

# Planck 2015 results. XVIII. Background geometry and topology of the Universe

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## ABSTRACT

Maps of cosmic microwave background (CMB) temperature and polarization from the 2015 release of *Planck* data provide the highest-quality full-sky view of the surface of last scattering available to date. This allows us to detect possible departures from a globally isotropic cosmology. We present the first searches using CMB polarization for correlations induced by a possible non-trivial topology with a fundamental domain intersecting, or nearly intersecting, the last scattering surface (at comoving distance  $\chi_{\text{rec}}$ ), both via a direct scan for matched circular patterns at the intersections and by an optimal likelihood calculation for specific topologies. We specialize to flat spaces with cubic toroidal (T3) and slab (T1) topologies, finding that explicit searches for the latter are sensitive to other topologies with antipodal symmetry. These searches yield no detection of a compact topology with a scale below the diameter of the last scattering surface. The limits on the radius  $\mathcal{R}_i$  of the largest sphere inscribed in the fundamental domain (at log-likelihood-ratio  $\Delta \ln \mathcal{L} > -5$  relative to a simply-connected flat *Planck* best-fit model) are:  $\mathcal{R}_i > 0.97 \chi_{\text{rec}}$  for the T3 cubic torus; and  $\mathcal{R}_i > 0.56 \chi_{\text{rec}}$  for the T1 slab. The limit for the T3 cubic torus from the matched-circles search is numerically equivalent,  $\mathcal{R}_i > 0.97 \chi_{\text{rec}}$  at 99% confidence level from polarization data alone. We also perform a Bayesian search for an anisotropic global Bianchi VII<sub>h</sub> geometry. In the non-physical setting where the Bianchi cosmology is decoupled from the standard cosmology, *Planck* temperature data favour the inclusion of a Bianchi component with a Bayes factor of at least 2.3 units of log-evidence. However, the cosmological parameters that generate this pattern are in strong disagreement with those found from CMB anisotropy data alone. Fitting the induced polarization pattern for this model to the *Planck* data requires an amplitude of  $-0.1 \pm 0.04$  compared to the value of +1 if the model were to be correct. In the physically motivated setting where the Bianchi parameters are coupled and fitted simultaneously with the standard cosmological parameters, we find no evidence for a Bianchi VII<sub>h</sub> cosmology and constrain the vorticity of such models to  $(\omega/H)_0 < 7.6 \times 10^{-10}$  (95% CL).

**Key words.** cosmology: observations — cosmic background radiation — cosmological parameters — Gravitation — Methods: data analysis — Methods: statistical

## 1. Introduction

This paper, one of a series associated with the 2015 release of *Planck*<sup>1</sup> data, will present limits on departures from the global isotropy of spacetime. We will assess anisotropic but homogeneous Bianchi cosmological models and non-trivial global topologies in the light of the latest temperature and polarization data.

In [Planck Collaboration XXVI \(2014\)](#), the limits came from the 2013 *Planck* cosmological data release: cosmic microwave background (CMB) intensity data collected over approximately one year. This work will use the 2015 *Planck* data: CMB intensity from the whole mission along with a subset of polarization data. The greater volume of intensity data will allow more restrictive limits on the possibility of topological scales slightly larger than the volume enclosed by the last-scattering surface (roughly the Hubble volume), probing the excess anisotropic correlations that would be induced at large angular scales were such a model to obtain. For cubic torus topologies, we can therefore observe explicit repeated patterns (matched circles) when the comoving length of an edge is less than twice the distance to the recombination surface,  $\chi_{\text{rec}} \approx 3.1H_0^{-1}$  (using units with  $c = 1$  here and throughout). Polarization, on the other hand, largely generated during recombination itself, can provide a more sensitive probe of topological domains smaller than the Hubble volume.

Whereas the analysis of temperature data in multiply connected universes has been treated in some depth in the literature (see [Planck Collaboration XXVI 2014](#), and references therein), the discussion of polarization has been less complete. This paper will therefore extend our previous likelihood analysis to polarized data, update the direct search for matched circles ([Cornish et al. 2004](#)) as discussed in [Bielewicz et al. \(2012\)](#), and use these to present the first limits on global topology from polarized CMB data.

The cosmological properties of Bianchi models ([Collins & Hawking 1973](#); [Barrow et al. 1985](#)), were initially discussed in the context of CMB intensity ([Barrow 1986](#); [Jaffe et al. 2006c,a](#); [Pontzen 2009](#)). As discussed in [Planck Collaboration XXVI \(2014\)](#) it is by now well known that the large-scale intensity pattern mimics that of a particular Bianchi VII<sub>h</sub> model, albeit one with cosmological parameters quite different from those needed to reproduce other CMB and cosmological data. More recently the induced polarization patterns have been calculated ([Pontzen & Challinor 2007](#); [Pontzen 2009](#); [Pontzen & Challinor 2011](#)). In this paper, we analyse the complete *Planck* intensity data, and compare the polarization pattern induced by that anisotropic model to *Planck* polarization data.

We note that the lack of a strong detection of cosmic *B*-mode polarization already provides some information about the Bianchi models: the induced geometrical focusing does not distinguish between *E* and *B* and thus should produce comparable amounts of each (e.g., [Pontzen 2009](#)). Note that this does *not* apply to topological models: the linear evolution of primordial perturbations guarantees that a lack of primordial tensor pertur-

bations results in a lack of *B*-mode polarization—the transfer function is not altered by topology.

In [Sect. 2](#), we discuss previous limits on anisotropic models from *Planck* and other experiments. In [Sect. 3](#) we discuss the CMB signals generated in such models, generalized to both temperature and polarization. In [Sect. 4](#) we describe the *Planck* data and simulations we use in this study, the different methods we apply to those data, and the validation checks performed on those simulations. In [Sect. 5](#) we discuss the results and conclude in [Sect. 6](#) with the outlook for application of these techniques to future data and broader classes of models.

## 2. Previous results

The first searches for non-trivial topology on cosmic scales looked for repeated patterns or individual objects in the distribution of galaxies ([Sokolov & Shvartsman 1974](#); [Fang & Sato 1983](#); [Fagundes & Wichoski 1987](#); [Lehoucq et al. 1996](#); [Roukema 1996](#); [Weatherley et al. 2003](#); [Fujii & Yoshii 2011](#)). Searches for topology using the CMB began with COBE ([Bennett et al. 1996](#)) and found no indications of a non-trivial topology on the scale of the last-scattering surface (e.g., [Starobinskij 1993](#); [Sokolov 1993](#); [Stevens et al. 1993](#); [de Oliveira-Costa & Smoot 1995](#); [Levin et al. 1998](#); [Bond et al. 1998, 2000](#); [Rocha et al. 2004](#); but see also [Roukema 2000b,a](#)). With the higher resolution and sensitivity of WMAP, there were indications of low power on large scales which could have had a topological origin ([Jarosik et al. 2011](#); [Luminet et al. 2003](#); [Caillerie et al. 2007](#); [Aurich 1999](#); [Aurich et al. 2004, 2005, 2006, 2008](#); [Aurich & Lustig 2013](#); [Lew & Roukema 2008](#); [Roukema et al. 2008](#); [Niarchou et al. 2004](#)), but this possibility was not borne out by detailed real- and harmonic-space analyses in two dimensions ([Cornish et al. 2004](#); [Key et al. 2007](#); [Bielewicz & Riazuelo 2009](#); [Dineen et al. 2005](#); [Kunz et al. 2006](#); [Phillips & Kogut 2006](#); [Niarchou & Jaffe 2007](#)). Most studies, including this work, have emphasized searches for fundamental domains with antipodal correlations; see, e.g., [Aurich & Lustig \(2014\)](#) for a recent discussion of another possible topology.

For a more complete overview of the field, we direct the reader to [Planck Collaboration XXVI \(2014\)](#). In that work, we applied various techniques to the *Planck* 2013 intensity data. For topology, we showed that a fundamental topological domain smaller than the Hubble volume is strongly disfavoured. This was done in two ways: first, a direct likelihood calculation of specific topological models; and second, a search for the expected repeated “circles in the sky” ([Cornish et al. 2004](#)), calibrated by simply-connected simulations. Both of these showed that the scale of any possible topology must exceed roughly the distance to the last scattering surface,  $\chi_{\text{rec}}$ . For the cubic torus, we found that the radius of the largest sphere inscribed in the topological fundamental domain must be  $\mathcal{R}_i > 0.92\chi_{\text{rec}}$  (at log-likelihood ratio  $\Delta \ln \mathcal{L} > -5$  relative to a simply-connected flat *Planck* 2013 best-fit model). The matched-circle limit on topologies predicting back-to-back circles was  $\mathcal{R}_i > 0.94\chi_{\text{rec}}$  at the 99% confidence level.

Prior to the present work, there have been some extensions of the search for cosmic topology to polarization data. In particular, [Bielewicz et al. \(2012\)](#) extended the direct search for matched circles to polarized data and found that the available WMAP data had insufficient sensitivity to provide useful constraints.

For Bianchi VII<sub>h</sub> models, in [Planck Collaboration XXVI \(2014\)](#) a full Bayesian analysis of the *Planck* 2013 temperature data was performed, following the methods of [McEwen et al.](#)

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(2013). It was concluded that a physically-motivated model was not favoured by the data. If considered as a phenomenological template (for which the parameters common to the standard stochastic CMB and the deterministic Bianchi VII<sub>h</sub> component are not linked), it was shown that an unphysical Bianchi VII<sub>h</sub> model is favoured, with a log-Bayes factor between  $1.5 \pm 0.1$  and  $2.8 \pm 0.1$ —equivalent to an odds ratio of between approximately 1:4 and 1:16—depending of the component separation technique adopted. Prior to the analysis of [Planck Collaboration XXVI \(2014\)](#), numerous analyses of Bianchi models using COBE ([Bennett et al. 1996](#)) and WMAP ([Jarosik et al. 2011](#)) data have been performed ([Bunn et al. 1996](#); [Kogut et al. 1997](#); [Jaffe et al. 2005, 2006a,c,b](#); [Cayón et al. 2006](#); [Land & Magueijo 2006](#); [McEwen et al. 2006](#); [Bridges et al. 2007](#); [Ghosh et al. 2007](#); [Pontzen & Challinor 2007](#); [Bridges et al. 2008](#); [McEwen et al. 2013](#)), and a similar Bianchi template was found in the WMAP data, first by [Jaffe et al. \(2005\)](#) and then subsequently by others ([Bridges et al. 2007](#); [Bridges et al. 2008](#); [McEwen et al. 2013](#)). [Pontzen & Challinor \(2007\)](#) discussed the CMB polarization signal from Bianchi models, and showed some incompatibility with WMAP data due to the large amplitude of both *E*- and *B*-mode components. For a more detailed review of the analysis of Bianchi models we refer the reader to [Planck Collaboration XXVI \(2014\)](#).

### 3. CMB signals in anisotropic and multiply-connected universes

#### 3.1. Topology

There is a long history of studying the possible topological compactification of Friedmann-Robertson-Walker (FRW) cosmologies; we refer readers to overviews such as [Levin \(2002\)](#), [Lachieze-Rey & Luminet \(1995\)](#), and [Riazuelo et al. \(2004a,b\)](#) for mathematical and physical detail. The effect of a non-trivial topology is equivalent to considering the full (*simply-connected*) three dimensional spatial slice of the manifold (the *covering space*) as being tiled by identical repetitions of a shape which is finite in one or more directions, the *fundamental domain*. In flat universes, to which we specialize here, there are a finite number of possibilities, each described by one or more continuous parameters describing the size in different directions.

In this paper, we pay special attention to topological models in which the fundamental domain is a right-rectangular prism (the three-torus, also referred to as “T3”), possibly with one or two infinite dimensions (the T2 “chimney” or “rod”, and T1 “slab” models). We limit these models in a number of ways. We explicitly compute the likelihood of the length of the fundamental domain for the cubic torus. Furthermore, we consider the slab model as a proxy for other models in which the matched circles (or excess correlations) are antipodally aligned, similar to the “lens” spaces available in manifolds with constant positive curvature. These models are thus sensitive to tori with varying side lengths, including those with non-right-angle corners. In these cases, the likelihood would have multiple peaks, one for each of the aligned pairs; their sizes correspond to those of the fundamental domains and their relative orientation to the angles. These non-rectangular prisms will be discussed in more detail in [Jaffe & Starkman \(2015\)](#).

#### 3.1.1. Computing the correlation matrices

In [Planck Collaboration XXVI \(2014\)](#) we computed the temperature-temperature (*TT*) correlation matrices by summing

up all modes  $\mathbf{k}_n$  that are present given the boundary conditions imposed by the non-trivial topology. For a cubic torus, we have  $\mathbf{k}_n = (2\pi/L)\mathbf{n}$  for a triplet of integers  $\mathbf{n}$ , and the harmonic-space correlation matrix

$$C_{\ell\ell'}^{mm'}(TT) \propto \sum_{\mathbf{n}} \Delta_{\ell}^{(T)}(k_n, \Delta\eta) \Delta_{\ell'}^{(T)}(k_n, \Delta\eta) P(k_n) Y_{\ell m}(\hat{\mathbf{n}}) Y_{\ell' m'}^*(\hat{\mathbf{n}}), \quad (1)$$

where  $\Delta_{\ell}^{(T)}(k, \Delta\eta)$  is the temperature radiation transfer function (see, e.g., [Bond & Efstathiou 1987](#) and [Seljak & Zaldarriaga 1996](#)).

It is straightforward to extend this method to include polarization, since the cubic topology affects neither the local physics that governs the transfer functions, nor the photon propagation. The only effect is the discretization of the modes. We can therefore simply replace the radiation transfer function for the temperature fluctuations with the one for polarization, and obtain

$$C_{\ell\ell'}^{mm'}(XX') \propto \sum_{\mathbf{n}} \Delta_{\ell}^{(X)}(k_n, \Delta\eta) \Delta_{\ell'}^{(X')} (k_n, \Delta\eta) P(k_n) Y_{\ell m}(\hat{\mathbf{n}}) Y_{\ell' m'}^*(\hat{\mathbf{n}}), \quad (2)$$

where  $X, X' = E, T$ . We are justified in ignoring the possibility of *B*-mode polarization as it is sourced only by primordial gravitational radiation even in the presence of nontrivial topology. In this way we obtain three sets of correlation matrices: *TT*, *TE*, and *EE*. In addition, since the publication of [Planck Collaboration XXVI \(2014\)](#) we have optimized the cubic torus calculation by taking into account more of the symmetries. The resulting speed-up of about an order of magnitude allowed us to reach a higher resolution of  $\ell_{\max} = 64$ .

The fiducial cosmology assumed in the calculation of the correlation matrices is a flat  $\Lambda$ CDM FRW universe with Hubble constant  $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ , where  $h = 0.6719$ ; scalar spectral index  $n_s = 0.9635$ ; baryon density  $\Omega_b h^2 = 0.0221$ ; cold dark matter density  $\Omega_c h^2 = 0.1197$ ; and neutrino density  $\Omega_\nu h^2 = 0.0006$ .

#### 3.1.2. Relative information in the matrices

To assess the information content of the correlation matrices, we consider the Kullback-Leibler (KL) divergence (see, e.g., [Kunz et al. 2006, 2008](#) and [Planck Collaboration XXVI 2014](#) for further applications of the KL divergence to topology). The KL divergence between two probability distributions  $p_1(x)$  and  $p_2(x)$  is given by

$$d_{\text{KL}} = \int p_1(x) \ln \frac{p_1(x)}{p_2(x)} dx. \quad (3)$$

If the two distributions are Gaussian with correlation matrices  $\mathbf{C}_1$  and  $\mathbf{C}_2$ , this expression simplifies to

$$d_{\text{KL}} = -\frac{1}{2} \left[ \ln |\mathbf{C}_1 \mathbf{C}_2^{-1}| + \text{Tr}(\mathbf{I} - \mathbf{C}_1 \mathbf{C}_2^{-1}) \right], \quad (4)$$

and is thus an asymmetric measure of the discrepancy between the correlation matrices. The KL divergence can be interpreted as the ensemble average of the log-likelihood ratio  $\Delta \ln \mathcal{L}$  between realizations of the two distributions. Hence, it enables us to probe the ability to tell if, on average, we can distinguish realizations of  $p_1$  from a fixed  $p_2$  without having to perform a brute-force Monte Carlo integration. Thus, the KL divergence is related to ensemble averages of the likelihood-ratio plots that we present for simulations (Sect. 4.4.1) and real data (Sect. 5.1) but can be calculated from the correlation matrices alone. Note that with this definition, the KL divergence is *minimized* for cases with the best match (*maximal* likelihood).

In [Planck Collaboration XXVI \(2014\)](#) we used the KL divergence to show that the likelihood is robust to differences in the cosmological model and small differences in the topology.

In [Fig. 1](#) we plot the KL divergence relative to an infinite universe for the slab topology as a function of resolution  $\ell_{\max}$  (upper panel) and fundamental domain size (lower panel). Our ability to detect a topology with a fundamental domain smaller than the distance to the last scattering surface (approximately at the horizon distance  $\chi_{\text{rec}} = 3.1H_0^{-1}$ , so with sides of length  $L = 2\chi_{\text{rec}} = 6.2H_0^{-1}$ ) grows significantly with the resolution even beyond the cases that we studied. For the noise levels of the 2015 lowP data considered here and defined in [Planck Collaboration XIII \(2015\)](#), polarization maps do not add much information beyond that contained in the temperature maps, although, as also shown in [Sect. 4.4.1](#), the higher sensitivity achievable by the full *Planck* low- $\ell$  data over all frequencies should be able to give even stronger constraints on these small fundamental domains.

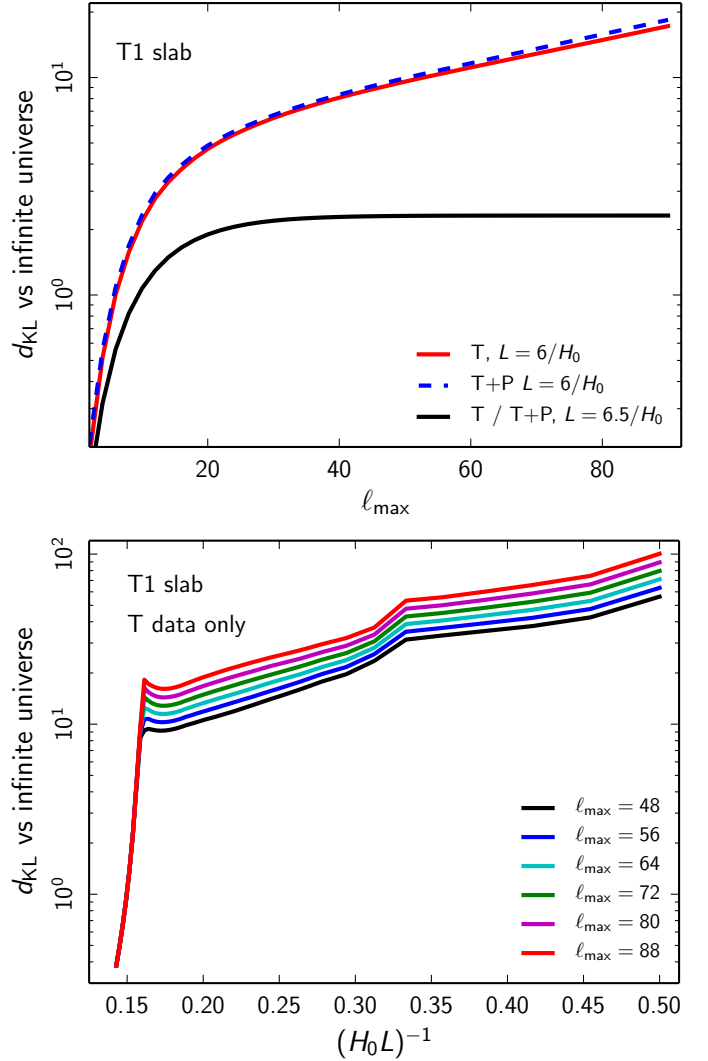
If, however, the fundamental domain is larger than the horizon (as is the case for  $L = 6.5H_0^{-1}$ ) then the relative information in the correlation matrix saturates quite early and a resolution of  $\ell_{\max} \simeq 48$  is actually sufficient. The main goal is thus to ensure that we have enough discriminatory power right up to the horizon size. In addition, polarization does not add much information in this case, irrespective of the noise level. This is to be expected: polarization is generated only for a short period of time around the surface of last scattering. Once the fundamental domain exceeds the horizon size, the relative information drops rapidly towards zero, and the dependence on  $\ell_{\max}$  becomes weak.

In [Fig. 2](#) we plot the KL divergence as a function of the size of the fundamental domain for fixed cube (T3), rod (T2), and slab (T1) topologies, each with fundamental domain size  $L = 5.5H_0^{-1}$ , compared to the slab. Each shows a strong dip at  $L = 5.5H_0^{-1}$ , indicating the ability to detect this topology (although note the presence of a weaker dip around half the correct size,  $L \simeq 2.75H_0^{-1}$ ). The figure also shows that  $\ell_{\max} = 40$  still shows the dip at the correct location, although somewhat more weakly than  $\ell_{\max} = 80$ .

Note that the shape of the curves is essentially identical, with the slab likelihood able to detect one or more sets of antipodal matched circles (and their related excess correlations at large angular scales) present in each case. [Figure 2](#) therefore shows that using the correlation matrix for a slab (T1) topology also allows detection of rod (T2) and cubic (T3) topologies: this is advantageous as the slab correlation matrix is considerably easier to calculate than the cube and rod, since it is only discretized in a single direction. [Figure 3](#) shows the KL divergence as a function of the relative rotation of the fundamental domain, showing that, despite the lack of the full set of three pairs of antipodal correlations, we can determine the relative rotation of a single pair. This is exactly how the matched-circles tests work. Furthermore, as we will demonstrate in [Sect. 4.4.1](#), slab likelihoods are indeed separately sensitive to the different sets of antipodal circles in cubic spaces. We can hence adopt the slab as the most general tool for searching for spaces with antipodal circles.

### 3.2. Bianchi models

The polarization properties of Bianchi models were first derived in [Pontzen & Challinor \(2007\)](#) and extensively categorized in [Pontzen \(2009\)](#) and [Pontzen & Challinor \(2011\)](#). In these works it was shown that advection in Bianchi universes leads to efficient conversion of *E*-mode polarization to *B* modes; evidence for a significant Bianchi component found in temperature data

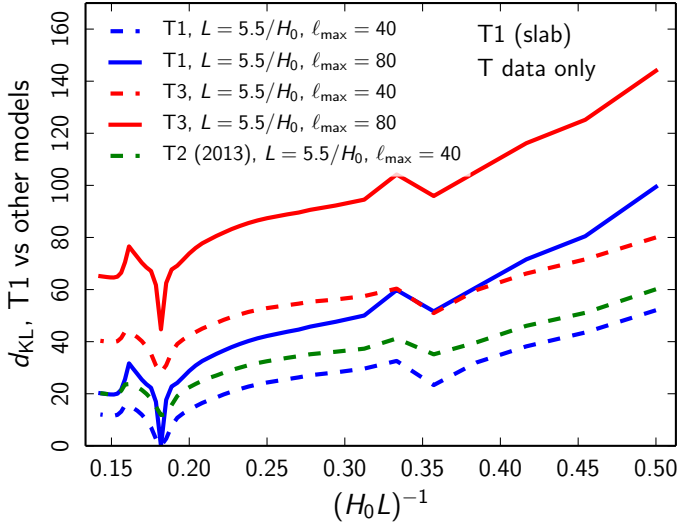


**Fig. 1.** The Kullback-Leibler divergence of slab (T1) topologies relative to an infinite universe as a function of  $\ell_{\max}$  with sizes  $L = 6H_0^{-1}$  and  $L = 6.5H_0^{-1}$  (top), and as a function of size  $L$  of the fundamental domain for various  $\ell_{\max}$  (bottom). Note that the information for  $L = 6H_0^{-1}$  continues to rise with  $\ell_{\max}$  whereas it levels off for the slightly larger  $L = 6.5H_0^{-1}$  case. The corresponding figures for cubic (T3) topologies look qualitatively similar except that all  $d_{\text{KL}}$  values are three times larger.

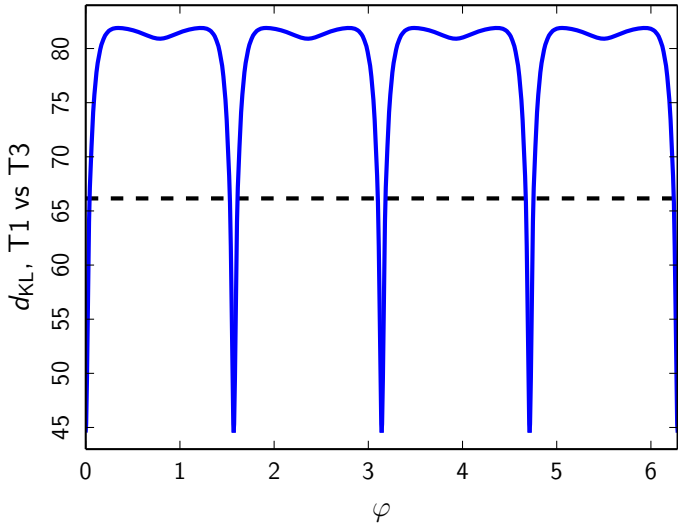
would therefore suggest a large *B*-mode signal (but not necessarily require it; see [Pontzen 2009](#)). For examples of the temperature and polarization signatures of Bianchi VII<sub>h</sub> models we refer the reader to figure 1 of [Pontzen \(2009\)](#). Despite the potential for CMB polarization to constrain the Bianchi sector, a full polarization analysis has not yet been carried out. The analysis of [Pontzen & Challinor \(2007\)](#) remains the state-of-the-art, where WMAP *BB* and *EB* power spectra were used to demonstrate (using a simple  $\chi^2$  analysis) that a Bianchi VII<sub>h</sub> model derived from temperature data was disfavoured compared to an isotropic model.

The subdominant, deterministic CMB contributions of Bianchi VII<sub>h</sub> models can be characterized by seven parameters: the matter and dark energy densities,  $\Omega_m$  and  $\Omega_\Lambda$ , respectively; the present dimensionless vorticity,  $(\omega/H)_0$ ; the dimensionless length-scale parameter,  $x$ , which controls the “tight-





**Fig. 2.** The Kullback-Leibler divergence of fixed cubic, rod, and slab topologies with fundamental domain side  $L = 5.5H_0^{-1}$  compared to a slab of variable fundamental domain size  $L$ . The chimney space T2 dates from the 2013 analysis (Planck Collaboration XXVI 2014) and was computed for the best-fit parameters of that release.



**Fig. 3.** The Kullback-Leibler divergence of a slab space relative to a cubic topology, as a function of rotation angle of the slab space (blue curve). Both spaces have  $L = 5.5H_0^{-1}$  and  $\ell_{\max} = 80$ . The horizontal black dashed line gives the Kullback-Leibler divergence of an infinite universe relative to the cubic topology and illustrates how much better the slab space fits with the correct orientation relative to the cubic torus.

ness” of the characteristic Bianchi spirals; and the Euler angles<sup>2</sup>,  $(\alpha, \beta, \gamma)$ , describing their orientation (i.e., the choice of coordinate system), where  $H$  is the Hubble parameter. For further details see Planck Collaboration XXVI (2014), McEwen et al. (2013), Pontzen (2009), Pontzen & Challinor (2007), Jaffe et al. (2006c), Jaffe et al. (2005), and Barrow et al. (1985).

<sup>2</sup> The active  $zyz$  Euler convention is adopted, corresponding to the rotation of a physical body in a *fixed* coordinate system about the  $z$ ,  $y$ , and  $z$  axes by  $\gamma$ ,  $\beta$ , and  $\alpha$ , respectively.

## 4. Methods

### 4.1. Data

In this work we use data from the *Planck* 2015 release. This includes intensity maps from the full mission, along with a subset of polarization data. Specifically, for the likelihood calculations discussed below (Sect. 4.3.1 for application to topology and Sect. 4.3.3 for Bianchi models) which rely on HEALPix maps at  $N_{\text{side}} = 16$ , we use the data designated “lowT,P”, as defined for the low- $\ell$  *Planck* likelihood for isotropic models (Planck Collaboration XI 2015; Planck Collaboration XIII 2015): lowP polarization maps based on the LFI 70 GHz channel and lowT temperature maps created by the Commander component separation method, along with the appropriate mask and noise correlation matrix. As in Planck Collaboration XXVI (2014), the intensity noise contribution is negligible on these scales, and diagonal regularizing noise with variance  $\sigma_I^2 = 4 \mu\text{K}^2$  has therefore been added to the intensity portion of the noise correlation matrix. We cut contaminated regions of the sky using the low- $\ell$  mask defined for the *Planck* isotropic likelihood code (Planck Collaboration XI 2015), retaining 94 % of the sky for temperature, and the lowT,P polarization mask, cleaned with the templates created from *Planck* 30 GHz and 353 GHz data, retaining 47 % of the sky for polarization.

The matched circles search (Sects. 4.2 and 5.1.1) uses four component-separated maps (Planck Collaboration IX 2015) which effectively combine both intensity and polarization information from different scales. The maps are smoothed with a Gaussian filter of 30’ and 50’ full width at half maximum (FWHM) for temperature and polarization, respectively, and degraded to  $N_{\text{side}} = 512$ . Corresponding temperature and polarization common masks for diffuse emission, with a point source cut for the brightest sources, downgraded analogously to the maps, are used. After degradation, and accounting for the needed expansion of the polarization mask due to the conversion of  $Q$  and  $U$  to  $E$ , the temperature map retains 74 % of the sky and the polarization map 40 %. These  $E$ -mode maps are calculated using the method of Bielewicz et al. (2012) (see also Kim 2011) and correspond to the spherical Laplacian of the scalar  $E$ , consequently filtering out power at large angular scales. The frequentist significance of the matched-circle results are determined by a suite of 300 Monte Carlo simulations of the *Planck* SMICA map.

Because the matched circles method uses a much higher value of  $N_{\text{side}}$ , it is sensitive out to a much higher maximum multipole,  $\ell_{\max}$ . From the results of Sect. 3.1.2, this indicates that it uses more of the information available when confronting models with fundamental domains within the last-scattering surface compared to our implementation of the likelihood, limited to  $\ell_{\max} \approx 40$ . As we show in Sect. 4.4.1, it also allows the use of high-pass filtered component-separated maps (as defined in Planck Collaboration IX 2015) without a significant decrease in the ability to detect a multiply-connected topology.

### 4.2. Topology: matched circles

As in Planck Collaboration XXVI (2014), we use the circle comparison statistic of Cornish et al. (1998), optimized for small-scale anisotropies (Cornish et al. 2004), to search for correlated circles in sky maps of the CMB temperature and polarization anisotropy.

Because a polarization map at small angular scales can be considered a snapshot of the last scattering surface, theoretically the polarization provides a better opportunity for the de-

tection of multi-connected topology signatures than a temperature anisotropy map. There is a small subtlety here: whereas the intensity is a scalar and thus is unchanged when viewed from different directions, the polarization is a tensor which behaves differently under rotation. Hence, we need to use the coordinate-independent quantities,  $E$  and  $B$ , which are scalars (or pseudo-scalars) and are thus unchanged when viewed from different directions. The decomposition into  $E$  and  $B$  of an arbitrary masked CMB polarization map, contaminated by noise, foregrounds, and systematic errors, is itself a computationally demanding task, non-local on the sky. Assuming negligible initial  $B$  polarization, we use only the  $E$  maps produced from (or as part of) the process of separating out the different astrophysical components from the *Planck* intensity and polarization data (Planck Collaboration IX 2015; Planck Collaboration X 2015).

Compared with the likelihood method described below, the circles search uses information on a wider variety of scales and is thus less sensitive to large-scale systematic errors, and uses information at higher multipoles  $\ell$ . The statistic is defined by

$$S_{i,j}^+(\alpha, \phi_*) = \frac{2 \sum_m |m| X_{i,m} X_{j,m}^* e^{-im\phi_*}}{\sum_n |n| (|X_{i,n}|^2 + |X_{j,n}|^2)}, \quad (5)$$

where  $X_{i,m}$  and  $X_{j,m}$  denote the Fourier coefficients of the temperature or  $E$ -mode fluctuations around two circles of angular radius  $\alpha$  centred at different points on the sky,  $i$  and  $j$ , respectively, with relative phase  $\phi_*$ . The  $m$ th harmonic of the field anisotropies around the circle is weighted by the factor  $|m|$ , taking into account the number of degrees of freedom per mode. Such weighting enhances the contribution of small-scale structure relative to large-scale fluctuations.

The  $S^+$  statistic corresponds to pairs of circles, with the points ordered in a clockwise direction (phased). For alternative ordering, when the points are ordered in an anti-clockwise direction (anti-phased along one of the circles), the Fourier coefficients  $X_{i,m}$  are complex conjugated, defining the  $S^-$  statistic. This allows the detection of both orientable and non-orientable topologies. For orientable topologies the matched circles have anti-phased correlations, while for non-orientable topologies they have a mixture of anti-phased and phased correlations.

The  $S^\pm$  statistics take values over the interval  $[-1, 1]$ . Circles that are perfectly matched have  $S = 1$ , while uncorrelated circles will have a mean value of  $S = 0$ . To find matched circles for each radius  $\alpha$ , the maximum value  $S_{\max}^\pm(\alpha) = \max_{i,j,\phi_*} S_{i,j}^\pm(\alpha, \phi_*)$  is determined.

Because general searches for matched circles are computationally very intensive, we restrict our analysis to a search for pairs of circles centred around antipodal points, so called back-to-back circles. The maps are also downgraded as described in Sect. 4.1. This increases the signal-to-noise ratio and greatly speeds up the computations required, but with no significant loss of discriminatory power. Regions most contaminated by Galactic foreground were removed from the analysis using the common temperature or polarization mask. More details on the numerical implementation of the algorithm can be found in Bielewicz & Banday (2011) and Bielewicz et al. (2012).

To draw any conclusions from an analysis based on the statistic  $S_{\max}^\pm(\alpha)$ , it is very important to correctly estimate the threshold for a statistically significant match of circle pairs. We used 300 Monte Carlo simulations of CMB maps, described in detail in Sect. 4.4.1, to establish the threshold such that fewer than 1 % of simulations would yield a false event. Note that we perform the entire analysis, including the final statistical calibration, separately for temperature and polarization.

### 4.3. Likelihood

#### 4.3.1. Topology

For the likelihood analysis of the large angle intensity and polarization data we have generalized the method implemented in Planck Collaboration XXVI (2014) to include polarization. The likelihood, i.e., the probability to find a combined temperature and polarization data map  $\mathbf{d}$  with associated noise matrix  $\mathbf{N}$  given a certain topological model  $T$  is then given by

$$P(\mathbf{d}|\mathbf{C}[\Theta_C, \Theta_T, T], A, \varphi) \propto \frac{1}{\sqrt{|\mathbf{AC} + \mathbf{N}|}} \exp \left[ -\frac{1}{2} \mathbf{d}^* (\mathbf{AC} + \mathbf{N})^{-1} \mathbf{d} \right], \quad (6)$$

where now  $\mathbf{d}$  is a  $3N_{\text{pix}}$ -component data vector obtained by concatenation of the  $(I, Q, U)$  datasets while  $\mathbf{C}$  and  $\mathbf{N}$  are  $3N_{\text{pix}} \times 3N_{\text{pix}}$  theoretical signal and noise correlation matrices, arranged in the block form as

$$\mathbf{C} = \begin{pmatrix} C_{II} & C_{IQ} & C_{IU} \\ C_{QI} & C_{QQ} & C_{QU} \\ C_{UI} & C_{UQ} & C_{UU} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} N_{II} & N_{IQ} & N_{IU} \\ N_{QI} & N_{QQ} & N_{QU} \\ N_{UI} & N_{UQ} & N_{UU} \end{pmatrix}. \quad (7)$$

Finally,  $\Theta_C$  is the set of standard cosmological parameters,  $\Theta_T$  is the set of topological parameters (e.g., the size,  $L$ , of the fundamental domain),  $\varphi$  is the orientation of the topology (e.g., the Euler angles), and  $A$  is a single amplitude, scaling the signal correlation matrix (this is equivalent to an overall amplitude in front of the power spectrum in the isotropic case). Working in pixel space allows for the straightforward application of an arbitrary mask, including separate masks for intensity and polarization parts of the data. The masking procedure can also be used to limit the analysis to intensity or polarization only.

Since  $\mathbf{C} + \mathbf{N}$  in pixel space is generally poorly conditioned, we again (following the 2013 procedure) project the data vector and correlation matrices onto a limited set of orthonormal basis vectors, select  $N_m$  such modes for comparison, and consider the likelihood marginalized over the remainder of the modes

$$p(\mathbf{d}|\mathbf{C}[\Theta_C, \Theta_T, T], \varphi, A) \propto \frac{1}{\sqrt{|\mathbf{AC} + \mathbf{N}|_M}} \exp \left[ -\frac{1}{2} \sum_{n=1}^{N_m} d_n^* (\mathbf{AC} + \mathbf{N})_{nn}^{-1} d_n \right], \quad (8)$$

where  $\mathbf{C}$  and  $\mathbf{N}$  are restricted to the  $N_m \times N_m$  subspace.

The choice of the basis modes and their number  $N_m$  used for analysis is a compromise between robust invertibility of  $\mathbf{C} + \mathbf{N}$  and the amount of information retained. All the models for which likelihoods are compared must be expanded in the same set of modes. Thus, in Planck Collaboration XXVI (2014) we used the set of eigenmodes of the cut-sky correlation matrix of the fiducial best-fit simply-connected universe,  $\mathbf{C}_{\text{fid}}$ , as the analysis basis, limiting ourselves to the  $N_m$  modes with the largest eigenvalues. For comparison with the numbers we use, a full-sky temperature map with maximum multipole  $\ell_{\max}$  has  $(\ell_{\max} + 1)^2 - 4$  independent modes (four are removed to account for the unobserved monopole and dipole).

The addition of polarization data, with much lower signal-to-noise than the temperature, raises a new question: how is the temperature and polarization data mix reflected in the limited basis set we project onto? The most natural choice is the set of eigenmodes of the signal-to-noise matrix  $\mathbf{C}_{\text{fid}} \mathbf{N}^{-1}$  for the fiducial model, and a restriction of the mode set based on signal-to-noise eigenvalues (see, e.g., Bond et al. 1998). This, however, requires robust invertibility of the noise correlation matrix, which, again,

is generally not the case for the smoothed data. Moreover, such a ranking by  $S/N$  would inevitably favour the temperature data, and we wish to explore the effect of including polarization data at an equal footing with temperature. We therefore continue to use the eigenmodes of the cut-sky fiducial correlation matrix as our basis. By default, we select the first  $N_m = 1085$  eigenmodes (corresponding to  $\ell_{\max} = 32$ ), though we vary the mode count where it is informative to do so.

In Fig. 4 we show  $I$ ,  $Q$ , and  $U$  maps of the highest-eigenvalue (i.e., highest contribution to the signal correlation) mode for our fiducial simply-connected model. Note that the scale is different for temperature compared to the two polarization maps: the temperature contribution to the mode is much greater than that of either polarization component. We show modes for the masked sky, although in fact the structure at large scales is similar to the full-sky case, rotated and adjusted somewhat to account for the mask. In Fig. 5 we show the structure of mode 301, with much lower signal amplitude. Temperature remains dominant, although polarization begins to have a larger effect. Note that at this level of signal amplitude, the pattern is aligned with the mask, and shows a strong correlation between temperature and polarization.

#### 4.3.2. Evaluating the topological likelihood

The aim of the topological likelihood analysis is to calculate the likelihood as a function of the parameters pertaining to a particular topology,  $p(\mathbf{d}|\Theta_T, T)$ . To do so, we must marginalize over the other parameters appearing in Eq. (8), namely  $\Theta_C$ ,  $\varphi$ , and  $A$ , as

$$p(\mathbf{d}|\Theta_T, T) = \int d\Theta_C d\varphi dA p(\mathbf{d}|\mathbf{C}[\Theta_C, \Theta_T, T], \varphi, A) p(\Theta_C, \varphi, A). \quad (9)$$

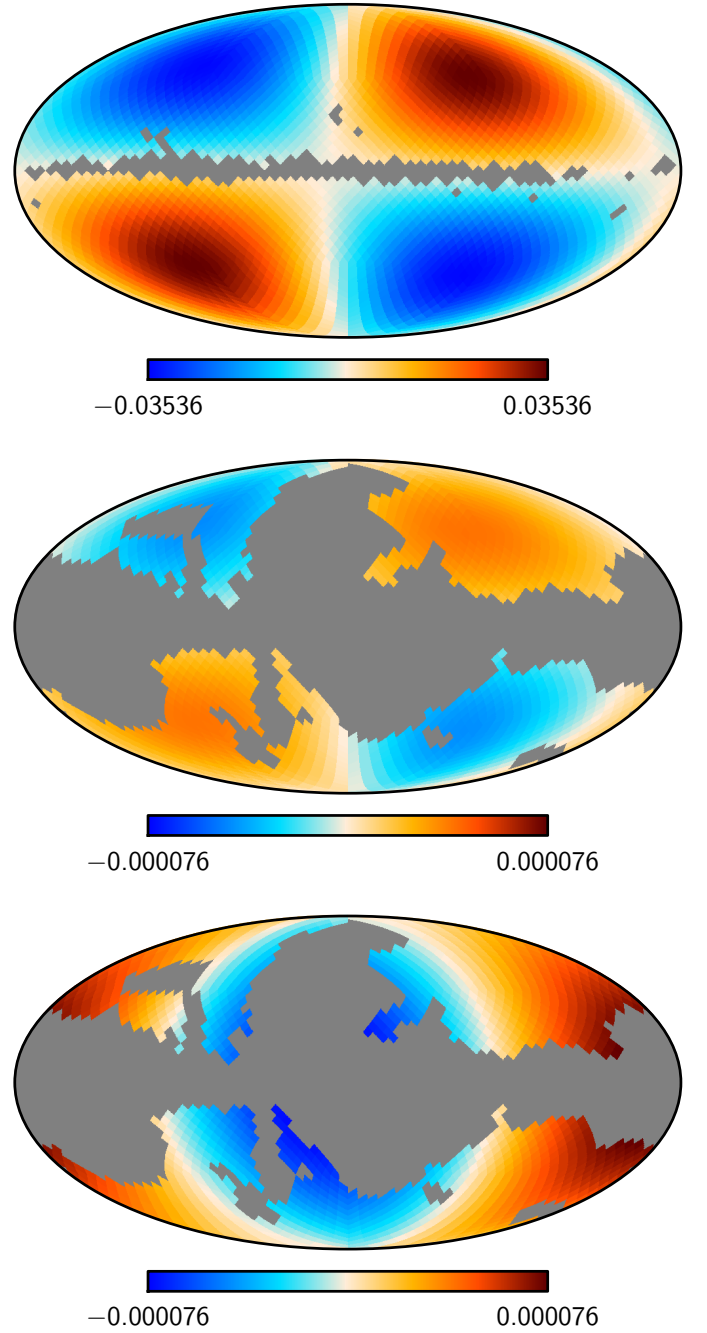
The complexity of the topological covariance matrix calculation precludes a joint examination of the full cosmological and topological parameter spaces. Instead, we adopt the delta-function prior  $p(\Theta_C) = \delta(\Theta_C - \Theta_C^*)$  to fix the cosmological parameters at their fiducial values,  $\Theta_C^*$  (as defined in Sect. 3.1.1), and evaluate the likelihood on a grid of topological parameters using a restricted set of pre-calculated covariance matrices. We note that, as discussed in Planck Collaboration XXVI (2014), the ability to detect or rule out a multiply connected topology is insensitive to the values of the cosmological parameters adopted for the calculation of the correlation matrices.

In the setting described above, Eq. (9) simplifies to

$$p(\mathbf{d}|\Theta_{T,i}, T) = \int d\varphi dA p(\mathbf{d}|\mathbf{C}[\Theta_C^*, \Theta_{T,i}, T], \varphi, A) p(\varphi, A), \quad (10)$$

where the likelihood at each gridpoint in topological parameter space,  $\Theta_{T,i}$ , is equal to the probability of obtaining the data given fixed cosmological and topological parameters and a compactification (i.e., fundamental domain shape and size), marginalized over orientation and amplitude. The calculation therefore reduces to evaluating the Bayesian evidence for a set of gridded topologies. As we focus on cubic torus and slab topologies in this work, we note that the sole topological parameter of interest is the size of the fundamental domain,  $L$ .

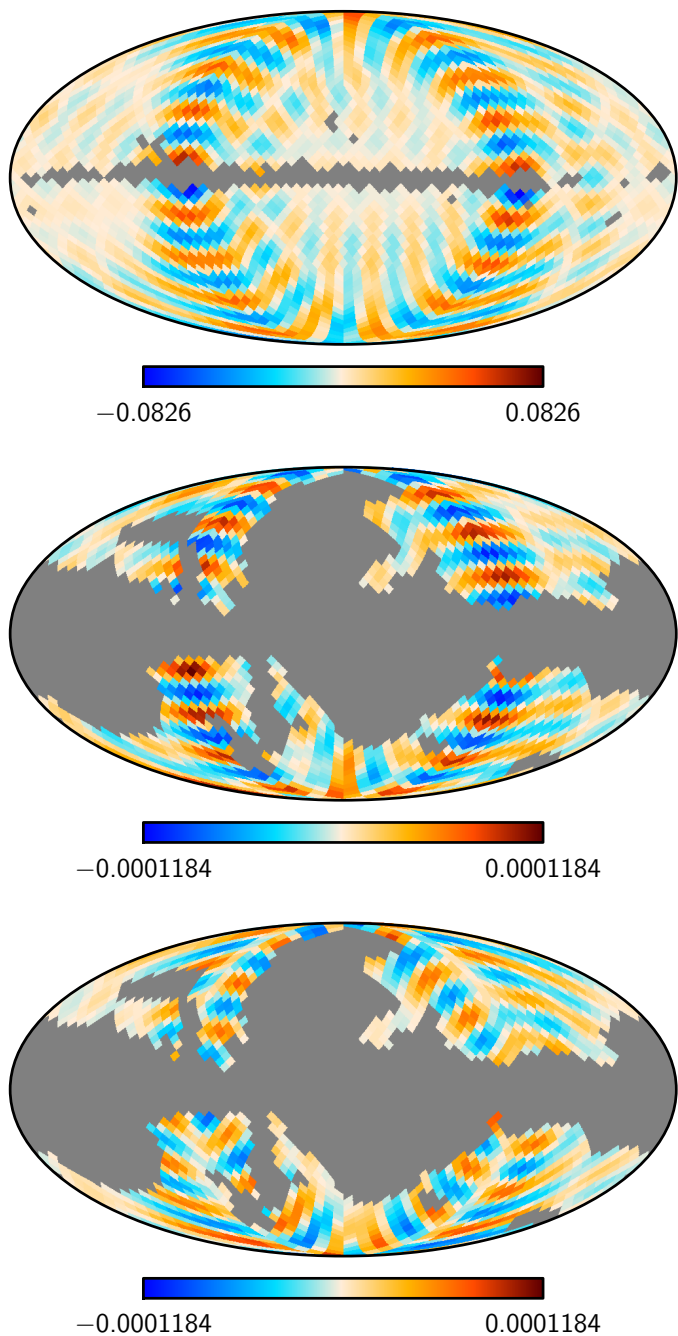
Even after fixing the cosmological parameters, calculating the Bayesian evidence is a time-consuming process, and is further complicated by the multimodal likelihood functions typical in non-trivial topologies. We therefore approach the problem on two fronts. We first approximate the likelihood function using a ‘‘profile likelihood’’ approach, as presented in



**Fig. 4.** Mode structure plotted as maps for the eigenvector corresponding to the highest-signal eigenvalue of the fiducial simply-connected model. The top map corresponds to temperature, middle to  $Q$  polarization, and bottom to  $U$  polarization. Masked pixels are plotted in grey.

Planck Collaboration XXVI (2014), in which the marginalization in Eq. (10) is replaced with maximization in the four-dimensional space of orientation and amplitude parameters. Specifically, we maximize the likelihood over the three angles defining the orientation of the fundamental domain using a three-dimensional Amoeba search (e.g., Press et al. 1992), where at each orientation the likelihood is separately maximized over the amplitude. Due to the complex structure of the likelihood surface in orientation space, we repeat this procedure five times





**Fig. 5.** Mode structure plotted as maps for the eigenvector corresponding to the 301st-highest-signal eigenvalue of the fiducial simply-connected model. The top map corresponds to temperature, middle to  $Q$  polarization, and bottom to  $U$  polarization. Masked pixels are plotted in grey.

with different starting orientations. To ensure uniform and non-degenerate coverage, the orientation space is traversed in a cartesian projection of the northern hemisphere of the three-sphere  $S^3$  representation of rotations.

The profile likelihood calculation allows rapid evaluation of the likelihood and testing of different models compared with a variety of data and simulations, but it is difficult to interpret in a Bayesian setting. As we show below, however, the numerical results of profiling over this limited set of parameters agree nu-

merically very well with the statistically correct marginalization procedure.

Our second approach explicitly calculates the marginalized likelihood, Eq. (10), allowing full Bayesian inference at the cost of increased computation time. We use the public MultiNest<sup>3</sup> code (Feroz & Hobson 2008; Feroz et al. 2009, 2013)—optimized for exploring multimodal probability distributions in tens of dimensions—to compute the desired evidence values via nested sampling (Skilling 2004). MultiNest is run in its importance nested sampling mode (Feroz et al. 2013) using 200 live points, with tolerance and efficiency set to their recommended values of 0.5 and 0.3, respectively. The final ingredient needed to calculate the evidence values are priors for the marginalized parameters. We use a log prior for the amplitude, truncated to the range  $0.1 \leq A \leq 10$ , and the Euler angles are defined to be uniform in  $0 \leq \alpha < 2\pi$ ,  $-1 \leq \cos\beta \leq 1$ , and  $0 \leq \gamma < 2\pi$ , respectively; MultiNest is able to wrap the priors on  $\alpha$  and  $\gamma$ . The combined code will be made public as part of the AniCosmo<sup>4</sup> package (McEwen et al. 2013).

It is worth noting that this formalism can be extended to compare models with different compactifications (or the simply connected model) using Bayesian model selection: the only additional requirements are priors for the topological parameters. Taking the current slab and cubic torus topologies as examples, by defining a prior on the size of the fundamental domain one can calculate the evidence for each model. Assuming each topology is equally likely a priori, i.e., that  $p(T_{\text{slab}}) = p(T_{\text{cub}})$ , one can then write down the relative probability of the two topologies given the data:

$$\frac{p(T_{\text{slab}}|\mathbf{d})}{p(T_{\text{cub}}|\mathbf{d})} \approx \frac{\sum_i p(L_i|T_{\text{slab}}) p(\mathbf{d}|L_i, T_{\text{slab}})}{\sum_j p(L'_j|T_{\text{cub}}) p(\mathbf{d}|L'_j, T_{\text{cub}})}. \quad (11)$$

Unfortunately, it is difficult to provide a physically-motivated proper prior distribution for the size of the fundamental domain. Even pleading ignorance and choosing a “naïve” uniform prior would require an arbitrary upper limit to  $L$  whose exact value would strongly influence the final conclusion. For this reason, we refrain from extending the formalism to model selection within this manuscript.

#### 4.3.3. Bianchi models

While physically the cosmological densities describing Bianchi models should be identified with their standard  $\Lambda$ CDM counterparts, in previous analyses unphysical models have been considered in which the densities are allowed to differ. The first coherent analysis of Bianchi VII<sub>h</sub> models was performed by McEwen et al. (2013), where the  $\Lambda$ CDM and Bianchi densities are coupled and all cosmological and Bianchi parameters are fit simultaneously. In the analysis of Planck Collaboration XXVI (2014), in order to compare with all prior studies both coupled and decoupled models were analysed. We consider the same two models here: namely, the physical *open-coupled-Bianchi* model where an open cosmology is considered (for consistency with the open Bianchi VII<sub>h</sub> models), in which the Bianchi densities are coupled to their standard cosmological counterparts; and the phenomenological *flat-decoupled-Bianchi* model where a flat cosmology is considered and in which the Bianchi densities are decoupled.

We firstly carry out a full Bayesian analysis for these two Bianchi VII<sub>h</sub> models, repeating the analysis

<sup>3</sup> <http://www.mrao.cam.ac.uk/software/multinest/>

<sup>4</sup> <http://www.jasonmcewen.org/>



performed in [Planck Collaboration XXVI \(2014\)](#) with updated *Planck* temperature data. The methodology is described in detail in [McEwen et al. \(2013\)](#) and summarized in [Planck Collaboration XXVI \(2014\)](#). The complete posterior distribution of all Bianchi and cosmological parameters is sampled and Bayesian evidence values are computed to compare Bianchi VII<sub>h</sub> models to their concordance counterparts. Bianchi temperature signatures are simulated using the Bianchi2<sup>5</sup> code ([McEwen et al. 2013](#)), while the AniCosmo code is used to perform the analysis, which in turn uses MultiNest to sample the posterior distribution and compute evidence values.

To connect with polarization data, we secondly analyse polarization templates computed using the best-fit parameters from the analysis of temperature data. For the resulting small set of best-fit models, polarization templates are computed using the approach of [Pontzen & Challinor \(2007\)](#) and [Pontzen \(2009\)](#), and have been provided by [Pontzen \(2015\)](#). These Bianchi VII<sub>h</sub> simulations are more accurate than those considered for the temperature analyses performed here and in previous works (see, e.g., [Planck Collaboration XXVI 2014](#), [McEwen et al. 2013](#), [Bridges et al. 2008](#), [Bridges et al. 2007](#), [Jaffe et al. 2005, 2006a,b,c](#)), since the recombination history is modelled. The largest effect of incorporating the recombination history is its impact on the polarization fraction, although the amplitude of the temperature component can also vary by approximately 5% (which is calibrated in the current analysis, as described below).

Using the simulated Bianchi VII<sub>h</sub> polarization templates computed following [Pontzen & Challinor \(2007\)](#) and [Pontzen \(2009\)](#), and provided by [Pontzen \(2015\)](#), we perform a maximum-likelihood fit for the amplitude of these templates using *Planck* polarization data (a full Bayesian evidence calculation of the complete temperature and polarization data-set incorporating the more accurate Bianchi models of [Pontzen & Challinor 2007](#) and [Pontzen 2009](#) is left to future work). The likelihood in the Bianchi scenario is identical to that considered in [Planck Collaboration XXVI \(2014\)](#) and [McEwen et al. \(2013\)](#); however, we now consider the Bianchi and cosmological parameters fixed and simply introduce a scaling of the Bianchi template. The resulting likelihood reads:

$$P(\mathbf{d} | \lambda, \mathbf{t}) \propto \exp\left[-\frac{1}{2}(\mathbf{d} - \lambda\mathbf{t})^\dagger(\mathbf{C} + \mathbf{N})^{-1}(\mathbf{d} - \lambda\mathbf{t})\right], \quad (12)$$

where  $\mathbf{d}$  denotes the data vector,  $\mathbf{t} = \mathbf{b}(\Theta_{\mathbf{b}}^*)$  is the Bianchi template for best-fit Bianchi parameters  $\Theta_{\mathbf{b}}^*$ ,  $\mathbf{C} = \mathbf{C}(\Theta_{\mathbf{c}})$  is the cosmological covariance matrix for the best-fit cosmological parameters  $\Theta_{\mathbf{c}}^*$ ,  $\mathbf{N}$  is the noise covariance, and  $\lambda$  is the introduced scaling parameter (the effective vorticity of the scaled Bianchi component is simply  $\lambda(\omega/H)_0$ ).

In order to effectively handle noise and partial sky coverage the data are analysed in pixel space. We restrict to polarization data only here since temperature data are used to determine the best-fit Bianchi parameters. The data and template vectors thus contain unmasked  $Q$  and  $U$  Stokes components only and, correspondingly, the cosmological and noise covariance matrices are given by the polarization ( $Q$  and  $U$ ) subspace of Eq. (7), and again contain unmasked pixels only.

The maximum-likelihood (ML) estimate of the template amplitude is given by  $\lambda^{\text{ML}} = \mathbf{t}^\dagger(\mathbf{C} + \mathbf{N})^{-1}\mathbf{d} / [\mathbf{t}^\dagger(\mathbf{C} + \mathbf{N})^{-1}\mathbf{t}]$  and its dispersion by  $\Delta\lambda^{\text{ML}} = [\mathbf{t}^\dagger(\mathbf{C} + \mathbf{N})^{-1}\mathbf{t}]^{-1/2}$  (see, e.g., [Kogut et al. 1997](#); [Jaffe et al. 2005](#)). If *Planck* polarization data support the

best-fit Bianchi model found from the analysis of temperature data we would expect  $\lambda^{\text{ML}} \simeq 1$ . A statistically significant deviation from unity in the fitted amplitude can thus be used to rule out the Bianchi model using polarization data.

As highlighted above, different methods are used to simulate Bianchi temperature and polarization components, where the amplitude of the temperature component may vary by a few percent between methods. We calibrate out this amplitude mismatch by scaling the polarization components by a multiplicative factor fitted so that the temperature components simulated by the two methods match, using a maximum-likelihood template fit again, as described above.

#### 4.4. Simulations and validation

##### 4.4.1. Topology

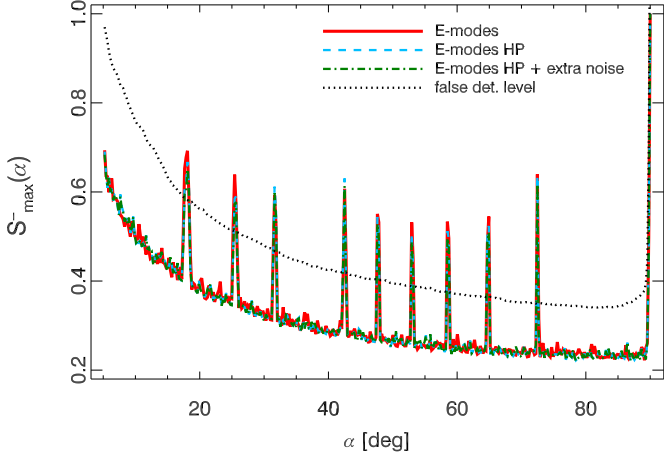
Matched circles. Before beginning the search for pairs of matched circles in the *Planck* data, we validate our algorithm using the same simulations as employed for the [Planck Collaboration XXVI \(2014\)](#) and [Bielewicz et al. \(2012\)](#) papers, i.e., the CMB sky for a universe with a three-torus topology for which the dimension of the cubic fundamental domain is  $L = 2H_0^{-1}$ , well within the last-scattering surface. We computed the  $a_{\ell m}$  coefficients up to the multipole of order  $\ell = 500$  and convolved them with the same smoothing beam profile as used for the *Planck* SMICA map. To the map was added noise corresponding to the SMICA map. In particular, we verified that our code is able to find all pairs of matched circles in such a map. The statistic  $S_{\text{max}}^-(\alpha)$  for the  $E$ -mode map is shown in Fig. 6.

Because for the baseline analysis we use high-pass filtered maps, we also show the analysis of the SMICA  $E$ -mode map high-pass filtered so that the lowest order multipoles ( $\ell < 20$ ) are removed from the map (the multipoles in the range  $20 \leq \ell \leq 40$  are apodized between 0 and 1 using a cosine as defined in [Planck Collaboration IX 2015](#)). The high-pass filtering does not decrease our ability to detect a multiply-connected topology using the matched circles method. This is consistent with the negligible sensitivity of the matched circle statistic to the reionization signal, studied by [Bielewicz et al. \(2012\)](#). This is a consequence of the weighting of the polarization data by a factor proportional to  $\ell^2$  employed in the transformation from the Stokes parameters  $Q$  and  $U$  to an  $E$ -mode map, which effectively filters out the largest multipoles from the data. This test shows that the matched circle method, contrary to the likelihood method, predominantly exploits the topological signal in the CMB anisotropies at moderate angular scales.

We also checked robustness of detection with respect to noise level in order to account for small discrepancies between the noise level in the *Planck* FFP8 simulations and the 2015 data ([Planck Collaboration XII 2015](#)). We repeated the analysis for the high-pass filtered map with added noise with 5% larger amplitude than for the original map. As we can see in Fig. 6, the statistic changes negligibly.

The intersection of the peaks in the matching statistic with the false detection level estimated for the CMB map corresponding to the simply-connected universe defines the minimum radius of the correlated circles that can be detected for this map. Extrapolating the height of the peak with radius of  $18^\circ$  seen in Fig. 6 towards smaller radii one can estimate the minimum radius to be  $\alpha_{\text{min}} \simeq 15^\circ$ .

<sup>5</sup> <http://www.jasonmcewen.org/>



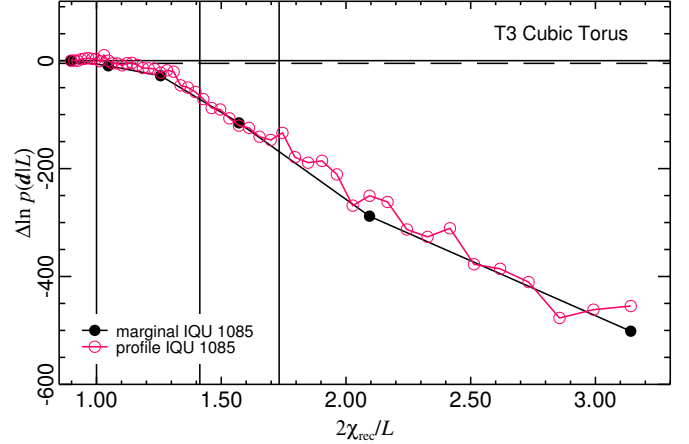
**Fig. 6.** An example of the  $S_{\max}^-$  statistic as a function of circle radius  $\alpha$  for a simulated CMB  $E$ -mode map of a universe with the topology of a cubic 3-torus with dimensions  $L = 2H_0^{-1}$ . To this map noise was added corresponding to the SMICA map. The thick overlapping curves show the statistic for simulated polarization maps with angular resolution and noise level corresponding to the *Planck* SMICA map for three cases: without high-pass filtering (solid, red); with filtering (dashed, blue); and with a 5% larger amplitude of noise (dot-dashed, green). The dotted line shows the false detection level established such that fewer than 1% out of 300 Monte Carlo simulations of the high-pass filtered *Planck* SMICA polarization maps, smoothed and masked in the same way as the data, would yield a false event for the full sky and cut sky analyses, respectively.

**Likelihood.** To validate and compare the performance of the two likelihood methods, we perform two sets of tests: a null test using a simulation of a simply connected universe, and a signal test using a simulation of a toroidal universe with  $L = 4H_0^{-1}$ . The two test maps are generated at  $N_{\text{side}} = 16$  and are band-limited using a  $640'$  Gaussian beam. Diagonal (white) noise is added with pixel variances  $\sigma_l^2 = 0.04 \mu\text{K}^2$  and  $\sigma_{Q/U}^2 = 0.16 \mu\text{K}^2$ , comparable to the expected eventual level of *Planck*'s 143 GHz channel. For clarity of interpretation, no mask is used in these tests; in this setting, the eigenmodes of the fiducial correlation matrix are linear combinations of the spherical harmonics at fixed wavenumber  $\ell$ . As fully exploring the likelihood is much more time-consuming than profiling it, we generate a complete set of test results—analyses of the two test maps using cubic torus and slab covariance matrices on a fine grid of fundamental domain scales—using the profile-likelihood code, and aim to verify the main cubic torus results using the marginalized likelihood generated with AniCosmo. Note that to speed up the calculation of the marginalized likelihood we use a slightly smaller band-limit ( $\ell_{\max} = 32$ ) than in the profile-likelihood calculation ( $\ell_{\max} = 40$ ); with our choice of smoothing scale and mode count, and considering the full sky for validation purposes, we obtain the same eigenbasis (and therefore analyze the same projected data) in both cases.

The results for the null test—in the form of the likelihood function for the fundamental domain scale of the assumed topology—are plotted in Fig. 7 for cubic tori and Fig. 8 for slabs. Concentrating initially on the cubic tori, we see that the likelihoods derived from the two codes agree. In both cases, the likelihood is found to be maximal for fundamental domain

scales larger than the horizon, and the small- $L$  cubic tori are very strongly disfavoured:  $p(d|L = 7H_0^{-1})/p(d|L = 2H_0^{-1}) \sim 10^{217}$ . Note that the AniCosmo likelihood curve contains errors on the likelihood at each  $L$  considered, but these are orders of magnitude smaller than the changes in likelihood between points (typical MultiNest uncertainties yield errors of order 0.1 in log-likelihood).

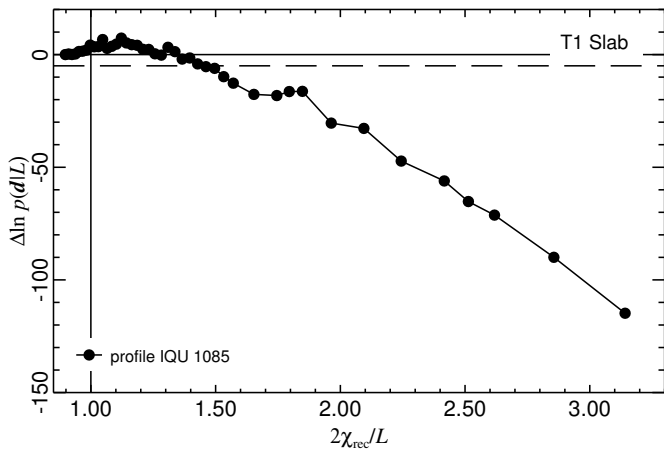
In both cases, the profile likelihood exhibits a mild rise around the horizon scale, due to chance alignments along the matched faces of the fundamental domain. It is slightly stronger in the slab case since the probability of such alignments is greater with only a single pair of faces.



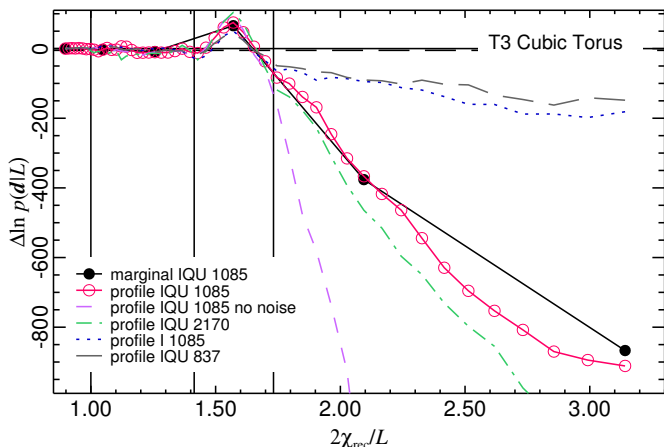
**Fig. 7.** The likelihood function for the fundamental domain scale of a cubic torus derived from simulations of a simply-connected universe, calculated through marginalization (black, filled circles) and profiling (pink, empty circles). The horizontal axis gives the inverse of the length of a side of the fundamental domain, relative to the distance to the last scattering surface. The vertical lines mark the positions where  $\chi_{\text{rec}}$  is equal to various characteristic sizes of the fundamental domain, namely the radius of the largest sphere that can be inscribed in the domain,  $\mathcal{R}_i = L/2$ , the smallest sphere in which the domain can be inscribed,  $\mathcal{R}_u = \sqrt{3}L/2$ , and the intermediate scale  $\mathcal{R}_m = \sqrt{2}L/2$ .

The results for the tests on the toroidal simulation are shown in Fig. 9 for toroidal covariance matrices and Fig. 10 for slab covariance matrices. Concentrating first on the results employing toroidal covariance matrices, the correct fundamental domain scale is clearly picked out by both the profile and full likelihood codes, with the simply connected case strongly disfavoured at a likelihood ratio of  $p(d|L = 4H_0^{-1})/p(d|L = 7H_0^{-1}) \sim 10^{28}$ . Turning to the results derived using slab covariance matrices, we see that—as expected from the Kullback-Leibler divergence analysis of Sect. 3.1.2—the correct fundamental domain scale is also found using the slab profile likelihood. Although, as also expected, the peak is not quite as pronounced when using the wrong covariance matrix, the simply connected universe is still overwhelmingly disfavoured at a ratio of  $p(d|L = 4H_0^{-1})/p(d|L = 7H_0^{-1}) \sim 10^{11}$ .

The speed of the profile-likelihood analysis allows for the effects of changing the mode count, composition and noise level to be investigated. We repeat the toroidal test using intensity-only ( $I$ ) and full ( $IQU$ ) correlation matrices, retaining between 837 and 2170 modes at a time. For the smoothing scale employed in our tests, the 838th mode is the first to be dominated

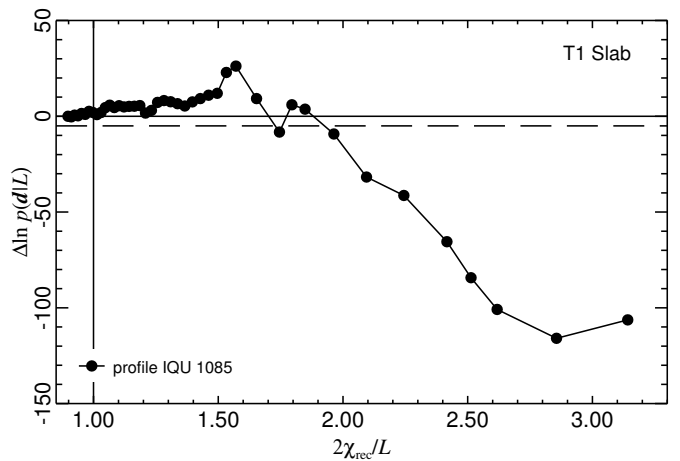


**Fig. 8.** The profile likelihood function for the fundamental domain scale of a slab topology derived from simulations of a simply-connected universe. The vertical line marks the position where  $\chi_{\text{rec}}$  is equal to the radius of the largest sphere that can be inscribed in the domain,  $\mathcal{R}_i = L/2$  (for slab spaces, the other two characteristic sizes are infinite).



**Fig. 9.** The likelihood function for the fundamental domain scale of a cubic torus derived from a simulation of a toroidal universe with  $L = 4H_0^{-1}$  or  $2\chi_{\text{rec}}/L = 1.5$ . The results from the profile-likelihood analysis (pink, clear circles) closely match those from the full marginalized likelihood (black, filled circles). Overlaid are additional profile likelihoods demonstrating the effects of changing the mode count and composition. In order of increasing constraining power, they utilize 837 *IQU* modes (grey, long dashed), 1085 *I* modes (blue, dotted), 1085 *IQU* modes (pink, clear circles), 2170 *IQU* modes (green, dot dashed) and finally 1085 noiseless *IQU* modes (purple, dashed). Adding low- $\ell$  (ideally low-noise) polarization greatly increases the constraining power of the data.

by polarization; runs using up to 837 *IQU* modes are therefore dominated by intensity information. The results of this investigation are contained in Fig. 9. The most striking conclusion is that the impact of adding temperature modes to the analysis is dwarfed by the impact of adding low- $\ell$  polarization information, even though the temperature modes are effectively noiseless. This conclusion is supported by the observation from Fig. 1 that the Kullback-Leibler divergence grows most rapidly at low  $\ell$ .



**Fig. 10.** The profile likelihood function for the fundamental domain scale of a slab derived from a simulation of a toroidal universe with  $L = 4H_0^{-1}$  or  $2\chi_{\text{rec}}/L = 1.5$ .

#### 4.4.2. Bianchi

The Bayesian analysis of Bianchi VII<sub>h</sub> models using temperature data is performed using the AniCosmo code, which has been extensively validated by McEwen et al. (2013), and was used to perform the Bianchi analysis of Planck Collaboration XXVI (2014). The maximum-likelihood template fitting method used to analyse polarization data is straightforward and has been validated on simulations, correctly recovering the amplitude of templates artificially embedded in simulated CMB observations.

## 5. Results

### 5.1. Topology

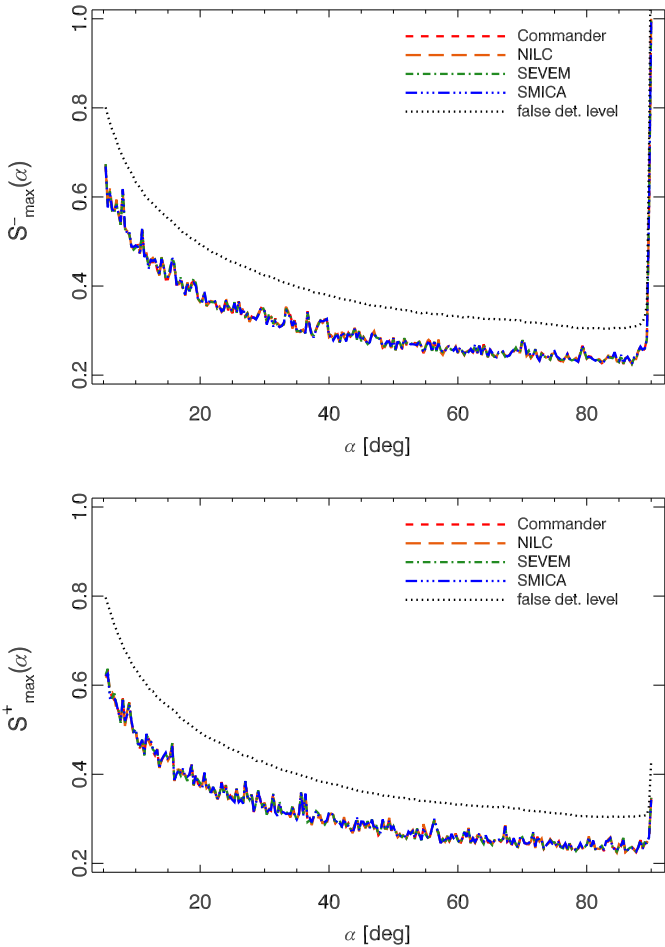
#### 5.1.1. Matched circles

We show the matched circle statistic for the CMB temperature and *E*-mode maps in Fig. 11 and 12, respectively. We do not find any statistically significant correlation of circle pairs in any map. Results for the temperature maps are consistent with the Planck Collaboration XXVI (2014) results. As seen in Fig. 6, the minimum radius at which the peaks expected for the matching statistic are larger than the false detection level for the polarization map is  $\alpha_{\text{min}} \approx 15^\circ$ . Thus, we can exclude, at the confidence level of 99%, any topology that predicts matching pairs of back-to-back circles larger than this radius, assuming that the relative orientation of the fundamental domain and mask allows its detection. This implies that in a flat universe described otherwise by the fiducial  $\Lambda$ CDM model, a 99% confidence-limit lower bound on the size of the fundamental domain is  $\mathcal{R}_i = L/2 \gtrsim \chi_{\text{rec}} \cos(\alpha_{\text{min}}) = 0.97\chi_{\text{rec}}$  or  $L \gtrsim 6.0H_0^{-1}$ . This is slightly stronger than the constraint obtained for the analysis of the 2013 *Planck* temperature maps i.e.,  $0.94\chi_{\text{rec}}$  (Planck Collaboration XXVI 2014). Note that the limits from polarization are at least as strong as those from temperature despite the considerably smaller amount of sky considered in the polarization analysis (40% compared to 74%).

#### 5.1.2. Likelihood

The results of applying the two likelihood codes to the *Planck* low- $\ell$  data are plotted in Fig. 13 for cubic tori and Fig. 14 for slabs. As with the null test, small-*L* topologies are strongly ruled



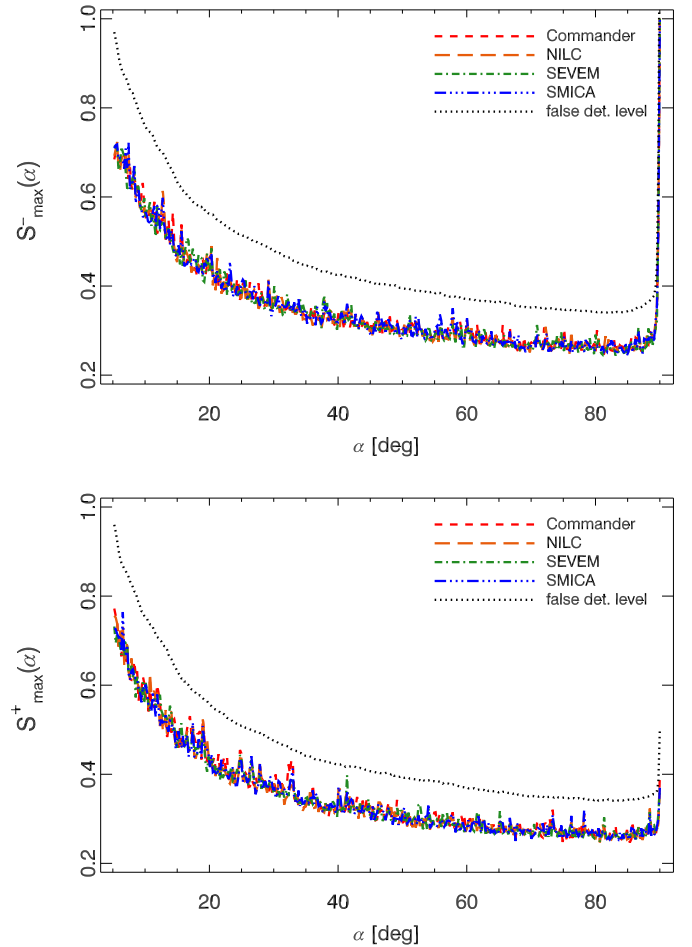


**Fig. 11.**  $S_{\max}^-$  (upper) and  $S_{\max}^+$  (lower) statistics as a function of circle radius  $\alpha$  for the *Planck* Commander (short-dashed red line), NILC (long dashed orange line), SEVEM (dot-dashed green line), and SMICA (three dot-dashed blue line) 2015 temperature maps. The dotted line shows the false detection level established such that fewer than 1 % out of 300 Monte Carlo simulations of the SMICA CMB temperature map, smoothed and masked in the same way as the data, would yield a false event. The peak at  $90^\circ$  corresponds to a match between two copies of the same circle of radius  $90^\circ$  centred around two antipodal points.

out, and the likelihoods are maximized at scales approaching or exceeding the horizon. In the marginalized case, we find that the maximum-likelihood fundamental domain scale of a cubic torus is  $L = 7H_0^{-1}$ , with scales of  $L \leq 6H_0^{-1}$  disfavoured at greater than  $3.2\sigma$ ; for slabs, we find the likelihood to be peaked at  $L = 6H_0^{-1}$  (just inside the last-scattering surface), though  $L = 7H_0^{-1}$  is allowed at  $1.9\sigma$ .

We have investigated the shape of the likelihood as a function of the slab orientation, and find that it is strongly peaked at an orientation such that the induced matched circles lie partially within the large polarization mask (retaining only 47 % of the sky). These orientations therefore do not benefit from the extra discriminatory power of the polarization and its correlation with temperature.

As noted when analyzing simulations of simply-connected models (Figs. 7–8), the profile likelihoods also show a mild rise around the horizon scale for the *Planck* data: this rise therefore cannot be interpreted as evidence for a multiply connected topol-



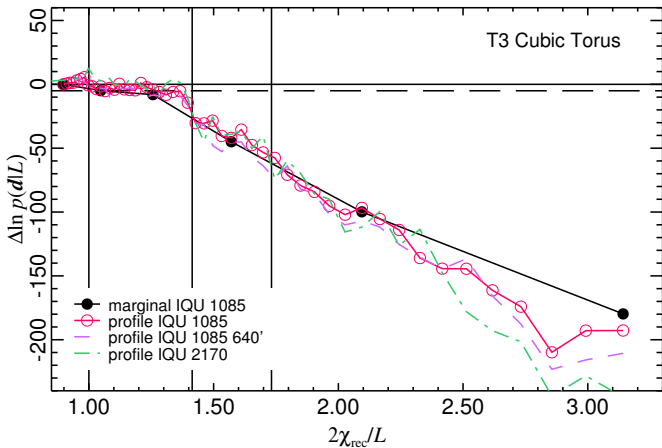
**Fig. 12.**  $S_{\max}^-$  (upper) and  $S_{\max}^+$  (lower) statistics as a function of circle radius  $\alpha$  for the *Planck* Commander (short-dashed red line), NILC (long dashed orange line), SEVEM (dot-dashed green line), and SMICA (three dot-dashed blue line) *E*-mode maps. The dotted line shows the false detection level established such that fewer than 1 % out of 300 Monte Carlo simulations of the SMICA CMB *E*-mode map, smoothed and masked in the same way as the data, would yield a false event. The peak at  $90^\circ$  corresponds to a match between two copies of the same circle of radius  $90^\circ$  centred around two antipodal points.

ogy. We found a similar effect with both profile and marginalized likelihoods in [Planck Collaboration XXVI \(2014\)](#) using simulations of simply connected universes and the *Planck* temperature data, though this peak was considerably more pronounced for the profile likelihood. Further investigation is required to determine whether this extra-horizon rise is still present in the marginalized likelihood with the present data.

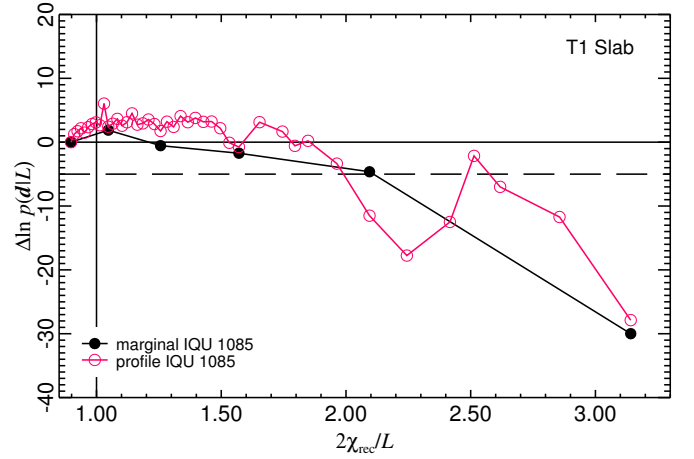
Though the small- $L$  topologies are strongly constrained by the data, we note that the dropoff in likelihood is not as sharp as that observed in our null test (or, indeed, the signal test) presented in Sect. 4.4.1. There are several reasons for this behaviour. The default smoothing scale used in the low- $\ell$  *Planck* dataset (440' Gaussian for intensity, no smoothing for polarization) means all of the first 1085 modes are in fact temperature-dominated. Exploiting once more the speed of the profile likelihood code, we therefore explore the dependence of the likelihoods on the number of modes and their composition in Fig. 13. We introduce polarization-dominated eigenmodes into the anal-

ysis in two ways: by simply doubling the number of modes and by retaining the mode count but applying additional smoothing (to bring the effective Gaussian smoothing FWHM to  $640'$ ). Though the constraints on small- $L$  topologies do become tighter when polarization-dominated modes are included, their impact remains weaker than in testing. This is because the noise in the low- $\ell$  data is significantly higher than in our tests—the typical diagonal covariance matrix element is  $\sigma_{lQU}^2 = 4.0 \mu\text{K}^2$ —and, finally, because we must use a sky cut, which can hide excess correlations. This strongly motivates repeating this analysis with the full multifrequency *Planck* polarization data when they become available.

Converting these likelihoods into Bayesian constraints on the size of the fundamental domain is not straightforward. Absent a proper prior giving an upper limit on the size of the fundamental domain, there is always infinite parameter volume available for ever-larger fundamental domains. Hence, these likelihood plots should be considered the full summary of the 2015 *Planck* data for these models. Nonetheless, it is often useful to consider a fall-off in the likelihood of  $\Delta \ln \mathcal{L} < -5$  as roughly equivalent to a  $3\sigma$ —99% confidence level—limit, the location of which we approximate by interpolating between calculated likelihood points. For the cubic T3 torus, we find that the marginal likelihoods of the combined temperature and polarization lowT,P data require that the length of an edge of the fundamental domain satisfies  $L > 6H_0^{-1}$  at this significance, or equivalently that the radius of the largest inscribed sphere in the fundamental domain is  $\mathcal{R}_i > 0.97\chi_{\text{rec}}$  (recall that  $\chi_{\text{rec}} \approx 3.1H_0^{-1}$  is the comoving distance to the last-scattering surface.) The profile likelihood gives the somewhat weaker limit,  $\mathcal{R}_i > 0.79\chi_{\text{rec}}$ . For the T1 slab, we have  $L > 3.5H_0^{-1}$  or  $\mathcal{R}_i > 0.56\chi_{\text{rec}}$ . Because the temperature data on the relevant scales are still dominated by the cosmological signal, and the polarization noise remains large, these results are only slightly stronger than those presented in [Planck Collaboration XXVI \(2014\)](#).



**Fig. 13.** The likelihood for the fundamental domain scale of a cubic torus derived from the *Planck* low- $\ell$  dataset. As in testing, the likelihoods calculated via profiling (pink, clear circles) and marginalization (black, solid circles) agree well. The impact of increasing the polarization content through additional smoothing (purple, dashed) or modes (green, dot dashed) is diminished compared to the test setting due to boosted noise.



**Fig. 14.** The likelihood for the fundamental domain scale of a slab topology derived from the *Planck* low- $\ell$  dataset via profiling (pink, clear circles) and marginalization (black, solid circles).

## 5.2. Bianchi

*Planck* temperature data are masked and analysed for evidence of a Bianchi VII<sub>h</sub> component, using the prior parameter ranges adopted in [McEwen et al. \(2013\)](#) and [Planck Collaboration XXVI \(2014\)](#). Cleaned temperature maps for each of the four component separation techniques are examined, where the mask defined for each technique is applied. The natural log-Bayes factors for the Bianchi models relative to their standard cosmological counterparts are shown in [Table 1](#). The Bayes factors are broadly consistent across the component separation methods. Most Bayes factors are similar to the values computed in [Planck Collaboration XXVI \(2014\)](#), with the exception of the analysis of the left-handed flat-decoupled-Bianchi model with SEVEM data, which has increased, but which is now more consistent with the other component separation methods.

For the phenomenological flat-decoupled-Bianchi model, evidence in support of a left-handed Bianchi template is again found ([Table 1](#)). The Bayes factors providing evidence for this model range between the values  $2.3 \pm 0.2$  and  $3.2 \pm 0.2$ , correspond to odds ratios of approximately 1:10 and 1:25, respectively (which on the Jeffreys scale are categorized as significant and strong, respectively; [Jeffreys 1961](#)). Recovered posterior distributions of the Bianchi parameters of this model for each component separation technique are shown in [Fig. 15a](#). The posterior distributions are consistent across component separation techniques, are similar to those recovered in [Planck Collaboration XXVI \(2014\)](#), and are reasonably well constrained (with the exception of the known  $\Omega_m^B - \Omega_\Lambda^B$  Bianchi parameter degeneracy ([Jaffe et al. 2006c](#); [Bridges et al. 2007](#))). Recall that the Bianchi densities ( $\Omega_m^B$  and  $\Omega_\Lambda^B$ ) are decoupled from the standard cosmology in the flat-decoupled-Bianchi model considered here and, as found previously, are inconsistent with standard estimates of the densities. The maximum-a-posteriori (MAP) and mean-posterior Bianchi parameter estimates for this model are given in [Table 2](#), while the corresponding MAP best-fit Bianchi temperature maps are shown in [Fig. 16](#). The best-fit maps for the left-handed flat-coupled-Bianchi model are consistent across component separation techniques and similar to the best-fit maps found in previous *Planck* ([Planck Collaboration XXVI 2014](#)) and WMAP (see, e.g., [Jaffe et al. 2005](#) and [McEwen et al. 2013](#)) temperature data.

**Table 1.** Natural log-Bayes factors of Bianchi models relative to equivalent  $\Lambda$ CDM model (positive favours Bianchi model).

Model	SMICA	SEVEM	NILC	Commander
Flat-decoupled-Bianchi (left-handed)	$2.5 \pm 0.1$	$3.1 \pm 0.1$	$2.3 \pm 0.2$	$3.2 \pm 0.2$
Flat-decoupled-Bianchi (right-handed)	$0.5 \pm 0.1$	$0.5 \pm 0.1$	$0.5 \pm 0.1$	$0.3 \pm 0.1$
Open-coupled-Bianchi (left-handed)	$0.2 \pm 0.1$	$0.0 \pm 0.1$	$0.0 \pm 0.1$	$0.0 \pm 0.1$
Open-coupled-Bianchi (right-handed)	$-0.3 \pm 0.1$	$-0.5 \pm 0.1$	$-0.2 \pm 0.1$	$-0.4 \pm 0.1$

For the physical open-coupled-Bianchi model where the Bianchi VII<sub>h</sub> model is coupled to the standard cosmology, there is again no evidence in support of a Bianchi contribution (Table 1). Recovered posterior distributions of the parameters of this model for each component separation technique are shown in Fig. 15b. The posterior distributions show some similarity across component separation techniques and with the distributions recovered in Planck Collaboration XXVI (2014). However, significant differences exist since the parameters are in general poorly constrained and the model is not favoured by the Bayesian evidence. The MAP and mean-posterior Bianchi parameter estimates for this model are given in Table 3, while the corresponding MAP best-fit Bianchi temperature maps are shown in Fig. 17. While the posterior distributions show differences between component separation techniques, the estimated parameters are consistent. Note that, although the mean parameter estimates have not changed markedly from Planck Collaboration XXVI (2014), the MAP estimates have indeed changed. Intriguingly, for each component separation technique, the MAP best-fit maps for the open-coupled-Bianchi model (Fig. 17), for which there is no evidence in support of a Bianchi contribution, show a similar but not identical morphology to the MAP best-fit maps for the flat-decoupled-Bianchi model (Fig. 16), for which the Bayesian evidence supports the inclusion of a Bianchi component. This was not the case in the previous *Planck* analysis (Planck Collaboration XXVI 2014, Fig. 22). A parameter combination is found for the physical open-coupled-Bianchi model that is broadly consistent with a standard open cosmology and that produces a Bianchi temperature map similar to the one found in the unphysical flat-decoupled-Bianchi model. This parameter combination lies on the known  $\Omega_m^B - \Omega_\Lambda^B$  Bianchi parameter degeneracy (Jaffe et al. 2006c; Bridges et al. 2007) and lies close but not directly on the well-known CMB geometric degeneracy (determined by an independent CMB analysis), since both the cosmological and Bianchi components are fitted simultaneously. It is important to stress that *Planck* temperature data do not favour such a model; however, neither is it possible to conclusively discount Bianchi VII<sub>h</sub> cosmologies in favour of  $\Lambda$ CDM cosmologies. An overall constraint on the vorticity of Bianchi VII<sub>h</sub> models of  $(\omega/H)_0 < 7.6 \times 10^{-10}$  (95% confidence level) is obtained from the analysis of the physical open-coupled-Bianchi model, which is consistent across all component separation techniques, as illustrated in Table 4.

To further constrain Bianchi VII<sub>h</sub> models using *Planck* polarization data we simulate Bianchi polarization maps, computed using the approach of Pontzen & Challinor (2007) and Pontzen (2009), and provided by Pontzen (2015) for the best-fit Bianchi parameters determined from the temperature analysis. *E*- and *B*-mode Bianchi maps for the best-fit flat-decoupled-Bianchi model and the open-coupled-Bianchi model are displayed in Fig. 18 and Fig. 19, respectively. As described in Sect. 4.3.3, we estimate the maximum-likelihood amplitude of

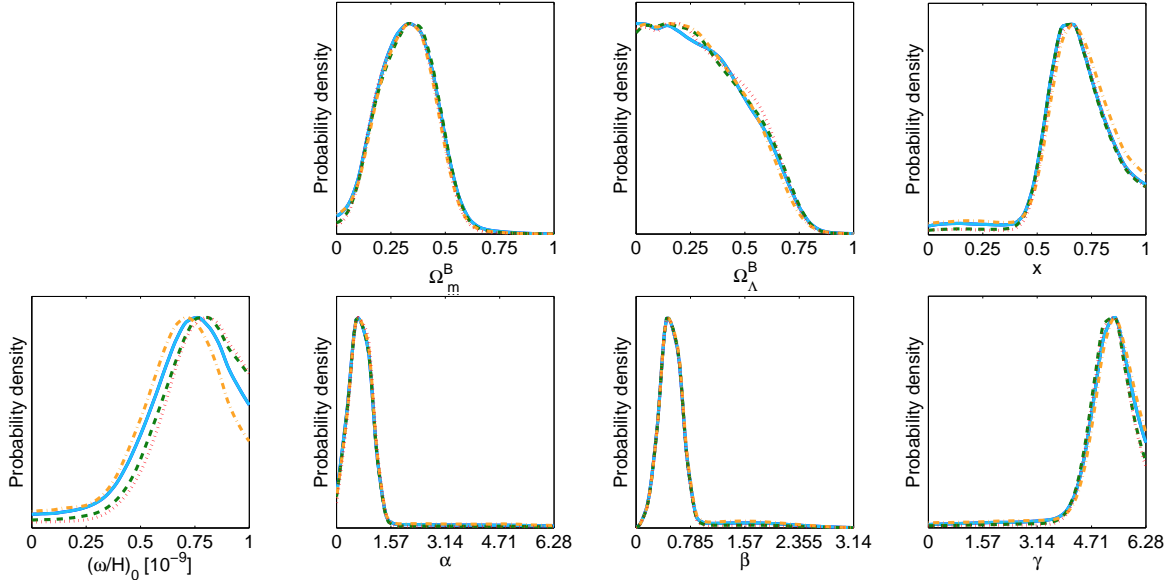
these Bianchi polarization templates in *Planck* polarization data, performing the analysis in the pixel space defined by the *Q* and *U* Stokes components, where noise and partial sky coverage can be handled effectively. The maximum-likelihood amplitudes estimated for each component separation technique are given in Table 5. The estimated amplitudes are close to zero and consistent across component separation techniques. The difference between the estimated amplitudes and zero is more likely due to small residual foreground contamination than a Bianchi component. Indeed, the amplitude estimates are at least 24 standard deviations from unity, the expected value for the best-fit Bianchi models determined from the temperature analysis. Both the best-fit flat-decoupled-Bianchi model and the open-decoupled-Bianchi model are thus strongly disfavoured by the *Planck* polarization data. This is not surprising since these models produce relatively large *E*- and *B*-mode contributions (see Fig. 18 and Fig. 19), as highlighted already by Pontzen & Challinor (2007). However, the full freedom of Bianchi models remains to be explored using temperature and polarization data simultaneously, for example through a complete Bayesian analysis, which is left to future work.

## 6. Discussion

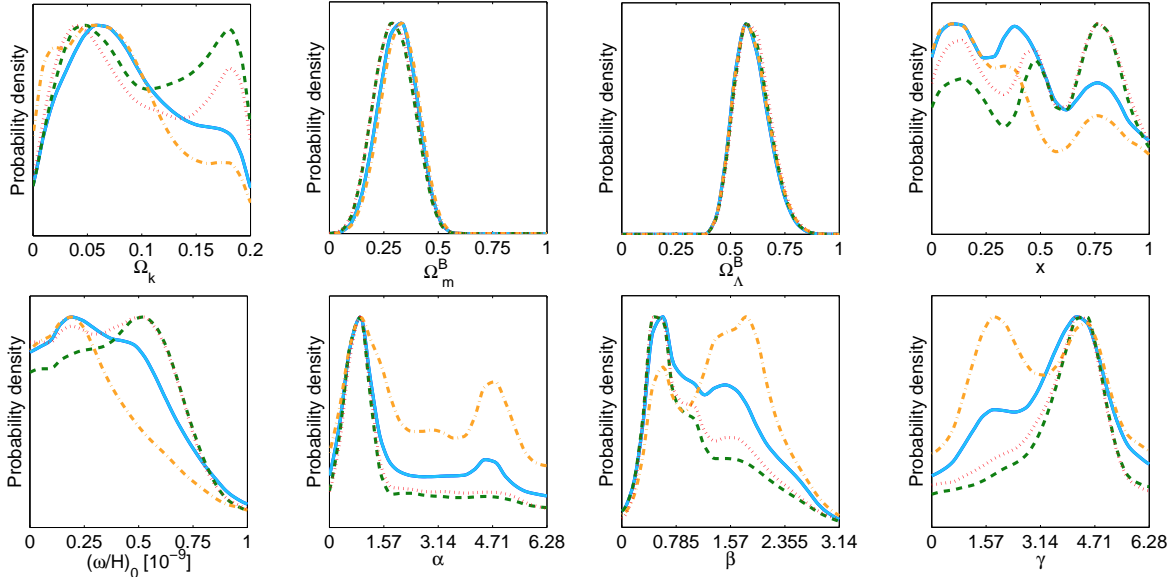
We have used *Planck* intensity and polarization data to evaluate specific departures from the large-scale isotropy of the Universe. Using both frequentist and Bayesian methods applied for the first time to polarization data, we find no evidence for a multi-connected topology with a scale less than roughly the distance to the last-scattering surface. Specifically, a frequentist search for antipodal matched circles on  $N_{\text{side}} = 512$  maps finds a lower bound on the size of the fundamental domain of  $0.97 \chi_{\text{rec}}$  from polarization data alone. Using Bayesian methods applied to low-resolution ( $N_{\text{side}} = 16$ ) maps of both temperature and polarization, we also find a lower limit of  $0.97 \chi_{\text{rec}}$  for the T3 cubic torus (for the T1 slab, the limit is  $0.56 \chi_{\text{rec}}$ ). These results are both consistent and complementary, giving coincidentally identical limits but with very different statistical foundations and data selections. The addition of polarization data at current levels of accuracy does not significantly improve the limits from intensity alone, but we have found that the polarization sensitivity of the full set of *Planck* detectors should give quantitative improvements in the limits, decreasing the likelihood of fundamental domains with scales smaller than the distance to the last-scattering surface by many orders of magnitude.

We also find no evidence for a Bianchi VII<sub>h</sub> model which departs from global isotropy via the presence of both rotation and shear. Although the large-scale temperature pattern measured by *Planck* has some similar features to that induced by focusing in a Bianchi VII<sub>h</sub> universe, it requires unphysical parameters. Fixed to those parameters, we have further shown that the polarization pattern induced by such models is strongly disfavoured by the *Planck* data.





(a) Flat-decoupled-Bianchi model.



(b) Open-coupled-Bianchi model.

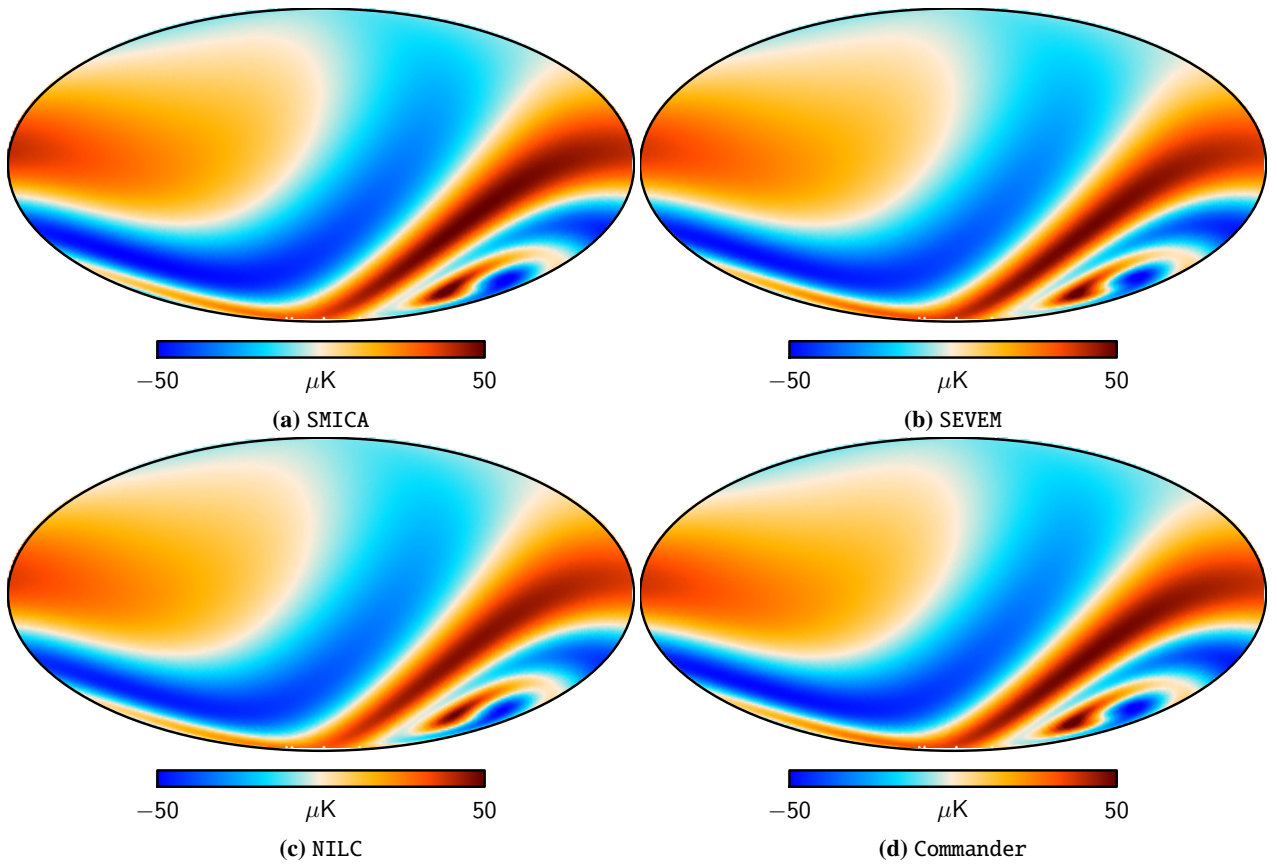
**Fig. 15.** Posterior distributions of Bianchi parameters recovered from *Planck* SMICA (solid blue curves), SEVEM (dashed green curves), NILC (dot-dashed yellow curves), and Commander (dotted red curves) component-separated data for left-handed models. *Planck* data provide evidence in support of a Bianchi component in the phenomenological flat-decoupled-Bianchi model (panel a) but not in the physical open-coupled-Bianchi model (panel b). Significant differences exist between the posterior distributions shown in panel b for each component separation method; this model is not favoured by data and parameters are in general poorly constrained.

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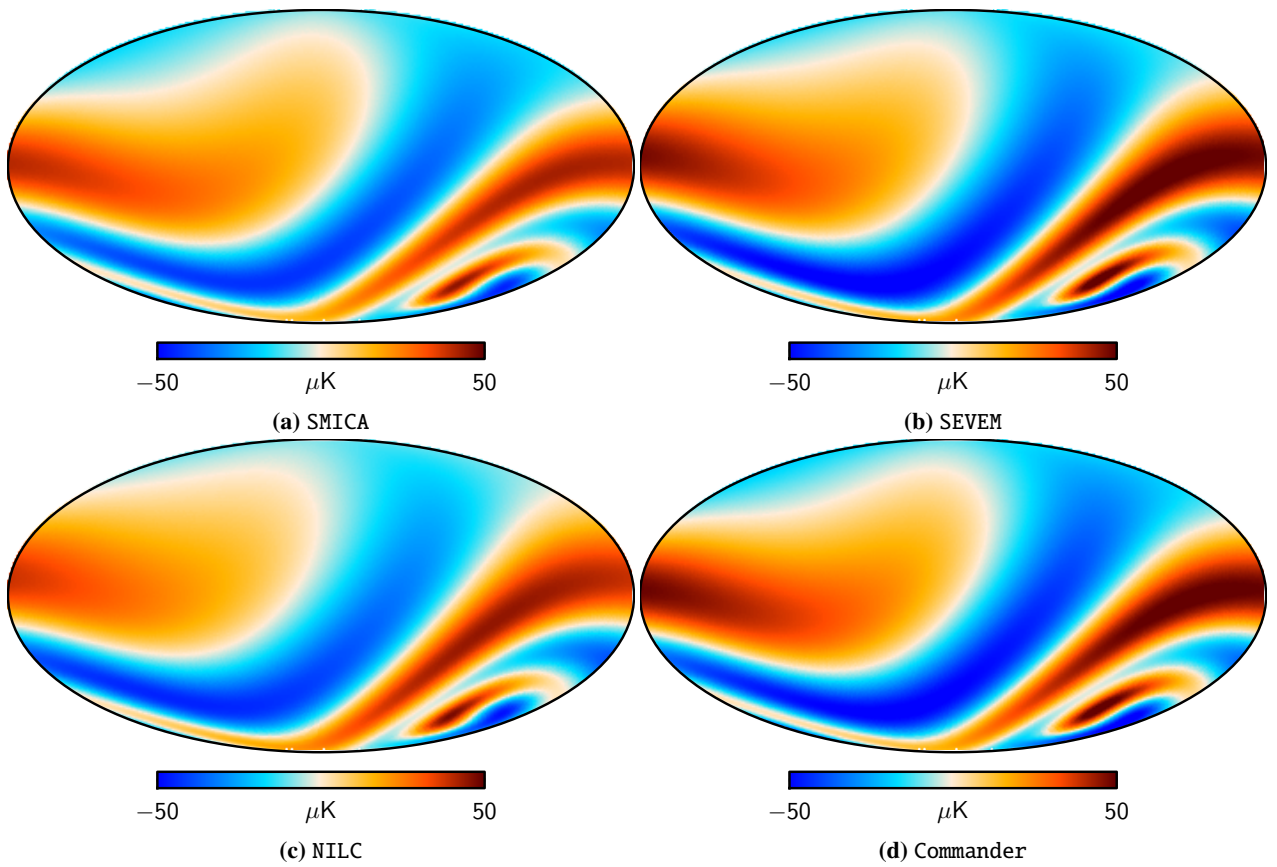
of the University of Geneva, as well as the Carver IBM iDataPlex, Hopper Cray XE6 and the Edison Cray XC30 at NERSC, and on the GPC supercomputer at the SciNet HPC Consortium. SciNet is funded by: the Canada Foundation for Innovation under the auspices of Compute Canada; the Government of Ontario; Ontario Research Fund - Research Excellence; and the University of Toronto.

## References

Aurich, R., The Fluctuations of the Cosmic Microwave Background for a Compact Hyperbolic Universe. 1999, *ApJ*, 524, 497, [arXiv:astro-ph/9903032](https://arxiv.org/abs/astro-ph/9903032)  
 Aurich, R., Janzer, H. S., Lustig, S., & Steiner, F., Do we live in a 'small universe'? 2008, *Classical and Quantum Gravity*, 25, 125006, [arXiv:0708.1420](https://arxiv.org/abs/0708.1420)



**Fig. 16.** Best-fit temperature maps for the left-handed flat-decoupled-Bianchi model.



**Fig. 17.** Best-fit temperature maps for the left-handed open-coupled-Bianchi model.

**Table 2.** Parameters recovered for the left-handed flat-decoupled-Bianchi model. *Planck* data favour the inclusion of a Bianchi component in this phenomenological model.

Bianchi parameter	SMICA		SEVEM		NILC		Commander	
	MAP	Mean	MAP	Mean	MAP	Mean	MAP	Mean
$\Omega_m^B$	0.34	$0.32 \pm 0.12$	0.34	$0.33 \pm 0.12$	0.42	$0.32 \pm 0.12$	0.23	$0.32 \pm 0.12$
$\Omega_\Lambda^B$	0.30	$0.31 \pm 0.20$	0.26	$0.31 \pm 0.20$	0.12	$0.30 \pm 0.19$	0.48	$0.31 \pm 0.20$
$x$	0.66	$0.67 \pm 0.17$	0.63	$0.68 \pm 0.15$	0.63	$0.68 \pm 0.18$	0.74	$0.69 \pm 0.15$
$(\omega/H)_0 \times 10^{10}$	8.6	$6.9 \pm 2.0$	9.2	$7.3 \pm 1.8$	8.2	$6.6 \pm 2.0$	8.2	$7.4 \pm 1.7$
$\alpha$	$39^\circ 4$	$52^\circ 7 \pm 49^\circ 7$	$39^\circ 3$	$48^\circ 0 \pm 39^\circ 5$	$41^\circ 1$	$57^\circ 1 \pm 56^\circ 0$	$41^\circ 9$	$47^\circ 8 \pm 36^\circ 1$
$\beta$	$27^\circ 8$	$34^\circ 4 \pm 21^\circ 1$	$28^\circ 2$	$32^\circ 2 \pm 16^\circ 8$	$28^\circ 9$	$35^\circ 8 \pm 22^\circ 8$	$27^\circ 6$	$31^\circ 5 \pm 15^\circ 4$
$\gamma$	$302^\circ 1$	$291^\circ 5 \pm 53^\circ 4$	$309^\circ 2$	$293^\circ 3 \pm 44^\circ 7$	$297^\circ 6$	$291^\circ 2 \pm 59^\circ 0$	$303^\circ 5$	$295^\circ 3 \pm 41^\circ 3$

**Table 3.** Parameters recovered for the left-handed open-coupled-Bianchi model. *Planck* data do not favour the inclusion of a Bianchi component in this model and some parameters are not well constrained.

Bianchi parameter	SMICA		SEVEM		NILC		Commander	
	MAP	Mean	MAP	Mean	MAP	Mean	MAP	Mean
$\Omega_k$	0.20	$0.09 \pm 0.05$	0.18	$0.10 \pm 0.06$	0.16	$0.08 \pm 0.05$	0.19	$0.10 \pm 0.06$
$\Omega_m^B$	0.23	$0.31 \pm 0.07$	0.20	$0.30 \pm 0.07$	0.16	$0.32 \pm 0.07$	0.24	$0.30 \pm 0.08$
$\Omega_\Lambda^B$	0.57	$0.60 \pm 0.07$	0.62	$0.60 \pm 0.07$	0.67	$0.60 \pm 0.07$	0.57	$0.61 \pm 0.07$
$x$	0.74	$0.45 \pm 0.28$	0.87	$0.51 \pm 0.28$	0.93	$0.42 \pm 0.29$	0.78	$0.49 \pm 0.28$
$(\omega/H)_0 \times 10^{10}$	6.2	$3.8 \pm 2.4$	6.5	$4.1 \pm 2.3$	5.4	$3.3 \pm 2.3$	6.8	$3.9 \pm 2.3$
$\alpha$	$39^\circ 0$	$136^\circ 8 \pm 100^\circ 6$	$41^\circ 0$	$116^\circ 1 \pm 96^\circ 6$	$43^\circ 4$	$161^\circ 2 \pm 101^\circ 7$	$40^\circ 1$	$121^\circ 5 \pm 96^\circ 8$
$\beta$	$27^\circ 6$	$72^\circ 3 \pm 38^\circ 8$	$29^\circ 4$	$63^\circ 2 \pm 39^\circ 1$	$28^\circ 4$	$83^\circ 3 \pm 37^\circ 4$	$28^\circ 4$	$66^\circ 6 \pm 38^\circ 8$
$\gamma$	$264^\circ 7$	$194^\circ 1 \pm 87^\circ 3$	$272^\circ 0$	$210^\circ 2 \pm 80^\circ 1$	$289^\circ 7$	$177^\circ 7 \pm 90^\circ 9$	$262^\circ 6$	$201^\circ 5 \pm 82^\circ 2$

**Table 4.** Upper-bounds on vorticity  $(\omega/H)_0$  (at 95 % confidence level).

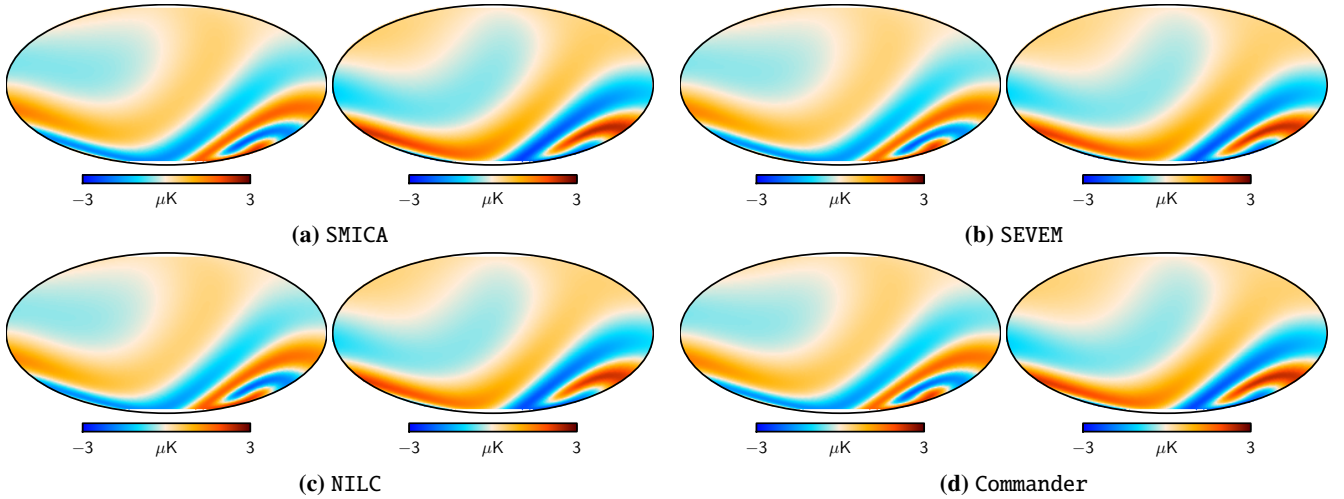
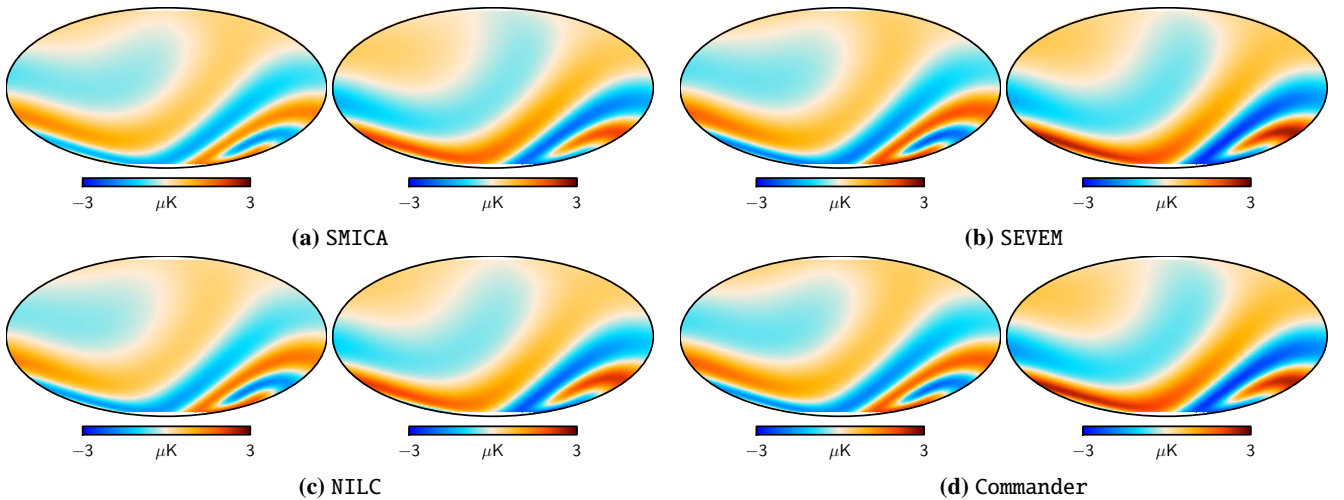
Model	SMICA	SEVEM	NILC	Commander
Open-coupled-Bianchi (left-handed)	$7.6 \times 10^{-10}$	$7.6 \times 10^{-10}$	$7.6 \times 10^{-10}$	$7.6 \times 10^{-10}$
Open-coupled-Bianchi (right-handed)	$7.6 \times 10^{-10}$	$7.6 \times 10^{-10}$	$7.6 \times 10^{-10}$	$7.1 \times 10^{-10}$

- Aurich, R. & Lustig, S., A search for cosmic topology in the final WMAP data. 2013, MNRAS, 433, 2517, [arXiv:1303.4226](#)
- Aurich, R. & Lustig, S., The Hantzsche-Wendt manifold in cosmic topology. 2014, Classical and Quantum Gravity, 31, 165009
- Aurich, R., Lustig, S., & Steiner, F., CMB anisotropy of the Poincaré dodecahedron. 2005, Classical and Quantum Gravity, 22, 2061, [arXiv:astro-ph/0412569](#)
- Aurich, R., Lustig, S., & Steiner, F., The circles-in-the-sky signature for three spherical universes. 2006, MNRAS, 369, 240, [arXiv:astro-ph/0510847](#)
- Aurich, R., Lustig, S., Steiner, F., & Then, H., Hyperbolic universes with a horned topology and the cosmic microwave background anisotropy. 2004, Classical and Quantum Gravity, 21, 4901, [arXiv:astro-ph/0403597](#)
- Barrow, J. D., General relativistic cosmological models and the cosmic microwave background radiation. 1986, Canadian Journal of Physics, 64, 152
- Barrow, J. D., Juszkiewicz, R., & Sonoda, D. H., Universal rotation – how large can it be? 1985, MNRAS, 213, 917
- Bennett, C. L., Banday, A. J., Gorski, K. M., et al., Four-Year COBE DMR Cosmic Microwave Background Observations: Maps and Basic Results. 1996, ApJ, 464, L1, [arXiv:astro-ph/9601067](#)
- Bielewicz, P. & Banday, A. J., Constraints on the topology of the Universe derived from the 7-yr WMAP data. 2011, MNRAS, 412, 2104, [arXiv:1012.3549](#)
- Bielewicz, P., Banday, A. J., & Górski, K. M., Constraining the topology of the Universe using the polarized cosmic microwave background maps. 2012, MNRAS, 421, 1064, [arXiv:1111.6046](#)
- Bielewicz, P. & Riazuelo, A., The study of topology of the Universe using multipole vectors. 2009, MNRAS, 396, 609, [arXiv:0804.2437](#)
- Bond, J. R. & Efstathiou, G., The statistics of cosmic background radiation fluctuations. 1987, MNRAS, 226, 655
- Bond, J. R., Jaffe, A. H., & Knox, L. E., Estimating the power spectrum of the cosmic microwave background. 1998, Phys. Rev. D, 57, 2117
- Bond, J. R., Pogosyan, D., & Souradeep, T., Computing CMB anisotropy in compact hyperbolic spaces. 1998, Classical and Quantum Gravity, 15, 2671, [arXiv:astro-ph/9804041](#)
- Bond, J. R., Pogosyan, D., & Souradeep, T., CMB anisotropy in compact hyperbolic universes. II. COBE maps and limits. 2000, Phys. Rev. D, 62, 043006, [arXiv:astro-ph/9912144](#)
- Bridges, M., McEwen, J. D., Cruz, M., et al., Bianchi VII<sub>h</sub> models and the cold spot texture. 2008, MNRAS, 390, 1372, [arXiv:0712.1789](#)
- Bridges, M., McEwen, J. D., Lasenby, A. N., & Hobson, M. P., Markov chain Monte Carlo analysis of Bianchi VII<sub>h</sub> models. 2007, MNRAS, 377, 1473, [arXiv:astro-ph/0605325](#)
- Bunn, E. F., Ferreira, P. G., & Silk, J., How anisotropic is our Universe? 1996, Phys. Lett., 77, 2883, [arXiv:astro-ph/9605123](#)
- Caillerie, S., Lachièze-Rey, M., Luminet, J.-P., et al., A new analysis of the Poincaré dodecahedral space model. 2007, A&A, 476, 691
- Cayón, L., Banday, A. J., Jaffe, T., et al., No Higher Criticism of the Bianchi-corrected Wilkinson Microwave Anisotropy Probe data. 2006, MNRAS, 369, 598, [arXiv:astro-ph/0602023](#)
- Collins, C. B. & Hawking, S. W., The rotation and distortion of the Universe. 1973, MNRAS, 162, 307
- Cornish, N. J., Spergel, D. N., & Starkman, G. D., Circles in the sky: finding topology with the microwave background radiation. 1998, Classical and Quantum Gravity, 15, 2657, [arXiv:gr-qc/9602039](#)
- Cornish, N. J., Spergel, D. N., Starkman, G. D., & Komatsu, E., Constraining the Topology of the Universe. 2004, Phys. Rev. Lett., 92, 201302, [arXiv:astro-ph/0310233](#)
- de Oliveira-Costa, A. & Smoot, G. F., Constraints on the Topology of the Universe from the 2 Year COBE Data. 1995, ApJ, 448, 477, [arXiv:astro-ph/9412003](#)



**Table 5.** Maximum-likelihood amplitude estimates  $\lambda^{\text{ML}}$  and  $1\sigma$  errors computed using polarization data.

Model	SMICA	SEVEM	NILC	Commander
Flat-decoupled-Bianchi (left-handed)	$-0.10 \pm 0.04$	$-0.10 \pm 0.04$	$-0.12 \pm 0.04$	$-0.11 \pm 0.04$
Open-coupled-Bianchi (left-handed)	$-0.11 \pm 0.04$	$-0.09 \pm 0.04$	$-0.11 \pm 0.04$	$-0.09 \pm 0.04$

**Fig. 18.** Polarization maps for the best-fit left-handed flat-decoupled-Bianchi model fitted to temperature data. In each panel  $E$ - (left) and  $B$ -mode (right) maps are shown. These polarization maps are simulated using the approach of Pontzen & Challinor (2007) and Pontzen (2009), and provided by Pontzen (2015).**Fig. 19.** Polarization maps for the best-fit left-handed open-coupled-Bianchi model fitted to temperature data. In each panel  $E$ - (left) and  $B$ -mode (right) maps are shown. These polarization maps are simulated using the approach of Pontzen & Challinor (2007) and Pontzen (2009), and provided by Pontzen (2015).

Dineen, P., Rocha, G., & Coles, P., Non-random phases in non-trivial topologies. 2005, MNRAS, 358, 1285

Fagundes, H. V. & Wichoski, U. F., A search for QSOs to fit a cosmological model with flat, closed spatial sections. 1987, Nature, 322, L5

Fang, L.-Z. & Sato, H., Is the periodicity in the distribution of quasar redshifts an evidence of multiply connected Universe? 1983, Communications in Theoretical Physics, 2, 1055

Feroz, F. & Hobson, M. P., Multimodal nested sampling: an efficient and robust alternative to Markov Chain Monte Carlo methods for astronomical data analyses. 2008, MNRAS, 384, 449, [arXiv:0704.3704](#)

Feroz, F., Hobson, M. P., & Bridges, M., MultiNest: an efficient and robust Bayesian inference tool for cosmology and particle physics. 2009, MNRAS, 398, 1601, [arXiv:0809.3437](#)

Feroz, F., Hobson, M. P., Cameron, E., & Pettitt, A. N., Importance Nested Sampling and the MultiNest Algorithm. 2013, ArXiv e-prints, [arXiv:1306.2144](#)

Fujii, H. & Yoshii, Y., An improved cosmic crystallography method to detect holonomies in flat spaces. 2011, A&A, 529, A121, [arXiv:1103.1466](#)

Ghosh, T., Hajian, A., & Souradeep, T., Unveiling hidden patterns in CMB anisotropy maps. 2007, Phys. Rev. D, 75, 083007, [arXiv:astro-ph/0604279](#)

Jaffe, A. H. & Starkman, G. 2015, in prep.

Jaffe, T. R., Banday, A. J., Eriksen, H. K., Górski, K. M., & Hansen, F. K., Evidence of Vorticity and Shear at Large Angular Scales in the WMAP Data: a Violation of Cosmological Isotropy? 2005, ApJ, 629, L1, [arXiv:astro-ph/0503213](#)

- Jaffe, T. R., Banday, A. J., Eriksen, H. K., Górski, K. M., & Hansen, F. K., Bianchi type VII<sub>h</sub> models and the WMAP 3-year data. 2006a, *A&A*, 460, 393, [arXiv:astro-ph/0606046](#)
- Jaffe, T. R., Banday, A. J., Eriksen, H. K., Górski, K. M., & Hansen, F. K., Fast and Efficient Template Fitting of Deterministic Anisotropic Cosmological Models Applied to WMAP Data. 2006b, *ApJ*, 643, 616, [arXiv:astro-ph/0603844](#)
- Jaffe, T. R., Hervik, S., Banday, A. J., & Górski, K. M., On the Viability of Bianchi Type VII<sub>h</sub> Models with Dark Energy. 2006c, *ApJ*, 644, 701, [arXiv:astro-ph/0512433](#)
- Jarosik, N., Bennett, C. L., Dunkley, J., et al., Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Sky Maps, Systematic Errors, and Basic Results. 2011, *ApJS*, 192, 14, [arXiv:1001.4744](#)
- Jeffreys, H. 1961, *Theory of probability*, 3rd edn. (Oxford: Oxford University Press)
- Key, J. S., Cornish, N. J., Spergel, D. N., & Starkman, G. D., Extending the WMAP bound on the size of the Universe. 2007, *Phys. Rev. D*, 75, 084034, [arXiv:astro-ph/0604616](#)
- Kim, J., How to make a clean separation between CMB E and B modes with proper foreground masking. 2011, *A&A*, 531, A32
- Kogut, A., Hinshaw, G., & Banday, A. J., Limits to global rotation and shear from the COBE-DMR four-year sky maps. 1997, *Phys. Rev. D*, 55, 1901, [arXiv:astro-ph/9701090](#)
- Kunz, M., Aghanim, N., Cayon, L., et al., Constraining topology in harmonic space. 2006, *Phys. Rev. D*, D73, 023511, [arXiv:astro-ph/0510164](#)
- Kunz, M., Aghanim, N., Riazuelo, A., & Forni, O., Detectability of nontrivial topologies. 2008, *Phys. Rev. D*, 77, 23525
- Lachieze-Rey, M. & Luminet, J., Cosmic topology. 1995, *Physics Reports*, 254, 135
- Land, K. & Magueijo, J., Template fitting and the large-angle cosmic microwave background anomalies. 2006, *MNRAS*, 367, 1714, [arXiv:astro-ph/0509752](#)
- Lehoucq, R., Lachieze-Rey, M., & Luminet, J. P., Cosmic crystallography. 1996, *A&A*, 313, 339, [arXiv:gr-qc/9604050](#)
- Levin, J., Topology and the cosmic microwave background. 2002, *Physics Reports*, 365, 251
- Levin, J., Scannapieco, E., & Silk, J., Is the Universe infinite or is it just really big? 1998, *Nature*, 58, 103516
- Lew, B. & Roukema, B., A test of the Poincaré dodecahedral space topology hypothesis with the WMAP CMB data. 2008, *A&A*, 482, 747, [arXiv:0801.1358](#)
- Luminet, J.-P., Weeks, J. R., Riazuelo, A., Lehoucq, R., & Uzan, J.-P., Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background. 2003, *Nature*, 425, 593
- McEwen, J. D., Hobson, M. P., Lasenby, A. N., & Mortlock, D. J., Non-Gaussianity detections in the Bianchi VII<sub>h</sub> corrected WMAP 1-year data made with directional spherical wavelets. 2006, *MNRAS*, 369, [arXiv:astro-ph/0510349](#)
- McEwen, J. D., Josset, T., Feeney, S. M., Peiris, H. V., & Lasenby, A. N., Bayesian evidence of anisotropic cosmologies: Bianchi VII<sub>h</sub> and WMAP. 2013, *MNRAS*, 436, 3680, [arXiv:1303.3409](#)
- Niarchou, A. & Jaffe, A. H., Imprints of Spherical Nontrivial Topologies on the Cosmic Microwave Background. 2007, *Phys. Rev. Lett.*, 99, 81302
- Niarchou, A., Jaffe, A. H., & Pogosian, L., Large-scale power in the CMB and new physics: An analysis using Bayesian model comparison. 2004, *Phys. Rev. D*, 69, 063515, [arXiv:astro-ph/0308461](#)
- Phillips, N. G. & Kogut, A., Constraints on the Topology of the Universe from the Wilkinson Microwave Anisotropy Probe First-Year Sky Maps. 2006, *ApJ*, 645, 820, [arXiv:astro-ph/0404400](#)
- Planck Collaboration ES. 2015, The Explanatory Supplement to the Planck 2015 results, [http://wiki.cosmos.esa.int/planckpla/index.php/Main\\_Page](http://wiki.cosmos.esa.int/planckpla/index.php/Main_Page) (ESA)
- Planck Collaboration XXVI, *Planck* 2013 results. XXVI. Background geometry and topology of the Universe. 2014, *A&A*, 571, A26, [arXiv:1303.5086](#)
- Planck Collaboration I, *Planck* 2015 results. I. Overview of products and results. 2015, in preparation
- Planck Collaboration II, *Planck* 2015 results. II. Low Frequency Instrument data processing. 2015, in preparation
- Planck Collaboration III, *Planck* 2015 results. III. LFI systematic uncertainties. 2015, in preparation
- Planck Collaboration IV, *Planck* 2015 results. IV. LFI beams and window functions. 2015, in preparation
- Planck Collaboration V, *Planck* 2015 results. V. LFI calibration. 2015, in preparation
- Planck Collaboration VI, *Planck* 2015 results. VI. LFI maps. 2015, in preparation
- Planck Collaboration VII, *Planck* 2015 results. VII. High Frequency Instrument data processing: Time-ordered information and beam processing. 2015, in preparation
- Planck Collaboration VIII, *Planck* 2015 results. VIII. High Frequency Instrument data processing: Calibration and maps. 2015, in preparation
- Planck Collaboration IX, *Planck* 2015 results. IX. Diffuse component separation: CMB maps. 2015, in preparation
- Planck Collaboration X, *Planck* 2015 results. X. Diffuse component separation: Foreground maps. 2015, in preparation
- Planck Collaboration XI, *Planck* 2015 results. XI. CMB power spectra, likelihood, and consistency of cosmological parameters. 2015, in preparation
- Planck Collaboration XII, *Planck* 2015 results. XII. Simulations. 2015, in preparation
- Planck Collaboration XIII, *Planck* 2015 results. XIII. Cosmological parameters. 2015, in preparation
- Planck Collaboration XIV, *Planck* 2015 results. XIV. Dark energy and modified gravity. 2015, in preparation
- Planck Collaboration XV, *Planck* 2015 results. XV. Gravitational lensing. 2015, in preparation
- Planck Collaboration XVI, *Planck* 2015 results. XVI. Isotropy and statistics of the CMB. 2015, in preparation
- Planck Collaboration XVII, *Planck* 2015 results. XVII. Constraints on primordial non-Gaussianity. 2015, in preparation
- Planck Collaboration XVIII, *Planck* 2015 results. XVIII. Background geometry and topology of the Universe. 2015, in preparation
- Planck Collaboration XIX, *Planck* 2015 results. XIX. Constraints on primordial magnetic fields. 2015, in preparation
- Planck Collaboration XX, *Planck* 2015 results. XX. Constraints on inflation. 2015, in preparation
- Planck Collaboration XXI, *Planck* 2015 results. XXI. The integrated Sachs-Wolfe effect. 2015, in preparation
- Planck Collaboration XXII, *Planck* 2015 results. XXII. A map of the thermal Sunyaev-Zeldovich effect. 2015, in preparation
- Planck Collaboration XXIII, *Planck* 2015 results. XXIII. Thermal Sunyaev-Zeldovich effect–cosmic infrared background correlation. 2015, in preparation
- Planck Collaboration XXIV, *Planck* 2015 results. XXIV. Cosmology from Sunyaev-Zeldovich cluster counts. 2015, in preparation
- Planck Collaboration XXV, *Planck* 2015 results. XXV. Diffuse, low-frequency Galactic foregrounds. 2015, in preparation
- Planck Collaboration XXVI, *Planck* 2015 results. XXVI. The Second Planck Catalogue of Compact Sources. 2015, in preparation
- Planck Collaboration XXVII, *Planck* 2015 results. XXVII. The Second Planck Catalogue of Sunyaev-Zeldovich Sources. 2015, in preparation
- Planck Collaboration XXVIII, *Planck* 2015 results. XXVIII. The Planck Catalogue of Galactic Cold Clumps. 2015, in preparation
- Pontzen, A., Rogues' gallery: The full freedom of the Bianchi CMB anomalies. 2009, *Phys. Rev. D*, 79, 103518, [arXiv:0901.2122](#)
- Pontzen, A. 2015, Private Communication
- Pontzen, A. & Challinor, A., Bianchi model CMB polarization and its implications for CMB anomalies. 2007, *MNRAS*, 380, 1387, [arXiv:0706.2075](#)
- Pontzen, A. & Challinor, A., Linearization of homogeneous, nearly-isotropic cosmological models. 2011, *Classical and Quantum Gravity*, 28, 185007, [arXiv:1009.3935](#)
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, *Numerical Recipes in C (2nd Ed.)*: The Art of Scientific Computing (New York, NY, USA: Cambridge University Press)
- Riazuelo, A., Uzan, J.-P., Lehoucq, R., & Weeks, J., Simulating cosmic microwave background maps in multiconnected spaces. 2004a, *Phys. Rev. D*, 69, 103514, [arXiv:astro-ph/0212223](#)
- Riazuelo, A., Weeks, J., Uzan, J.-P., Lehoucq, R., & Luminet, J.-P., Cosmic microwave background anisotropies in multiconnected flat spaces. 2004b, *Phys. Rev. D*, 69, 103518, [arXiv:astro-ph/0311314](#)
- Rocha, G., Cayón, L., Bowen, R., et al., Topology of the Universe from COBE-DMR - a wavelet approach. 2004, *MNRAS*, 351, 769
- Roukema, B. F., On Determining the Topology of the Observable Universe via three-dimensional Quasar Positions. 1996, *MNRAS*, 283, 1147, [arXiv:astro-ph/9603052](#)
- Roukema, B. F., A counterexample to claimed COBE constraints on compact toroidal universe models. 2000a, *Classical and Quantum Gravity*, 17, 3951, [arXiv:astro-ph/0007140](#)
- Roukema, B. F., COBE and global topology: an example of the application of the identified circles principle. 2000b, *MNRAS*, 312, 712, [arXiv:astro-ph/9910272](#)
- Roukema, B. F., Buliński, Z., & Gaudin, N. E., Poincaré dodecahedral space parameter estimates. 2008, *A&A*, 492, 657, [arXiv:0807.4260](#)
- Seljak, U. & Zaldarriaga, M., A Line-of-Sight Integration Approach to Cosmic Microwave Background Anisotropies. 1996, *ApJ*, 469, 437
- Skilling, J. 2004, in *American Institute of Physics Conference Series*, Vol. 735, American Institute of Physics Conference Series, ed. R. Fischer, R. Preuss, & U. V. Toussaint, 395–405

- Sokolov, D. D. & Shvartsman, V. F., An estimate of the size of the universe from a topological point of view. 1974, Soviet Journal of Experimental and Theoretical Physics, 39, 196
- Sokolov, I. Y., Topologically nontrivial nature of the universe in connection with the anisotropy of the background radiation. 1993, Soviet Journal of Experimental and Theoretical Physics Letters, 57, 617
- Starobinskiĭ, A. A., New restrictions on spatial topology of the universe from microwave background temperature fluctuations. 1993, Soviet Journal of Experimental and Theoretical Physics Letters, 57, 622, [arXiv:gr-qc/9305019](https://arxiv.org/abs/gr-qc/9305019)
- Stevens, D., Scott, D., & Silk, J., Microwave background anisotropy in a toroidal universe. 1993, Phys. Rev. Lett., 71, 20
- Weatherley, S. J., Warren, S. J., Croom, S. M., et al., Ghosts of the Milky Way: a search for topology in new quasar catalogues. 2003, Nature, 342, L9
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