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Analytical Rectification

Data from analytic aerotriangulation are used to determine the rectifier settings for near-vertical photos.

Introduction

Distortions in near-vertical aerial photographs caused by tilt can be removed by rectification. In some applications, e.g., the production of highway photo plans, the rectifier can also be used to lessen the amount of distortions caused by topography.

Although the theory in this article may be applied to most rectifiers, the formulas were specifically derived for the Wild E4 Rectifier. Five independent manual settings are on this instrument: the x and y-table tilt handwheels, the course (motor switch) and fine (foot wheel) scale setting, and the x and y negative displacement dials.

Abstract: *Following the analytical triangulation and adjustment of a strip or block of photographs, both photo and ground coordinates of pass points are available. These may be employed to solve for the coefficients in the projective transformation equations. The coefficients in turn may be substituted into formulas which yield the settings of the rectifier. Formulas applicable to the Wild E4 Rectifier are derived, and a practical test is applied to assess the method of analytical rectification.*

The conventional procedure for the production of rectified photographs is time consuming, laborious and requires a well trained technician to operate the rectifier and perform the required calculations (Wild 1965). On completion of the stereo triangulation and adjustment of a strip of photographs, the ground coordinates of control and pass points whose images appear in the photographs, are available. The planimetric ground coordinates of all points appearing on each photograph are then plotted on templates. The template is then placed on the projection table of the rectifier while the operator attempts to fit the projected point images of the negative onto the plot by varying the five rectifier elements in an empirical, trial and error sequence. Often the full potential of the rectifier is not realized because of production short-cuts, and the calculations required in arriving at estimates for x and y negative displacements.

It has been proposed that the conventional procedure of rectification be replaced by one in which the elements of rectification would be determined by the computer. The computer would output these parameters in such a form that an operator could simply *dial* the values of the parameters into the rectifier to expose a rectified photograph. The need for the templates and the time-consuming trial and error procedure would be eliminated.

As described later in this report, formulas for the five input elements of rectification for the Wild E4 Rectifier have been derived. These formulas express the five

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manual settings of the rectifier as a function of the focal length of the rectifier and the eight coefficients of the projective transformation equations. Thus, before the rectification formulas can be applied, values of the eight coefficients of the projective transformation Equations 1 are required (Gagnon, 1969):

$$
\begin{aligned}
X &= (a_1X' + b_1Y' + c_1)/(a_0X' + b_0Y' + 1) \\
Y &= (a_2X' + b_2Y' + c_2)/(a_0X' + b_0Y' + 1)
\end{aligned}
\tag{1}
$$

where

X, Y = photo coordinates
X', Y' = map coordinates.

The eight coefficients $a_1 \cdots c_2$ of the projective transformation equations are computed by substituting the photo coordinates X, Y and map coordinates X', Y' of four or more points per photograph into the equations. Values for the eight coefficients can then be found from the linear equations. The principal of least squares can be employed to provide the values of the eight coefficients in the redundant case.

In order to acquire planimetric ground coordinates, an aerial triangulation of the strip is usually required. Instrumental and semi-analytical methods of triangulation do not provide photo coordinates of the required points as does the analytical solution. Analytical aerotriangulation is therefore more compatible with analytical rectification.

Theory of Rectification and Derivation of Formulas

In Figure 1, planes I and II intersect at angle ν. Point S represents the center of a bundle of rays, and f' and h' are rays from S to vanishing points H' and G. Points J' and J are isocenters in planes I and II respectively, and coordinate systems $x'J'y'$ and xJy are projectively related. $J'x'$ and Jx lie parallel to the line of intersection of planes I and II. Coordinate systems $X'U'Y'$ and XUY are arbitrarily rotated and translated from $x'J'y$ and xJy by the amounts indicated on the figure. It can be shown that the coefficients to the projective transformation Equations 1 are functions of the eight parameters: $h', f', X_c, Y_c, \mathrm{K}, y_c', {}_c', \mathrm{K}'$ (Von Gruber 1932). These functions are described by the following equations:

$$
\begin{aligned}
a_1 &= + [h' \cos(\mathrm{K} + \mathrm{K}') - X_c \sin \mathrm{K}']/[f' - y_c'] \\
a_2 &= + [h' \sin(\mathrm{K} + \mathrm{K}') - Y_c \sin \mathrm{K}']/[f' - y_c'] \\
b_1 &= - [h' \sin(\mathrm{K} + \mathrm{K}') + X_c \cos \mathrm{K}']/[f' - y_c'] \\
b_2 &= + [h' \cos(\mathrm{K} + \mathrm{K}') - Y_c \cos \mathrm{K}']/[f' - y_c'] \\
c_1 &= + \{[h'(x_c' \cos K - y_c' \sin \mathrm{K})]/[f' - y_c']\} + X_c \\
c_2 &= + \{[h'(x_c' \sin \mathrm{K} + y_c' \cos \mathrm{K})]/[f' - y_c']\} + Y_c \\
a_0 &= - (\sin \mathrm{K}')/(f' - y_c') \\
b_0 &= - (\cos \mathrm{K}')/(f' - y_c').
\end{aligned}
\tag{2}
$$

Figure 2 is an orthogonal view of the principal plane $GJSJ'H'$ in Figure 1. On the rectifier, plane II is the negative plane and plane I is the exposure plane. GK is the focal length, f_r, of the rectifier lens. The angles β and α are those that the exposure and negative planes make with the plane of the lens, respectively. Line BSB' is normal to the lens plane through S.

The five elements of the Wild E4 Rectifier are tilt in both the x and y directions, scale, and negative displacements in the x and y directions. The x and y tilts, β_x and β_y, are displayed as tangents of the x and y components of the exposure plant tilt β

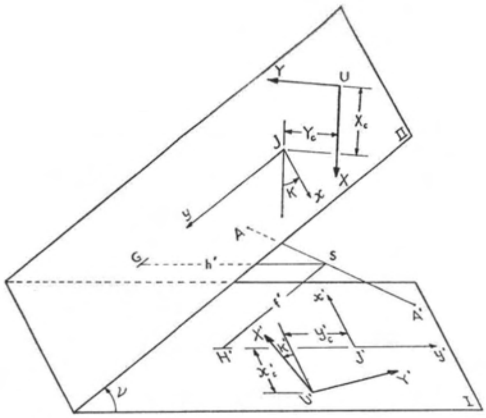

FIG. 1. Relation between the two independent coordinate systems.

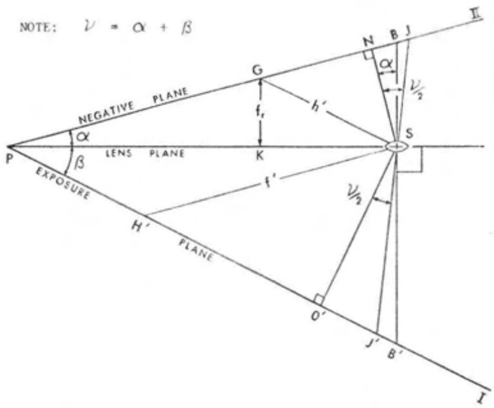

FIG. 2. Geometry of the Wild E4 Rectifier.

(relative to the plane of the lens). From Figure 1, if the coordinate system XUY is taken as that of the negative, the angle between the X-axis of the negative and the direction of maximum tilt is K. Thus,

$$\tan \beta_x = - \tan \beta \sin \mathrm{K}$$
$$\tan \beta_y = + \tan \beta \cos \mathrm{K}.$$

From Figure 2,

$$\sin \beta = \frac{f_r}{h'} \cdot$$

Therefore,

$$\tan \beta = f_r/\sqrt{(h'^2 - f_r^2)}$$

Thus,

$$\begin{aligned} \tan \beta_x &= - (f_r \sin \mathrm{K})/\sqrt{(h'^2 - f_r^2)} \\ \tan \beta_y &= + (f_r \cos \mathrm{K})/\sqrt{(h'^2 - f_r^2)}. \end{aligned} \tag{3}$$

The scale factor V is read from a graduated scale opposite a sliding marker. It expresses the dividend of distance $B'S$ over SB (Figure 2). But

$$B'S/SB = \tan \beta / \tan \alpha$$

and

$$\tan \beta = f_r/\sqrt{(h'^2 - f_r^2)}$$
$$\tan \alpha = f_r/\sqrt{(f'^2 - f_r^2)}$$

Thus,

$$V = \sqrt{[(f'^2 - f_r^2)/(h'^2 - f_r^2)]}. \tag{4}$$

Both x and y negative displacements e_x and e_y are measured as distances in mm from B (where the vertical through the center of the lens meets the negative plane). In Figure 1, X_c and Y_c are the coordinates of the isocenter J in the system XUY of the negative. Therefore, to express e_x and e_y in terms of X_c and Y_c, the x and y components of the distance from B to J are needed.

But

$$SN = h' \sin \nu$$
$$NJ = SN \tan \nu/2 = h' \sin \nu \tan \nu/2$$
$$NB = SN \tan \alpha = h' \sin \nu \tan \alpha$$

Therefore

$$BJ = NJ - NB = h' \sin \nu(\tan \nu/2 - \tan \alpha)$$

Also

$$\tan \alpha = f_r/\sqrt{(f'^2 - f_r^2)}$$
$$\tan \beta = f_r/\sqrt{(h'^2 - f_r^2)}$$

and

$$\nu = \alpha + \beta.$$

By trigonometric calculation, the distance BJ is computed in terms of f_r, f' and h':

$$BJ = \frac{f_r^2(h' - f')[\sqrt{(h'^2 - f_r^2)} + \sqrt{(f'^2 - f_r^2)} + h' + f']}{\sqrt{(f'^2 - f_r^2)}\{[f' + \sqrt{(f'^2 - f_r^2)}][h' + \sqrt{(h'^2 - f_r^2)}] - f_r^2\}}$$

Once again, the angle between the X-axis of the negative and the direction of maximum tilt is K. Thus, the x and y components of BJ are:

$$BJ_x = - BJ \sin \mathrm{K}$$
$$BJ_y = + BJ \cos \mathrm{K}.$$

Combining the x and y components of BJ with the displacements X_c and Y_c and applying proper signs, we arrive at the following formulas for the negative displacements:

$$\begin{aligned} e_x &= X_c - BJ \sin \mathrm{K} \\ e_y &= Y_c + BJ \cos \mathrm{K}. \end{aligned} \tag{5}$$

At this point, the five input parameters of rectification (β_x, B_y, V, e_x, e_y,) with the Wild E4 Rectifier are formulated as functions of the parameters X_c, Y_c, f', h' and K (Figure 1), and the focal length of the rectifier lens f_r. The remaining three parameters x_c', y_c', and K′ shown in Figure 1 define the location and orientation of the photographic material on the exposure table. These values are not required because the operator simply places the photo sensitive material in the desired area of the projection while a red filter over the lens protects the emulsion from exposure.

It can be shown that the following five equations result from Equations 4:

$$\begin{aligned} h' &= \sqrt{[(a_0b_1 - a_1b_0)^2 + (a_0b_2 - a_2b_0)^2]}/(a_0^2 + b_0^2) \\ f' &= \frac{\sqrt{(a_0^2 + b_0^2)}\,ABS[(a_2b_1 - a_1b_2) + c_2(a_1b_0 - a_0b_1) + c_1(a_0b_2 - a_2b_0)]}{(a_0b_1 - a_1b_0)^2 + (a_0b_2 - a_2b_0)^2} \\ \tan \mathrm{K} &= (a_0b_2 - a_2b_0)/(a_0b_1 - a_1b_0) \\ X_c &= [a_0(a_1 - b_2) + b_0(a_2 + b_1)]/(a_0^2 + b_0^2) \\ Y_c &= [a_0(a_2 + b_1) - b_0(a_1 - b_2)]/(a_0^2 + b_0^2). \end{aligned} \tag{6}$$

On substituting these expressions into the rectification formulae for the Wild E4 Rectifier Equations 3, 4 and 5, the following equations result:

$$
\begin{aligned}
\tan \beta_x &= -\ \text{SUB20 SUB4} \\
\tan \beta_y &= +\ \text{SUB20 SUB3} \\
V &= (\text{SUB13 SUB1})/(\text{SUB12 SUB10}) \\
e_x &= \frac{a_0(a_1 - b_2) + b_0(a_2 + b_1)}{\text{SUB1}} - \frac{\text{SUB25 SUB4}}{\sqrt{\text{SUB10}}} \\
e_y &= \frac{a_0(a_2 + b_1) - b_0(a_1 - b_2)}{\text{SUB1}} + \frac{\text{SUB25 SUB3}}{\sqrt{\text{SUB10}}}
\end{aligned}
\tag{7}
$$

where

$\text{SUB1} = a_0^2 + b_0^2$

$\text{SUB2} = a_2 b_1 - a_1 b_2$

$\text{SUB3} = a_0 b_1 - a_1 b_0$

$\text{SUB4} = a_0 b_2 - a_2 b_0$

$\text{SUB10} = \text{SUB3}^2 + \text{SUB4}^2$

$\text{SUB11} = ABS(\text{SUB2} - c_2\ \text{SUB3} + c_1\ \text{SUB4})$

$\text{SUB12} = \sqrt{}(\text{SUB10} - f_r^2\ \text{SUB1}^2)$

$\text{SUB13} = \sqrt{}(\text{SUB1 SUB11}^2 - f_r^2\ \text{SUB10}^2)$

$\text{SUB15} = \text{SUB11}\ \sqrt{\text{SUB1}^3}$

$\text{SUB16} = \sqrt{\text{SUB10}^3}$

$\text{SUB20} = f_r\ \text{SUB1}/(\text{SUB12}\sqrt{\text{SUB10}})$

$$\text{SUB25} = \frac{f_r^2(\text{SUB16} - \text{SUB15})(\text{SUB10 SUB12} + \text{SUB1 SUB13} + \text{SUB16} + \text{SUB15})}{\text{SUB1 SUB13}[(\text{SUB11}\ \sqrt{\text{SUB1}} + \text{SUB13})(\sqrt{\text{SUB10}} + \text{SUB12}) - f_r^2\ \text{SUB1 SUB10}]}$$

NOTE:

$\text{SUB25} = BJ.$

Thus, values for the five parameters of rectification may be found with the knowledge of photo and map coordinates of at least four points (in one photograph), the focal length of the rectifier lens, and the use of Equations 1 and 7.

A Practical Test

In order to assess the analytical method of rectification in terms of efficiency and accuracy, a practical test was conducted at the Department of Highways of Ontario. The analytical rectification and enlargement to 50 ft./in. of 12 photographs in a strip of normal angle-photography, taken with a 12 in. camera at a scale of approximately 300 ft./in., constituted the test. The attempt at conventional rectification of three of the above photographs made possible a partial comparison of the two methods.

The plates were prepared and measured for the purpose of analytical triangulation using a Kelsh KPP-3 Stereo-plotter fitted with a Coradograph coordinatograph. The analytical triangulation program used was Version III of the Coast and Geodetic Survey's Three-Photo Aerotriangulation (Keller & Tewinkel 1966), modified somewhat for the purposes of the Department of Highways. Next, the Strip and Block adjustment program of the National Research Council (Schut 1966) was employed to adjust the strip to 14 well placed ground control points. Finally, the Analytical Rectification program written by the author was run to generate the five input parameters of rectification for each photo to be rectified. Data from this program was then used to expose rectified and enlarged negative films on the Wild E4 Rectifier from the diapositives.

Although templates for each photograph were not required in the rectification process, a base control sheet covering the area of the mosaic was required. This sheet, on which a ten-inch (or 500 ft.) grid and horizontal ground control points were drawn, controlled the mosaic.

On joining adjacent photographs, it was found that images agreed quite well. The discrepancies in positions did not exceed 2 mm. and averaged less than 1 mm. at enlarged scale. Also, on assembling the mosaic and fitting to the base control sheet, it took little effort to match the photographic images to the plotted ground control.

The test indicated that the analytical method of rectification results in rectified photographs at least as good as those obtained from conventional methods, and required fewer man- and instrument-hours per photograph. For these reasons it is being adopted as the standard procedure for rectification of near vertical aerial photographs at the Department of Highways of Ontario.

Acknowledgements

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