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UNIVALENCE CRITERIA AND ANALOGUES OF THE JOHN CONSTANT

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Abstract

Let $p(z) = zf'(z)/f(z)$ for a function $f(z)$ analytic on the unit disc $|z| < 1$ in the complex plane and normalised by $f(0) = 0, f'(0) = 1$. We provide lower and upper bounds for the best constants δ_0 and δ_1 such that the conditions $e^{-\delta_0/2} < |p(z)| < e^{\delta_0/2}$ and $|p(w)/p(z)| < e^{\delta_1}$ for $|z|, |w| < 1$ respectively imply univalence of f on the unit disc.

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1. Introduction

For a nonconstant analytic function f on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, set

$$M(f) = \sup_{z\in\mathbb{D}} |f'(z)| \quad \text{and} \quad m(f) = \inf_{z\in\mathbb{D}} |f'(z)|.$$

Note that $M(f)$ is a positive number (possibly $+\infty$) whereas $m(f)$ is a finite nonnegative number. In 1969, John [7] proved the following result.

THEOREM A (John). *There exists a number $\gamma \in [\pi/2, \log(97 + 56\sqrt{3})]$ with the following property: if a nonconstant analytic function f on $\mathbb{D}$ satisfies the condition $M(f) \le e^{\gamma} m(f)$, then f is univalent on $\mathbb{D}$.*

We remark that $\log(97 + 56\sqrt{3}) = 5.2678\ldots$. The largest possible number γ with the property in the theorem is called the (logarithmic) John constant and will be denoted by γ_1. (In the literature, the John constant refers to e^{γ_1}. We adopt, however, the logarithmic one for our convenience in this note.) Yamashita [12] improved John's result by showing that $\gamma_1 \le \pi$. Gevirtz [3, 4] further proved that $\gamma_1 \le \lambda\pi$ and conjectured that $\gamma_1 = \lambda\pi$, where $\lambda = 0.6278\ldots$ is the number determined by a transcendental equation.

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We could consider a similar problem for $zf'(z)/f(z)$ instead of $f'(z)$ for an analytic function f on $\mathbb{D}$ with $f(0)=0$, $f'(0)\neq 0$. Let

$$L(f)=\sup_{z\in\mathbb{D}}\left|\frac{zf'(z)}{f(z)}\right| \quad \text{and} \quad l(f)=\inf_{z\in\mathbb{D}}\left|\frac{zf'(z)}{f(z)}\right|$$

for such a function f. Here, the value of $zf'(z)/f(z)$ at $z=0$ will be understood as $\lim_{z\to 0} zf'(z)/f(z)=1$ as usual. Note that $0\le l(f)\le 1\le L(f)\le +\infty$. It is easy to see that $l(f)=1$ (or $L(f)=1$) precisely when $f(z)=az$ for a nonzero constant a. Since $zf'(z)/f(z)$ is unchanged under the dilation $f\mapsto af$ for a nonzero constant a, we can restrict our attention to analytic functions $f(z)$ on $\mathbb{D}$ normalised by $f(0)=0$, $f'(0)=1$. Denote by $\mathcal{A}$ the class of those normalised analytic functions on $\mathbb{D}$. Thus the problem can be formulated as follows.

Problem 1.1. Find a number $\delta>0$ with the following property: if a function $f\in\mathcal{A}$ satisfies the condition $L(f)\le e^{\delta} l(f)$ then f is univalent on $\mathbb{D}$.

Since the value 1 plays a special role in the study of $zf'(z)/f(z)$, it is also natural to consider the following problem.

Problem 1.2. Find a number $\delta>0$ with the following property: if a function $f\in\mathcal{A}$ satisfies the condition $e^{-\delta/2}<|zf'(z)/f(z)|<e^{\delta/2}$ on $\mathbb{D}$, then f is univalent on $\mathbb{D}$.

Let δ_1 and δ_0 be the largest possible numbers δ in Problems 1.1 and 1.2, respectively (if they exist).

The authors proved in [9] that $\pi/6=0.523\ldots\le\delta_0\le\pi=3.14\ldots$. Obviously, $\delta_1\le\delta_0\le 2\delta_1$. Therefore, we already have the estimates $\pi/12=0.261\ldots\le\delta_1\le\pi$.

The purpose of the present note is to improve the estimates.

Theorem 1.3. *The constant δ_0 satisfies*

$$\frac{\pi}{3}=1.04719\ldots<\delta_0<\frac{5\pi}{7}=2.24399\ldots.$$

Theorem 1.4. *The constant δ_1 satisfies*

$$\frac{7\pi}{25}=0.87964\ldots<\delta_1<\frac{5\pi}{7}=2.24399\ldots.$$

We remark that the above results are not optimal. Indeed, more elaborate numerical computations would yield slightly better bounds, as will be suggested at the end of Section 2.

In order to obtain a lower bound, we need a univalence criterion due to Becker [1] with numerical computations as we will explain in Section 2. On the other hand, to give an upper bound, we should construct a nonunivalent function satisfying the condition in Problems 1.1 or 1.2. The function $F_a\in\mathcal{A}$ determined by the differential equation

$$\frac{zF_a'(z)}{F_a(z)}=\left(\frac{1-iz}{1+iz}\right)^{ai} \tag{1.1}$$

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is a candidate for an extremal one, where a is a positive constant and i is the imaginary unit $\sqrt{-1}$. As will be seen later, $L(F_a)/l(F_a) = e^{\pi a}$. We will give a detailed account on this function and provide the upper bound in the above theorems in Section 3. The proof involves matrices of large order. Therefore, we made use of Mathematica 8.0 to carry out symbolic computations.

The most interesting problem is to determine the values of δ_0 and δ_1. However, this seems to the authors very hard. We end this section with some open questions which may be easier to solve. Let a^* be the supremum of the numbers a such that F_a is univalent on $\mathbb{D}$. Likewise let a_* be the infimum of the numbers a such that F_a is not univalent on $\mathbb{D}$. Obviously, $\delta_0 \le \pi a_* \le \pi a^*$. In the proof of the above theorems, we indeed show that $a_* < 5/7$.

(1) Is it true that $a_* = a^*$?
(2) Is it true that $\delta_0 = \pi a_*$?
(3) Is it true that $\delta_0 = \delta_1$?

2. Obtaining lower bounds: univalence criteria

We recall the basic hyperbolic geometry of the unit disc $\mathbb{D}$. The hyperbolic distance between two points z_1, z_2 in $\mathbb{D}$ is defined by

$$d(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2},$$

where the infimum is taken over all rectifiable paths γ joining z_1 and z_2 in $\mathbb{D}$. The Schwarz–Pick lemma asserts that

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \le \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}, \tag{2.1}$$

for any analytic map $\omega : \mathbb{D} \to \mathbb{D}$. In particular, for an analytic automorphism T of $\mathbb{D}$, we have $|T'(z)|/(1 - |T(z)|^2) = 1/(1 - |z|^2)$ and therefore $d(T(z_1), T(z_2)) = d(z_1, z_2)$ for $z_1, z_2 \in \mathbb{D}$. It is well known that the above infimum is attained by the circular arc (possibly a line segment) joining z_1 and z_2 whose whole circle is perpendicular to the unit circle. By using these facts, one can compute the hyperbolic distance: $d(z_1, z_2) = \operatorname{arctanh} |(z_1 - z_2)/(1 - \bar{z}_1 z_2)|$. Here, we recall that $\operatorname{arctanh} r = \frac{1}{2} \log((1 + r)/(1 - r))$.

The following is a useful univalence criterion due to Becker [1].

LEMMA 2.1. *Let f be a nonconstant analytic function on $\mathbb{D}$. If*

$$(1 - |z|^2) \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \quad z \in \mathbb{D},$$

then f is univalent on $\mathbb{D}$.

Sometimes, it is more convenient to consider the pre-Schwarzian norm

$$\|f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right|$$

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because it has several nice properties (see [8], for example). By Becker's theorem above, we see that the condition $\|f\| \le 1$ implies univalence of f on $\mathbb{D}$. We used this norm to deduce the estimate $\pi/6 \le \delta_0$. In this note, however, we do use the original form (Lemma 2.1) of Becker's theorem to improve the estimate.

For a nonnegative number c, we consider the quantity

$$\Phi(c) = \sup_{0<r<1} \{r + c(1-r^2)\operatorname{arctanh} r\} = c \sup_{0<r<1} \{c^{-1}r + (1-r^2)\operatorname{arctanh} r\}.$$

We see that $\Phi(c)$ is nondecreasing in c and that $c^{-1}\Phi(c)$ is nonincreasing in c. In terms of this function, we will prove the following technical lemma which yields lower bounds for δ_0 and δ_1 as corollaries.

Lemma 2.2. *Let $f \in \mathcal{A}$. If $L(f)/l(f) < +\infty$ and if the inequality*

$$\frac{2}{\pi}\Phi(L(f)) \log \frac{L(f)}{l(f)} \le 1$$

holds, then f is univalent on $\mathbb{D}$.

The lemma immediately yields the following results.

Corollary 2.3. *Let $\delta > 0$ be given. If*

$$\frac{2\delta}{\pi}\Phi(e^{\delta/2}) \le 1, \tag{2.2}$$

then $\delta \le \delta_0$. If

$$\frac{2\delta}{\pi}\Phi(e^{\delta}) \le 1, \tag{2.3}$$

then $\delta \le \delta_1$.

To show the corollary, we first assume (2.2) and consider a function $f \in \mathcal{A}$ satisfying $e^{-\delta/2} < |zf'(z)/f(z)| < e^{\delta/2}$. Then $L(f) \le e^{\delta/2}$ and $\log L(f)/l(f) \le \delta$ so that

$$\frac{2}{\pi}\Phi(L(f)) \log \frac{L(f)}{l(f)} \le \frac{2\delta}{\pi}\Phi(e^{\delta/2}) \le 1.$$

We now apply Lemma 2.2 to conclude univalence of f. Secondly, we assume (2.3) and consider a function $f \in \mathcal{A}$ satisfying $L(f) \le e^{\delta} l(f)$. Then $L(f) \le e^{\delta}$ and the conclusion follows similarly.

Let us prepare for the proof of Lemma 2.2. We note that the function $\arctan z = (1/2i)\log((1+iz)/(1-iz))$ maps the unit disc $\mathbb{D}$ conformally onto the vertical parallel strip $|\operatorname{Re} w| < \pi/4$. Therefore, for a constant $a > 0$, the function

$$Q_a(z) = \exp(2a \arctan z) = \left(\frac{1-iz}{1+iz}\right)^{ai} \tag{2.4}$$

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is the universal covering projection of $\mathbb{D}$ onto the annulus $e^{-\pi a/2} < |w| < e^{\pi a/2}$. We note that the function Q_a satisfies $Q_a(0) = 1$ and

$$\frac{Q_a'(z)}{Q_a(z)} = \frac{2a}{1+z^2}.$$

PROOF OF LEMMA 2.2 Let $p(z) = zf'(z)/f(z)$ for a function $f \in \mathcal{A}$. If p is a constant, then f is clearly univalent. We can thus assume that p is not a constant so that $l(f) < 1 < L(f)$. Let $\delta = \log L(f)/l(f) < +\infty$ and $m = \sqrt{L(f)l(f)}$. We consider the universal covering map $Q = mQ_a$ of $\mathbb{D}$ onto the annulus $W = \{w : l(f) < |w| < L(f)\} = \{w : me^{-\delta/2} < |w| < me^{\delta/2}\}$, where Q_a is given in (2.4) with $a = \delta/\pi$. Note that $p(\mathbb{D}) \subset W$ by assumption. Since the real interval $(-1, 1)$ is mapped onto $(l(f), L(f))$ by Q, we can choose an $\alpha \in (-1, 1)$ so that $Q(\alpha) = 1$. Then, $P = Q \circ T$ is a universal covering map of $\mathbb{D}$ onto W with $P(0) = 1$, where $T(z) = (z + \alpha)/(1 + \alpha z)$. Since $P : \mathbb{D} \to W$ is a covering map, we can take a lift ω of p with respect to P so that $\omega(0) = 0$ and $p = P \circ \omega$. We write $w = \omega(z)$. Note here that the Schwarz lemma implies $|w| \leq |z|$. We now have

$$\frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + p(z) - 1 = \frac{z\omega'(z)P'(w)}{P(w)} + P(w) - 1.$$

Set $\tau = T(w) \in \mathbb{D}$. Since T is a hyperbolic isometry of $\mathbb{D}$, one has the relation $(1 - |w|^2)|T'(w)| = 1 - |\tau|^2$. Therefore, by using (2.1),

$$\begin{aligned}(1-|z|^2)\left|\frac{\omega'(z)P'(w)}{P(w)}\right| &\leq (1-|w|^2)\left|\frac{Q'(\tau)T'(w)}{Q(\tau)}\right| \\ &= (1-|\tau|^2)\left|\frac{Q'(\tau)}{Q(\tau)}\right| \\ &= \frac{2a(1-|\tau|^2)}{|1+\tau^2|} \\ &\leq 2a.\end{aligned}$$

Let γ be the image of the line segment $(0, w)$ under the Möbius mapping T. Then

$$P(w) - 1 = \int_0^w P'(t)\,dt = \int_0^w Q'(T(t))T'(t)\,dt = \int_\gamma Q'(u)\,du = \int_\gamma \frac{2aQ(u)}{1+u^2}\,du.$$

Since $|Q(u)| \leq L(f)$,

$$\begin{aligned}|P(w) - 1| &\leq 2aL(f)\int_\gamma \frac{|du|}{1-|u|^2} = 2aL(f)\int_0^w \frac{|du|}{1-|u|^2} \\ &= 2aL(f)d(0, w) \leq 2aL(f)\,\mathrm{arctanh}\,|z|.\end{aligned}$$

Therefore,

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 2a|z| + 2aL(f)(1-|z|^2)\,\mathrm{arctanh}\,|z|. \tag{2.5}$$

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Hence,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 2a\Phi(L(f)) = \frac{2\delta}{\pi}\Phi(L(f)).$$

Lemma 2.1 now implies the required assertion. □

The above method also gives a norm estimate of the pre-Schwarzian derivative. Though we do not use it in this note, we record it for possible future reference.

Proposition 2.4. *Suppose that $L(f)/l(f) < +\infty$ for a function $f \in \mathcal{A}$. Then the pre-Schwarzian norm of f is estimated as*

$$\|f\| \le \frac{2}{\pi}\,(1+L(f))\log\frac{L(f)}{l(f)}.$$

Proof. Let $a = (1/\pi)\log(L(f)/l(f))$. By (2.5),

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le 2a + 2aL(f)(1-r^2)\frac{\operatorname{arctanh} r}{r}$$

for $|z| = r < 1$. Since $(1-r^2)\operatorname{arctanh} r/r$ is decreasing in $0 < r < 1$, the inequality $(1-r^2)\operatorname{arctanh} r/r \le 1$ holds. Hence,

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le 2a + 2aL(f),$$

completing the proof. □

To prove Theorems 1.3 and 1.4, the following technical result is helpful. To state it, we introduce the auxiliary function

$$H(x,c) = \frac{1-c}{2}x + \frac{1+c}{2}x^{-1}.$$

Lemma 2.5. *Let $c > 1$. If a number $x_1 \in (0,1)$ satisfies the inequality $x_1\operatorname{arctanh} x_1 < (1+c)/2c$, then $\Phi(c) < H(x_1, c)$.*

Proof. Let $g(x) = x + c(1-x^2)\operatorname{arctanh} x$. Then $g'(x) = 1 + c - 2cx\operatorname{arctanh} x$. Since $x\operatorname{arctanh} x$ (strictly) increases from 0 to $+\infty$ when x moves from 0 to 1, there exists a unique zero $x_0 \in (0,1)$ of $g'(x)$ such that $g'(x) > 0$ in $0 < x < x_0$ and $g'(x) < 0$ in $x_0 < x < 1$. Note here that the assumption implies that $0 < x_1 < x_0$. We see now that $g(x)$ takes its maximum at $x = x_0$ and therefore

$$\Phi(c) = g(x_0) = \frac{1-c}{2}x_0 + \frac{1+c}{2}x_0^{-1} = H(x_0, c).$$

Since $H_x(x,c) = (1-c)/2 - (1+c)/(2x^2) < 0$, the function $H(x,c)$ is decreasing in $x > 0$ for a fixed $c > 1$. Hence, $x_1 < x_0$ implies that $H(x_0,c) < H(x_1,c)$, which proves the assertion. □

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Proof of Theorem 1.3. Let $\delta = \pi/3$ and set $c = e^{\delta/2} = e^{\pi/6}$. If we take $x_1 = 17/22$, then

$$\frac{1+c}{2c} - x_1 \operatorname{arctanh} x_1 = \frac{1+e^{-\pi/6}}{2} - \frac{17}{44}\log\frac{39}{5} = 0.00255\ldots > 0.$$

Therefore, Lemma 2.5 yields

$$\frac{2\delta}{\pi}\Phi(e^{\delta/2}) = \frac{2}{3}\Phi(c) < \frac{2}{3}H(x_1, c) = \frac{773 + 195e^{\pi/6}}{1122} = 0.982\ldots < 1.$$

We now apply Corollary 2.3 to obtain $\pi/3 < \delta_0$. □

Proof of Theorem 1.4. We proceed along the same lines as above. Let $\delta = 7\pi/25$ and set $c = e^{\delta}$. Taking $x_1 = 20/27$, we have

$$\frac{1+c}{2c} - x_1 \operatorname{arctanh} x_1 = \frac{1+e^{-7\pi/25}}{2} - \frac{10}{27}\log\frac{47}{7} = 0.00219\ldots > 0.$$

Lemma 2.5 now implies that

$$\frac{2\delta}{\pi}\Phi(e^{\delta}) = \frac{14}{25}\Phi(c) < \frac{14}{25}H(x_1, c) = \frac{7903 + 2303e^{7\pi/25}}{13\,500} = 0.9965\ldots < 1.$$

We again apply Corollary 2.3 to obtain $7\pi/25 < \delta_1$. □

Remark 2.6. We can slightly improve Theorems 1.3 and 1.4 by changing the choice of δ and x_1 in the above proofs. For instance, concerning Theorem 1.3, we can take

$$(\delta, x_1) = \left(\frac{22\pi}{65}, \frac{17}{22}\right), \quad \left(\frac{87\pi}{257}, \frac{2765}{3578}\right),$$

to have lower bounds $22\pi/65 = 1.06330\ldots$ and $87\pi/257 = 1.06349\ldots$, respectively, for δ_0. Numerical computations with Mathematica 8 suggest that the solution to the equation $(2\delta/\pi)\Phi(e^{\delta/2}) = 1$ is about $\delta = 1.0635213$. Therefore, it seems that we would obtain at most this value as a lower bound for δ_0 by the above method.

Similarly, concerning Theorem 1.4, we can take

$$(\delta, x_1) = \left(\frac{25\pi}{89}, \frac{622}{839}\right), \quad \left(\frac{127\pi}{452}, \frac{321}{433}\right),$$

to have lower bounds $25\pi/89 = 0.882469\ldots$ and $127\pi/452 = 0.882704\ldots$, respectively, for δ_1.

We see that the numerical solution to the equation $(2\delta/\pi)\Phi(e^{\delta}) = 1$ is about $\delta = 0.8827139$. Therefore, the above method seems to give only a lower bound for δ_1 not better than this value.

3. Obtaining upper bounds: nonunivalence of a specific function

We will provide upper bounds for δ_0 by checking nonunivalence of the function $F_a \in \mathcal{A}$ defined by (1.1) for a suitably chosen positive constant a. Since F_a cannot be expressed in a simple form, it is not easy to determine its univalence. In this note, we will observe its Grunsky coefficients to examine univalence, whereas we used Gronwall's area theorem (or its refinement by Prawitz) in [9] to see that $a \le 1$ is necessary for F_a to be univalent.

Let $f \in \mathcal{A}$. The Grunsky coefficients $c_{j,k}$ of f are defined by the series expansion

$$\log \frac{f(z) - f(w)}{z - w} = -\sum_{j,k=0}^{\infty} c_{j,k} z^j w^k \tag{3.1}$$

in $|z| < \varepsilon, |w| < \varepsilon$ for a small enough $\varepsilon > 0$. We remark here that the obvious symmetry relation $c_{j,k} = c_{k,j}$ holds. Note also that $c_{j,0}$ $(j = 1, 2, \ldots)$ are the logarithmic coefficients of $f(z)/z$, in other words, $-\log(f(z)/z) = c_{1,0}z + c_{2,0}z^2 + \cdots$ as we can see by letting $w = 0$ in (3.1). Grunsky's theorem was strengthened by Pommerenke as follows (see [10, Theorem 3.1]).

Lemma 3.1. *Let $f \in \mathcal{A}$ and $\{c_{j,k}\}$ be its Grunsky coefficients. If f is univalent on $|z| < 1$ then*

$$\sum_{m=1}^{\infty} m \left| \sum_{k=1}^{n} c_{m,k} t_k \right|^2 \le \sum_{m=1}^{n} \frac{|t_m|^2}{m}$$

holds for arbitrary $n \ge 1$ and $t_1, \ldots, t_n \in \mathbb{C}$.

We remark that the Grunsky coefficients are usually defined for the function $g(\zeta) = 1/f(1/\zeta)$. This change affects only the coefficients $c_{j,0} = c_{0,j}$, which do not involve the Grunsky inequalities. See [5] for more information.

From Lemma 3.1, the inequality

$$\sum_{m=1}^{n} m \left| \sum_{k=1}^{n} c_{m,k} t_k \right|^2 \le \sum_{m=1}^{n} \frac{|t_m|^2}{m} \tag{3.2}$$

follows for every n and $t_1, \ldots, t_n \in \mathbb{C}$. This implies that the Hermitian matrix $G_f(n) = (\gamma_{j,k}^{(n)})$ of order n is positive semidefinite; in other words, $\mathbf{t}G_f(n)\mathbf{t}^* \ge 0$ for any $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{C}^n$, where

$$\gamma_{j,k}^{(n)} = \frac{\delta_{j,k}}{j} - \sum_{m=1}^{n} m c_{m,j} \overline{c_{m,k}} \quad (1 \le j, k \le n),$$

$\delta_{j,k}$ means Kronecker's delta and $\mathbf{t}^*$ is the conjugate transpose of $\mathbf{t}$ as a matrix.

Letting $t_k = \delta_{j,k}$ in (3.2), we have $\sum_{m=1}^{n} m|c_{m,j}|^2 \le 1/j$ for $j \le n$, which implies that $|c_{m,j}| \le 1/\sqrt{mj} \le 1$ for $m, j \ge 1$. This guarantees that the series expansion in (3.1) is convergent in $|z| < 1, |w| < 1$, and therefore, that f is univalent on $\mathbb{D}$. We shall call $G_f(n)$ the *Grunsky matrix* of order n. We have observed the following assertion.

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Corollary 3.2. *A function $f \in \mathcal{A}$ is univalent on $\mathbb{D}$ if and only if its Grunsky matrix $G_f(n)$ of order n is positive semidefinite for every $n \geq 1$.*

In order to compute the Grunsky coefficients of $F_a(z)$, it is convenient to have recursion formulas for relevant coefficients. The following elementary lemma gives a recursion formula for exponentiation.

Lemma 3.3. *Let $g(z) = b_1 z + b_2 z^2 + \cdots$ be a given function analytic around $z = 0$ and let $h(z) = e^{g(z)} = c_0 + c_1 z + c_2 z^2 + \cdots$. Then c_n can be computed recursively by $c_0 = 1$ and*

$$c_n = \frac{1}{n} \sum_{k=0}^{n-1} (n-k) b_{n-k} c_k \quad (n \geq 1).$$

Proof. Compare the coefficients of the power series expansions of both sides of $h'(z) = g'(z)h(z)$. □

We turn to the function $F_a(z)$ for a fixed $a > 0$. In view of (2.4), we see that the relation (1.1) can also be expressed by $zF_a'(z)/F_a(z) = Q_a(z) = \exp(2a \arctan z)$. In particular, the range of the function $zF_a'(z)/F_a(z)$ is the annulus $e^{-\pi a/2} < |w| < e^{\pi a/2}$ and, in particular, $l(F_a) = e^{-\pi a/2}$, $L(F_a) = e^{\pi a/2}$ and $L(F_a)/l(F_a) = e^{\pi a}$, as already stated in the Introduction. Using the formula

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

together with Lemma 3.3, we can compute the Taylor coefficients b_n of $Q_a(z)$ recursively. (See also [11] for additional information about the coefficients.) In this way,

$$\begin{aligned} \frac{zF_a'(z)}{F_a(z)} &= Q_a(z) = \sum_{n=0}^{\infty} b_n z^n \\ &= 1 + 2az + 2a^2 z^2 + \frac{2}{3} a(2a^2 - 1) z^3 + \frac{2}{3} a^2 (a^2 - 2) z^4 + \cdots. \end{aligned}$$

Dividing by z and integrating with respect to z,

$$\log \frac{F_a(z)}{z} = \sum_{n=1}^{\infty} \frac{b_n}{n} z^n = 2az + a^2 z^2 + \frac{2}{9} a(2a^2 - 1) z^3 + \frac{1}{6} a^2 (a^2 - 2) z^4 + \cdots.$$

We again use Lemma 3.3 to compute the Taylor coefficients of $F_a(z)/z$ recursively and finally arrive at the representation

$$\begin{aligned} F_a(z) &= z \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} z^n \right) \\ &= z + 2az^2 + 3a^2 z^3 + \frac{2}{9} a(17a^2 - 1) z^4 + \frac{1}{9} a^2 (38a^2 - 7) z^5 + \cdots. \end{aligned}$$

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In order to compute the Grunsky coefficients, we use the following formulas. These formulas are essentially known; see [6] and [2, formula (2.13)], for example. However, since we could not find exactly the same formula in the literature, we state it as a lemma and give a proof.

LEMMA 3.4. *The Grunsky coefficients $c_{j,k}$ of a function $f(z) = z + a_2 z^2 + \cdots$ in $\mathcal{A}$ satisfy the recursion formula*

$$c_{j,k} = \sum_{l=1}^{k-1} \frac{l}{k} a_{k-l} c_{j+1,l} - \sum_{m=1}^{j} a_{m+1} c_{j-m,k} - \frac{a_{j+k+1}}{k} \tag{3.3}$$

for $j \geq 0$ and $k \geq 1$.

PROOF. Differentiating both sides of (3.1) with respect to w, we obtain the relation

$$wf'(w) - w\frac{f(z) - f(w)}{z - w} = (f(z) - f(w)) \sum_{j,k=0}^{\infty} kc_{j,k} z^j w^k.$$

Letting $a_1 = 1$, we compute first the left-hand side of the relation:

$$\begin{aligned}
(\text{LHS}) &= \sum_{n=1}^{\infty} a_n (nw^n - w(z^{n-1} + \cdots + zw^{n-2} + w^{n-1})) \\
&= \sum_{n=1}^{\infty} a_n ((n-1)w^n - z^{n-1}w - \cdots - zw^{n-1}).
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
(\text{RHS}) &= \sum_{n=1}^{\infty} \sum_{j,k=0}^{\infty} ka_n c_{j,k} (z^{j+n} w^k - z^j w^{k+n}) \\
&= \sum_{l,m=0}^{\infty} \left(\sum_{n=1}^{l} ma_n c_{l-n,m} - \sum_{n=1}^{m} (m-n) a_n c_{l,m-n} \right) z^l w^m.
\end{aligned}$$

Comparing the coefficients of the term $z^l w^m$,

$$-a_{l+m} = mc_{l-1,m} + \sum_{n=2}^{l} ma_n c_{l-n,m} - \sum_{n=1}^{m} (m-n) a_n c_{l,m-n}$$

for $l \geq 1$ and $m \geq 1$. We now let $(j, k) = (l - 1, m)$ to obtain the required relation. □

We can now compute $c_{j,k}$ recursively. Indeed, first we apply (3.3) with $k = 1$ to compute $c_{j,1}$ recursively in $j \geq 0$:

$$c_{j,1} = -\sum_{m=1}^{j} a_{m+1} c_{j-m,1} - a_{j+2}, \quad j \geq 0.$$

If we determine $c_{l,m}$ for all $l \geq 0$ and $1 \leq m < k$, then we use (3.3) to give $c_{j,k}$ recursively in $j \geq 0$. In practice, to determine $c_{j,k}$, it is enough to start with $c_{l,1}$ for $0 \leq l \leq j+k-1$, which determine $c_{l,2}$ for $0 \leq l \leq j+k-2$, and so on. In this way, we can compute the Grunsky matrix $G(n) = G_{F_a}(n)$. For instance, $G(1) = [1 - a^4]$ and

$$G(2) = \frac{1}{81} \begin{bmatrix} 81 - 8a^2 - 97a^4 - 8a^6 & -14a^3(1+a^2)^2 \\ -14a^3(1+a^2)^2 & 81/2 - 4a^2 - 10a^4 + 10a^6 - 49a^8/2 \end{bmatrix}.$$

We are now ready to give the upper bound in Theorems 1.3 and 1.4.

Computer-assisted proof of $\delta_0 < 5\pi/7$. We consider the Grunsky matrix $A_a = G(18)$ of order 18 for the function $f = F_a$. We computed A_a symbolically with the help of Mathematica 8 but we will not give a list of the elements of A_a due to limitations of space. Let $a_0 = 5/7$. We will show that F_a is not univalent for a close enough to a_0.

We see that A_{a_0} is a square matrix of order 18 with rational elements. Mathematica 8 can compute its eigenvalues and corresponding eigenvectors numerically. In this way, we found that one eigenvalue of A_{a_0} was apparently negative. Since numerical computations might not be reliable enough, we will make this observation rigorous. By approximating an eigenvector corresponding to the negative eigenvalue, we find that the rational vector

$$\mathbf{v} = \left(-\tfrac{1}{3}, -\tfrac{1}{6}, \tfrac{3}{10}, \tfrac{3}{10}, -\tfrac{1}{6}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{1}{6}, -\tfrac{1}{5}, -\tfrac{1}{5}, \tfrac{1}{10}, \tfrac{1}{5}, \tfrac{1}{10}, -\tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{6}\right)$$

satisfies

$$\mathbf{v}A_{a_0}\mathbf{v}^* = -\frac{37 \cdot 61 \cdot 102353087 \cdot 29977321169 \cdot N}{3^{49} \cdot 5^{16} \cdot 7^{92} \cdot 11^{12} \cdot 13^4 \cdot 17^3 \cdot 19^4 \cdot 23^4 \cdot 29^2 \cdot 31^4} < 0.$$

Here, $N = 76346348854682571404146112285557118341692971860401383400032$ $36561014990492155539274861647761359966219067479516880182420828 3713$ is an integer with 125 digits, which cannot be factorised by Mathematica 8. Therefore, A_{a_0} is not positive semidefinite. Since $\mathbf{v}A_a\mathbf{v}^* < 0$ still holds for a close enough to a_0, we have $a_* < a_0$ by Corollary 3.2, where a_* is the number defined in the Introduction. We have thus seen that $\delta_1 \leq \delta_0 \leq \pi a_* < \pi a_0 = 5\pi/7$. $\square$

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