline{R}^{\pi(x)})^{\text{cont},x}$. But then there is a Euclidean neighborhood of x on which γ does not vanish, so that $1/\gamma$ is a continuous function on this neighborhood, and, if $y = \pi(x)$, on the intersection of this neighborhood

with X_y we have

$$g^y = \sum_{i=1}^h \frac{\alpha_i(y)}{\gamma(y)} u_i^y,$$

which shows that $g^y \in (I\overline{R}^y)^{\text{cont},x}$, a contradiction. □

10. MIXED NATURAL CLOSURE AND CONTINUOUS CLOSURE
FOR MONOMIAL IDEALS IN POLYNOMIAL RINGS

We know that for an ideal I of a finitely generated reduced \mathbb{C} -algebra, $I^\natural \subseteq I^{\text{cont}} \subseteq I^{\text{ax}}$, and that if I is unmixed, the conditions that it be naturally closed, continuously closed, and axes closed are all equivalent. It turns out that when I is mixed, even if it is a monomial ideal, a naturally closed ideal need not be continuously closed and a continuously closed ideal need not be axes closed.

Mixed natural closure is intended to make up for the following defect of natural closure: when $f \in R$, it is possible that fg is not in the natural closure of fI even though g is in the natural closure of I . (Note that it is obvious that if g is in the continuous closure of I , then fg is in the continuous closure of fI .)

In fact, in the polynomial ring $\mathbb{C}[u, v, x]$ the ideal (u^3x, v^3x) is naturally closed but not continuously closed, since u^2v^2x is in the continuous closure. The point is that u^2v^2 is in the natural closure of (u^3, v^3) , and so it is in the continuous closure. It follows that u^2v^2x is in the continuous closure of (u^3x, v^3x) . To see that $\mu = u^2v^2x$ is not in the natural closure, note that for I monomial both I^\natural and $I_{>1}$ are monomial. Thus, it suffices to show that $\mu \notin I$ and $\mu \notin I_{>1}$. The first is clear. The second is true since it is obviously true even after we localize at the prime ideal (x) .

We note also that in $\mathbb{C}[u, v, x]$ the ideal (u^2, v^2, uvx^2) is continuously closed but not axes closed; see Example 9.2 of the preceding section.

In this section, we first define a modified version of natural closure for ideals I in any ring, which we refer to as *mixed natural closure*. We let $I^{\natural^{\text{mix}}}$ denote the mixed natural closure of I . If $I = I^{\natural^{\text{mix}}}$, we say that I is *MN-closed*. It will turn out that $I^\natural \subseteq I^{\natural^{\text{mix}}}$, that a primary ideal is MN-closed if and only if it is naturally closed. The definition will force it to be true that if $fJ \subseteq I$ and $u \in J^{\natural^{\text{mix}}}$, then $fu \in I^{\natural^{\text{mix}}}$.

We shall eventually use the fiber criterion to prove that for monomial ideals in polynomial rings over \mathbb{C} , continuous closure always equals mixed natural closure.

Definition 10.1. We shall say that an ideal I of a ring R is *MN-closed* if whenever $fJ \subseteq I$ and $u \in J^\natural$, then $fu \in I$. An obviously equivalent condition is that for all $f \in R$, $f((I :_R f)^\natural) \subseteq I$. Since f may be taken to be 1, an MN-closed ideal is naturally closed. An intersection of MN-closed ideals is clearly MN-closed. We define the *mixed natural closure* of I , denoted $I^{\natural^{\text{mix}}}$, to be the intersection of all the MN-closed ideals containing I , which is evidently the smallest MN-closed ideal containing I . It is easy to see that $u \in I^{\natural^{\text{mix}}}$ if and only if (\dagger) there are finite sequences of ideals $I = I_0, I_1, \dots, I_h, J_0, \dots, J_{h-1}$ and finite sequences of elements u_1, \dots, u_h and f_1, \dots, f_h in R such that $u \in I_h$ and such that for every $i, 0 \leq i < h$, $f_{i+1}J_i \subseteq I_i$, $u_{i+1} \in J_i^\natural$, and $I_{i+1} = I_i + f_{i+1}u_{i+1}R$. We note that we may take the ideals J_i to be finitely generated, and if $u_i \in J_i^\natural$ there is a subring R_0 of R finitely generated over \mathbb{Z} and containing u_i , and an ideal J_0 of R_0 generated by finitely many elements of J_i such that $u_i \in J_0^\natural$. It follows, for example, that $I^{\natural^{\text{mix}}}$ is

the union of the ideals $(I \cap S)^{\natural^{\text{mix}}}$ (in S) as S runs through any directed family of subrings of R whose union is R . In particular, we may let S run through subrings of R finitely generated over \mathbb{Z} .

We give a third description of $I^{\natural^{\text{mix}}}$. Let $\mathfrak{A}_1(I) := \mathfrak{A}(I) := \sum_{f \in R} f(I :_R f)^{\natural}$. Recursively, define $\mathfrak{A}_{n+1}(I) = \mathfrak{A}(\mathfrak{A}_n(I))$. Note that $I \subseteq I^{\natural} \subseteq \mathfrak{A}(I)$ since we may take $f = 1$. Then it is easy to see $I = I^{\natural^{\text{mix}}}$ if and only if $I = \mathfrak{A}(I)$ and that $I^{\natural^{\text{mix}}} = \bigcup_{n=1}^{\infty} \mathfrak{A}_n(I)$.

The following two propositions are immediate from the definition and discussion above.

Proposition 10.2. *Let (R_λ, I_λ) be a directed family of rings and ideals (so that the map $R_\lambda \rightarrow R_\mu$ takes I_λ into I_μ). Let (R, I) be the direct limit, where $I \subseteq R$ is an ideal. Then $I^{\natural^{\text{mix}}}$ is the direct limit of the ideals $I_\lambda^{\natural^{\text{mix}}}$ (calculated in the respective R_λ).* \square

Proposition 10.3. *If $I \subseteq J$ are ideals of R , then $I^{\natural^{\text{mix}}} \subseteq J^{\natural^{\text{mix}}}$ and $(I^{\natural^{\text{mix}}})^{\natural^{\text{mix}}} = I^{\natural^{\text{mix}}} \supseteq I$. In other words, mixed natural closure is a closure operation in the sense of Definition 1.1.* \square

We also have the following.

Proposition 10.4. *If $I \subseteq R$, $I^{\natural} \subseteq I^{\natural^{\text{mix}}}$. If $-\#$ is any closure operation on ideals of R such that $I^{\natural} \subseteq I^{\#}$ for all ideals $I \subseteq R$ and $f \cdot (I^{\#}) \subseteq (fI)^{\#}$ for all $f \in R$ and all ideals $I \subseteq R$, then $I^{\natural^{\text{mix}}} \subseteq I^{\#}$ for all I . In particular, $I^{\natural^{\text{mix}}} \subseteq I^-$, $I^{\natural^{\text{mix}}} \subseteq I^{\text{ax}}$ whenever I^{ax} is defined and $I^{\natural^{\text{mix}}} \subseteq I^{\text{cont}}$ when I^{cont} is defined. Consequently, if $I = I^{\text{cont}}$, then $I = I^{\natural^{\text{mix}}}$. Moreover, if f is a nonzerodivisor in a seminormal ring, then fR is MN-closed.*

Proof. The statements are immediate except for the last statement. This statement reduces to the case where R is finitely generated over the integers and, hence, excellent. The result now follows from Theorem 7.17, by which fR is axes closed. \square

Proposition 10.5. *If $R \rightarrow S$ is a homomorphism and I is an ideal of R , then $I^{\natural^{\text{mix}}}$ maps into $(IS)^{\natural^{\text{mix}}}$ in S . That is, mixed natural closure is persistent. Hence, if J is MN-closed in S , its contraction to R is MN-closed in R . If W is any multiplicative system in R , the mixed natural closure of $IW^{-1}S$ in $W^{-1}S$ is $I^{\natural^{\text{mix}}}W^{-1}S$.*

Proof. The first statement follows at once from the recursive construction of the elements in the mixed natural closure and the fact that natural closure is persistent, and the second statement is immediate from the first. The fact that $I^{\natural^{\text{mix}}}W^{-1}S \subseteq (IW^{-1}S)^{\natural^{\text{mix}}}$ is a special case of persistence. For the converse, we use the final construction of $I^{\natural^{\text{mix}}}$ in Definition 10.1. It suffices to show that $\mathfrak{A}(IW^{-1}S) \subseteq W^{-1}\mathfrak{A}(IS)$. This follows at once because for all $f \in S$ and $w \in W$,

$$IW^{-1}S :_{W^{-1}S} (f/w) = IW^{-1}S :_{W^{-1}S} (f/1) = (IS :_S f)W^{-1}S;$$

moreover, localization commutes with natural closure (see Proposition 5.11) and the sum of ideals. \square

Corollary 10.6. *Let $I \subseteq R$ and let $u \in R$. Then $u \in I^{\natural^{\text{mix}}}$ if and only if for every prime (respectively, maximal) ideal m of R , $u/1 \in (IR_m)^{\natural^{\text{mix}}}$.* \square

Proposition 10.7. *If I is primary to a prime ideal P of R , then $I = I^{\natural\text{mix}}$ if and only if $I = I^{\natural}$.*

Proof. It suffices to show that if $I = I^{\natural}$, then $I = I^{\natural\text{mix}}$. For this we must show that if $fJ \subseteq I$ and $u \in J^{\natural}$, then $fu \in I$. The condition that $u \in J^{\natural}$ can be expressed in terms of finitely many elements of J and R with certain equations holding among them. Therefore, we can choose a subring R_0 of R that is finitely generated over \mathbb{Z} that contains u, f , and these elements. Let I_0 be the contraction of I to R_0 , J_0 the contraction of J , and let P_0 be the contraction of P . Then it is still true that I_0 is primary to P_0 , that I_0 is naturally closed, that $fJ_0 \subseteq I_0$, and that u is in the natural closure of J_0 in R_0 . It will suffice to show that $fu \in I_0$ in R_0 . Therefore, it suffices to consider the case where R is a finitely generated algebra over \mathbb{Z} . In particular, we may assume without loss of generality that R is excellent.

The result is now immediate from Theorem 7.8 and Proposition 10.4, for we have $I = I^{\natural}$ for a primary ideal implies $I = I^{\text{ax}}$ as well in the excellent case, and $I^{\natural\text{mix}} \subseteq I^{\text{ax}} = I$. □

Proposition 10.8. *Let J, I be ideals of the ring R and let $f \in R$. Suppose that $I = I^{\natural\text{mix}}$. Then $fJ \subseteq I$ implies $f \cdot (J^{\natural\text{mix}}) \subseteq I$.*

Proof. If $u \in J^{\natural\text{mix}}$, then there are sequences of ideals and elements as in condition (†) of Definition 10.1 such that J is successively enlarged by one additional generator at a time until one reaches an ideal containing u . It will suffice to show that each of the successive enlargements J_i of J has the property that $fJ_i \subseteq I$. Since the base step of the induction is simply that $fJ \subseteq I$, it will suffice to show that if $fJ_i \subseteq I$ and $J_{i+1} = J_i + Rgv$, where there is an ideal \mathfrak{A} of R and $g \in R$ such that $g\mathfrak{A} \subseteq J_i$ and $v \in \mathfrak{A}^{\natural}$, then $fgv \in I$. Since $fg\mathfrak{A} \subseteq fJ_i \subseteq I$, the result follows at once from the fact that $I = I^{\natural\text{mix}}$. □

Remark. If an ideal I is graded or multigraded, the same holds for its closure or another ideal derived from it in many instances. This is true for I^{cont} by Proposition 2.10, $I_{>1}$ and I^{\natural} by Proposition 5.14, as well as for I^{ax} by Proposition 4.14. In particular, in each of these cases, if I is a monomial ideal in a polynomial ring over a field, so is I^{cont} , $I_{>1}$, I^{\natural} , and I^{ax} . This is also true of $I^{\natural\text{mix}}$ if the ring contains an infinite field. We conjecture that this hypothesis is unnecessary.

Proposition 10.9. *Let R be a \mathbb{Z}^h -graded ring, let $h > 0$, and let I be a homogeneous ideal with respect to this grading. Assume that R contains an infinite field. Then $I^{\natural\text{mix}}$ is graded. If R is a polynomial ring over a ring A , then the MN-closure of every monomial ideal is a monomial ideal. In this latter case, if I is an ideal of A that is MN-closed, then IR is MN-closed.*

Proof. All of the statements but the last are immediate from Discussion 2.9. For the final statement, by a direct limit argument, we may assume that the number of variables is finite: say $R = A[x_1, \dots, x_h]$. Let F be a polynomial in the mixed natural closure of IR . It suffices to show that each term $a\mu$ of F , where μ is monomial in x_i , is such that $a \in I$. Since $(IR)^{\natural\text{mix}}$ is multigraded, we have that $a\mu$ is in $(IR)^{\natural\text{mix}}$. There is a unique A -homomorphism $R \rightarrow A$ that maps all the variables to 1. Since mixed natural closure is persistent by Proposition 10.5, the image of $a\mu$, which is a , is in the mixed natural closure of I , which is I . □

Remark. By Discussion 2.9, mixed natural closure preserves \mathbb{Z}^h -gradings whenever the ring contains arbitrarily large families of units such that the difference of any two distinct elements is also a unit. We conjecture that the results of Proposition 10.9 hold without any hypothesis on R . However, we have not been able to prove in general that (*) if I is MN-closed in A , then $IA[x]$ is MN-closed in the polynomial ring $A[x]$. (Of course, if one knows this for one variable one gets the result for arbitrary sets of variables.) If we simply knew this fact, it would imply that mixed natural closure commutes with \mathbb{Z}^h -gradings in general. To see this, suppose that R is \mathbb{Z}^h -multigraded and that $F \in I^{\text{mix}}$ for a graded ideal I . By Discussion 2.9, if s is a sufficiently large positive integer, t_1, \dots, t_s are new indeterminates of degree $(0, \dots, 0)$, and $g = \prod_{i=1}^s t_i \prod_{1 \leq j < k \leq s} (t_j - t_k)$, then each component F_λ of F is in the mixed natural closure of IS , where $S = R[t_1, \dots, t_s][1/g]$. Since mixed natural closure commutes with localization by Proposition 10.5, there is an integer $N > 0$ such that $g^N F_\lambda$ is in the mixed natural closure of $IR[t_1, \dots, t_s]$. Let J be the mixed natural closure of I in R . If we know (*), we can conclude that when $g^N F_\lambda$ is viewed as a polynomial in t_i , every coefficient is in J . Hence, we have the desired result if some coefficient of $g^N F_\lambda$ is F_λ . It is therefore sufficient to note that for lexicographic order with $t_1 > \dots > t_n$, the highest order term in g (and hence in g^N) has coefficient 1 (this term is $(\prod_{i=1}^s t_i) \prod_{j=1}^s t_j^{s-j}$). \square

Lemma 10.10. *Let R be a polynomial ring over \mathbb{C} in variables x_1, \dots, x_n . Let I_1 be a naturally closed monomial ideal primary to the homogeneous maximal ideal \mathfrak{m}_1 in $R_1 = \mathbb{C}[x_1, \dots, x_k]$ for some $k < n$, and suppose that*

(#) ν is a monomial in $I_1 :_{R_1} \mathfrak{m}_1$ such that $I_1 + \nu R_1$ is naturally closed in R_1 .

Let J be a continuously closed monomial ideal of $R_2 = \mathbb{C}[x_{k+1}, \dots, x_n]$. Then $I_1 R + \nu J R$ is continuously closed in R .

Proof. The continuous closure of $I_1 R + \nu J R$ will be monomial. Since ν is in the integral closure of I_1 , $I_1 + \nu R_1$ is naturally closed and primary in R_1 (or, if $\nu = 1$, it is all of R_1), and so its extension to R is either R or a primary naturally closed ideal. Hence, its extension to R is continuously closed. It follows that the continuous closure of $I_1 + \nu J$ must have the form $I_1 + \nu J'$, where $J' \supseteq J$ is a monomial ideal of R_2 (when ν is multiplied by any variable in R_1 , the product is in $I_1 R$ by hypothesis).

Let f be a monomial in $J' - J$ such that $f\nu \in (I_1 + JR)^{\text{cont}}$. Let $B = R_2$ and let $g = \nu$. We apply the fiber criterion Theorem 9.3 of the preceding section. The fiber over a point of B is simply the result of specializing the x_j for $j > k$ to complex constants given by the point. Thus, all of the fibers may be identified with R_1 , and the image of $g = \nu$ is simply ν . In this case, $\nu \notin I_1^{\text{cont},v}$ in all fibers R_1 for all v in $\text{MaxSpec}(R)$. To see this, observe that I_1 is primary to the maximal ideal $(x_1, \dots, x_k)R_1$ of R_1 . We have that $\nu \notin I_1 = I_1^\natural$, which is the same as $I_1^{\text{cont}} = I_1^{\text{ax}}$ by Corollary 7.15, and we may apply Proposition 7.16 to conclude that $\nu \notin I_1^{\text{cont},(0, \dots, 0)}$. But then $f \notin I_1 R + \nu J R$ by Theorem 9.3. \square

Definition 10.11. Let K be any field, and let $R = K[x_1, \dots, x_n]$ be a polynomial ring over K . We shall say that an ideal I of R is *MN-basic* if there exist, for some positive integer k , mutually disjoint nonempty subsets S_1, \dots, S_k of the variables, monomial ideals I_1, \dots, I_k such that for $1 \leq j \leq k$, I_j is naturally closed and primary to the homogeneous maximal ideal \mathfrak{m}_j of $R_j = K[x_t : x_t \in S_j]$, and for

each j , where $1 \leq j \leq k - 1$, there is a monomial μ_j such that $(\#_j) \mu_j \in I_j :_{R_j} \mathfrak{m}_j$ and $I_j + \mu_j R_j$ is naturally closed; moreover,

$$(\dagger) \quad I = (I_1 + \mu_1 I_2 + \mu_1 \mu_2 I_3 + \cdots + \mu_1 \mu_2 \cdots \mu_{k-1} I_k)R.$$

Note that if $k = 1$ this simply means that I is a naturally closed primary monomial ideal. An alternative recursive definition is the following. We allow among the MN-basic ideals the primary naturally closed monomial ideals and those ideals obtained by the following recursive rule. If R_1 and R_2 are polynomial subrings of R generated by mutually disjoint subsets of the variables, I_1 is a naturally closed monomial ideal primary to the homogeneous maximal ideal \mathfrak{m}_1 of R_1 , μ_1 is a monomial of R_1 such that $(\#_1) \mu_1 \in I_1 :_{R_1} \mathfrak{m}_1$ and $I_1 + \mu_1 R_1$ is naturally closed, and J is an MN-basic ideal in R_2 , then $(I_1 + \mu_1 J)R$ is an MN-basic ideal of R .

We shall say that an MN-basic ideal as above is *maximal* MN-basic for the monomial μ if S_1, \dots, S_k is a partition of all of the variables $\{x_1, \dots, x_n\}$ and if there exist $\mu_j \in R_j$ such that $\mu = \mu_1 \cdots \mu_k$ and for all j , $1 \leq j \leq k$, I_j is maximal among naturally closed \mathfrak{m}_j -primary ideals in R_j that do not contain μ_j .

Remark. Let K be any field, let $R = K[x_1, \dots, x_n]$ be a polynomial ring, and let $R_1 = K[x_1, \dots, x_k]$ with homogeneous maximal ideal \mathfrak{m}_1 . Let I_1 be \mathfrak{m}_1 -primary and naturally closed. Suppose that $\nu \in I_1^- - I_1$. Then it is automatic that condition $(\#)$ from Lemma 10.10 holds, i.e., that $\nu \in I_1 :_{R_1} \mathfrak{m}_1$ and that $I'_1 = I_1 + \nu R_1$ is naturally closed. The first part of the condition holds because $\mathfrak{m}_1 = \text{Rad}(I_1)$, so that $\mathfrak{m}_1 \nu \subseteq \text{Rad}(I_1)I_1^- \subseteq (I_1)_{>1} \subseteq I_1$ (cf. Corollary 5.2), and the second because $(I'_1)_{>1} \subseteq (I_1^-)_{>1} = (I_1)_{>1} \subseteq I_1 \subseteq I'_1$. Also note that in the definition ν can be 1, but only if $I_1 = \mathfrak{m}_1$. Finally, note that if $k = 1$, the condition $(\#)$ holds precisely when $I_1 = x_1^{a+1}$ for $a \in \mathbb{N}$ and $\mu_1 = x_1^a$.

From the recursive version of MN-basic monomial ideals and Lemma 10.10 we have at once the following theorem.

Theorem 10.12. *If R is the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, every MN-basic monomial ideal is continuously closed.* □

Corollary 10.13. *Let K be a field of characteristic 0 and let R be the polynomial ring $K[x_1, \dots, x_n]$. Then for every MN-basic monomial ideal I , $I^{\text{h}^{\text{mix}}} = I$.*

Proof. R is the directed union of the rings $K_0[x_1, \dots, x_n]$, where K_0 runs through all subfields of K finitely generated over \mathbb{Q} . It therefore suffices to prove the result for a field K that is finitely generated over \mathbb{Q} . Any such field K is isomorphic to a subfield of \mathbb{C} , and so there is no loss of generality in assuming that $K \subseteq \mathbb{C}$. The given MN-basic ideal I , expanded to $S = \mathbb{C}[x_1, \dots, x_n]$, is obviously still MN-basic. Hence, IS is continuously closed by the preceding result, and so IS is MN-closed by Proposition 10.4. Hence, $IS \cap R = I$ is MN-closed by Proposition 10.5. □

Remark. We conjecture that the result just above and the result just below hold for every field K , not only those of characteristic 0.

In Theorem 10.15 below we show that MN-closed monomial ideals maximal with respect to not containing a given monomial are maximal MN-basic monomial ideals. It will be convenient to have the following result.

Lemma 10.14. *Let K be a field of characteristic 0 and let $R = K[x_1, \dots, x_n]$ be a polynomial ring. Let μ be a monomial. Let I be a monomial ideal with $I^{\text{h}^{\text{mix}}} = I$*

such that $\mu \notin I$. Let S be the set of variables that do not occur in μ , and let $J = I + (x_j : x_j \in S)$. Then $J^{\text{mix}} = J$ and $\mu \notin J$. Hence, a monomial ideal I which is maximal with respect to the conditions that $I^{\text{mix}} = I$ and $\mu \notin I$ contains all variables that do not occur in μ .

Proof. After renumbering we may assume that x_1, \dots, x_k occur in μ , so that $S = \{x_{k+1}, \dots, x_n\}$. Consider the K -algebra homomorphism $R \rightarrow R_1 = K[x_1, \dots, x_k]$ that fixes x_j for $j \leq k$ and kills x_j for $j \geq k + 1$. The image I_1 of I in R_1 is the monomial ideal spanned over K precisely by those monomials in I that are in R_1 . We claim that $I_1^{\text{mix}} = I_1$ in R_1 , for if $\nu \notin I_1$ were in the mixed natural closure of I_1 in R_1 , then ν would be in $I^{\text{mix}} = I$, since $I_1 R \subseteq I$ (we may expand using the inclusion $R_1 \subseteq R$). This shows that $J^{\text{mix}} = J$, for J is the contraction of I_1 under the surjection $R \rightarrow R_1$. The final statement is now clear. \square

Theorem 10.15. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K of characteristic 0. Then a monomial ideal $I \subseteq R$ is MN-closed if and only if it is an intersection of MN-basic monomial ideals. Moreover, these may be taken to be maximal MN-basic monomial ideals.*

Proof. Let μ be a fixed monomial and let I be an MN-closed monomial ideal that does not contain μ . Evidently, I can be enlarged to a maximal MN-closed monomial ideal that does not contain μ . Consequently, it will suffice to show that if I is a maximal MN-closed monomial ideal for μ , then it is a maximal MN-basic monomial ideal for μ . We shall prove this by induction on n . If $\mu = 1$, then the only possible choice of I is the homogeneous maximal ideal, and this is maximal MN-basic for 1. Henceforth, we assume that $\mu \neq 1$.

We next handle the case where all of the x_i occur in μ .

For each monomial $\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[x_1, \dots, x_n]$, let $h(\alpha) = (a_1, \dots, a_n) \in \mathbb{N}^n$. Consider the convex hull C of points of \mathbb{N}^n corresponding to all monomials in I together with $h(\mu)$. $h(\mu)$ must be a boundary point of C , or else μ would be in $I_{>1}$. By a hyperplane in \mathbb{R}^n we mean the translate of a vector subspace of dimension $n - 1$. We may choose a supporting hyperplane of C that contains $h(\mu)$. That is, there is a hyperplane through $h(\mu)$ such that C lies entirely in one of the closed half-spaces it determines. There is a nonzero real linear form L over \mathbb{R} and $c \in \mathbb{R}$ such that this hyperplane is defined by the equation $L = c$. Let y_1, \dots, y_n be real variables, and renumber the y_i so that y_1, \dots, y_k are the real variables that have nonzero coefficients in L ; at least one coefficient is nonzero.

Thus, we may assume that the equation of the hyperplane is $m_1 y_1 + \cdots + m_k y_k = c$, where the $m_i \in \mathbb{R} - \{0\}$.

By multiplying by -1 if necessary, we may assume that $c \geq 0$. We may assume that all the coefficients m_i are positive. (To see this, note that all points with sufficiently large coordinates represent elements in I and will lie on one side of this hyperplane. If m_i is positive (respectively, negative), choose a large value $N \in \mathbb{N}$ for the y_j , $j \neq i$, and a very large positive integer value B for the value of m_i . The value of $L = m_1 y_1 + \cdots + m_k y_k$ will be $> c$ (respectively, $< c$) for $B \gg 0$. Thus, if there are coefficients with different signs, not all points with large coordinates are on the same side of the hyperplane.) Thus, we may assume that all the m_i have the same sign. By multiplying by -1 we may assume that all of them are positive. Write $\mu = \mu_1 \theta$, where μ_1 involves x_1, \dots, x_k and θ involves the other variables.

Note that L evaluated at $h(\mu)$ or $h(\mu_1)$ is c (the value of L only depends on the first k entries of the vector). Since $h(\mu)$ has no nonzero entries, $c > 0$.

Let $I_1 \subseteq K[x_1, \dots, x_k] = R_1$ be generated by all monomials λ except μ_1 in x_1, \dots, x_k such that the value of L at $h(\lambda)$ is $\geq c$. Note that I_1 is primary to (x_1, \dots, x_k) : since each coefficient of L is positive, the functional will be $> c$ on Ne_i , $1 \leq i \leq k$, when $N \gg 0$.

Moreover, $I_1 + \mu_1 R_1$ is integrally closed: the set of exponent vectors contains all lattice points in its convex hull because it is the intersection of a half-space, the first orthant, and the set of lattice points. In particular, it is naturally closed. It is also the case that I_1 contains $x_i \mu_1$ for $1 \leq i \leq k$: the value of L is evidently $\geq c$ for each of these, and each of these is different from μ_1 . We also claim that I_1 is naturally closed. It suffices to show that μ_1 is not in the natural closure, and for this it suffices to see that $\mu_1 \notin (I_1)_{>1}$. But if $\mu_1^t \in I_1^{t+1}$, then $th(\mu_1)$ is the sum of $t + 1$ values of h on points of I_1 . When we apply L , we find that $tL(h(\mu_1))$ is the sum of $t + 1$ real numbers, each of which is $\geq c$. This implies that $tc \geq (t + 1)c$, a contradiction.

Moreover, $I_1 R + \mu_1 R$ is the ideal of R generated by all monomials in R on which the value of L is $\geq c$. Clearly, $I \subseteq I_1 R + \nu R$. Given any monomial in I , the value of L on the associated vector is $\geq c$, and this value depends only on that part of the monomial involving x_1, \dots, x_k ; the latter must be in $(I_1, \nu)K[x_1, \dots, x_k]$. The monomials in I that are not in $I_1 R$ must be monomials of the form $\alpha\beta$, where $\alpha \in K[x_1, \dots, x_k]$ and $\beta \in K[x_{k+1}, \dots, x_n]$. Since they are in I , the value of the functional on $h(\alpha)$ must be $\geq c$, which means that $\alpha \in I$ unless $\alpha = \nu$. Thus, $I \subseteq I_1 R + J_0 \nu R$, where $J_0 \subseteq K[x_{k+1}, \dots, x_n]$ contains those monomials whose product with ν is in I . θ cannot be in the mixed natural closure of J , or else $\mu = \nu\theta$ will be in the mixed natural closure of $\nu J R \subseteq I$. Thus, we may enlarge J_0 to a maximal mixed naturally closed ideal J of $K[x_{k+1}, \dots, x_n]$ maximal with respect to not containing θ .

By the induction hypothesis we have that J is a maximal MN-basic monomial ideal of $K[x_{k+1}, \dots, x_n]$ maximal with respect to not containing θ . It follows that $I_1 R + J R$ is a maximal MN-basic ideal for $\mu = \mu_1 \theta$. Since I is contained in this ideal, I must be equal to $I_1 + J R$.

This completes the treatment of the case where all of the x_i occur in μ . In the general case, by renumbering, suppose that x_1, \dots, x_h are the variables that do not occur in μ . Giving a maximal monomial MN-closed ideal I such that $\mu \notin I$ is equivalent to giving such an ideal \mathfrak{A} for μ in $K[x_{h+1}, \dots, x_n]$ and then enlarging it to include x_1, \dots, x_h , by Lemma 10.14. The enlarged ideal will still be maximal MN-basic: if I_1 is the initial primary ideal in the representation (\dagger) of \mathfrak{A} as in Definition 10.11, one can simply enlarge I_1 to contain x_1, \dots, x_h . □

We now come to the motivating result for this section.

Theorem 10.16. *For a monomial ideal I in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, $I^{\text{cont}} = I^{\natural^{\text{mix}}}$.*

Proof. We observed earlier that $I^{\natural^{\text{mix}}} \subseteq I^{\text{cont}}$. Thus, it will suffice to show that $I^{\natural^{\text{mix}}}$ is continuously closed. But by Theorem 10.15, it is an intersection of MN-basic ideals, and by Theorem 10.12 every MN-basic ideal is continuously closed. □

Example 10.17. Note that it is not true that continuous closure agrees with mixed natural closure in reduced affine algebras over \mathbb{C} . In $R = \mathbb{C}[x^2, x^3, xy, y]$, a subring of the polynomial ring $S = \mathbb{C}[x, y]$, we have that the principal ideal $I = yR$ is such that $I = I^{\natural^{\text{mix}}}$. (We know that $I^{\natural^{\text{mix}}}$ is monomial and is contained in $(y, xy)R$, since this is the contraction of $y\mathbb{C}[x, y]$. Hence, it will suffice to show that xy is not in $f(I : f)^{\natural}$ for any choice of f . Since xy is irreducible in R , the only possibilities for f are, up to multiplication by nonzero elements of \mathbb{C} , 1 and xy . The choice $f = 1$ does not work since $I^{\natural} = I$, and the choice $f = xy$ does not work since $I :_R f$ is proper monomial, and so its natural closure is contained in the homogeneous maximal ideal \mathfrak{m} of R , while $xy \notin xym$.) However, $xy \in I^{\text{cont}}$.

11. A BIGGER AXES CLOSURE

In deciding on a generalization of Brenner’s notion of axes closure to arbitrary Noetherian rings, we had a choice between whether we would base it on seminormal rings or so-called *weakly normal* rings.

Definition 11.1 ([AB69]). Let R be a reduced Noetherian ring, and let R' be the integral closure of R in its fraction field. Then the *weak normalization* R^{wn} of R is the set of all elements $x \in R'$ that satisfy the following property for all $\mathfrak{p} \in \text{Spec } R$.

If R/\mathfrak{p} has prime characteristic $p > 0$, let $\pi(\mathfrak{p}) := p$; otherwise let $\pi(\mathfrak{p}) := 1$. Then there is some positive integer n such that $(x/1)^{\pi(\mathfrak{p})^n} \in R_{\mathfrak{p}} + \text{Jac}(R'_{\mathfrak{p}})$.

We say that R is *weakly normal* if $R = R^{\text{wn}}$.

It is clear that if R is weakly normal, it is also seminormal. Moreover, if R has equal characteristic 0, the weak normalization is of course the same as the seminormalization of R , and in particular in the finitely generated \mathbb{C} -algebra case, they agree. Therefore, with the following definition, one has that $I^{\text{ax}} = I^{\text{AX}}$ for any ideal I in a \mathbb{C} -algebra.

Definition 11.2. Let R be a Noetherian ring, let I be an ideal of R , and let $f \in R$. We write $f \in I^{\text{AX}}$ if for every map from R to an excellent one-dimensional weakly normal ring S , the image of f is in IS .

It is quite straightforward to verify that $I \mapsto I^{\text{AX}}$ is a closure operation.

Proposition 11.3. For every ideal $I \subseteq R$, a Noetherian ring, $I^{\text{ax}} \subseteq I^{\text{AX}} \subseteq I^-$.

Proof. This is clear, since normal \implies weakly normal \implies seminormal. □

One has the following parallel to Proposition 3.1. Items (6) and (7) follow in this case for the same reasons their analogues did in the seminormal case.

Proposition 11.4. Suppose R, S are reduced Noetherian rings. Let R' be the integral closure of R in its total quotient ring.

- (1) Suppose R is seminormal. Then R is weakly normal if and only if for any prime integer p and any $x \in R'$ such that $x^p, px \in R$, we have $x \in R$. [Ito83, Proposition 1]
- (2) If $g : R \rightarrow S$ is faithfully flat and S is weakly normal, then R is weakly normal. [Man80, Corollary II.2]
- (3) If R is weakly normal and W is a multiplicative set, then $W^{-1}R$ is weakly normal. [Man80, Corollary IV.2]

- (4) Suppose the integral closure of R in its total quotient ring is module-finite over R . The following are equivalent: [Man80, Corollary IV.4]
- (a) R is weakly normal.
 - (b) $R_{\mathfrak{m}}$ is weakly normal for all $\mathfrak{m} \in \text{Max Spec } R$.
 - (c) $R_{\mathfrak{p}}$ is weakly normal for all $\mathfrak{p} \in \text{Spec } R$.
 - (d) $R_{\mathfrak{p}}$ is weakly normal for all $\mathfrak{p} \in \text{Spec } R$ such that $\text{depth } R_{\mathfrak{p}} = 1$.
- (5) Suppose $g : R \rightarrow S$ is flat with all fibers geometrically reduced and generic fibers normal (e.g., a normal homomorphism) and R' is module-finite over R . If R is weakly normal, then so is S . [Man80, Proposition III.3] In particular, if S is smooth over R , which includes the case where S is étale over R , and R is weakly normal, then S is weakly normal.
- (6) A directed union of weakly normal rings is weakly normal.
- (7) If R is local and weakly normal, then the Henselization of R and the strict Henselization of R are weakly normal.
- (8) Suppose R is excellent and local. R is weakly normal $\iff \hat{R}$ is weakly normal. [Man80, Proposition III.5]
- (9) Let X be an indeterminate over R . R is weakly normal $\iff R[[X]]$ is weakly normal. [Man80, Proposition III.7]

For reasons parallel to observations in the $^{\text{ax}}$ case, it suffices to consider only maps where S is local, or even complete local. Many properties which hold for I^{ax} have analogies in I^{AX} . To see how this works, we offer the following analogue to Theorem 3.3.

Theorem 11.5. *Let k be a field of prime characteristic $p > 0$, and let L_1, \dots, L_n be finite algebraic extension fields of k such that under the diagonal embedding $k \rightarrow L_1 \times \dots \times L_n$, the image of k is p th-root closed. Let (V_i, \mathfrak{m}_i) be discrete valuation rings such that $V_i/\mathfrak{m}_i \cong L_i$. Let S be the subring of $\prod_{i=1}^n V_i$ consisting of all n -tuples (v_1, \dots, v_n) such that there exists $\alpha \in k$ such that $v_i \equiv \alpha \pmod{\mathfrak{m}_i}$ for all i . Then S is weakly normal.*

Conversely, let (R, \mathfrak{m}, k) be a complete one-dimensional weakly normal Noetherian local ring, where k has prime characteristic $p > 0$. Then there exist such extension fields L_i and such DVRs V_i (which moreover are complete) such that R is isomorphic to the ring S described above.

Proof. If R is weakly normal, then it is seminormal, so it has the form given in Theorem 3.3. One must only check that k is p th-root closed in $L_1 \times \dots \times L_n$. So let p be the characteristic of k , which must therefore agree with the characteristics of all the L_i . Let $c = (c_1, \dots, c_n) \in \prod_{i=1}^n L_i$ such that $c^p \in k$. Let $v = (v_1, \dots, v_n) \in \prod V_i$ such that c_i is the residue class of $v_i \pmod{\mathfrak{m}_i}$, for each i . Then $\overline{pv} = p\overline{v} = 0 \in k$ so that $pv \in R$, and $\overline{v^p} = c^p \in k$ so that $v^p \in R$. Since R is weakly normal, it follows that $v \in R$.

Conversely, suppose R is constructed in the way outlined in the statement of the theorem. Say p is the characteristic of k . Let q be a prime integer and let $v \in \prod_{i=1}^n V_i$ be such that $v^q, qv \in R$. If $q \neq p$, then since q is a unit in all of the L_i , it follows that it is invertible in $\prod L_i$. So $q\overline{v} \in k$ implies that $\overline{v} \in k$, which then implies that $v \in R$ by the description of R . On the other hand, if $v = (v_1, \dots, v_n) \in \prod V_i$ is such that $v^p, pv \in R$, then, in particular, $\overline{v^p} \in k$, and since k is p th-root closed in $\prod L_i$, it follows that $\overline{v} \in k$, so that $v \in R$. \square

It follows, for example, that a local or complete ring of axes over any field is weakly normal, since the diagonal embedding in question is the usual one, $k \rightarrow k \times \cdots \times k$, in which it is clear that the image of k is p th-root closed.

We also have the following parallel to Proposition 3.4.

Proposition 11.6. *Let L be an algebraically closed field.*

- (a) *A complete axes ring over L is weakly normal.*
- (b) *Every complete local one-dimensional weakly normal ring of equal characteristic with algebraically closed residue class field L is isomorphic with a complete ring of axes over L .*
- (c) *Every affine ring of axes over L is weakly normal.*
- (d) *A one-dimensional affine L -algebra R is weakly normal if and only if there are finitely many étale L -algebra maps $\theta_i : R \rightarrow A_i$, where the A_i are affine rings of axes over L , and every maximal ideal of R lies under a maximal ideal of some A_i .*

Combining this with Proposition 3.4, it follows that for complete one-dimensional rings with algebraically closed residue field and for finitely generated one-dimensional algebras over an algebraically closed field, there is no difference between weak normality and seminormality.

Proof. For part (a), we use the characterization in Theorem 11.5, noting that one obtains no extra p th roots from the diagonal embedding $L \rightarrow L \times \cdots \times L$. For part (b), any complete local one-dimensional weakly normal ring of equal characteristic with residue field L is in particular seminormal, so by Proposition 3.4(b), it is isomorphic to a complete ring of axes over L . Part (c) then follows immediately from part (a) and from parts (4)(b) and (8) of Proposition 11.4.

The proof of part (d) follows from the argument in the proof of Proposition 3.4(d), using the corresponding parts of Proposition 11.4 in place of where we had previously used Proposition 3.1. □

However, ${}^{\text{AX}}$ is really too big for our purposes here. Consider the following.

Example 11.7. Let k be a field of characteristic $p > 0$, let t, x be analytic indeterminates, and let $R = k(t^p)[[x, tx]]$. Then we claim that R is seminormal, but also that $R' = R^{\text{wn}} = k(t)[[x]]$. The statement about the weak normalization follows from the fact that $t^p, pt = 0 \in R$. The resulting ring is obviously normal, so $R' = R^{\text{wn}} = k(t)[[x]]$. To see that R is seminormal, take any $f \in R'$ such that $f^2, f^3 \in R$. Then if f_0 is the constant term of the power series f , we have $f_0^2, f_0^3 \in k(t^p)$, whence $f_0 = f_0^3/f_0^2 \in k(t^p)$. But R is exactly the set of all $f \in R'$ whose constant term is in $k(t^p)$.

Now let $\mathfrak{m} = (x, tx)R$ be the unique maximal ideal of R , and let $I = xR$. Note that I is \mathfrak{m} -primary. Then $I^{\text{AX}} = xR^{\text{wn}} \cap R = (x, tx)R = \mathfrak{m}$, but $I^{\natural} = I^{\text{ax}} = I$ because R is a one-dimensional complete seminormal ring. Hence:

- (1) ${}^{\text{ax}}$ and ${}^{\text{AX}}$ do not always agree, even for \mathfrak{m} -primary ideals in one-dimensional complete local domains, and
- (2) ${}^{\natural}$ and ${}^{\text{AX}}$ do not always agree, even for \mathfrak{m} -primary ideals in one-dimensional complete local domains.

Property (1) is perhaps not surprising, but property (2) means that the closure is too big to apply our methods *mutatis mutandis*: since one lacks the property

that $I^{\text{AX}} = I^{\natural}$ for primary ideals, it is not clear how one would prove an analogue of Theorem 7.12 or of the crucial Theorem 7.17 in this new context (substituting I^{AX} for I^{ax} and weakly normal for seminormal everywhere). Thus, this bigger axes closure does not appear to be as suitable for our main purpose here as the smaller one. However, we have provided some of the fundamentals in this section because it may be useful in other situations.

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