*Photogrammetric Determination of Lunar Motions Essential for Navigation**

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ABSTRACT: *The first men to reach the Moon's surface will be required to establish their position in order to return to Earth. Celestial navigation is the most practical method of position determination on the Moon. The stars observed, prior to launch, must be transformed to a selenocentric frame of reference. The selenocentric transformation requires a knowledge of the instantaneous celestial orientation of the Moon's pole and the short-term rotational rate. Earthbound measures of these quantities are not sufficiently accurate. This paper describes a method of obtaining more accurate values of the orientation of the Moon's pole and short-term rotational rate by an application of analytical photogrammetry to a fixed star space correlated with a moving lunar surface space with time referenced satellite star-terrain photography. The method is independent of previously established control-points on the Moon or orbital parameters. This paper also describes a procedure for simulating the satellite system with Earthbound photography and a procedure for verifying and improving the satellite system with exposures made at the Moon's surface.*

A. REQUIREMENT AND PROBLEM

IN ORDER to be able to land safely on the Moon, to carry out lunar explorations, and to return to Earth, man must navigate accurately. This requires that position be determined accurately with respect to a universally usable, meaningful reference system. Moon referenced star observations provide the most efficient input data to single or sequential lunar position determination. The reasons that Moon referenced star observations provide the most practical method of position determination are briefly enumerated.

1. Stars will be visible at all times without cloud or sunlight obscuration.
2. Sharp curvature of the Moon's surface and rugged terrain limit the visibility of identifiable surface features to several miles; and resection of these identifiable surface features is not practical where the terrain is strange or may have position errors equal to 25 per cent of the range of visibility.
3. The normal local reconnaissance done when a man is lost on Earth is not practical with the apparel and equipment necessary to survive on the Moon.

On Earth, celestial navigation is simplified by a one-to-one correspondence between the right ascension and declination of the stars and the sidereal

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longitude and latitude of points on the Earth's surface and an accurate knowledge of the Earth's rotational rate. For this reason, the latitude and longitude of a point may be established by observation of the zenith angles of two or more stars referenced to time. Celestial navigation on Earth is only complicated by cloud coverage and daylight. These problems do not exist on the Moon but the one-to-one correspondence between star places and a frame of reference based on the celestial-orientation of the Moon's pole and the Moon's rotational rate must be established by a suitable transformation. The transformation of star places to a lunar frame of reference may be accomplished with three time-dependent numbers illustrated in Figure 1.

ι = co-declination of the Moon's pole.
Ω' = right ascension of the Moon's pole minus 90°.
m = rotational rate of the Moon.

The definitions of ι and Ω' are taken from the American Ephemeris. These quantities may be derived from measures of the Moon's orbit and certain assumptions embraced in Cassini's laws:

(1) The Moon rotates uniformly about an axis which is fixed with respect to the Moon itself and the period of rotation is identical to the sidereal period of its orbit.
(2) The inclination I of the Moon's pole to the pole of the Earth's orbit, or the ecliptic, is a constant.
(3) The pole of the Moon and the pole of the Moon's orbit lie on opposite sides of the pole of the Earth's orbit on the same great circle. The geometric significance of these laws is illustrated in Figure 2. If the second and third of Cassini's laws are valid,

$$I = \text{a constant, and}$$

Ω, the ecliptical-longitude of the descending node of the Moon's orbit, is equal to the ecliptical longitude of the ascending node of the Moon's equator, in which case ι and Ω' could be obtained from the following transformation:

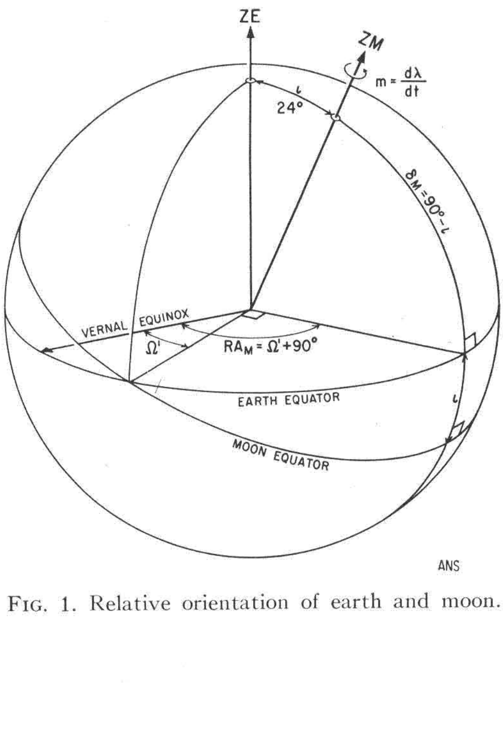

FIG. 1. Relative orientation of earth and moon.

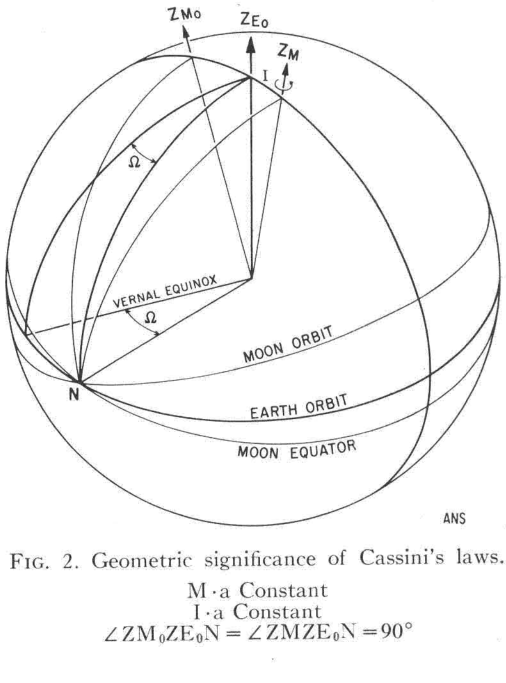

FIG. 2. Geometric significance of Cassini's laws.
M·a Constant
I·a Constant
$\angle ZM_0ZE_0N = \angle ZMZE_0N = 90°$

$$\sin \iota \cos \Omega' = -\cos I \sin e + \sin I \cos e \cos \Omega$$
$$\sin \iota \sin \Omega' = \sin I \sin \Omega$$
$$\cos \iota = \cos I \cos e + \sin I \sin e \sin \Omega$$

and conversely,

$$\sin I \cos \Omega = \cos \iota \sin e + \sin \iota \cos e \cos \Omega'$$
$$\sin I \sin \Omega = \sin \iota \sin \Omega'$$
$$\cos I = \cos \iota \cos e - \sin \iota \sin e \cos \Omega'$$

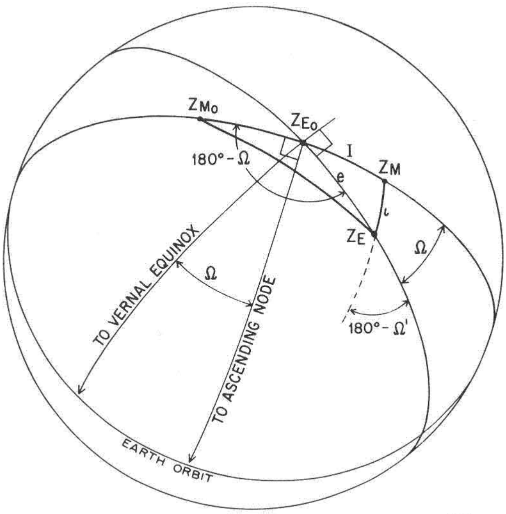

FIG. 3. Relation of I and Ω to e, ι and Ω' as Expressed by Cassini's law.

where e is the obliquity of the ecliptic. These transformations may be derived from the spherical relations shown in Figure 3. Ω is accurately determined from observations of the Moon's orbit and e is regarded as an accurately determined quantity. Assume for the moment, that I is an accurately known constant, that the short period rotational rate is known and constant, and that the ecliptical-longitude of the descending node of the Moon's orbit is equal to the ascending node of the Moon's equator. Then the quantities ι and Ω' could be computed for a range of times covering the lunar explorer's visit to the Moon. With these values, the corresponding selenocentric latitudes and longitudes referenced to suitable intervals of time would follow for any star a:

$$\sin \phi_a = \sin \delta_a \cos \iota + \cos \delta_a \sin \iota \sin (RA_a - \Omega')$$
$$\cos Z_0a = \sin \delta_a \cos \iota_0 + \cos \delta_a \sin \iota_0 \sin (RA_a - \Omega_0')$$
$$\cos Z_0Z = \cos \iota \cos \iota_0 + \sin \iota \sin \iota_0 \cos (\Omega' - \Omega_0')$$
$$\sin \lambda_x = \frac{\sin \iota \sin (\Omega' - \Omega_0')}{\sin Z_0Z}$$
$$\sin (\lambda_{a_0} - \lambda_x) = \frac{\cos Z_0a - \cos Z_0Z \sin \phi_a}{\sin Z_0Z \cos \phi_a}$$
$$\lambda_a = (\lambda_{a_0} - \lambda_x) + \lambda_x + \int m dt$$

λ_a and ϕ_a are the selenocentric directions of a star at time T referred to a meridian defined by the ascending node of the Moon's equator at time T_0. T_0 is some arbitrarily selected epoch to which all subsequent times refer. The transformation of star places to a fixed selenocentric frame of reference is illustrated in Figure 4. Z_0 and Z are two positions of the Moon's pole at times T_0 and T. For any two positions of the Moon's pole there is a third point x. Point x is the pole of the fundamental plane defined by

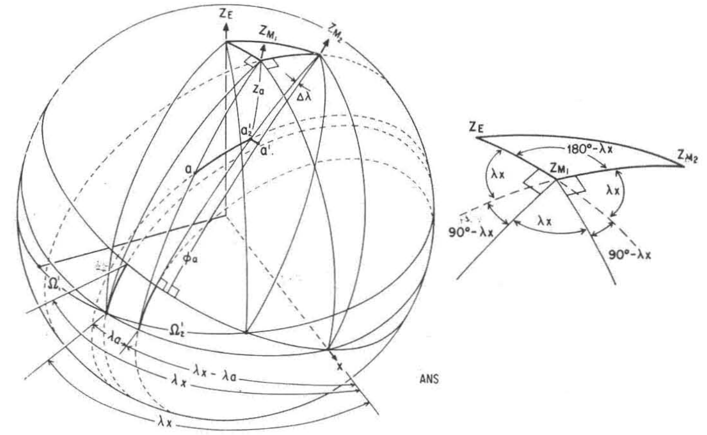

FIG. 4. Motion of a point as a consequence of nutation and rotation of the moon.

the motion $Z_0 Z$ for which the longitude relation of a surface point is fixed in the interval $T-T_0$ excluding rotation. Exploitation of the point of zero motion makes it possible to correlate a moving point to a fixed fictitious meridian at different times. If Cassini's laws were rigorously correct and the corresponding input values were sufficiently accurate, there would be no more to preparing a star almanac of selenocentric star places referred to the Moon's pole and an arbitrarily fixed selenocentric meridian than outlined.

This, however, is not the case. Variation of Cassini's laws established by theory are correlated by physical libration which is, among other things, an actual periodic nodding of the Moon's pole. The problem arises therefore not from inadequate theory but from inadequate measures of physical libration to provide inputs to the theoretical equations whence accurate time referenced values of ι, Ω', and m may be determined.

Physical librations are measured with a heliometer. These measures are employed to determine values of I. According to a translation of a Russian work entitled *The Moon* by N. P. Barabashov, V. A. Bronshten, *et al.*, the root mean square errors of heliometer measures are approximately equal to the magnitude of physical libration. This is further supported by a list of values of I compiled by Kopal in his *Mathematical Elements of Lunar Topography*:

Date	*Authority*	I
1914	Hayn	1°32′20″
1948–49	Koziel	1°31′36″ ± 23″
1949	Belkovitch	1°31′10″ ± 22″
1950	Nefedzev	1°32′ 4″ ± 15″
1950	Yakovkin	1°33′48″ ± 17″
1955	Watts	1°33′50″ ± 19″

The large probable error in the determinations and the large spread in the separate values suggest that I is not constant or may have an absolute error of several minutes. According to Kopal, there is some encouragement in the uniformity of the rotational rate. The small spread in the determination of the selenographic coordinates of Mösting A, a small crater near the zero meridian, from heliometric measures during the past one hundred years indicates that the rotational rate is synchronized with the sidereal period to ±0.1 second of time.

The requirement therefore is the transformation of star places to a selenocentric frame of reference. The problem is to provide accurate measures of the instantaneous orientation of the Moon's pole (ι, Ω') and the instantaneous rotational rate as inputs to the transformation.

B. Determination with Star-Terrain Satellite Photography

Recoverable unmanned flights will orbit or land on the Moon. It is suggested that a photogrammetric reduction of data acquired from a polar orbiting package consisting of an accurate clock, a star camera, and a terrain camera making time referenced synchronized exposures to provide 60 per cent endlap with nominal $\pm 3°$ stability at an altitude of 100 kilometers can provide sufficiently accurate inputs to the selenocentric transformation of star places to be employed by the first manned flights.

Earthbound measures cannot provide sufficiently accurate values of orientation of the Moon's pole. Near-polar orbiting star-terrain satellite photography can provide accurate values by a beautiful application of photogrammetry to a fixed and moving object space. Assume that the interior and relative orientation of the star and terrain camera system have been determined on the ground and further that the clock time of each exposure is recorded on the film. Assume further that star images, fiducial marks and conjugate terrain-images are measured. Initially, coefficients of film distortion, measuring machine swing, and film-tilt are determined with orthographic transformation of the following form with film and calibrated fiducial mark coordinates as the given data:

$$x'A_x + y'B_x + C_x = x$$

$$x'A_y + y'B_y + C_y = y$$

where the primed values are film fiducial-coordinates and the unprimed values are calibrated fiducial-coordinates. Having determined the coefficients with three or more measures, the process is reversed in the transformation of every star and terrain image measured. The stars selected and identified are reduced to their apparent places with added terms to remove the effects of aberration arising from the Moon's orbital velocity about the earth and the satellites orbital velocity about the Moon. The additional terms to the standard apparent place equations have the form

$$\Delta RA'' = (l \sin RA_0 + m \cos RA_0) \sec \delta_0$$

$$\Delta\delta'' = (l' \sin RA_0 + m' \cos RA_0) \sin \delta_0 + n' \cos \delta_0$$

l, m, n, l', m', n' are computed with approximate values of the Moon and satellite orbital parameters and are determined once for each camera station. The maximum angular aberrations from the orbital velocities of the Moon and of the satellite are $0''.7$ and $1''.0$ respectively. Therefore orbital parameters with a possible error of one part in 10 are sufficiently accurate. However, the terrain-images also suffer from satellite aberration. It seems more practical to correct the coordinates of the terrain image-directly. If the y axis lies in the direction of motion,

$$dx = 0$$

$$dy = -K(x^2 + y^2 + f^2)^{1/2}$$

where

$$K = \frac{1}{V}\sqrt{\frac{GM}{a(1-e^2)}\left[1 + e^2 + 2e \cos(\mu - \omega)\right]}$$

$$V = \text{velocity of light}$$

Again the orbital parameters contained under the radical can be approximate since the maximum value for a camera with 150-millimeter focal-length is

$$dy = 0.0012 \text{ millimeter.}$$

The displacements are negative and therefore must be applied positively to restore an image to its correct position.

With star-image coordinates adjusted to the calibrated standard and star places adjusted for Moon and satellite orbital aberration, the celestial directions of the camera axes are determined with a tangential solution of the form:

$$a \cdot \Delta s + b \cdot \Delta RA_z + c \cdot \Delta\delta_z = \Delta\theta$$

where

(1) Δs, ΔRA_z and $\Delta\delta_z$ are differential corrections to the initial estimates of the astronomic swing of the y axis and the celestial direction of the z axis;
(2) $\Delta\theta$ is the difference between the film angle of a star image and the same angle computed with star data and approximate star orientation data; and
(3) a, b, c are the coefficients computed with star and approximate orientation data.

n equations are normalized and re-solved with each revision of the approximate orientation, coefficients, and $\Delta\theta$ values until

$$\Sigma P\theta^2 = \text{minimum}$$

whence

$$s = s' + \Sigma\Delta s$$
$$RA_z = RA_z' + \Sigma\Delta RA_z$$
$$\delta_z = \delta_z' + \Sigma\Delta RA_z$$

The fundamental angles are employed to compute the celestial direction cosines of the star camera axes which by a matrix multiplication of the ground determined relative orientation angles gives the celestial direction cosines of the terrain camera axes. The θ reduction of orientation is not a classical photogrammetric reduction. The tests have demonstrated 300 per cent improvement in axes orientation accuracy arising from the fact that 90 per cent of the systematic errors in interior-orientation data are radial. The θ solution is independent of radial errors. For this reason, the θ solution is employed wherever axes orientation is involved.

Star-orientation imposed on the conjugate imagery of two over-lapping pairs gives rise to an inequality of the form:

$$\tan\theta \neq \tan\theta'$$

or

$$\frac{xa}{ya} \neq \frac{Xa'}{Ya'}$$

owing to the motion of, and rotation about, the Moon's pole in the time elapsed between the two exposures. Taking the first differentials of the five fundamental orientation angles of the leading exposure with respect to θ of the trailing exposure produces an iterative equation of the form:

$$a(\Delta s_2) + b(\Delta RA_{z_2}) + c(\Delta\delta_{z_2}) + d(\Delta RA_{u_2}) + e(\Delta\delta_{u_2}) = \Delta\theta_1$$

where the coefficients, differential corrections, and $\Delta\theta$ terms have the geometric significance as the $\Delta\theta$ equation employed for the determination of the celestial-axes orientation of the star camera.

The numerical subscripts 1 and 2 denote trailing and leading exposures. With five or more conjugate images, five equations may be solved by successive approximations, uniquely until

$$\Sigma\Delta\theta = 0$$

or more than five by Least Squares until

$$\Sigma\Delta\theta^2 = \text{a minimum.}$$

Then the changes during the time $T_2 - T_1$ owing to the combined motion of the Moon's pole and rotation are

$$\Sigma\Delta s$$
$$\Sigma\Delta RA_z$$
$$\Sigma\Delta\delta_z$$
$$\Sigma\Delta RA_u$$
$$\Sigma\Delta\delta_u$$

The right ascension and declination of each conjugate-image are computed with the celestial orientation cosines of the terrain camera axes and the image-coordinates of the terrain camera. A second group is computed with celestial orientation cosines adjusted for the Moon's motion in the interval $T_2 - T_1$, giving rise to an array of differences for each leading image

$$RA - RA' = \Delta RA$$
$$\delta - \delta' = \Delta\delta$$

These changes are likely to be very small for an exposure interval of 32 seconds. Larger changes can be obtained from adjacent passes which, for reasons of excessive coverage, are separated by a little more than 14 hours. Since these changes are a direct consequence of the fundamental motions of the Moon, operating on appropriate selenocentric transformations produces the corresponding orientation functions (ι, Ω') and changes ($\Delta\iota$, $\Delta\Omega$, $\Delta\lambda$) of the Moon itself. The relation of $\Delta\Omega$, $\Delta\iota$ and $\Delta\lambda$ to $\Delta\delta$ and ΔRA is illustrated in Figure 5.

It is assumed in the following reduction that the mechanical rotation axis of the Moon is a fixed point with respect to the Moon's physical surface in which case

$$\frac{d\phi}{dt} = 0.$$

Taking the first-order differentials of the variables in the equation

$$\sin\phi = \cos\iota\sin\delta + \sin\iota\cos\delta\sin(RA - \Omega')$$

we obtain, after some manipulation, a condition equation of the form:

$$\begin{aligned}\cos\delta\Delta\delta = {} & (\tan\iota\Delta\iota)\sin\delta - (\tan\iota\sin\Omega')[\cos\delta\cos RA\Delta\delta + \cos\delta\sin RA\Delta RA] \\ & + (\tan\iota\cos\Omega')[\sin\delta\sin RA\Delta\delta - \cos\delta\cos RA\Delta RA] \\ & + (\tan\iota\sin\Omega'\Delta\Omega - \cos\Omega'\Delta\iota)\cos\delta\sin RA \\ & + (\sin\Omega'\Delta\iota + \tan\iota\cos\Omega'\Delta\Omega)\cos\delta\cos RA\end{aligned}$$

Forming five or more equations with data from five or more conjugate-images a set of orientation functions ι, Ω', $\Delta\iota$ and $\Delta\Omega$ are obtained from each over-lapping pair. Values obtained from a single overlapping pair are expected to contain large errors although, owing to the larger scale, far more accurate values than those obtained with

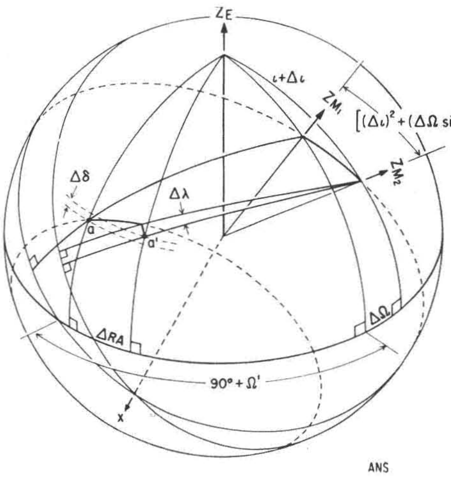

FIG. 5. Relation of $\Delta\Omega$, $\Delta\iota$ and $\Delta\lambda$ to $\Delta\delta$ and ΔRA of a lunar surface point.

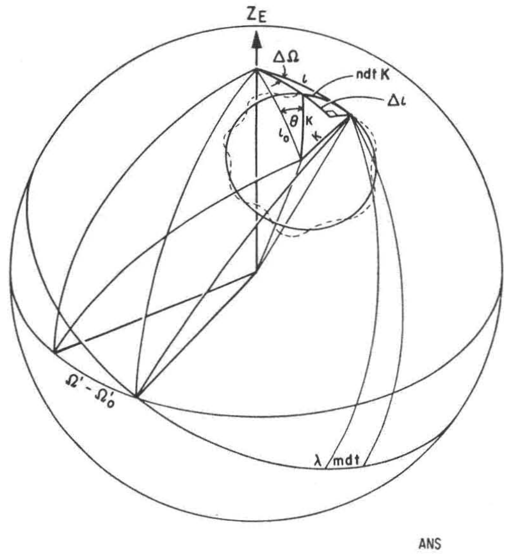

FIG. 6. Elements of angular velocity.

heliometer measures. The number of observations for any reduction of I by any authority has not exceeded 300. 7,000 exposures are required to obtain complete coverage of the Moon in a 27-day period. To obtain a good statistical reduction, let it be assumed that values of ι and Ω' are approximately correct. Then let

$$\Delta\lambda = m \cdot \Delta T$$

$$\Delta\iota = nK \sin\theta \cdot \Delta T$$

$$\Delta\Omega = nK \cos\theta \csc \iota \Delta T$$

$$\Delta T = (T_n - T_{n-1}) \text{ exposure interval.}$$

These relations are illustrated in Figure 6.

$$\sin\theta = (T_0 - T_n)n - \frac{(T_0 - T_n)^3 n^3}{3!} + \frac{(T_0 - T_n)^5 n^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{(T_0 - T_n)^2 n^2}{2!} + \frac{(T_0 - T_n)^4 n^4}{4!} \cdots$$

In the above equations,

n is the angular velocity of the instantaneous pole about the short term mean pole.
m is the angular velocity of rotation about the Moon's mechanical axis.
T_0 is the time the mean pole and instantaneous pole lie on the same hour circle.
k is the angle subtended by the mean pole and the instantaneous pole.
θ is the astronomic azimuth of the instantaneous pole at the mean pole.

Now

$$\Delta\delta = \sin\iota \cos(RA - \Omega')\Delta\lambda + \sin(RA - \Omega')\Delta\iota$$

$$\cos\delta\Delta RA = [\cos\iota\cos\delta - \sin\iota\sin\delta\sin(RA - \Omega')]\Delta\lambda + \sin\delta\cos(RA - \Omega')\Delta\iota + \cos\delta\Delta\Omega$$

Let the total angular motion of a line be N.

$$N^2 = \Delta\delta^2 + \cos^2 \delta \cdot \Delta RA^2$$

Substituting the elements of $\Delta\iota$, $\Delta\Omega$, and $\Delta\lambda$ in the equation above, yields after some reduction, an equation of the form:

$$A(m) + B\left(1 - \frac{n^2T_0}{3!}\right)n^2KT_0 + C\left(\frac{n^2T_0^2}{3!} - 1\right)n^2K - D(n^4KT_0) + E(n^4K)$$
$$+ F\left(1 - \frac{n^2T_0^2}{2!}\right)nK + G(n^3KT_0) - H(n^3k) = \frac{N^2}{\Delta T}$$

where the coefficients A, B, C, D, E, F, G, H, and $N_2/\Delta T$ are known quantities obtained in the substitution. With equations of condition in this form every conjugate image of every exposure of every pass may be combined into normal equations whereby a Least Squares value of n, m, T_0 and K may be obtained. Two more constants ι_0 and Ω_0' are required before the reduction is complete. With values of ι and Ω' obtained from each pair and K, T_0 and n obtained by a Least Squares reduction of all pairs, a Least Squares value of ι_0 and Ω_0' from all pairs may be obtained from

$$\cos \iota = \cos K \cos \iota_0 + \sin K \sin \iota_0 \cos \theta$$
$$\sin \iota \sin (\Omega' - \Omega_0') = \sin K \sin \theta$$
$$\sin \iota \cos (\Omega' - \Omega_0') = \cos K \sin \iota_0 - \sin K \cos \iota_0 \cos \theta$$

or

$$\cos K = \cos \iota \cos \iota_0 + \sin \iota \sin \iota_0 \cos (\Omega' - \Omega_0')$$
$$\sin K \sin \theta = \sin \iota \sin (\Omega' - \Omega_0')$$
$$\sin K \cos \theta = \cos \iota \sin \iota_0 - \sin \iota \cos \iota_0 \cos (\Omega' - \Omega_0')$$

where

$$\theta = n(T_0 - T)$$

Having constants K, n, m, T_0, ι_0, and Ω_0' for the life of the satellite, the selenocentric transformation of any star a for anytime T follows in the reverse simple order:

$$\theta = n(T_0 - T)$$
$$\cos \iota = \cos K \cos \iota_0 + \sin K \sin \iota_0 \cos \theta$$
$$\tan (\Omega' - \Omega_0') = \frac{\sin K \sin \theta}{\cos K \sin \iota_0 - \sin K \cos \iota_0 \cos \theta}$$
$$\Omega' = (\Omega' - \Omega_0') + \Omega_0'$$
$$\sin \phi_a = \cos \delta_a \cos \iota + \cos \delta_a \sin \iota \sin (RA_a - \Omega') \tag{1}$$
$$\cos Z_0a = \sin \delta_a \cos \iota_0 + \cos \delta_a \sin \iota_0 \sin (RA_a - \Omega_0')$$
$$\sin \lambda_x = \frac{\sin \iota \sin (\Omega' - \Omega_0')}{\sin K}$$
$$\sin (\lambda_{a_0} - \lambda_x) = \frac{\cos Z_0a - \cos K \sin \phi_a}{\sin K \cos \phi_a}$$
$$\lambda_a = (\lambda_{a_0} - \lambda_x) + \lambda_x + \int mdt \tag{2}$$

C. Test and Simulation of Satellite Reduction Equations with Earthbound Photography

It has become more or less standard procedure to construct and prepare numerical models of new concepts. However, costly efforts which have no precedent require more insight than that obtainable from a numerical model. Under such circumstances, physical simulation should be performed if possible. Analysis shows that the determination of the Moon's fundamental motions with satellite star-terrain photography can be completely simulated with earthbound photography with two minor exceptions in the data reduction procedure. These minor exceptions are: (1) the correction of star, peripheral and conjugate-images for atmospheric refraction; (2) a preliminary reduction of the selenocentric nadir. Realistic simulation is possible because of the generality of the data reduction equations insofar as the determinations are independent of:

(1) the coordinates of the camera station, and
(2) the coordinates of previously determined points on the Moon's surface,
(3) the parameters of the Moon's orbit, and
(4) the parameters of the satellite orbit.

For these reasons, we are not concerned with orbital perturbations or the accuracy of previously established selenocentric coordinates. Similarly, reductions made with earthbound photography are independent of the geocentric coordinates of the camera station and the Moon's ephemeris data.

With regard to the two minor exceptions to the data reduction procedure, the refraction correction would be omitted if the differential errors were zero. While the differential refraction errors are not zero, only approximate exterior-orientation data are required for an approximate absolute correction to obtain an exact differential-correction. The coordinate corrections are computed as follows:

$$dx = (xz - xn)(xn^2 + yn^2 + f^2)^{1/2}(A - B\tan^2 Zn)\sec Zn$$

$$dy = (yz - yn)(xn^2 + yn^2 + f^2)^{1/2}(A - B\tan \lambda Zn)\sec Zn$$

where xz, and yz are determined with a 3×3 employing the approximate coordinates of the camera station and the camera and astronomic coordinates of three stars falling in the field surrounding the Moon's disk, and

$$\cos Zn = \frac{xzxn + yzyn + f^2}{LzLn}$$

$$A = \frac{980.1 \cdot b \cdot \sin 1''}{(459.7 + t)f\dfrac{t}{Lz}}$$

$$B = \frac{9.76 \cdot b \cdot \sin 1''}{(459.7 + t)f\dfrac{f}{Lz}}$$

b = barometric pressure in inches.

t = temperature in Fahrenheit degrees.

The tangential solution for axes orientation is again accomplished followed by the second minor data reduction exception which is the determination of the selenocentric nadir image-coordinates. The horizon and the geometric center of a spherical object subtend constant angles at an exterior point. This property is employed to write n

equations with peripheral image coordinates of the form:

$$\frac{xp}{Lp}\left(\frac{xn}{Ln\cos K}\right)+\frac{yp}{Lp}\left(\frac{yn}{Ln\cos K}\right)+\frac{f}{Lp}\left(\frac{f}{Ln\cos K}\right)=1$$

Reduction of the quantities in parentheses by a solution of normal equations gives

$$\frac{xn}{Ln},\quad \frac{yn}{Ln},\quad \frac{f}{Ln}$$

and cos K. The geometry of the solution is shown in Figure 7. A systematic error enters here insofar as the geometric center of the Moon does not coincide with the center of mass and further the separation of the geometric centers varies with the libration of the Moon in latitude and longitude. The separation of the geometric centers is attributed to the bias produced by a large range of mountains in the south. In any case, each exposure is transformed to the selenocentric nadir. One exposure is selected as a reference exposure whereby relative orientation equations are solved with nonperipheral conjugate-images to determine the relative orientation of every other exposure with respect to the selected reference exposure. Aside from the separation of geometric centers, the relative tilts so obtained are the selenocentric angles enclosed at the center of the Moon by the surface features and the corresponding selenocentric nadirs. A classical reduction of relative orientation angles would produce little or nothing in the way of relative orientation angles owing to the narrow cone subtended by the Moon and the weak intersection angles between exposures. The maximum intersection angle of 15° arises directly from the separation of the observer from the Moon's geocentric coordinates, the inclination of the Moon's pole to its orbit, and the periodic inequality between the Moon's rotation and revolution. The difficulty is partially surmounted by use of the tangential reciprocal orientation equations applied to overlapping pairs of satellite photography. The relative orientation angles are employed to determine the angles subtended at the center of the Moon by the selenocentric nadirs and the conjugate-images with sum and difference tangent formulas. These data combined with camera subtended angles and the celestial direction of the selenocentric nadirs and conjugate-images are employed to determine the celestial selenocentric direction of each conjugate-image for each exposure time. While the latitudes of a common point are fixed, the celestial directions are affected only by physical libration and the axial

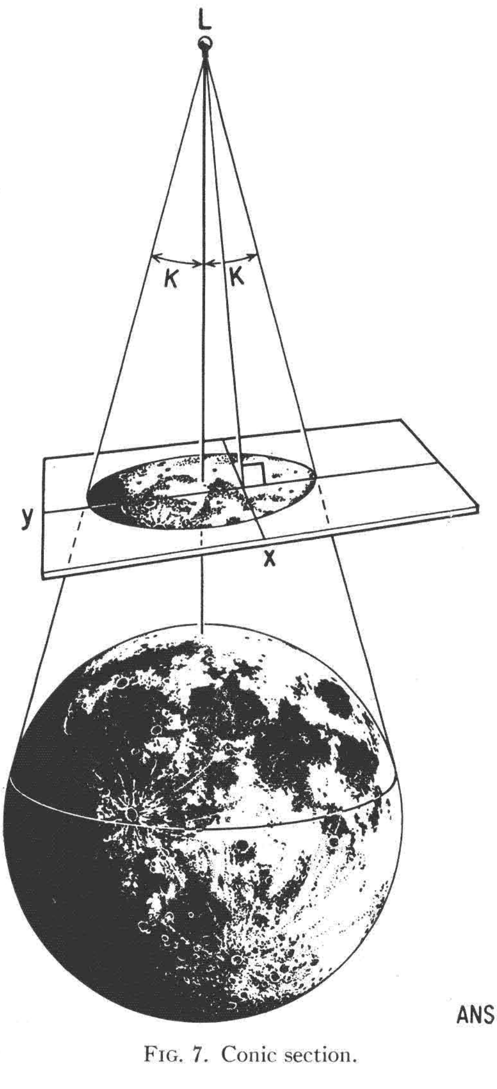

FIG. 7. Conic section.

rotation of the Moon, which in an elapsed time of one month, produces significant changes. The property of fixed latitude and the celestial changes in direction are employed to derive the orientation of the Moon's pole and rotational rate with the previously described data reduction procedure.

To this end, the Autometric Corporation has fabricated a Moon camera for use on the Earth specifically to simulate the data reduction procedures to be employed in the determination of the time referenced orientation of the Moon's pole and rotational rate with satellite photography. The camera is shown in Figure 8. The camera has a 12-inch, F 2.5 lens with a circular field of 10° and a 10-exposure film magazine. Film distortion and disorientation are controlled with a gridded plano-concave element with the planar side in coincidence with the focal-plane. A plano-parallel element containing a circular 3° neutral density filter is situated in air near the plano-concave element. The filter has a transmission of 1:400. The camera is equatorially mounted and driven with a sidereal clock to provide star-image-motion compensation. The right ascension and the declination circles are graduated and viewed in an eyepiece. The finder telescope axis lies on the polar axis of the mount and is consequentially stationary. These features allow the observer to preset and intermittently reset the celestial direction of an exposure with no motion of his head in viewing the object and little motion in viewing the circles. Both circles may be moved rapidly or clamped and moved slowly. A between-the-lens shutter pulses a timer when fully open and the instant the shutter starts to close.

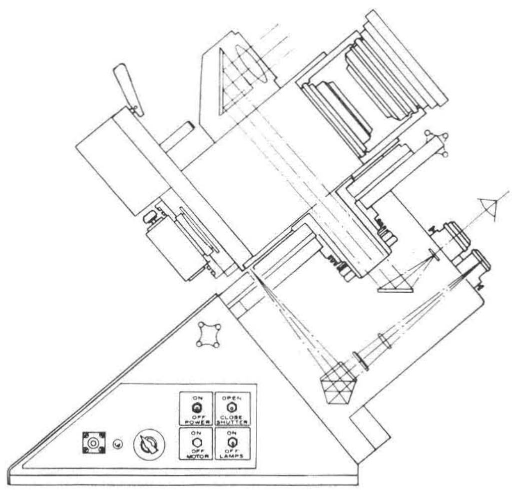

FIG. 8. Star moon Camera.

The maximum parallax angle arising from geometric libration is 15°; approximately six months are required to achieve maximum positive and negative values in either latitude or longitude. Actually a somewhat longer period is required since the extremes of the two geometric librations do not occur at the same time. Assuming cloud coverage will eliminate some observations, a series of 10 or more 10-second time-referenced exposures will be made at each local lunar opposition for a period of one year on either 2G spectrographic or Tri X film. These exposures will simulate the data obtained from one pass. The observatory from which the exposures will be made is shown in Figure 9. The accuracy of the data is not expected to be at all comparable to the satellite reduction. A directional accuracy of $0''.5$ at an earthbound camera station corresponds to 2 minutes at the center of the Moon. A directional accuracy of $10''.0$ at a satellite station corresponds to $0''.2$ at the center of the Moon. The intersection angles at an earthbound camera station vary from 0 to 15° whereas the intersection angles of conjugate satellite exposures are around 40°. At most, the earthbound exposures will not exceed 120 of one side of the Moon whereas the satellite exposures will approximate 7,000 geometrically distributed over the Moon's surface. Finally, the Moon subtends an angle of $27'$ with earthbound photography compared with a diagonal cone angle of 90° of an area of the Moon with satellite photography.

D. DETERMINATION OF THE FUNDAMENTAL MOTIONS OF THE MOON WITH MEASURES MADE AT THE MOON'S SURFACE

To complete the analysis, suppose an observer on the Moon is equipped with a wide-angle metrical camera and an accurate chronometer. We shall see the problem of

FIG. 9. The observatory from which exposures will be made.

measures of the Moon's fundamental motions limited in accuracy with earthbound photography, complicated with satellite photography is no problem at all with moon-bound photography. Unfortunately, the first man on the Moon cannot afford the time required for a preliminary fundamental motion reduction. However, he may have an opportunity to make a series of time exposures of the region of the sky surrounding the Moon's pole that could be reduced upon his return to Earth. The camera need not be leveled, only firmly fixed to the surface for the duration of the n time referenced exposures.

The angular relation between the camera axes and the Moon's axes is rigidly fixed for all exposures. The camera x, y, z axes define constant angles ϕ_x, ϕ_y, ϕ_z with the Moon's pole for all exposures. Therefore, three cones, x, y, and z, are generated whose axis is the Moon's pole. Thus as the relation between the orientation of the Moon with respect to the celestial sphere changes owing to the motion of the Moon's pole and rotation, the camera axes trace out the Moon's angular motion on the celestial sphere. The geometry is illustrated in Figure 10.

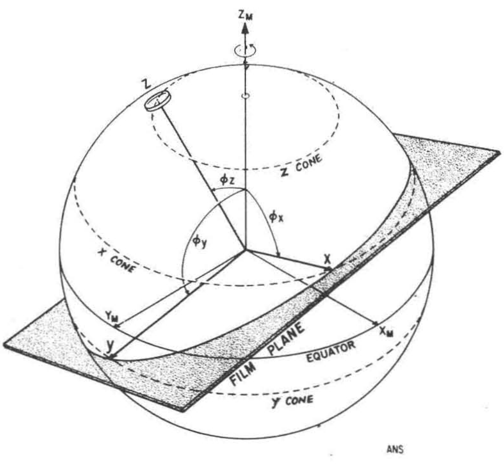

FIG. 10. The cones generated by the fixed camera axes on the moon.

The image-coordinates are corrected for film distortion and the apparent places of the stars reduced with the added aberration correction for the Moon's orbital velocity. The celestial axes orientation is reduced with the iterative tangential solution, previously described, of the form:

$$a \cdot \Delta s + b \cdot \Delta RA_z + c \cdot \Delta\delta_z = \Delta\theta$$

from which is obtained

$$s = s' + \Sigma\Delta s$$

$$RA_z = RA_z' + \Sigma\Delta RA_z$$

$$\delta\text{-} = \delta_z' + \Sigma\Delta\delta_z$$

With these fundamental angles, the axes direction angles are determined for each exposure:

	X	Y	Z
x	α_x	β_x	γ_x
y	α_y	β_y	γ_y
f	α_z	β_z	γ_z

The reduction to this point is identical to the reduction of the earthbound and satellite photography. An equation of the following form may be written for each axis:

$$\cos \alpha_x \left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_x} \right) + \cos \beta_x \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_x} \right) + \cos \gamma_x \left(\frac{\cos \iota_0}{\sin \phi_x} \right) = 1 \cdots \tag{1}$$

$$\cos \alpha_y \left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_y} \right) + \cos \beta_y \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_y} \right) + \cos \gamma_y \left(\frac{\cos \iota_0}{\sin \phi_y} \right) = 1 \cdots \tag{2}$$

$$\cos \alpha_z \left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_z} \right) + \cos \beta_z \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_z} \right) + \cos \gamma_z \left(\frac{\cos \iota_0}{\sin \phi_z} \right) = 1 \cdots \tag{3}$$

There are three such equations for each exposure. Equations (1) from all exposures are combined to form an array of normal equations in the quantities shown in parentheses; equations (2) form a second array of normal equations; and equations (3) form a third array of normal equations. With the corresponding data produced from each exposure, the first array yields Least Squares components of the sin ϕ_x function; the second array, Least Squares components of the sin ϕ_y function; and the third array, Least Squares components of the sin ϕ_z function. Then,

$$\sin^2 \phi_x K^2 = \frac{1}{\left[\left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_x} \right)^2 + \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_x} \right)^2 + \left(\frac{\cos \iota_0}{\sin \phi_x} \right) \right]^2}$$

$$\sin^2 \phi_y K^2 = \frac{1}{\left[\left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_y} \right)^2 + \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_y} \right)^2 + \left(\frac{\cos \iota_0}{\sin \phi_y} \right) \right]^2}$$

$$\sin^2 \phi_z K^2 = \frac{1}{\left[\left(\frac{\sin \iota_0 \cos \Omega_0'}{\sin \phi_z} \right)^2 + \left(\frac{\sin \iota_0 \sin \Omega_0'}{\sin \phi_z} \right)^2 + \left(\frac{\cos \iota_0}{\sin \phi_z} \right) \right]^2}$$

$$K^2 = \sin^2 \phi_x K^2 + \sin^2 \phi_y K^2 + \sin^2 \phi_z K^2$$

Having a Least Squares value of the invariant angles ϕ_x, ϕ_y, ϕ_z, the particular orientation of the Moon's pole is obtained for each exposure time, T_n.

$$\tan \Omega_n' = \frac{\cos \beta_{xn} \sin \phi_x + \cos \beta_{yn} \sin \phi_y + \cos \beta_{zn} \sin \phi_z}{\cos \alpha_{xn} \sin \phi_x + \cos \alpha_{yn} \sin \phi_y + \cos \alpha_{zn} \sin \phi_z}$$

$$\cos \iota_n = \cos \gamma_{xn} \sin \phi_x + \cos \gamma_{yn} \sin \phi_y + \cos \gamma_{zn} \sin \phi_z$$

Having n values of ι and Ω' for n exposure times, the rotational rate may be obtained for any pair of times $(T_n - T_{n+1})$.

$$\Delta\lambda_{n,n+1} = \frac{\sin \delta_{z_n}[\Delta\delta_z \sin (RA_{z_n} - \Omega_n') - \Delta\iota] + \cos \delta_{z_n} \cos (RA_{z_n} - \Omega_n')(\Delta RA_z - \Delta\Omega')}{\cos \iota_n \cos (RA_{z_n} - \Omega_n')}$$

where

$$\Delta\delta_z = \delta_{z_{n+1}} - \delta_{z_n}$$
$$\Delta RA_z = RA_{z_{n+1}} - RA_{z_n}$$
$$\Delta\Omega = \Omega'_{n+1} - \Omega_n'$$
$$\Delta\iota = \iota_{n+1} - \iota_n$$

Then

$$m_{n,n+1} = \frac{\Delta\lambda_{n,n+1}}{T_{n+1} - T_n}$$

Thus, without level, measures of the orientation of the Moon's pole and rotational rate have been found in the simplest possible manner.

This concludes the photogrammetric determination of the Moon's fundamental motions with satellite photography, simulation of these determinations with earth-bound photography, and the verification of these determinations with exposures made at the Moon's surface.

Ancient Indian Fishtraps in the Potomac River

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An important "missing link" in archaeological studies in the Potomac River Basin was recently closed. In the course of the author's continuing efforts to develop aerial tactics in the war against water pollution, under a Research Grant sponsored by the Division of Water Supply and Pollution Control, U. S. Public Health Service, a series of ancient Indian fishtraps were discovered. The structures were discovered on 24 January, 1962, while on a flight from Washington National Airport to Martinsburg, West Virginia, in company with Mr. C. B. Diamond, of East Coast Flying Service.

A total of seven fishtraps were sighted. Two are located near Mason Island, just south of the Dickerson, Maryland, Potomac Electric Power Company thermal electric plant, in that part of the river adjacent to the Virginia shore. An additional five were sighted between Mason Island, and the B&O Railroad bridge at Brunswick, Maryland. Two were sighted immediately downstream from the railroad bridge.

At that time, the Potomac River was quite low, and the water much clearer than usual. Very little silt was evident. Conditions were more favorable than usual for observation of subsurface features.

The fishtraps appear to be made of stone. They are in the form of elongated "V"s, open end upstream. The wings of each structure lie at an angle to the normal bedding of the tilted sedimentary rock formations characteristic of the area. In form, they correspond to primitive fishtraps previously sighted by the author, and familiar to most photo-interpreters who have studied photography taken of islands in the Southwestern Pacific Ocean area.

Contact was made with the U. S. Army Corps of Engineers personnel who are currently conducting an extensive study of the entire Potomac River Basin. They knew of no structures of the form described in the river, which at least confirmed that the structures are not weirs of recent origin.

Subsequent investigation confirmed that the structures were fishtraps, most of them of ancient origin. Mr. Nicholas Yinger, Presi-