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AN INVERSE MAPPING THEOREM IN FRECHET SPACES

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ABSTRACT. Within the framework of σ-differentiability, introduced by H. R. Fischer in locally convex spaces, sufficient conditions for an inverse mapping theorem between Fréchet spaces are established.

RESUME. En se basant sur les propriétés de la σ-différentiabilité introduite par H. R. Fischer dans les espaces localement convexes, les auteurs établissent des conditions suffisantes pour obtenir un théorème "d'application inverse" entre deux espaces de Fréchet.

1. **Introduction**. In a Banach space, the notion of differentiable mapping is well established (cf. Henri Cartan [1]). In fact, let E and F be Banach spaces, f a mapping from an open set U in E to F, and $a \in U$. f is (Fréchet) differentiable at a if there exists $g \in \mathcal{L}(E, F)$ such that

$$\|f(x) - f(a) - g(x - a)\| = 0(\|x - a\|).$$

We shall write $f'(a)$ for g.

We then have the well known inverse mapping theorem for Banach spaces, namely: If f is strictly differentiable at a ([1]), and if $f'(a)$ is an isomorphism, then there exist an open neighborhood V of a, and an open neighborhood W of $f(a)$, such that f is a homeomorphism from V onto W, and the inverse homeomorphism is strictly differentiable at $f(a)$.

Different extensions of this theorem to some classes of locally convex spaces have been proposed ([5], [6], [7]), using different notions of a differentiable mapping. In fact, there is no unique natural definition of a differentiable mapping in these spaces (see for example [8]).

In this paper, we shall prove a generalization of the inverse mapping theorem for Fréchet spaces, using an extension of the definition of differentiability introduced by H. R. Fischer in [2].

Let E and F be two locally convex spaces with topology generated by sets of semi-norms Γ_E and Γ_F.

DEFINITION 1. *A calibration is a mapping* σ *from* Γ_F *to* Γ_E.

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DEFINITION 2. *Let* σ *be a calibration. We define* $\mathcal{L}_\sigma(E, F)$ *as the set of those* $f \in \mathcal{L}(E, F)$ *such that* $\|f\|_q^\sigma < \infty$ *for every* q *in* Γ_F *where*

$$\|f\|_q^\sigma = \sup_{\sigma(q)(x) \leq 1} q(f(x)).$$

Let f be a mapping from an open set U of E to F, $a \in U$ and σ a calibration.

DEFINITION 3. *We call* f H_σ*-differentiable at* a *if there exists* $A \in \mathcal{L}_\sigma(E, F)$ *such that, for each* $\epsilon > 0$ *and* $q \in \Gamma_F$, *there exists* $\delta(\epsilon, q) > 0$ *such that*

$$a + h \in U \text{ and } \sigma(q)(h) \leq \delta \Rightarrow q(r(a, h)) \leq \epsilon\, \sigma(q)(h)$$

where $r(a, h) = f(a + h) - f(a) - A(h)$.

Obviously, when such an A exists, it is unique. In general, we shall write $A = f'(a)$.

DEFINITION 4. *We call* f *strictly* H_σ*-differentiable at* a *if there exists* $B \in \mathcal{L}_\sigma(E, F)$ *such that, for all* $\epsilon > 0$ *and* $q \in \Gamma_F$, *there exists* $\delta(\epsilon, q) > 0$ *such that*

$$\sigma(q)(x - a) \leq \delta \text{ and } \sigma(q)(y - a) \leq \delta \Rightarrow q(r_1(x, y) \leq \epsilon\, \sigma(q)(x - y)$$

where $r_1(x, y) = f(x) - f(y) - B(x - y)$.

It is clear that definition 4 implies definition 3 with $B = A$.

DEFINITION 5. *We call* f M_σ*-differentiable at* a *if there exists* $A \in \mathcal{L}_\sigma(E, F)$ *such that, for each* $\epsilon > 0$ *and* $q \in \Gamma_F$, *there exists* $V_{\epsilon,q}(a)$, *a open neighborhood of* a, *such that*

$$a + h \in V_{\epsilon,q}(a) \Rightarrow q(r(a, h)) \leq \epsilon\, \sigma(q)(h)$$

where $r(a, h) = f(a + h) - f(a) - A(h)$.

Here again, when such an A exists, it is unique, and we shall write $A = f'(a)$.

DEFINITION 6. *If* f *is* M_σ*-differentiable at* a *and if we can find an open neighborhood* $V_\epsilon(a)$ *of* a *such that, for all* $q \in \Gamma_F$

$$a + h \in V_\epsilon(a) \Rightarrow q(r(a, h)) \leq \epsilon\, \sigma(q)(h),$$

we call f *uniformly* M_σ*-differentiable at* a.

DEFINITION 7. *We call* f *strictly uniformly* M_σ*-differentiable at* a *if there exists* $B \in \mathcal{L}_\sigma(E, F)$ *such that, for each* $\epsilon > 0$, *there exists* $V_\epsilon(a)$, *an open neighborhood of* a, *such that, for all* $q \in \Gamma_F$

$$x, y \in V_\epsilon(a) \Rightarrow q(r_1(x, y)) \leq \epsilon\, \sigma(q)(x - y),$$

where $r_1(x, y) = f(x) - f(y) - B(x - y)$.

Here again, it is clear that definition 7 implies definition 6 with $B = A$.

REMARK. It is possible to find non trivial examples of strictly uniformly M_σ-differentiable functions.

EXAMPLE 1. *If* $\sigma(\Gamma_F)$ *is a finite subset of* Γ_E, *and if* f *is strictly* H_σ*-differentiable at* a, *then* f *is strictly uniformly* M_σ*-differentiable at* a.

EXAMPLE 2. *Let* $E = \{x = (x_1, x_2, \ldots, x_n, \ldots): x_n \in \mathbb{R}\}$ *the space of sequences with the set of semi-norms* $\{p_n : p_n(x) = |x_n|\}_{n \in \mathbb{N}}$, *and let* $f: x \in E \to f(x) = (\ldots, x_n + 1/n \cos x_n, \ldots) \in E$.

Then f is strictly uniformly M_{Id}-differentiable at $o \in E$. In fact, we have

$$f(x) - f(y) = (\ldots, x_n - y_n, \ldots) + \left(\ldots, \frac{1}{n}(\cos x_n - \cos y_n), \ldots\right)$$

$$= Id(x - y) + \left(\ldots, \frac{1}{n}(\cos x_n - \cos y_n), \ldots\right)$$

$$p_n(r(x, y)) = \frac{1}{n}|\cos x_n - \cos y_n| = \frac{2}{n}\left|\sin \frac{x_n + y_n}{2}\right|\left|\sin \frac{x_n - y_n}{2}\right|.$$

Since $\forall \epsilon > 0\ \exists k(\epsilon)$ such that $n > k \Rightarrow 1/n < \epsilon$, we have

$$(*)\ n > k \Rightarrow p_n(r(x, y)) \leq \epsilon|x_n - y_n| = \epsilon p_n(x - y).$$

Furthermore, from the continuity of $2/n \sin (x_n + y_n)/2$ at (o, o), we have $\forall \epsilon > 0$ $\exists \delta(n, \epsilon) > 0$ such that

$$|x_n| \leq \delta \text{ and } |y_n| \leq \delta \Rightarrow \left|\frac{1}{n} \sin \frac{x_n + y_n}{2}\right| \leq \epsilon.$$

For $x, y \in B_n = \{z: p_n(z) < \delta\}$, we have $|x_n| < \delta$ and $|y_n| < \delta$; then

$$(**)\ p_n(r(x, y)) \leq \epsilon|x_n - y_n| = \epsilon p_n(x_n - y_n).$$

Let $\epsilon > 0$ be given; we then can find $k = k(\epsilon)$. The set $V = \cap_{1 \leq n \leq k} B_n$ is an open neighborhood of o; furthermore, $\forall\ x, y \in V$ and $\forall\ n \in \mathbb{N}$, we have $p_n(r(x, y)) \leq \epsilon p_n(x - y)$.

In fact, if $n > k$, $p_n(r(x, y))$ satisfies $(*)$; and if $n \leq k$, then $x, y \in B_n$ and $p_n(r(x, y))$ satisfies $(**)$.

2. **Continuity of a differentiable mapping**. We have the following lemma.

LEMMA 1. *If* f *is uniformly* M_σ*-differentiable at* a, *then* f *is continuous at* a.

PROOF. Let $\epsilon > 0$ and $q \in \Gamma_F$ be given. If $\epsilon_1 < \epsilon$, then there exists a neighborhood V_{ϵ_1} of a, $p_1 \in \Gamma_E$ and $\delta_1 > 0$ such that the open semi-ball $B_{p_1}(a, \delta_1)$ is contained in V_{ϵ_1} and

$$p_1(h) < \delta_1 \Rightarrow q(r(a, h)) \leq \epsilon_1\, \sigma(q)\, (h).$$

Let $\delta_2 = \epsilon/\|f'(a)\|_q^\sigma + \epsilon_1$. Since $B_{p_1}(a, \delta_1) \cap B_{\sigma(q)}(a, \delta_2)$ is open, then there exists $p \in \Gamma_F$ and $\delta > 0$ such that $B_p(a, \delta) \subseteq B_{p_1}(a, \delta_1) \cap B_{\sigma(q)}(a, \delta_2)$. Since $p(h) < \delta \Rightarrow p_1(h) < \delta_1$ and $\sigma(q)\,(h) < \delta_2$, we have $q(f(a + h) - f(a)) < \epsilon$.

3. **Natural calibration**.

LEMMA 2. *Let* (E, Γ_E), (F, Γ_F) *be two locally convex spaces, and* $A \in \mathcal{L}(E, F)$. *Then there exists a calibration* σ *such that* $A \in \mathcal{L}_\sigma(E, F)$.

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PROOF. The assumption on A implies that, for every $q \in \Gamma_F$, there exists $p \in \Gamma_E$ and $c > 0$ such that

$$\forall x \in E \quad q(A(x)) \le c\, p(x)$$

We define a mapping $\sigma: \Gamma_F \to \Gamma_E$ by choosing, for a given semi-norm $q \in \Gamma_F$, a semi-norm $p \in \Gamma_E$ satisfying the above condition. It is then clear that $A \in \mathscr{L}_\sigma(E, F)$.

DEFINITION 8. *A natural calibration for $A \in \mathscr{L}(E, F)$ is a calibration σ such that $A \in \mathscr{L}_\sigma(E, F)$.*

We shall often write σ_A for such a calibration.

LEMMA 3. *Let (E, Γ_E), (F, Γ_F), (G, Γ_G) be locally convex spaces, $a \in E$ and f a mapping from E to F, H_σ-differentiable (uniformly M_σ-differentiable) at a. Let $u \in \mathscr{L}(F, G)$ and σ_u be a natural calibration for u. Then $u \circ f$* is *$H_{\sigma\circ\sigma_u}$-differentiable (uniformly $M_{\sigma\circ\sigma_u}$-differentiable) at a.*

PROOF. We know that $f'(a) \in \mathscr{L}_\sigma(E, F)$. Let us show that $u \circ f'(a) \in \mathscr{L}_{\sigma\circ\sigma_u}(E, G)$. For every $p \in \Gamma_G$ there exists $c_p > 0$ such that

$$p((u \circ f'(a))(h)) = p(u(f'(a)h)) \le c_p\, \sigma_u(p)(f'(a)h).$$

But $\sigma_u(p)(f'(a)h) \le \|f'(a)\|^{\sigma}_{\sigma_u(p)}((\sigma\circ\sigma_u)(p))(h)$. Consequently,

$$p((u \circ f'(a))(h)) \le c_p \|f'(a)\|^{\sigma}_{\sigma_u(p)}((\sigma\circ\sigma_u)(p))(h)$$

and

$$u \circ f'(a) \in \mathscr{L}_{\sigma\circ\sigma_u}(E, G).$$

Furthermore $(u \circ f)\,(a + h) - (u \circ f)(a) = u(f(a + h) - f(a)) = (u \circ f'(a))(h) + (u \circ r)(a, h)$ and $\forall p \in \Gamma_G$ we have $p((u \circ r)(a, h)) \le c_p\, \sigma_u(p)(r(a, h))$.

But $\sigma_u(p) \in \Gamma_F$ and f is H_σ-differentiable at a; hence for each $\epsilon/c_p > 0$ there exists $\delta > 0$ such that

$$(\sigma(\sigma_u(p)))(h) \le \delta \Rightarrow \sigma_u(p)(r(a, h)) \le \frac{\epsilon}{c_p}\,(\sigma(\sigma_u(p)))(h)$$

i.e. $((\sigma \circ \sigma_p(p))(h) \le \delta$ implies $p((u \circ r)(a, h)) \le \epsilon$. This means that $u \circ f$ is $H_{\sigma\circ\sigma_u}$-differentiable at a.

We have an analogous result if f is uniformly M_σ-differentiable at a. Indeed for each $\epsilon/c_p > 0$ there exists an open neighborhood V_ϵ of a such that

$$\forall q \in \Gamma_F \text{ we have } \quad a + h \in V_\epsilon \Rightarrow q(r(a, h)) \le \frac{\epsilon}{c_p}\,\sigma(q)(h).$$

In particular, if $q = \sigma_u(p)$, we have

$$a + h \in V_\epsilon \Rightarrow \sigma_u(p)(r(a, h)) \le \frac{\epsilon}{c_p}\,((\sigma\circ\sigma_u)(p))(h),$$

i.e. $a + h \in V_\epsilon$ implies $p((u \circ r)(a, h)) \leq \epsilon((\sigma \circ \sigma_u)(p))(h)$. Therefore $u \circ f$ is uniformly $M_{\sigma \circ \sigma_u}$-differentiable.

LEMMA 4. *With the assumptions of lemma 3, let us suppose in addition that f is strictly H_σ-differentiable (strictly uniformly M_σ-differentiable) at a. Then $u \circ f$ is strictly $H_{\sigma \circ \sigma_u}$-differentiable (strictly uniformly M_σ-differentiable) at a.*

PROOF. For all $x, y \in E$ we have

$$(u \circ f)(x) - (u \circ f)(y) = (u \circ f'(a))(x - y) + (u \circ f)(x, y)$$

where $u \circ f'(a) \in \mathscr{L}_{\sigma \circ \sigma_u}(E, G)$ and $p((u \circ r)(x, y)) \leq c_p\, \sigma_u(p)(r(x, y))\ \forall p \in \Gamma_G$.

If f is strictly H_σ-differentiable at a, then for each $\epsilon/c_p > 0$ there exists $\delta > 0$ such that

$$(\sigma(\sigma_u(p)))(x - a) \leq \delta \text{ and } (\sigma(\sigma_u(p)))(y - a) \leq \delta, \text{ which implies}$$

$$\sigma_u(p)(r(x, y)) \leq \frac{\epsilon}{c_p}(\sigma(\sigma_u(p)))(x - y).$$

Therefore $p((u \circ r)(x, y)) \leq \epsilon\, ((\sigma \circ \sigma_u)(p))(x - y)$.

If f is strictly uniformly M_σ-differentiable at a, then, for each $\epsilon/c_p > 0$, there exists an open neighborhood V_ϵ of a such that

$$\forall\, x, y \in V_\epsilon,\ \sigma_u(p)(r(x, y)) \leq \epsilon\, ((\sigma \circ \sigma_u)(p))(x - y).$$

Hence $p((u \circ r)(x, y)) \leq \epsilon\, ((\sigma \circ \sigma_u)(p))(x - y)$.

LEMMA 5. *Let (E, Γ_E), (F, Γ_F), (G, Γ_G) be locally convex spaces, $a \in E$. Let $u \in \mathscr{L}(E, F)$, σ_u be a natural calibration for u, and f a mapping from F to G, H_σ-differentiable (uniformly M_σ-differentiable) at $u(a)$. Then $f \circ u$ is $H_{\sigma_u \circ \sigma}$-differentiable (uniformly $M_{\sigma_u \circ \sigma}$-differentiable) at a.*

PROOF. We have

$$\begin{aligned}(f \circ u)(a + h) - (f \circ u)(a) &= f(u(a) + u(h)) - f(u(a))\\ &= f'_{u(a)}(u(h)) + r(u(a), u(h))\end{aligned}$$

where $f'_{u(a)} \in \mathscr{L}_\sigma(F, G)$. Let us show that $f'_{u(a)} \circ u \in \mathscr{L}_{\sigma_u \circ \sigma}(E, G)$. If $q \in \Gamma_G$, then

$$q(f'_{u(a)}(u(h))) \leq \|f'_{u(a)}\|^\sigma_{\sigma(q)}\sigma(q)(u(h)) \leq c_{\sigma(q)}\, \|f'_{u(a)}\|^\sigma_{\sigma(q)}(\sigma_u \circ \sigma)(q)(h).$$

Therefore $f'_{u(a)} \circ u \in L_{\sigma_u \circ \sigma}(E, G)$. Furthermore, the H_σ-differentiability of f at $u(a)$ implies

For every $\epsilon > 0$, there exists $\delta_1, \delta > 0$ such that

$$\sigma(q)(u(h)) \leq \delta_1 \Rightarrow q(r(u(a), u(h))) \leq \frac{\epsilon}{c_{\sigma(q)}}\sigma(q)(u(h))$$

$$\text{and } ((\sigma_u \circ \sigma)(q))(h) \leq \delta \Rightarrow \sigma(q)(u(h)) \leq \delta_1.$$

Consequently, $((\sigma_u \circ \sigma)(q))(h) \leq \delta \Rightarrow q(r(u(a), u(h))) \leq \epsilon((\sigma_u \circ \sigma)(q))(h)$, i.e. $(f \circ u)$ is $H_{\sigma_u \circ \sigma}$-differentiable at a.

We have an analogous result if f is uniformly M_σ-differentiable at $u(a)$. Indeed, for each $\epsilon > 0$, there exists an open neighborhood W_ϵ of $u(a)$ such that, for every $q \in \Gamma_G$, we have

$$u(a) + h' \in W_\epsilon \Rightarrow q(r(u(a), h')) \leq \frac{\epsilon}{c_{\sigma(q)}} \sigma(q)(h').$$

If $a + h \in V_\epsilon = u^{-1}(W_\epsilon)$, we have $u(a) + u(h) \in W_\epsilon$ and hence

$$q(r(u(a), u(h))) \leq \frac{\epsilon}{c_{\sigma(q)}} \sigma(q)(u(h)) \leq \epsilon((\sigma_u \circ \sigma)(q))(h).$$

3. **An inverse mapping theorem in E.** Consider now the case where E and F are Fréchet spaces. We shall first prove an inverse mapping theorem for a strictly uniformly M_σ-differentiable mapping from E to E with $f'(a) = 1_E$, and then extend it to a theorem for mappings from E to F. In the sequel we shall consider the convex enveloppe of Γ_E rather than Γ_E itself, but keep the same notation for it.

THEOREM 1. *Let (E, Γ_E) be a Fréchet space, $\sigma: \Gamma_E \to \Gamma_E$ a projective calibration (i.e. $\sigma^2 = \sigma \circ \sigma = \sigma$) and f a mapping from E to E, strictly uniformly M_σ-differentiable at $a \in E$ such that $f'(a) = 1_E$. Then there exists an open set U which contains a, and an open set V which contains $f(a)$ such that f is a homeomorphism from U to V. Furthermore f^{-1} is uniformly M_σ-differentiable at $f(a)$.*

PROOF. Let $0 < \epsilon < 1$. As f is strictly uniformly M_σ-differentiable at a, there exists an open neighborhood V_ϵ of a such that

$$\forall\, q \in \Gamma_E \text{ and } x, y \in V_\epsilon,\ q(f(x) - f(y) + (y - x)) \leq \epsilon\, \sigma(q)(x - y).$$

Let $U \subseteq V_\epsilon$ be an open set; we show that $f(U)$ is also an open set. If $b \in U$, there exists $q \in \Gamma_E$ and $\delta > 0$ such that the semi-ball $B_q(b, \delta)$ is contained in U. We shall show that there exists a $q' \in \Gamma_E$ and $\delta' > 0$ such that $B_{q'}(f(b), \delta') \subseteq f(U)$. For a given $y \in E$, we shall determine a $q' \in \Gamma_E$ and a $\delta' > 0$ such that

$$q'(f(b) - y) < \delta' \Rightarrow \exists x \in B_q(b, \delta) \text{ with } y = f(x).$$

Let $\delta' < \delta$. Consider the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by

$$x_o = b, \quad x_{n+1} = y + x_n - f(x_n).$$

We have

$x_1 - x_o = y - f(b) = x_1 - b$, hence $q(y - f(b)) = q(x_1 - b)$. If y is such that $q(y - f(b)) < \delta'$, then $x_1 \in B_q(b, \delta') \subseteq V_\epsilon$.
$x_2 - b = x_2 - x_1 + x_1 - b$ and $x_2 - x_1 = x_1 - b + f(b) - f(x_1)$, since f is strictly uniformly M_σ-differentiable at a, and $x_1 \in V_\epsilon$, we have

$$q(x_2 - x_1) = q(x_1 - b + f(b) - f(x_1)) \leq \epsilon\, \sigma(q)(x_1 - b)$$

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i.e. $q(x_2 - b) \leq (\epsilon\, \sigma(q) + \epsilon^o \sigma^o(q))(y - f(b))$ where $\sigma^o(q) = q$.
If y is such that $(\epsilon\, \sigma(q) + \epsilon^o\sigma^o(q))(y - f(b)) < \delta'$, then $x_2 \in B_q(b, \delta') \subseteq V_\epsilon$. By repeating the argument, we get

$$q(x_{n+1} - b) \leq \left(\sum_{i=0}^{n} \epsilon^i \sigma^i(q)(y - f(b)\right) \text{ for } n \geq 0$$

The series $\Sigma_{i=0}^{\infty} \epsilon^i\sigma^i(q)$ converges weakly to $q' = (\epsilon/1 - \epsilon)\, \sigma(q) + q \in \Gamma_E$. If y is such that $q'(y - f(b)) < \delta'$, then $x_n \in B_q(b, \delta') \subseteq V_\epsilon$ for all $n \in \mathbb{N}$. Let us now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. In fact, $x_{n+1} - x_n = x_n - f(x_n) - x_{n-1} + f(x_{n-1})$ and $p(x_{n+1} - x_n) \leq \epsilon\, \sigma(p)\,(\, x_n - x_{n-1}) \leq \ldots \leq \epsilon^n\sigma^n(p)(y - f(b))$ for all p in Γ_E. Since $p(x_n - x_m) \leq (\epsilon^{n-1}\sigma^{n-1}(p) + \ldots + \epsilon^m\sigma^m(p))(y - f(b)) = (\epsilon^{n-1} + \ldots + \epsilon^m)\sigma(p)(y - f(b))$, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Fréchet space E.

Therefore it converges to a certain $x \in \overline{B_q(b, \delta')} \subseteq B_q(b, \delta)$. The strict uniform M_σ-differentiability of f at a implies that f is continuous in V_ϵ, so that $y = f(x)$, as $y - f(x_n) = x_{n+1} - x_n$. This shows that every element in $B_{q'}(f(b), \delta')$ is the image by f of an element in $B_q(b, \delta)$, i.e. the semi-ball $B_{q'}(f(b), \delta')$ is included in $f(U)$, that is, f is an open mapping from V_ϵ to $f(V_\epsilon)$.

Let us now show that the restriction of f to V_ϵ is injective. Let x and y be elements of V_ϵ such that $f(x) = f(y)$. Since $x - y = f(x) - f(y) - r(x, y) = -r(x, y)$ where $q(-r(x, y)) = q(r(x, y)) \leq \epsilon\, \sigma(q)(x - y)$, then $q(x - y) \leq \epsilon\, \sigma(q)(x - y)$ for all q in Γ_E. Therefore $\sigma(q((x - y) \leq \epsilon\, \sigma^2(q)\,(x - y)$, i.e $q(x - y) \leq \epsilon^2\, \sigma^2(q)\,(x - y)$.

The repetition of the same argument leads to $q(x - y) \leq \epsilon^n\, \sigma^n(q)\,(x - y)$ for all q in Γ_E. Since $\lim_{n\to\infty} \epsilon^n\sigma(q)(x - y) = o$, we have $x = y$, so that f is an homeomorphism from V_ϵ onto $f(V_\epsilon)$.

Let us finally show that f^{-1} is uniformly M_σ-differentiable at $f(a)$, with $(f^{-1})'(f(a)) = 1_E$. First of all, $1_E = f'(a)$ is an element of $\mathscr{L}_\sigma(E, E)$. Furthermore $f^{-1}(f(x)) - f^{-1}(f(a)) = x - a$; in view of the strict uniform M_σ-differentiability of f at a, we have

$$x - a = f(x) - f(a) - r(x, a) \text{ where } q(-r(x, y)) \leq \epsilon\, \sigma(q)(x - a)$$

for all q in Γ_E.

Therefore

$$q(r(x, a)) \leq \epsilon(\sigma(q)(f(x) - f(a)) + \sigma(q)(r(x, a))) \leq \epsilon\, \sigma(q)(f(x) - f(a)) + \epsilon^2\sigma^2(q)(x - a).$$

By iteration, we obtain

$$q(r(x, a)) \leq \sum_{i=0}^{\infty} \epsilon^i\sigma^i(q)(f(x) - f(a)) = \frac{\epsilon}{1 - \epsilon}\, \sigma(q)(f(x) - f(a)),$$

which means that f^{-1} is uniformly M_σ-differentiable at $f(a)$. Indeed, $\forall\, \epsilon' > 0\ \exists\, \epsilon$ with $0 < \epsilon = \epsilon'/1 + \epsilon' < 1$, such that $f^{-1}(f(x) - f^{-1}(f(a)) = f(x) - f(a) - r(x, a)$ and $\forall\, f(x) \in f(V_\epsilon)$ we have $q(r(x, a)) \leq \epsilon'\sigma(q)(f(x) - f(a))$.

4. An inverse mapping theorem between Fréchet spaces.

THEOREM 2. *Let* (E, Γ_E) *and* (F, Γ_F) *be Fréchet spaces,* σ *a calibration and* f *a mapping from* E *to* F *strictly uniformly* M_σ*-differentiable at* $a \in E$ *such that* $f'(a) \in Isom(E, F)$. *Let* σ' *be a natural calibration for* $(f'(a))^{-1}$ *such that* $\sigma \circ \sigma'$ *is a projective calibration. Then there exists an open neighborhood* U *of* a, *and an open neighborhood* V *of* $f(a)$ *such that* f *is a homeomorphism from* U *onto* V. *Furthermore the inverse mapping* f^{-1} *is uniformly* $M_{\sigma' \circ \sigma \circ \sigma'}$*-differentiable at* $f(a)$.

PROOF. Consider the mapping $g = (f'(a))^{-1} \circ f$ from E to E. The mapping g is strictly uniformly $M_{\sigma \circ \sigma'}$-differentiable at a, with $g'(a) = 1_E$. By theorem 1, there exists an open set U' containing a, and an open set V' containing $g(a)$ such that g is a homeomorphism from U' onto V'. Consider now $V = f'(a)(V')$; V is an open set in F containing $f(a)$. Since $f'(a)$ is an isomorphism from E to F, $f = f'(a) \circ g$ is an homeomorphism from U' onto V. In view of theorem 1 and lemma 5, f^{-1} is uniformly $M_{\sigma' \circ \sigma \circ \sigma'}$-differentiable at $f(a)$.

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