DEMOGRAPHY & VITAL STATISTICS

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Content	**Page No.**

Population Statistics
or
Vital Statistics

What is vital statistics? [C.U]

Ans:→ Def 1. The term 'vital/population statistics' signifies the data or the techniques used in the analysis of data relating to the vital events occurring in a given community.

Def 2. Vital statistics are, conventionally, numerical records of marriage, births, sickness, migration, divorce and deaths by which the health and growth of a community may be studied.

What is Demography?

Ans:→ Demography = Demos + graphy

Demos means life and graphy means description.

The study where description of human life is discussed in details, can be considered as Demography.

Demography is concerned with the growth, development, and movement of human populations as aggregates. Its raw material ranges from the statistics of heights and weights or of blood pressure in men, or to the distribution of the rents they pay for their housing accommodation or to the classes of education they give their children.

What are the vital events? [C.U]

Ans:→ Vital events are the events related to human life, such as birth, death, marriage, divorce, separation, adoption, sickness, population growth, etc.

Collection of raw data of Vital Statistics :–

There are four important sources from which the raw data of vital statistics are generally obtained.

1) Census.
2) Vital Statistics Registers.
3) Hospital records.
4) Ad Hoc survey.

1) Census :– A census represents a comprehensive profile of a country's population. Census operations are conducted in almost all countries at intervals of ten years. In a census, the enumeration of every individuals of all habitational areas is carried out at a specific time. In a census, besides counting the number of persons living in a specific regions; information regarding to their age, sex, marital status, education level, occupation, region, familial characteristics of the individuals are also compiled. But this information is available for the census year only.

Hence, census data fails to produce vital statistics for intercensal years. More over, the data obtained in respect of births and deaths are not complete even for the census year. Hence, the census enumeration fails to provide data suitable for vital statistics.

2) Registration Data :– In a community there is a system of registering the occurances of several vital events to the proper authorities under legal requirements, specially for the vital events like birth, death, marriage, divorce. So from vital statistics registers we have data regarding the no. of these vital events.

V.I In case of birth, when a child is born, the information with regard to the date of birth, place, name of the father, sex of the new born, the age of mother, nationality, religion and profession of the parents has to be reported to the appropriate authority in that area.

In case of death of a person, information in respect of date of death, name of the deceased, his/her father's/husband's name, place of death, age, sex, marital status, religion, cause of death has to be supplied to the registering authority, because the disposal of the body requires a death certificate from the authorities.

4> What is live birth, still birth and foetal birth ?

"Live Birth means the complete explusion or extraction from its mother of a product of conception, irrespective of the duration of pregnancy, which after such separation breathes or shows some other sign of life. Each product of such birth is considered as live born."

"Still Birth is meant death of a product of conception that has completed 28 weeks of gestation prior to separation from its mother, the death being indicated by the fact that after seperation the foetus does not show any sign of life."

"Foetal Death means death of a foetus before separation from mother, may thus be either a still birth or a miscarriage (or abortion) according as it occurs after or before completion of 28 weeks of gestation. The death is indicated by the fact that after such seperation the foetus does not breathe or show any other sign of life, such as beating of the heart, pulsation of the umbilical cord, etc."

3> Hospital Records :– Every hospital as well as nursing home or health centre or any public health programme maintained records for each patients about such particulars as the age, sex, etc. of the patient along with the nature of illness, the type of treatment applied to the patient and also the outcome. So information regarding the cases of disease or group of diseases and also the cases of birth and death that occurs in hospitals or nursing home can be obtained from these hospital records.

4) Ad hoc surveys :- In countries with defective registration systems, occasionally surveys are conducted to collect data on vital events. In our country NSSO(National Sample Survey Organisation) occasionally conducts special surveys to collect raw data on some particulars vital events of some specified characters.

5) Principles in Census :-

i) The nature of population should be specified.

ii) census should be taken in a single day, at least with in a fixed period.

iii) Census should be taken every 10 yrs atleast.

iv) The census should necessarily include those information i.e. name, age, sex, relationship with the head of the family, data of birth, place of birth, civil standard of living, mother's tongue, marital status, knowledge of language, permanent residence, nationality phisical disability (if any) etc.

v) There are four stages to conduct :-

(a) planning stage,
(b) Enumeration,
(c) Procedure or compilation stage,
(d) Analysis and reporting stage.

vi) There are two procedures of census taking —

(a) Defacto Procedure : In this case the person will be counted at that place in that time of taking census.

(b) Dejure Procedure : In this case the person will be counted at his permanent residence.

The type of the method should be critically examined.

5) <u>Different errors in Census Data</u> :—

The errors that are found in Census are classified under two heads —

1) errors of coverage.
2) errors in response.

1) <u>Errors of Coverage</u> :— During a census some individuals, or even some families, may be left out of the count while some others may be erroneously included or included more than once.

These errors of coverage in census data may be taken care of by sample check. This sample gives an estimate of proportion of persons left out or included more than ones. On the basis of these estimates <u>the enumerated figure, may be adjusted.</u>

2) <u>Errors in Response</u> :— The most serious errors under this category are found in the information gathered on the age. These needs special attention as the age composition of the population is vital for most demographic studies.

i) <u>Generally in population statistics, ages are considered to be completed on last birthday</u>. For e.g. 'A man of age 31 years 3 months is said to be of 31 years. Similarly a man of age 31 years 11 months is said to be of age 31 years.'

Now an error occurs in age figures many census enumerators don't follow the meaning of <u>age 0 l.b.d.</u>. As a result of this many children of exact age below, they get recorded as of <u>age 1 l.b.d.</u>, while really they are of age 0 l.b.d. So it has been found that in most censuses children in the first few years of life are underenumerated. Hence, the population figure of age 0 l.b.d may be too low and that of age 1 l.b.d may be too high.

ii) People have a natural tendency to state their ages in round figures i.e. 0 and 5 (or multiple of 5) or even digits. This leads to considerable heaping at ages ending with 0 and 5, and some heaping at ages that end in even digits. This heaping occurs at the cost of other ages, which shows a deficiency of people, this being particularly noticeable at an age like 13 which number tends to be avoided especially in western countries. This type of error may be called '<u>careless error</u>'.

iii) Since people enjoy certain legal and social advantages on reaching majority (i.e. age 18), young people often tends to report their age as 18 before actually reaching that age. Again young people, specially women have a tendency to understate their age after attaining majority. Old people, on the other hand exaggerate their age to avail the benifits of senior citizens. This type of errors may be called "<u>wilful error</u>".

5) Methods of Guarding against and Adjusting errors in <u>Age returns of Census data</u> :—

1) The enumerators should be given a better type of training explaining the significance of all the term included in the census schedule.

2) Enumerators may put indirect questions to have an idea about the correct age figures i.e. to enquire about the death of birth of a person in addition to his or her age. This will be appropriate when the people being enumerated are illeterate.

3) Once the age returns have been received for a more or less correct age distribution, one may use instead of one year age group, broader age groups, so that the ages at which the clustering or heaping occurs fall somewhat at the middle of such age groups.

4) **Myers' Method/Test** :→ Myers developed a test for picking out the best 5-year grouping. The possible 5-year groupings may be distinguished as 1-5, 2-6, 3-7, 4-8 and 5-9; if we consider the end digits of the various ages and note that each of the given clusters determines a complementary cluster, e.g. 1-5 determines 6-10, 2-6 determines 7-11, etc. Now if we add the percentages at various individual digits for each 5-year grouping, then in each case we should get a sum close to 50%. According to Myers' test, that 5-year grouping is to be preferred for which the sum comes closest to 50%.

5) In census the age of some persons may not be recorded (the cases where the age is not available). In this situation the common practise is to distribute the no. of persons of this category among all age groups in proportion to the numbers already available against these age groups.

Total Population Size (N)

Pop^n size where ages are available (N_1)

Pop^n size where ages are unavailable (N_2)

Note that, → $N = N_1 + N_2$

Ages	Pop^n where age figures available	Prop. of people where ages are available	Pop^n where age figures unavailable	Total pop^n.	or / No. of people in whole pop^n.
0	f_0	$f_0/N_1 = p_0$	$N_2 \times p_0$	$f_0 + N_2 \cdot p_0$	$f_0' = f_0 + N_2 \cdot \frac{f_0}{N_1}$
1	f_1	$f_1/N_1 = p_1$	$N_2 \times p_1$	$f_1 + N_2 \cdot p_1$	$f_1' = f_1 + N_2 \cdot \frac{f_1}{N_1}$
2	f_2	$f_2/N_1 = p_2$	$N_2 \times p_2$	$f_2 + N_2 \cdot p_2$	$f_2' = f_2 + N_2 \cdot \frac{f_2}{N_1}$
3	f_3	$f_3/N_1 = p_3$	$N_2 \times p_3$	$f_3 + N_2 \cdot p_3$	$f_3' = f_3 + N_2 \cdot \frac{f_3}{N_1}$
⋮	⋮	⋮	⋮	⋮	⋮
k	f_k	$f_k/N_1 = p_k$	$N_2 \times p_k$	$f_k + N_2 \cdot p_k$	$f_k' = f_k + N_2 \cdot \frac{f_k}{N_1}$
TOTAL	$\sum_{i=1}^{k} f_i = N_1$	1	$N_2 = N_2 \cdot \sum p_i$	$N = N_1 + N_2$	$\sum f_i' = N$

where, $$\sum_{i=1}^{k} f_i' = (f_0 + f_1 + f_2 + \cdots + f_k) + \frac{N_2}{N_1}(f_0 + f_1 + \cdots + f_k)$$
$$= N_1 + N_2/\cancel{N_1} \times \cancel{N_1} = N_1 + N_2 = N$$

Errors in Registration Data :—

1) **Errors of Coverage :—** In all countries births are under registered when a child dies shortly after birth, the guardians don't care to report it. The position is some what better for deaths since the disposal of a body requires a death certificate.

The error of incompleteness may be estimated by sample check.

2) **Errors in Response :—**

a) **Errors in recording the place in which birth and death occurs :—**

A birth or death occuring in a city hospital may be recorded as an addition or loss to the city popn. although actually the mother or the the person dying may really belong to some other area.

b) **Errors in recording the cause of death :—**

i) The cause of death may be unknown to the informant.

ii) The cause of death may be willfully misstated because the cause of death may have associated with some social stigma (e.g. leprosy, tuberculosis, alcoholism or suicide).

iii) There may be a number of causes leading to the death of a person of which one will be the principle cause. However the principle cause may be wrongly recorded owing to a real difficulty in identifying the true principle cause.

Rates of Vital events :—

Def-1. The raw data on vital statistics which are generally classified according to different characters, say, age, sex, occupation, religion, etc., are given in the form of a frequency table. But, these frequency values alone will not be enough for a statistical study in Popn. Statistics. For e.g., the statement that in country A 1,00,000 people died in a certain years, where as in country B, 2,00,000 people died in the same year which convey no significant implication regarding the death pattern of these two countries A and B unless we know the popn size of each community. Therefore, it is also necessary at least to know the popn. size of each community to have an idea as to their relative mortality situation. So, by relating these two, mainly the no. of deaths to the popn. size, we have a rate (in this case a death rate). So, we define rate as:

$$\text{Rate of the vital event (E)} = \frac{\text{No. of cases in which vital event (E) occurs in a given region during a given period}}{\text{Total no. of exposers to the risk of occurance of vital event (E) in a given region during a time period}} \times K$$

According to the defn. a rate will be a proper fraction, so, for care of understanding the fraction is generally multiplied by a constant K which is usually 10^3 or 10^5 accordingly every rate is expressed 'per thousand of population' or 'per lakh of popn.'.

Now every rate relates to :

i) A particular vital event (i.e. birth or death)

ii) A given geographical region (i.e. India or West Bengal)

iii) A particular period of time.

The 2nd and the 3rd points may not always be expressed but will be understood from the context.

Or

* Alternative Def-2.

If N represents the number of occurances of a certain vital event E (such as birth or death, etc.) during a stated interval of time, and P represents the size of population within which vital occurrence took place (i.e. P represents the number of persons exposed to the risk of occurrence of the event E), then $\frac{N}{P}$ is regarded as the rate of that vital statistics.

The period of time is usually taken as one year. The size P of the popn is exact or approximate; usually mid year popn. size is taken. The rate is a number which lies between 0 and 1. This defn. of rate is similar to classical defn. of probability.

A multipler called radix is used to round off the decimals. Usually 1000 is taken as radix so that the rate is 'per thousand of popln.'. The rate is then expressed as $\frac{N}{P} \times k$.

The number of persons exposed to the risk of a vital event is usually the population of the given area during the given period or some segment of that population.

⇨ What should be the population of a period?/ What is the mean popn. of a period?

Ans:> As an estimate of the popln., we may take the popln. either at the beginning or at the end or at the middle of the given time period. However it is the best to take the average population of the community during the given period.

Let P_t denote the popln. at the time t. Assume P_t to be continuous function of t. Then, for the time interval (t_1, t_2) the average popln. will be

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P_t \, dt.$$

But popl^n. at a fractions of time is rarely available. In its absence, we take mid-point popl^n. If that is again not available, the average of the popl^n. at the two ends of the interval is also used. It is to be noted that popl^n. can be well approximated by a linear function of t, say $P_t = a + bt$ between t_1 and t_2, provided t_1 & t_2 are close enough i.e. provided (t_1, t_2) is a short interval.

Justification:—

Average / Mean Popl^n. in this situation will be -

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}(a+bt)\,dt = \frac{\left[at + \frac{bt^2}{2}\right]_{t_1}^{t_2}}{t_2 - t_1}$$

$$= a + b\left(\frac{t_1+t_2}{2}\right)$$

$$= P\left(\frac{t_1+t_2}{2}\right)$$

= Popl^n. at the mid point of time interval.

Also then, the average popl^n. at the two ends :

$$= \frac{P_{t_1} + P_{t_2}}{2}$$

$$= \frac{a + bt_1 + a + bt_2}{2}$$

$$= a + b\left(\frac{t_1+t_2}{2}\right)$$

$$= P_{\left(\frac{t_1+t_2}{2}\right)}$$

— Justified.

So, in this situation, with this assumption of linearity in the popl^n., all the approximation are exactly equal to the actual average of the popl^n.

Probability Interpretation of RATE :—

$$\lim_{n\to\infty} \frac{f_n(A)}{n} = \text{Probability of the event A.}$$

i.e. the observed value of $f_n(A)$ and n is a good estimate of $P(A)$ where $f_n(A)$ denotes the no. of cases in which the event A occurs. The experiment is repeated under some conditions, the major conditions remain same.

Now, the rate is defined as —

$$\text{Rate of the event} = \frac{\text{Total no. of cases in which vital events occur}}{\text{Total no. of persons exposed to the risk of occurance of the vital event (E).}}$$

$$= \frac{f_n(E)}{n},$$ where, $f_n(E)$ is the no. of times E occurs among these n cases.

In most cases n will be a very large number (Usually a few millions or few crores). Hence, the observed value of the rate may be taken to be a very good estimate of the probability that a person exposed to the risk of occurance of E will actually have E during the given period. However, this interpretation can't be given for such cases.

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Ratio of the vital event :— Here the purpose is to compare the importance of one vital event E_1 to that of other E_2. The defn of the ratio is given by—

Ratio of vital events

$$= \frac{\text{No. of cases in which the vital event } E_1 \text{ occurs}}{\text{No. of cases in which the vital event } E_2 \text{ occurs}} \times K$$

Here, k is also 10^3 or 10^5.

∴ Like rate, ratio is also expressed 'per thousand' or 'per lakh of popln'.

* Alternative Definiton :→ If N_1 represents the number of occurrences of an event (vital) E_1 which may or may not include an another vital event E_2 which occurs N_2 in a population at the same time, the quotient $\frac{N_1}{N_2}$ is described as a vital statistics ratio.

Using the radix K, the ratio is expressed as

$$\frac{N_1}{N_2} \times K.$$

The relative number of males and females in a popl^n. is mesured by sex ratio, which is defined as –

Sex-ratio $= \frac{N_1}{N_2} \times 1000$, where N_1, N_2 are the no.s of females and males in a popl^n. at time 't'.

Probability Interpretation of RATIO : —

Apart from the constant, ratio can be written in the form –

$$\frac{fn(E_1)}{fn(E_2)} = \frac{fn(E_1)/n}{fn(E_2)/n} = \frac{\text{Rate of } E_1}{\text{Rate of } E_2}.$$

Supposing, the total no. of exposers to the risk of occurance of vital events in each case is n.

Because of the probability interpretation for rate it can be said that in most cases these ratio may be taken as a very good estimate of the ratio of the $P(E_1)$ and $P(E_2)$, i.e.,

$$\text{Ratio} = \frac{\text{Prob. of } (E_1)}{\text{Prob. of } (E_2)} = \frac{P(E_1)}{P(E_2)}.$$

➪ MORTALITY RATES :—

1) **CRUDE DEATH RATE** :— The simplest type of rate used in the measurement of mortality is the Crude Death Rate (CDR), which is defined as :

$$CDR = \frac{D}{P} \times 1000 = m$$

where,

D = number of deaths from all causes which occured in the popl.n of the given region during the given period;

P = total popl.n size of the given region during the given period.

So, CDR gives the death rate per 1000 of popl.n.

~~NB # [It is the most widely used of vital statistics rates.~~

~~▨ CDR measures the relative frequency of death in a particular popl.n in a specified time interval, i.e. CDR has a simple interpretation, it gives the no. of deaths that occur, on the average, per 1000 people in the community]~~

▨ Advantages or Merits :—

i) It is the most widely used of vital statistics rate.

ii) It is easy to compute and to understand as well as it requires only the total no. of deaths and total popl.n size.

iii) It is simple to interprete because it gives the no. of deaths that occur, on the average, per 1000 people in the given community.

iv) It has a probability interpretation. If all the people of the region are taken to be equally exposed to the risk of death from some cause or other, during the interval, then CDR can be regarded as the probability rate of death.

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Drawbacks or Demerits :— (***)

i) Mortality depends very much on the composition of the popl^n. by age, sex, occupation, place of dwelling, race, etc. But CDR does not consider any of these facts.

ii) Because of the first defect, the CDR is unsuitable as an index of relative mortality in different places with different age and sex-distribution. For e.g. the regions with high proportion of old popl^n. will exhibit a relatively high CDR, since aged people do contribute many deaths, the effect of death rate is to increase the numerator without proportionately reducing the denominator, this means the popl^n. is abnormal by virtue of a shortage of young people.

C.U

iii) So, under the most circumstances, the CDR may well be used for comparing the mortality situations of the same place at different times provided the periods compared are not too far apart because in a stable large community the age and sex composition of the popl^n. changes very shortly.

SPECIFIC DEATH RATE :—

A Specific Death Rate (SDR) is a death rate computed for a specific segment of the community. Thus a SDR is given by :

$$SDR = \frac{\text{Total no. of deaths in a specific segment of the given popl}^n\text{. during the given period in the given region}}{\text{Total no. of persons in that specific segment of the popl}^n\text{. in the given period in the given region}} \times 1000$$

Death rates may be made specific w.r.t. age, sex, occupation, place of dwelling etc. or a combination of some of these factors. But generally they are made specific w.r.t. age, sex and combination of age and sex. The most important types of specific death rates are:

i) Age Specific Death Rate (ASDR),
ii) Sex Specific Death Rate (SSDR),
iii) Specific Death Rate for both Age and Sex.

i) Age Specific death rate (ASDR) :—

$\boxed{C.U}$ What is ASDR? The ASDR for the age group x to $\overline{x+n}$ is

$$ {}_n m_x = \frac{{}_n D_x}{{}_n P_x} \times 1000 \text{ , where —} $$

${}_n D_x$ is the no. of deaths between ages x to $\overline{x+n-1}$ last birthday (l.b.d) in a given period in a given region. and ${}_n P_x$ is the no. of persons in the same age group in the given period in the given region.

The annual age-specific death rate is written as — $m_x = \frac{D_x}{P_x} \times 1000$, where

D_x = number of deaths among persons aged x l.b.d

and P_x = number of persons aged x l.b.d.

5 years age group :—

Age group x to $(x+4)$	${}_5 D_x$	${}_5 P_x$	${}_5 m_x = 1000 \times \frac{{}_5 D_x}{{}_5 P_x}$
0–4			
5–9			
10–14			
15–19			
20–24			
⋮			

$\therefore$ ASDR for 5 years age group $x \to (x+4)$ l.b.d.

$$ {}_5 m_x = \frac{{}_5 D_x}{{}_5 P_x} \times 1000 $$

where ${}_5 P_x$ = Population of age $x \to (x+4)$ l.b.d.

${}_5 D_x$ = Total no. of death among age $x \to (x+4)$ l.b.d.

${}_5 m_x$ = Age specific death rate for $x \to (x+4)$ l.b.d.

Similarly, we can define age specific death rate for 10 year age group.

ii) Sex Specific death rate (SSDR) :—

In this case we have two types of specific death rates, one is for males and the other for females. By def^n.

$${}^{m}m = \frac{{}^{m}D}{{}^{m}P} \times 1000 \qquad\qquad {}^{f}m = \frac{{}^{f}D}{{}^{f}P} \times 1000$$

${}^{m}m$ = SDR for males
${}^{m}P$ = total no. of males
${}^{m}D$ = total no. of deaths among males

${}^{f}m$ = SDR for females
${}^{f}P$ = total no. of females.
${}^{f}D$ = total no. of deaths among females.

iii) Specific Death for age and sex :—

The male ASDR is —

$${}^{m}_{n}m_x = \frac{{}^{m}_{n}D_x}{{}^{m}_{n}P_x} \times 1000,$$

where, ${}^{m}_{n}P_x$, ${}^{m}_{n}D_x$ respectively denote the number of males aged x to $(x+n-1)$ and the number of deaths occuring to such males.

Advantages or Merits :—

i) The specific death rates are the true and best measures of mortality because they give more detailed information regarding the mortality pattern of the community than the CDR does.

ii) Since in case of SDR (especially ASDR) we have for different rates more or less homogeneous groups, i.e., groups are such that for all the members of a group the mortality may be supposed to be about the same, the SDRs are more meaningfull measures of mortality than the CDR. For the same reason it is more appropriate to give a probability interpretation to the SDR's than CDR i.e. the probability that a person of a certain specified time will die within the given period.

iii) Specificity by age and sex eliminates differences in death rates arising from variation in pop^n composition in respect of these characters. To this extent, such SDRs can be compared from one community to another community. i.e. SDR 's provide more appropriate measures of the relative mortality situation in the regions.

Disadvantages or Demerits :—

i) Computation of SDR's requires very elaborate data, e.g. to obtain ASDR, the dist. of deaths as well as the popl^n. by age and sex must be available.

ii) There will be a big mass of data, where one obtains specific death rates, specificity being achieved w.r.t. age, sex, etc. or some combination of these factors. The significance of such data may be difficult to comprehend.

iii) The comparison of mortality levels of two regions for the same period or those of the two periods for the same region with the help of SDR's may not be easy. It may so happen that one region has higher rates than the other for some segments while lower rates for the other segments. In such cases one can't say which of the two regions has higher overall mortality.

C.U

Remarks :→

1). If ${}_nM_x$ is the death rate at age group x to $(x+n)$ and ${}_nP_x$ is the popl^n. in the same age group, the total deaths will be $\sum {}_nM_x \cdot {}_nP_x$, where the sum is over all age groups.

Then the CDR is → $$m = \frac{\sum {}_nM_x \cdot {}_nP_x}{\sum {}_nP_x}.$$

Clearly, if the values ${}_nP_x$ are increased for older ages at the expense of the younger ages then not with standing the constancy of age rates of mortality, the CDR will rise. The weights used in such an average are therefore important.

Consider the CDRs for two communities A and B,

$$m^a = \frac{\sum {}_nM_x^a \cdot {}_nP_x^a}{\sum {}_nP_x^a}, \quad m^b = \frac{\sum {}_nM_x^b \cdot {}_nP_x^b}{\sum {}_nP_x^b}.$$

Even when two communities have same mortality situation at different age groups, then m^a and m^b may be unequal simply because the proportion ${}_nP_x^a / \sum {}_nP_x^a$, ${}_nP_x^b / \sum {}_nP_x^b$ may not be same, i.e. because the age-dist^n. of the two communities may not be identical. Hence, CDR can not be used to compare mortality situations in different places unless the popln of the places have identical age distribution/ sex distribution, a condition which is seldom fulfilled. ASDR can be used to compare mortality situations of two communities here.*

*Imp. Ques:→ **Relation between CDR and SDR :—**

$$CDR = \frac{D}{P} \text{ [Ignore multiplier 1000]}$$

$$= \frac{\sum_x D_x}{\sum_x P_x}$$

Again we know,

$$SDR = m_x = \frac{D_x}{P_x} \text{ [Ignoring multiplier 1000]}$$

i.e. $D_x = m_x P_x$.

Hence, $$CDR = \frac{\sum_x m_x P_x}{\sum_x P_x}$$

= weighted average of SDR's (m_x) where P_x be the weights.

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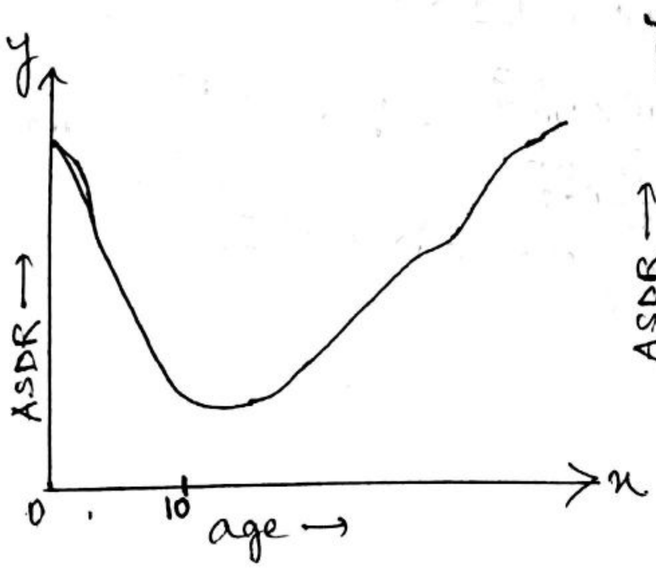
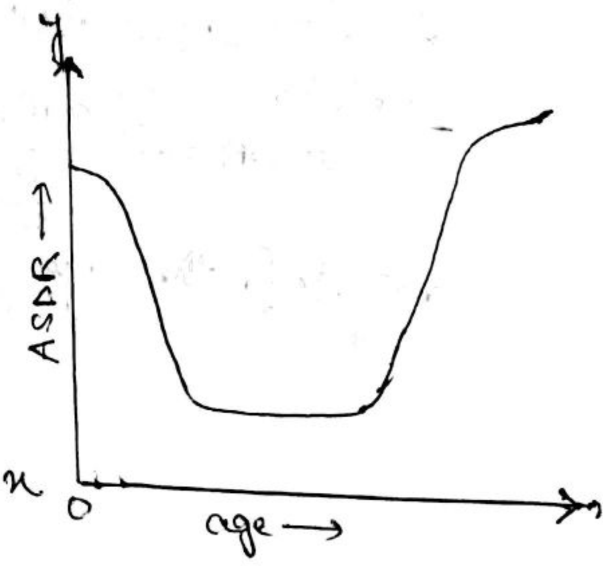
*Remark ②: — The ASDR's show a striking behaviour for communities in general. The ASDR's start at a rather high level in the first year of life, then gradually diminish and reach in a minimum around the age group 10-14, then again rise slowly until midlife and then rise rapidly at the older ages. The curve with ASDR along y-axis and age along x-axis is roughly U-shaped or bath-tub shaped.





C.U

3) STANDARDISED DEATH RATE (STDR) : —— (***)

IMPORTANCE :- In order to get a clear insight into the mortality situation of two communities or even of the same community over two different time periods, it is necessary to make comparisons of death rates according to different characteristics such as age, sex, etc. However, it is difficult mentally to assimilate a large number of rates and for many purposes, one series may have higher specific death rates (SDRs) than the other for some sections but lower SDRs for the other sections. In such a case, we can't make a general statement of the form "Mortality is higher or lower in A than B". Hence in many purposes it is desirable to have a summary measure to describe the whole experience. Because of these facts, it is desirable to have a single figure index — some sort of average of the death rates for various sections of the popl^n.

C.U

Direct Standardisation :→ The simplest single figure index of this type CDR, since $CDR = \frac{\sum {}_nm_x \cdot {}_nP_x}{\sum {}_nP_x}$; but it suffers from the defect that it is greatly influenced by the weights i.e. age-distribution of the locality. To eliminate the defect in using CDR, an index is constructed with some standard weights i.e. the same set of weights for the two communities. This is done by considering a third population, called a standard population. Let ${}_nP_x^S$ be the no. of persons in the age group x to $\overline{x+n-1}$ l.b.d. in the standard popl^n, then the age-adjusted death rate or age-standardised death is —

$$STDR = \frac{\sum_x {}_nm_x \cdot {}_nP_x^s}{\sum_x {}_nP_x^S}$$ where $\frac{{}_nP_x^S}{\sum {}_nP_x^S}$ are the weights for the age group x to $\overline{x+n-1}$.

This age-adjusted death rate is the CDR which would be observed in the standard popl^n if it experienced the age-specific death rates of the community in question.

A death rate may be adjusted for characters other than age and may be similarly interpreted. For instance, in case the death rates for A are specific for both age and sex, the STDR for A is —

$$STDR^{a} = \frac{\sum_n {}^{m}m_x^a \cdot {}^{m}_{n}P_x^s + \sum_n {}^{f}m_x^a \cdot {}^{f}_{n}P_x^s}{\sum_n {}^{m}P_x^s + \sum_n {}^{f}P_x^s}$$

Advantages :— A STDR is easy to compute and is also easy to interpret. If the age specific death rates of one community are higher than those of another, the fact would get reflected in the adjusted death rate.

Disadvantages :— One drawback of considering adjusted death rate as a reliable index is the subjective consideration in the choice of the standard population. If the standard population does not vary too much in its age-dist^n. from those of the community compared, then the difficulty may not be great. the usual procedure is to take as standard the actual population of a bigger community of which A and B are parts.

In some studies one uses the ratio of STDR ^{a} to CDR^S as an index of mortality, called the Comparative mortality factor (CMF). Thus

$$CMF^a = \sum_x m_x^a P_x^S \Big/ \sum_x m_x^S P_x^S = \sum_x \left(\frac{m_x^a}{m_x^S}\right) D_x^S \Big/ \sum_x D_x^S$$

A similar index is the weighted harmonic average of the ratios m_x^a / m_x^S, the weights being the death figures for the given population. It is called the Standardised mortality factor (SMF). Thus

$$SMF^a = \sum_x D_x^a \Big/ \sum_x D_x^a \left(\frac{m_x^S}{m_x^a}\right)$$

$$= \sum_x m_x^a P_x^a \Big/ \sum m_x^S P_x^a .$$

Indirect Standardization : → (***)

[C.U]

The direct standardisation involves ${}_nM_x$, the age specific death rates for the community as well as ${}_nP_x^s$, the age distn. of the standard popln. When ${}_nM_x$ are not available, the direct method can't be used and there is an indirect method.

Suppose that only the total no. of deaths, and hence the CDR is known for the community. Suppose, C is an adjustment factor such that —

$$CDR \times C = STDR$$

$$\Leftrightarrow \frac{\sum {}_nm_x^a \, {}_nP_x^a}{\sum {}_nP_x^a} \times C = \frac{\sum {}_nm_x^a \cdot {}_nP_x^s}{\sum {}_nP_x^s}$$

$$\Leftrightarrow C = \frac{\sum {}_nm_x^a \cdot {}_nP_x^s / \sum {}_nP_x^s}{\sum {}_nm_x^a \cdot {}_nP_x^a / \sum {}_nP_x^a}.$$

But this factor C can't be evaluated with the type of data we have in hand, since ${}_nm_x^a$ for each group, are not available. The usual practice is to replace ${}_nm_x^a$ by ${}_nm_x^s$. C is then approximated by —

$$C_1 = \frac{\sum {}_nm_x^s \, {}_nP_x^s / \sum {}_nP_x^s}{\sum {}_nm_x^s \, {}_nP_x^a / \sum {}_nP_x^a} \text{ and, correspondingly,}$$

the $\widehat{STDR}$ is approximated by → $CDR \times C_1$.

Comment : →

1) Generally the indirect method gives almost the same value of STDR as the standard method (direct) would.

2) If ASDR's of given popln. are proportional to ASDR's of standard popln., then the two methods are exactly equal to each other.

Justification : —

Let, $m_x^a \propto m_x^s \;\; \forall x$.

$$\Rightarrow m_x^a = \lambda \cdot m_x^s \;\; [\lambda \text{ is any constant}]$$

$$\hat{STDR} = CDR \times C_1$$

$$= \frac{\sum_x m_x^a P_x^a}{\sum_x P_x^a} \times \frac{\sum_x m_x^s P_x^s / \sum_x P_x^s}{\sum_x m_x^s P_x^a / \sum_x P_x^a}$$

$$= \frac{\sum_x m_x^a P_x^a}{\cancel{\sum_x P_x^a}} \times \frac{\frac{1}{\cancel{x}}\sum_x m_x^a P_x^s \Big/ \sum_x P_x^s}{\cancel{\frac{1}{x}\sum_x m_x^a P_x^a \Big/ \sum_x P_x^a}} = \frac{\sum_x m_x^a P_x^s}{\sum_x P_x^s}$$

$$= STDR^a$$

Comparative Mortality Index :→

NO

The popln at the start of the period been taken as standard in computation of STDR (or the CMF) which gives rise to difficulties in making a comparison of mortality overtimes. For the age-sex distn. of the current popln. may be widely different from that of the standard so STDR (or CMF) values give an unrealistic picture. To meet this objection, the Comparative Mortality Index (CMI) has been introduced to meet this objection. Here use is made of a shifting set of weights in taking a weighted average of SDRs. thus the CMI for a given period will be given by the formula

$$CMI = \sum_x w_x m_x \Big/ \sum_x w_x m_x^s,$$

where, $w_x = \frac{1}{2}\left[\frac{P_x^s}{\sum_x P_x^s} + \frac{P_x}{\sum_x P_x}\right]$,

P_x^s and P_x being the popln figures at age x for the standard and the given period, respectively, and m_x^s and m_x the SDRs at age x for the periods.

We may be required to compare the mortality of a community in successive years. this may be achieved by forming ratios of the corresponding CMIs.

4) **CAUSE OF DEATH RATE :—**

$\boxed{C.U}$ The (crude) cause of death rate for cause c, denoted by m^c, is defined as —

$$m^c = \frac{D^c}{P} \times 1,00,000 \, ,$$

where,

D^c = total no. of deaths from cause c occuring in the given period in the given community.

and P = total popl^n. of the given community in the given period.

This rate has the multiplier 100,000, instead of 1,000, in order that in any given case the computed rate does not appear as a small fraction.

This rate is used to measure the contribution to the total mortality of a community that is made by a specified cause of deaths, say a specified dis-ease or accidents.

$\boxed{\text{USE}}$:— This is the masure that serves as the basis for many public-heath programmes to reduce deaths from the cause, and also as an index of success or failure of such health programmes.

Merits :— i) It has a simple interpretation because it gives the no. of persons, per lakh people, dying from the cause c, during the given period in the given region.

ii) It is also simple to calculate although it is not as simple as CDR.

iii) It is a measure of reduction of the popl^n. as a whole from the given cause.

Drawbacks :— i) It can not be used to compare the mortality situations dues a cause-of-death for different community, since it does not take into account the age-sex composition of the community.

ii) Cause of death being subject to the great degree of reporting errors, the computed rate is also likely to be unreliable.

iii) It is not a probability rate, for the whole popl^n may not always be regarded as the popl^n exposed to the risk of death from a given cause.

5) MATERNAL MORTALITY RATE :—

The maternal mortality rate is defined as —

$MMR = m^P = \frac{D^P}{B} \times 1000$, where D^P and B are the number of deaths from puerperal causes among the female population and the number of live births occuring, in the given period in the given community, respectively.

The rate is intended to measure the proportion of mother deaths from puerperal causes among the female popln. that goes through conception sometime during the period.

Rationale :→ The part of the female popln. that goes through conception some time during the period, and not the whole popln., is exposed to the risk of dying from puerperal causes i.e. causes relating to child-birth. This popln. may be approximated by the no. of mothers giving birth to live-born children plus the no. of those delivered dead foetuses.

But, foetal death are almost universally poorly registered and most countries maintain data not on the no. of mothers rather on the no. of live-births. these are the reasons why maternal mortality rate has as its denominator the no. of live births.

Drawbacks :→ Firstly, puerperal causes of death are generally subject to a large margin of reporting errors.

Secondly, live births are generally under-registered than maternal deaths. As such, the maternal mortality rate will tend to be over-stated to some extent.

Lastly, It is not a probability rate. It may be noted that only a part of the female popln. that experience conception some time during the given period and not the whole popln. is exposed to the risk of death from the puerperal cause.

6) CASE FATALITY RATE :— c.u We have considered the case of a specific cause of death by cause of death rate.

Now we want to measure how fatal these particular cause of death is in the context of the given community and the given period of time. This type of rate is defined as —

$$CFR = \frac{D_i}{C_i} \times 1000,$$

where D_i = no. of deaths among cases of the disease i.

and C_i = total no. of cases of the disease i.

Merits :—

1) This may be regarded as the most refined specific death rate provided age, sex, occupation, etc. are taken into account in its computation.

2) It is truely a probability rate when hospital records are complete, because in true sense it indicates the no. of persons at the risk of dying of a particular cause are really those persons who are actually attacked. For example — the case fatality rate for the disease 'cancer' represents the probability that a person suffering from 'cancer' in a given period will die of that disease in that period.

Demerits :—

1) It is difficult to get the rate correctly because of the usual difficulties in correctly diagonizing the disease.

2) The computation of this rate is difficult because of the non-availibility of the relevant data. Generally these rates are computed by taking information from hospital records.

3) The case fatality rate for the cases of disease which are treated in hospital tends to be very high because those cases of disease treated in hospital are more serious in comparison to the cases treated outside.

7) INFANT MORTALITY RATE :— (***)

Definition :- [C.U] The infant mortality rate (IMR), too, is an alternative to, and in a ~~sense~~ an improvement upon, the age-specific death rate for age 0 l.b.d. — in other words, upon the death rate for infants (i.e. children under 1 year age). It is defined as —

$$IMR = \frac{D_0}{B} \times 1000$$

during a certain year

where D_0 = number of deaths among children of age 0 l.b.d. / the no. of infants dying before attaining the age one.

(***) B = total no. of live births during the same year.

Necessacity of Computing IMR :— [C.U] The age specific death rate for age 0 l.b.d, which has the same numerator, has for its denominator the number of infants obtained from popl^n. census. However, it is well known that infants are grossly under-enumerated in a popl^n. census. Therefore, the ASDR at age x (m0) tends to be highly over stated. Moreover, estimates of popl^n. classified according to the age are seldom obtainable. This is why the IMR is generally used, in lieu of the ASDR (m0), as the measure of infant mortality.

Merits or Advantages :— [C.U]

1) It is likely to be more correctly obtainable than m0 (ASDR of age 0 l.b.d.)

2) IMR does not require the data of popl^n. census or estimates, the IMR can be computed for any popl^n and for any time period provided only the no. of infant deaths and the number of live births are available.

3) The IMR besides giving an idea of the effect of mortality about infants, may be used as an index of the effect of health measure taken by the community. This is why as a rate of mortality, it is the most sensitive way to improve the system of public health and indirectly the system of mass awareness, proper education, etc.

Drawbacks or Demirits :—

1) The IMR is not a probability rate. the IMR as computed C.U above does not give an accurate measure of the risk of death of infants during the first year of life. Since, its numerator and denominator are not strictly related. The death under 1 year in a given calender year include those of some children born in the previous year; moreover, some of the deaths among the current year's births during the first year of life may occur in the following calender year. If fertility and mortality are stable, these two types of errors tend to cancel each other, but their effect may be considerable when fertility and mortality are changing fairly & rapidly.

2) The more serious drawback arises from the under-registration of live-births. there is also found a reluctance to register as live-born those infants who die immediately after birth. This leads to an over-estimation in IMR value than what it should be. This is why it has been said that it is impossible to lower the IMR without saving a single life simply by improving the birth registration system.

IMR as an index of the general healthiness :— ***

C.U The great risk of death under 1 year of age is not equalled at any other part of the life span, except at very old ages. But unlike deaths at very old ages, infant deaths are highly responsive to improvements in environmental and medical conditions. No wonder, then the IMR serves as an excellent index of the general healthiness of the community. Hence IMR is likely to be highly sensitive mortality rate (under standard social and climatic conditions) of the efficiency of the general healthiness of the community.

C.U

8) NEONATAL MORTALITY RATE :—

While estimating IMR based on the data of mortality of infants by months or by weeks during the first year of life, it has been found that infant deaths during the first to fourth week of life constitute a major segment of the total infants mortality. The mortality rate relating to this period of life namely upto four weeks or the first month is known as Neonatal Mortality rate.

Symbollically NMR is given by —

$$NMR = m^N = \frac{D_N^Z}{B^Z} \times 1000$$

where, D_N^Z = death of infants during the neonatal period i.e. 0 to 1 month in the calender year Z.

and B^Z = total no. of live births during the calender year Z.

Again like the def^n. of IMR, the present def^n. is also subject to error, because a child is born in december in $(Z-1)^{th}$ calender year may die in Z^{th} calender year. This makes an obvious over estimation of NMR. Hence an improved estimate of NMR is based on the following def^n.

$$NMR = \frac{D_Z^N}{\left[\frac{1}{2}\{\text{no. of live birth in december } (Z-1)\} + \frac{1}{2}\{\text{no. of live birth in december } Z\} + \{\text{no. of like birth from Jan. } Z \text{ to nov. } Z\}\right]}$$

Remarks :— However if there is no significant variation in the no. of births over months in two consecutive years $(Z-1)$ and Z then the above adjustment may be unnecessary, otherwise improvement in the estimated NMR using the adjustment can be made satisfactory.

C.U

* IMPT. QUES:— Distinguish between neonatal and perinatal mortality rates.

→ explained here.

9) [C.U] **PERINATAL MORTALITY RATE :—**

Still births represent not only personal tragedies but also a significant source of wastage of life, and therefore the object of serious attention by public health authorities.

There is a tendency to distinguish even more clearly the true natal deaths from those attributable to post-natal environmental influences by reference to deaths in the first week and such deaths per 1000 related live births provide an early neonatal mortality rate. These deaths combined with still births and rated to a 1,000 total births can be regarded as measuring mortality at a period of time surrounding birth and may be described as Perinatal mortality.

Hence, the perinatal mortality rate is defined as $m^P = \frac{D_P}{B} \times 1000$ where D_P is the no. of deaths in the first week of life plus still births during a certain year and B is the total no. of births (live and still) during the same year.

Advantages :— The perinatal mortality rate has the advantage that unlike its components, the still birth and early neonatal rates, it is not likely to be disturbed by variations in the practice of recording or in the actual timing of foetal deaths. If a foetus is regarded as surviving beyond intra-uterine existence, a death is transferred from still birth to the early neonatal category though the actual reality of the situation — death around the point of delivery — is unaffected. This is quite an important point as the precise fixation of the time of foetal death is often difficult.

[U.S]

LIFE TABLE :-

C.U [Another way of summarizing a mortality experience is by means of life table, i.e., a table which shows, on the basis of current mortality rates, the number out of 1000 births (or some other convenient starting number, termed 'radix' of the table) who survive to certain specified ages and the numbers dying between these successive points of age. <u>A full life table gives this information for each integral age from nought to an upper limit beyond which the number of survivors is negligible.</u> It gives a more vivid representation of a mortality situation <u>than is provided by ASDR's or the SDR's w.r.t. both age and sex.</u>]

Two types of life tables :-

i) [First, suppose we observe a certain number of persons all born at same time throughout their lifetime. Suppose also that we record in a life table the no. of persons dying at the first year of age, the no. of persons during the second year of age, and so on, and also all the survivors at the beginning of each year of age, in this way we get a life table which is called <u>generation or cohort life table.</u>]

<u>Demerits</u>:→ However such a life table, although gives real picture of mortality also suffers from certain disadvantages. U.S

<u>Firstly</u>, for constructing a table in this case, we shall have to wait 100 years, i.e., till the last person dies.-

<u>Secondly</u>, during this long period the mortality situation of the given community may change. Hence, the picture of mortality that we get from such table does not relate to the period of time.

ii) [Second, here we consider a hypothetical cohort of people all assume to be born at the same time. Then we see how much a cohort would behave if they were subject to the observed rate of mortality during a given calender period, usually a short period. Hence, this table gives us a hypothetical picture for the hypothetical cohort over the years of age. This life table is called <u>Current or instantaneous life table.</u>]

In constructing such a life-table we consider a given hypothetical cohort of births say of one lakh births then we ask the following questions if they were subject to the observed level of mortality.

1) How many of them would die during the first year of age and how many of them will remain alive at the end of the year?

2) How many would die at the second year of age and how many of them ..... will remain alive at the end of the second year and so on.....

Also when all the members of cohort die what would be there average longivity. Answers to all such questions are given in the current life table.

Description of complete life table :— (***)

A typical life table has the following columns —

(i) Age in years (in integer) x	(ii) l_x	(iii) d_x	(iv) q_x	(v) L_x	(vi) T_x	(vii) $\overset{\circ}{e}_x$
0						
1						
2						
3						
4						
⋮						

c.4

The various symbols entering the table are defined as follows :—

(i) x : the integral value of age in years.

(ii) l_x : the number of persons surviving to the exact age x.

(iii) d_x : the number of persons dying between exact age x and the next age $(x+1)$.

so, $d_x = l_x - l_{x+1}$.

(iv) q_x : the probability that a person of exact age x will die before attaining age $x+1$, hence — $q_x = \frac{d_x}{l_x}$.

Some tables include, beside q_x the function $p_x = 1 - q_x$ which gives the probability that a person of age x will survive till age $x+1$.

(v) $\underline{\underline{L_x}}$: the number of years lived, in the aggregate, by the cohort of l_0 persons between ages x and $(x+1)$. Hence

$$L_x = \int_x^{x+1} l_t\,dt, \Rightarrow L_x = \int_0^1 l_{x+t}\,dt \quad [\text{assuming } t \text{ be constant}]$$

Approximate formulae for L_x :$\longrightarrow$

▨ i) Assuming that d_x deaths occurring in the age-interval $(x, x+1)$ are uniformly distributed over this interval.

Equivalently, $\dfrac{(-l_{x+t} + l_x)}{(-l_{x+1} + l_x)} = t$ (constant), $0 < t < 1$, where l_{x+t} is the number of persons surviving to the exact age $(x+t)$, i.e. $l_{x+t} = l_x - t.d_x$, is a linear function of t, $0<t<1$.

then —

$$L_x = \int_0^1 l_{x+t}\,dt$$

$$= \int_0^1 (l_x - t.d_x)\,dt = l_x.\left|t\right|_0^1 - d_x\left|\frac{t^2}{2}\right|_0^1$$

$$= l_x - \frac{d_x}{2}$$

$$= l_x - \frac{(l_x - l_{x+1})}{2}$$

$$= \frac{l_x + l_{x+1}}{2}.$$

▨ ii) Assuming that the no. of persons in the age-interval $(x, x+1)$ is decreasing at a constant rate (b, say),

i.e., assuming $\dfrac{l_{x+t}}{l_x} = b^t$, $0 \le t \le 1$. then —

$$L_x = \int_0^1 l_{x+t}\,dt$$

$$= l_x \int_0^1 b^t\,dt = l_x.\left[\frac{b^t}{\ln b}\right]_0^1 = l_x.\frac{b-1}{\ln b}.$$

Note that — $l_{x+1} = l_x.b$ and $\ln b = \ln\left(\frac{l_{x+1}}{l_x}\right) = \ln l_{x+1} - \ln l_x$.

Hence, $L_x = \dfrac{l_{x+1} - l_x}{\ln l_{x+1} - \ln l_x} = -\dfrac{d_x}{\ln l_{x+1} - \ln l_x}$, based on the assumption $\dfrac{l_{x+t}}{l_x} = b^t$, $0 \le t \le 1$, $0<b<1 \Leftrightarrow l_x = ab^x$, $0<b<1$.

(*) (iii) Note that, out of the l_x members of the cohort alive at age x, l_{x+1} persons live one complete year in the age interval x to $x+1$ and the remaining d_x persons who die in that interval, live varying fraction of year denoting by a_x, the average of these fractions, we get

$$L_x = l_{x+1} + a_x . d_x$$

$$\boxed{L_x = l_x - d_x(1-a_x)}$$

Based on U. S. mortality rate Chiang found that

$$a_0 = \begin{cases} 0.10 \text{ for white} \\ 0.14 \text{ for non-white} \end{cases} \Bigg\rangle \text{ irrespective of sex.}$$

$$\left.\begin{aligned} a_1 &= 0.43 \\ a_2 &= 0.45 \\ a_3 &= 0.47 \\ a_4 &= 0.49 \end{aligned}\right\rangle \text{ irrespective of sex and race.}$$

When, the d_x deaths occurring in the age-interval $(x, x+1)$ are uniformly distributed over the interval, i.e., $a_x = 0.5$ for $x \geq 5$, irrespective of age, sex and race.

hence, from chiang's study, we can say that

$$L_x = l_x - d_x(1 - .5)$$

$$= l_x - \frac{1}{2} d_x \quad [x \geq 5]$$

$$= l_x - \frac{l_x - l_{x+1}}{2}$$

i.e. $$\boxed{L_x = \frac{l_x + l_{x+1}}{2}}$$

But in the age interval $(0, 1)$, it has been observed that within the very early weeks of life there is considerable variation in mortality; in fact, the risk of death at the time of birth is maximum. and then it is gradually decreases as age increases. Hence, deaths are not uniformly distributed over the interval $(0, 1)$ and consequently the formula

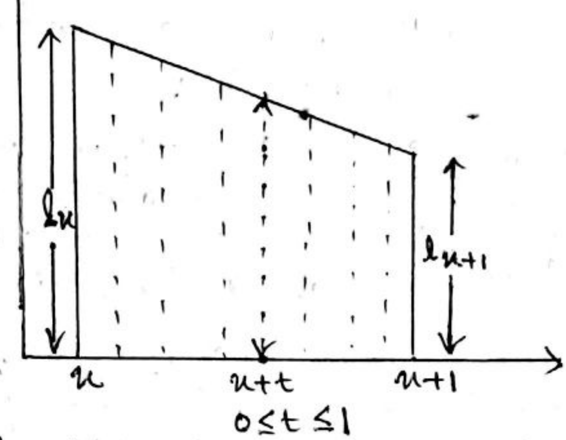
$L_x = \frac{l_x + l_{x+1}}{2}$ does not hold good for $x = 0$.

→ 1ST INTERPRETATION OF L_x.

(ii) If we assume that the death are uniformly distributed over the age interval x to $x+1$ then it is easy to note that l_{x+t} (i.e. the no. of persons alive at exact age $x+t$) will be a linear function of t for $0 \le t \le 1$.

Suppose, without loss of generality the function is

$$l_{x+t} = a + bt$$



Then as at $t = 0$

$$l_{x+t} = l_x$$

and at $t = 1$

$$l_{x+t} = l_{x+1}$$

then we get $-$ $a = l_x$ and $b = l_{x+1} - l_x$

$$\therefore \quad l_{x+t} = l_x + (l_{x+1} - l_x)t$$

hence we have $-$

$$L_x = \int_0^1 l_{x+t} \, dt$$

$$= \int_0^1 \left[l_x + (l_{x+1} - l_x)t \right] dt$$

i.e. $$\boxed{L_x = \frac{l_x + l_{x+1}}{2}}$$

Thus L_x may also be interpreted as the average size of the cohort between age x and $x+1$.

→ 2ND INTERPRETATION OF L_x.

C.U **Stationary Population :—** A popl^n is said to be stationary if it is of constant size and constant age and sex composition over time. Suppose in a popl^n, every year the number of births is exactly l_0 (say) and is equal to the number of deaths and these are distributed uniformly throughout the year, and the popl^n is not affected by emigration or immigration. Under the above conditions, in long run, the popl^n will be of the same size from year to year and will have the same age-distribution so that the popl^n becomes a stationary popl^n. C.U

Stable Population :— A popl^n is said to be stable if i) it has a fixed age and sex distribution, ii) constant mortality and fertility rates are experienced at each age, and iii) the popl^n is closed to emigration or immigration.

Hence, for a stable popl^n the overall rates of births and deaths remain constant and consequently such a popl^n increases at a constant rate, thus supporting the compound interest law of population growth.

C.U

* In a stable popl^n, mortality and fertility rates are constant but need not be equal. In particular, in a stable popl^n, if the constant overall birth and death rates are equal, then the popl^n size remains fixed and in this case stable popl^n becomes a stationary popl^n. A stationary popl^n is always stable but a stable popl^n needs to be stationary. C.U

Remark :→* Interpretation of L_x : Suppose in a popl^n, every year there are l_0 births and these being uniformly distributed over the year, and that the death rate at each age remains the same — same as the given by the q_x column of the life table. Let there be migration. Under the above conditions, in long run, the popl^n will be of the same size from year to year and will have same age distribution so that the number of persons between the ages x and $(x+1)$ denoted by L_x will always be the same. Thus the columns L_x (or T_x) of the life table may be interpreted as giving respectively the age-dist^n (or the no. of persons with age x or more) in a Stationary Population.

(vi) T_x : The total no. of years lived by the cohort after attaining age x or the total future years lived by l_x persons who have attained age x. We have, then —

$$T_x = \int_0^\infty l_{x+t}\,dt \quad \text{(or)} \quad T_x = \int_0^\infty l_{x+t}\,dt$$

$$= \int_0^1 l_{x+t}\,dt + \int_1^2 l_{x+t}\,dt + \cdots \qquad = \sum_{i=0}^{\infty} \int_{x+i}^{x+i+1} l_t\,dt$$

$$= \int_0^1 l_{x+t}\,dt + \int_0^1 l_{\overline{x+1}+t}\,dt + \cdots \qquad = \sum_{i=0}^{\infty} \int_0^1 l_{x+i+t}\,dt$$

$$= L_x + L_{x+1} + \cdots \qquad = \sum_{i=0}^{\infty} L_{x+i}$$

$$= L_x + L_{x+1} + L_{x+2} + \cdots$$

So that, T_x may be looked upon as a cumulative total of the L_x values, the value being summed by starting from the end of the L_x column.

$$\boxed{T_x = L_x + T_{x+1}}$$

(vii) $\mathring{e}_x$: The average no. of years lived after age x by each of the l_x persons who have attained age x.

(i.e) It gives the average no. of years a person of age x is likely to survive under the existing mortality rate. It is called the (complete) expectation of life or life expectancy at age x and is obtained from the relation

$$\boxed{\mathring{e}_x = \frac{T_x}{l_x} = \frac{\sum_{i=0}^{\infty} L_{x+i}}{l_x}}$$

$\mathring{e}_0$ is the expectation of life at age 0, is the average longevity of a person belonging to the given community.

(e.g.) Remark: → "The expectation of life at birth or at age '0' is 67 years, then all persons reaching age 65 have only two further years to live on the average" — This expectation at birth is calculated by averaging the life times of all persons born, some of whom die before 65; where as the expectation of life at

65 is calculated only in relation to those who die after age 65. Suppose our popln. consists of four persons who die at age 66, 55, 59, 88. Note that

$$e_0^o = \frac{1}{4}(66+55+59+88) = 67 \text{ years},$$

i.e., these four persons at the time of birth were expected to live on the average for 67 years. On the other hand, starting from age 65, we only consider those who are still alive, i.e., those who die at 66, 88 and they live 1 and 23 years, i.e., an average of 12 years beyond age 65. Hence, $e_{65}^o = 12$.

c.v. ▨ A closely related concept is that of the curtate expectation of life, denoted by e_x, which represents the average number of complete years of life lived after age x by any of the l_x persons who attain age x. We have —

(c.v.) $e_x = \dfrac{\sum_{t=1}^{\infty} l_{x+t}}{l_x}$, so that $\Rightarrow e_x^o \simeq e_x + \frac{1}{2}$.

▨ Relation between e_x^o and e_x : —

We know that —

$$e_x^o = \frac{T_x}{l_x}$$

$$= \frac{L_x + L_{x+1} + L_{x+2} + \cdots\cdots}{l_x}$$

$$\simeq \frac{\left\{\frac{1}{2}(l_x + l_{x+1}) + \frac{1}{2}(l_{x+1} + l_{x+2}) + \cdots\cdots\right\}}{l_x}$$

[assuming deaths (d_x) are uniformly distributed over the interval $(x, x+1)$]

$$\simeq \frac{1}{2}\cdot\frac{\cancel{l_x}}{\cancel{l_x}} + \sum_{t=1}^{\infty} l_{x+t} / l_x$$

i.e. $\boxed{e_x^o \simeq \frac{1}{2} + e_x}$

C.U

Remark :- The CDR of a life table stationary population, except the multiplier 1000, is equal to $(e_0^o)^{-1}$.

By definition, $e_0^o = \frac{T_0}{l_0}$, the expectation of life at age 0, is the average age at death. Note that

$$\frac{1}{e_0^o} = \frac{l_0}{T_0} = \frac{l_0}{\sum_{i=0}^{\infty} L_i}.$$

In a life table stationary popl^n, there are exactly l_0 births and deaths every year. Hence, in such a popl^n, every year there are l_0 death and in such a popl^n, the no. of persons between ages x and $x+1$ is L_x and $T_0 = L_0 + L_1 + L_2 + \cdots$, is the number of persons with age 0 or more; that is, T_0 is the total number of persons in a stationary popl^n.

Hence, $\frac{1}{e_0^o} = \frac{l_0}{T_0} = \frac{\text{Number of deaths in a year in a stationary popl}^n}{\text{Total number of persons in a stationary popl}^n \text{ in a year}}$

$$= \frac{CDR}{1000}, \text{ of a life table stationary popl}^n.$$

Some Important Results on Problems :→

1) Prove and interpret the relation:

$$e_x^o = \frac{T_x - T_{x+5}}{l_x} + \frac{l_{x+5}}{l_x} \cdot e_{x+5}^o = \frac{\sum_{t=x}^{x+4} L_t}{l_x} + \frac{l_{x+5}}{l_x} \cdot e_{x+5}^o.$$

Sol^n.→ $e_x^o = \frac{T_x}{l_x} = \frac{T_x + T_{x+5} - T_{x+5}}{l_x} = \frac{T_x - T_{x+5}}{l_x} + \frac{l_{x+5}}{l_x} \cdot \frac{T_{x+5}}{l_{x+5}}$

$$= \frac{T_x - T_{x+5}}{l_x} + \frac{l_{x+5}}{l_x} \cdot e_{x+5}^o$$

$$= \sum_{t=x}^{x+4} \frac{L_t}{l_x} + \frac{l_{x+5}}{l_x} \cdot e_{x+5}^o. \text{ (Proved)}$$

The complete expectation of life at age x (e_x^o) is the average years lived between x and $x+5$ $\left(\sum_{t=x}^{x+4} L_t / l_x\right)$ plus the complete expectation of life beyond age $(x+5)$ to those persons who live to $x+5$.

C.U

2) Derive, by starting from a suitable functional form for l_x, the formula —

(a) $L_x = \dfrac{l_x + l_{x+1}}{2}$

(b) $L_x = (l_x - l_{x+1}) / (\ln l_x - \ln l_{x+1}) = - \dfrac{d_x}{\ln p_x}$.

Soln. →

(a) $$L_x = \int_x^{x+1} l_t \, dt = \int_0^1 l_{x+t} \, dt$$

Let, $l_{x+t} = a + bt$.

$\Rightarrow l_x = a$ and $l_{x+1} = a + b \Rightarrow b = l_{x+1} - a = l_{x+1} - l_x$

Now, $$L_x = \int_0^1 l_{x+t} \, dt = \int_0^1 (a + bt) \, dt$$

$$= a + \frac{b}{2}$$

$$= l_x + \frac{l_{x+1} - l_x}{2}$$

i.e. $$\boxed{L_x = \frac{l_x + l_{x+1}}{2}}$$ (Proved)

(b) Let, $l_{x+t} = ab^t$

$\Rightarrow l_x = a$, $l_{x+1} = ab = l_x \cdot b$

$$\Rightarrow \boxed{\frac{l_{x+1}}{l_x} = b}.$$

So, $$L_x = \int_0^1 ab^t \, dt = a \cdot \left[\frac{b^t}{\ln b} \right]_0^1$$

$$= \frac{a}{\ln b} [b - 1]$$

$$= \frac{l_x}{\ln\left(\frac{l_{x+1}}{l_x}\right)} \left[\frac{l_{x+1}}{l_x} - 1 \right]$$

i.e. $$L_x = \frac{l_{x+1} - l_x}{\ln l_{x+1} - \ln l_x}$$

$$\Rightarrow \boxed{L_x = \frac{l_x - l_{x+1}}{\ln l_x - \ln l_{x+1}}}$$

i.e. $$\boxed{L_x = \frac{-d_x}{\ln p_x}}$$ (Proved)

3) Show that — $p_x = \dfrac{e_x}{1+e_{x+1}}$

<u>Sol^n.</u> → $1+e_{x+1} = 1 + \dfrac{\sum_{i=1}^{\infty} l_{x+1+i}}{l_{x+1}}$

$$= \frac{l_{x+1} + \sum_{i=1}^{\infty} l_{x+1+i}}{l_{x+1}}$$

$$= \frac{\sum_{i=1}^{\infty} l_{x+i}/l_x}{l_{x+1}/l_x}$$

i.e. $1+e_{x+1} = \dfrac{e_x}{p_x}$

$$\Rightarrow \boxed{p_x = \frac{e_x}{1+e_{x+1}}} \text{ (Proved)}$$

C.U

4) Show that —

i) $\dfrac{e_x e_{x+1} \cdots\cdots e_{x+n-1}}{(1+e_{x+1})(1+e_{x+2}) \cdots\cdots (1+e_{x+n})} = {}_nP_x$

<u>Sol^n.</u> → L.H.S = $\left(\dfrac{e_x}{1+e_{x+1}}\right)\left(\dfrac{e_{x+1}}{1+e_{x+2}}\right)\cdots\cdots\left(\dfrac{e_{x+n-1}}{1+e_{x+n}}\right)$

$$= \left(\frac{\dfrac{l_{x+1}+l_{x+2}+\cdots\cdots}{l_x}}{1+\dfrac{l_{x+2}+l_{x+3}+\cdots\cdots}{l_{x+1}}}\right)\left(\frac{\dfrac{l_{x+2}+l_{x+3}+\cdots\cdots}{l_{x+1}}}{1+\dfrac{l_{x+3}+l_{x+4}+\cdots\cdots}{l_{x+2}}}\right)\cdots\cdots\left(\frac{\dfrac{l_{x+n}+l_{x+n+1}+\cdots\cdots}{l_{x+n-1}}}{1+\dfrac{l_{x+n+1}+l_{x+n+2}+\cdots\cdots}{l_{x+n}}}\right)$$

$$= \frac{\cancel{l_{x+1}}}{l_x} \cdot \frac{l_{x+2}}{\cancel{l_{x+1}}} \cdots\cdots \frac{l_{x+n}}{l_{x+n-1}}$$

$$= \frac{l_{x+n}}{l_x} = {}_nP_x = \text{R.H.S (Proved)}$$

Or

L.H.S = $\left(\dfrac{e_x}{1+e_{x+1}}\right)\left(\dfrac{e_{x+1}}{1+e_{x+2}}\right)\cdots\cdots\left(\dfrac{e_{x+n-1}}{1+e_{x+n}}\right)$

$= p_x \cdot p_{x+1} \cdots\cdots p_{x+n-1}$ [From 3)]

$= \dfrac{l_{x+1}}{l_x} \cdot \dfrac{l_{x+2}}{l_{x+1}} \cdots\cdots \dfrac{l_{x+n}}{l_{x+n-1}} = \dfrac{l_{x+n}}{l_x} = {}_nP_x = $ R.H.S

(Proved)

$\boxed{\text{C.U}}$

ii) $\left({}_{\frac{1}{2}}p_x + {}_{\frac{3}{2}}p_x + {}_{\frac{5}{2}}p_x + \cdots \right) - \left({}_1p_x + {}_2p_x + {}_3p_x + \cdots \right)$

$= \frac{1}{2}$, approximately.

Soln. $L.H.S = \left(\frac{l_{x+1/2}}{l_x} + \frac{l_{x+3/2}}{l_x} + \cdots \right) - \left(\frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \cdots \right)$

Under uniform distn. of deaths over each age interval, we have $l_{x+1/2} = (l_x + l_{x+1})/2$.

$$= \frac{1}{l_x}\left\{ \left(\frac{l_x + l_{x+1}}{2} \right) + \left(\frac{l_{x+1} + l_{x+2}}{2} \right) + \cdots \right\}$$

$$- \frac{1}{l_x}\left\{ l_{x+1} + l_{x+2} + \cdots \right\}$$

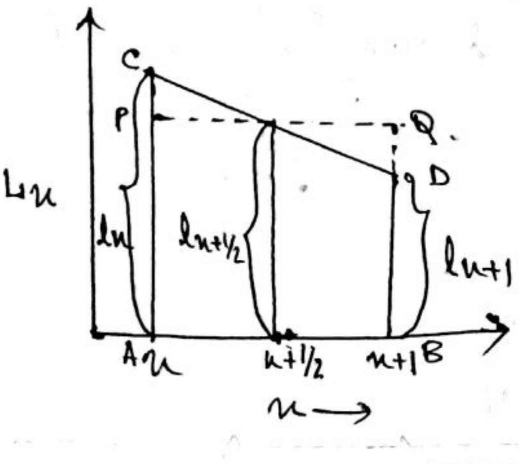
$$= \frac{1}{l_x}\left(\frac{l_x}{2} + l_{x+1} + l_{x+2} + \cdots \right)$$

$$- \frac{1}{l_x}\left(l_{x+1} + l_{x+2} + \cdots \right)$$

$$= \frac{1}{2} + \frac{1}{l_x}\left(\cancel{l_{x+1} + l_{x+2} + \cdots} \right) - \frac{1}{l_x}\left(\cancel{l_{x+1} + l_{x+2} + \cdots} \right)$$

$$= \frac{1}{2} = RHS \text{ (Proved)}$$

Lemma: $\boxed{l_{x+\frac{1}{2}} = \frac{l_x + l_{x+1}}{2}}$



L_x = Area of the trapezium ABCD

$= \frac{1}{2}(l_x + l_{x+1})$

= Area of rectangle ABPQ

$= l_{x+1/2} \cdot 1$

$= l_{x+1/2}$

$\therefore \boxed{l_{x+1/2} = \frac{1}{2}(l_x + l_{x+1})}$

$\boxed{\text{C.U}}$

5) Show that,

$$l_{x+1} + \int_0^1 t\left(-\frac{dl_{x+t}}{dt}\right)dt = \int_0^1 l_{x+t}\,dt$$

Soln.

$$L_x = \int_0^1 l_{x+t}\,dt = \int_0^1 (l_{x+t}\cdot 1)\,dt$$

$$= \left[l_{x+t}\cdot t\right]_0^1 - \int_0^1 \frac{d}{dt}(l_{x+t})\cdot t\,dt$$

$$= l_{x+1} + \int_0^1 t\left(-\frac{dl_{x+t}}{dt}\right)dt \quad [\text{Proved}]$$

Alt. method :→

$$l_{x+1} + \int_0^1 t\left(-\frac{d\,l_{x+t}}{dt}\right)dt$$

$$= l_{x+1} + \left[-t\cdot l_{x+t}\right]_0^1 - \int_0^1 \frac{d}{dt}(t)(-l_{x+t})\,dt$$

$$= \cancel{l_{x+1}} - \cancel{l_{x+1}} + \int_0^1 l_{x+t}\,dt$$

$$= \int_0^1 l_{x+t}\,dt \quad [\text{Proved}]$$

$\boxed{\text{C.U}}$

6) Show that — $T_x = \sum_{r=x}^{\infty} l_{r+1} + \int_0^1 t\left\{-\frac{d}{dt}\left(\sum_{r=x}^{\infty} l_{r+t}\right)\right\}dt$

Soln. $T_x = L_x + L_{x+1} + \cdots\cdots$

$$= \sum_{r=x}^{\infty} L_r$$

$$= \sum_{r=x}^{\infty}\left[\int_0^1 l_{r+t}\,dt\right]$$

$$= \sum_{r=x}^{\infty}\left\{\left[l_{r+t}\cdot t\right]_0^1 - \int_0^1 \frac{d}{dt}(l_{r+t})\cdot t\,dt\right\}$$

$$= \sum_{r=x}^{\infty} l_{r+1} + \int_0^1 t\left\{-\frac{d}{dt}\left(\sum_{r=x}^{\infty} l_{r+t}\right)\right\}dt$$

[Proved]

7) $\boxed{C.U}$ $e_x^0 = 1 - \frac{q_x}{2} + \sum_{i=1}^{\infty} \frac{l_{x+i}}{l_x} - \frac{1}{2}\sum_{i=1}^{\infty} \frac{l_{x+i}\, q_{x+i}}{l_x}$

$$\sum_{i=0}^{\infty} \frac{l_{x+i}}{l_x}\left(1 - \frac{q_{x+i}}{2}\right)$$

$$= \sum_{i=0}^{\infty} \frac{l_{x+i}}{l_x}\left(1 - \frac{d_{x+i}}{2l_{x+i}}\right)$$

$$= \sum_{i=0}^{\infty} \frac{\cancel{l_{x+i}}}{l_x} \cdot \frac{l_{x+i} + (l_{x+i} - d_{x+i})}{2\cancel{l_{x+i}}}$$

$$= \sum_{i=0}^{\infty} \left(\frac{l_{x+i} + l_{x+i+1}}{2}\right) \Big/ l_x$$

$$= \sum_{i=0}^{\infty} L_{x+i}/l_x$$ [Under uniform distribution of death]

$$= \frac{T_x}{l_x}$$

$$= e_x^0$$ [Proved]

$\boxed{C.U}$

8) $e_x^0 = \frac{1}{2} + \sum_{i=1}^{\infty} \frac{i\,d_{x+i}}{l_x}$ under the assumption that $l_{x+t} = a + bt$.

Sol$\underline{n}$

$$\frac{1}{2} + \sum_{i=1}^{\infty} \frac{i\,d_{x+i}}{l_x}$$

$$= \frac{1}{2} + \frac{1}{l_x}\left\{(l_{x+1} - l_{x+2}) + 2(l_{x+2} - l_{x+3}) + 3(l_{x+3} - l_{x+4}) + \cdots\cdots\right\}$$

$$= \frac{1}{2} + \frac{1}{l_x}\left(l_{x+1} + l_{x+2} + \cdots\cdots\right)$$

$$= \frac{1}{2} + e_x$$

$$\simeq e_x^0$$ (Proved)

[C.V.]

9) Given a complete life table, obtain the probability of the following events in terms of usual notations.
i) a randomly selected person is of age 25 l.b.d.
ii) A person of age 30 l.b.d will die between ages 45 and 50 l.b.d.
iii) At least one of two persons aged 25 l.b.d and 30 l.b.d. will survive till age 50 l.b.d.

Soln. →

i) Required Probability $= \dfrac{\text{number of persons of age 25 l.b.d}}{l_0}$

$= \dfrac{L_{25}}{l_0}$ [In stationary population or life-table population the number of persons in $[x, x+1)$ age-group is L_x]

ii) Required Probability =

$= \dfrac{\{\text{the number of persons dying between ages 45 and 50 l.b.d.}\}}{\text{the number of persons of aged 30}}$

$= \dfrac{\{\text{the number of persons who have reached age 45 but died before the age 51}\}}{l_{30}}$

$= \dfrac{l_{45} - l_{51}}{l_{30}}$.

iii) Define, A = the event that a person aged 25 l.b.d will survive till age 50 l.b.d. and
B = the event that a person aged 30 l.b.d. will survive till age 50 l.b.d.

Required Probability $= P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$$= \frac{l_{50}}{l_{25}} + \frac{l_{50}}{l_{30}} - \frac{l_{50}}{l_{25}} \cdot \frac{l_{50}}{l_{30}}.$$

(C.U)

10) If the average annual probability of dying between exact age 20 and 30 is 0·001, compute the value l_{30}/l_{20}.

Soln. →
$$\frac{\frac{d_{20}}{l_{20}} + \frac{d_{21}}{l_{20}} + \cdots + \frac{d_{29}}{l_{20}}}{10} = 0{\cdot}001$$

$$\Rightarrow \frac{d_{20} + d_{21} + \cdots + d_{29}}{l_{20}} = 0{\cdot}01$$

Now, $\frac{l_{30}}{l_{20}} = \frac{l_{20} - (d_{20} + d_{21} + \cdots + d_{29})}{l_{20}} = 1 - 0{\cdot}01$

$= 0{\cdot}99$ (Ans)

11) If $a_x = \frac{L_x - l_{x+1}}{l_x - l_{x+1}}$ and $m_x = \frac{l_x - l_{x+1}}{L_x}$ are both known, show that they together determine $\frac{l_{x+1}}{l_x}$.

Soln. → $L_x = \frac{l_x - l_{x+1}}{m_x}$ and $a_x(l_x - l_{x+1}) = \frac{l_x - l_{x+1}}{m_x} - l_{x+1}$

or, $l_{x+1} = \frac{l_x(1 - m_x a_x)}{1 + m_x(1 - a_x)}$

or, $\boxed{\frac{l_{x+1}}{l_x} = \frac{1 - m_x a_x}{1 + m_x(1 - a_x)}}$

12) If (a) $l_x = \left(\frac{1}{\frac{x}{5}+1}\right)^{0.1}$, (b) $l(x) = \sqrt{1-\frac{x}{100}}$,
what is e_0^o? what is $_5q_x$? In each life table l_x is defined for $0 \le x \le 100$.

Soln. →

(a) $l_x = \left(\frac{x}{5}+1\right)^{-0.1}$

By definition,

$$e_0^o = \frac{1}{l_0}\int_0^{100} l(x)\,dx$$

$$= 1.\int_0^{100} \left(\frac{x}{5}+1\right)^{-0.1} dx$$

$$= \left[\frac{\left(\frac{x}{5}+1\right)^{0.9}}{0.9}\right]_0^{100}$$

$$= \frac{(21)^{0.9}-1}{0.9}$$

Now, $_5q_x = 1 - {}_5p_x = 1 - \frac{l_{x+5}}{l_x}$

$$= 1 - \frac{\left(\frac{x+5}{5}+1\right)^{-0.1}}{\left(\frac{x}{5}+1\right)^{-0.1}}$$

$$= 1 - \frac{\left(\frac{x}{5}+2\right)^{-0.1}}{\left(\frac{x}{5}+1\right)^{-0.1}} = 1 - \left(\frac{x+5}{x+10}\right)^{0.1}$$

(b) $l(x) = \sqrt{1-\frac{x}{100}}$, $0 \le x \le 100$

By defn., $e_0^o = \frac{1}{l_0}\int_0^{100} l(x)\,dx = \frac{1}{1}\int_0^{100}\sqrt{1-\frac{x}{100}}\,dx$

$$= \left[\frac{\left(1-\frac{x}{100}\right)^{3/2}}{\frac{3}{2}\cdot\left(-\frac{1}{100}\right)}\right]_0^{100}$$

$$= 66.67 \text{ years}$$

U.9

Again, $_5q_x = 1 - \frac{l_{x+5}}{l_x}$

$$= 1 - \sqrt{\frac{100-(x+5)}{100-x}} = 1 - \sqrt{1-\left(\frac{5}{100-x}\right)} \; *$$

C.V

Rates of Mortality and Probabilities of Death:

The two functions q_x and m_x represent different concepts, q_x is the probability that a person of exact age x will die before reaching age $(x+1)$. It follows that $q_x = \frac{d_x}{l_x}$. Clearly —,

$$p_x = 1 - q_x = \frac{l_x - d_x}{l_x} = \frac{l_{x+1}}{l_x}$$

is the probability that a person of exact age x will survive till his/her next birthday.

m_x represents the average risk of dying a person belonging to the age group x to $x+1$, in that age group. Hence, we have —

$$m_x = \frac{d_x}{\int_0^1 l_{x+t}\,dt} = \frac{d_x}{L_x}$$

If the deaths are uniformly distributed over the year of age (as they are, approximately, except at birth and at extreme old age) then —

$$L_x = \int_0^1 l_{x+t}\,dt$$

$\therefore$ $L_x = l_x - \frac{1}{2} d_x$, which is approximately the population of the middle of the year of age. Then we may write.

$$m_x = \frac{d_x}{l_x - \frac{1}{2} d_x}$$

$$= \frac{d_x / l_x}{1 - 1/2 \cdot d_x / l_x} = \frac{q_x}{1 - \frac{1}{2} q_x} \text{. and}$$

C.V

$$\boxed{q_x = \frac{2m_x}{2+m_x}}, \quad \boxed{p_x = \frac{2-m_x}{2+m_x}} \text{ —— } (*)$$

* The quantities m_x are estimated by the observed ASDRs for the community.

[C.V] It has been pointed out earlier that deaths are not uniformly distributed over the early ages of life specially for age 0; mortality is generally very high in the first few weeks after birth and then it diminishes sharply. Then we have the formula —

$L_x = l_x - (1-a_x) d_x$. (explained earlier).

In general, we have — $m_x = \dfrac{d_x}{l_x - (1-a_x) d_x}$

$$= \frac{d_x / l_x}{\frac{l_x}{l_x} - \frac{(1-a_x) d_x}{l_x}}$$

$$\therefore \boxed{m_x = \frac{q_x}{1-(1-a_x) q_x}} \quad \forall x$$

i.e. $\boxed{q_x = \frac{m_x}{1+(1-a_x) m_x}}$ $\forall x$ (**)

* at $x=0$.

In case $a_x = \frac{1}{2}$, i.e., in case the deaths are uniformly distributed, we have the formula (*)

[C.V]

CONSTRUCTION OF LIFE TABLE*: —

Why is q_x called Pivotal column? [The values of q_x are obtained from (*) and from (**), where the corresponding $m_x = \frac{d_x}{L_x}$ are computed on the basis of census records and death registration data. It will be seen, as discussed below, that the complete life table can be constructed if we have the quantities q_x or p_x for all x; the only other data which is needed is the radix l_0. The q_x column is [C.V] thus called the pivotal column of the life table. Starting with radix l_0 and q_x, $x = 0, 1, 2, \ldots\ldots$, we have $d_0 = l_0 q_0 \Rightarrow l_1 = l_0 - d_0$; $d_1 = l_1 q_1 \Rightarrow l_2 = l_1 - d_1$ and so on. From these l_x values, we can compute the columns L_x, T_x and $\mathring{e}_x$ of the table, by using the relations: $L_x = \frac{1}{2}(l_x + l_{x+1})$, $T_x = \sum_{i=0}^{\infty} L_{x+i}$, $\mathring{e}_x = \frac{T_x}{l_x}$.

$$L_x = \frac{l_x}{m_x}\left\{1 - \frac{l_{x+1}}{l_x}\right\} = \frac{l_x}{m_x}\left\{1 - \frac{1 - a_x m_x}{1 + m_x(1-a_x)}\right\}$$

i.e. $$\boxed{L_x = \frac{l_x}{1 + m_x(1-a_x)}} \quad \text{———} (***)$$

If a_x and m_x values are given, then we can evaluate $\frac{l_{x+1}}{l_x}$, $\forall\, x = 0, 1, 2, \ldots$ and if l_0 is given, then $l_1 = \frac{l_1}{l_0} \cdot l_0$ and $l_2 = \frac{l_2}{l_1} \cdot l_1, \ldots$, can be obtained. After evaluating l_x', we can evaluate L_x from (* **). Hence the other columns can be filled up by usual procedure.]

$\boxed{\text{C.V}}$

ABRIDGED LIFE TABLE : [We have discussed so far what is called a complete life table in which the age interval is year throughout the table and the life table functions such as l_x, d_x, q_x, m_x, etc are given for all integral values of x, on the other hand in abridged life table, as the name suggests, the values of these functions are given for the age-interval $[x, x+n)$. A typical life table consists of the following columns:

i) l_x, the number of persons out of a cohort l_0, living at the beginning of the interval x to $x+n$.

ii) ${}_nq_x$, the probability that a person of the cohort at age x will die before reaching age $x+n$, is given by — ${}_nq_x = 1 - {}_np_x = 1 - \frac{l_{x+n}}{l_x}$.

iii) ${}_nd_x$, the number of deaths in the age interval x to $x+n$, is given by ${}_nd_x = {}_nq_x \times l_x$.

iv) ${}_nL_x$, the number of members of the life table stationary popln. in the age group $[x, x+n)$ OR the total number of years lived by the cohort while in the given age-group, is given by —

$${}_nL_x = \int_0^n l_{x+t}\, dt$$

(v) $\underline{T_x} = \int_0^\infty l_{x+t}\,dt$, is the number of years lived by the cohort while at age x and thereafter, or the number of members of the life-table stationary population of age x or above. Now $T_x = \int_0^n l_{x+t}\,dt + \int_n^\infty l_{x+t}\,dt$

$$= {}_nL_x + \int_0^\infty l_{x+n+t}\,dt$$

$$= {}_nL_x + T_{x+n}.$$

(vi) $\underline{\mathring{e}_x}$, the expectation of life at age x, which equals $\frac{T_x}{l_x}$.

For the early years of life, the mortality is generally very high in the first year and then it diminishes rapidly. Hence, we provided all the columns for first five years at one year age interval, then we consider age-interval x to $x+5$, in an abridged life table.]

Ex. 1) In a stationary popln. subject to $l(x) = \frac{e^{-.02x} + 1 - .01x}{2}$, $0 \le x \le 110$, What is the value of $\mathring{e}_0$. At what age is the absolute number of deaths a maximum?

Soln. → The last person dies at age 110.

$$\mathring{e}_0 = \frac{1}{l(0)} \cdot \int_0^{110} l(x)\,dx = \frac{1}{1} \cdot \int_0^{110} \left\{ \frac{e^{-0.02x} + 1 - 0.01x}{2} \right\} dx$$

$$= \frac{1}{2}\left[-\frac{e^{-0.02x}}{0.02} + x - 0.01\frac{x^2}{2} \right]_0^{110}$$

$$= \frac{1}{2}\left[-\frac{e^{-2.2}}{0.02} + 110 - \frac{121}{2} \right] \simeq 47$$

Now, $d(x) = l(x) - l(x+1) = \frac{1}{2}\left\{ e^{-0.02x} - e^{-0.02(x+1)} + 0.01 \right\}$

Now, $d'(x) = +\frac{1}{2}(-0.02)\left\{ e^{-0.02x} - e^{-0.02(x+1)} \right\}$

$$= (-0.01)(e^{-0.02x})(1 - e^{-0.02}) < 0, \ \forall x \ge 0$$

Hence $d(x)$ is decreasing in $x \in [0, 110]$

Therefore $d(x)$ is maximum at age $x = 0$.

Ex:- 2) In a population, the probability of dying over each 10-year interval equal to $\frac{1}{2}$, calculate a suitable formula for l_x, $6 \le x \le 66$. Find the change that of three persons aged 6 at least two will be alive 20 years later.

Soln →

$${}_{10}P_6 = \frac{l_{16}}{l_6} = \frac{1}{2}, \quad \frac{l_{26}}{l_{16}} = \frac{1}{2}, \text{ etc.}$$

Hence $\frac{l_x}{l_6} = \frac{l_{16}}{l_6} \cdot \frac{l_{26}}{l_{16}} \cdots\cdots \frac{l_x}{l_{x-10}} = \left(\frac{1}{2}\right)^{\frac{x-6}{10}}$

$$\Rightarrow l_x = l_6 \left(\frac{1}{2}\right)^{\frac{x-6}{10}}.$$

Required probability $= \left(\frac{l_{26}}{l_6}\right)^3 + \binom{3}{2}\left(\frac{l_{26}}{l_6}\right)^2\left(1 - \frac{l_{26}}{l_6}\right)$

$$= \left(\frac{1}{4}\right)^3 + \binom{3}{2}\left(\frac{1}{4}\right)^2 \cdot \frac{3}{4}$$

$$= \frac{5}{32}.$$

LIFE TABLE & NUMERICALS

1) x : Age of the person.

2) l_x : No. of persons survive into exact age x out of an assume no. l_0 live births.
(l_0 is usually taken as 10^5).
l_x is a decreasing function of x.

3) ${}_n d_x$: No. of life table deaths in the age group x to $x+n$. $\therefore {}_n d_x = l_x - l_{x+n}$

For $n=1$, $d_x = l_x - l_{x+1}$.

4) ${}_n q_x$: The probability that a person of exact age x will die before reaching age $x+n$.

$$\therefore \quad {}_n q_x = \frac{{}_n d_x}{l_x} = 1 - \frac{l_{x+n}}{l_x}$$

For $n=1$, $q_x = 1 - \frac{l_{x+1}}{l_x}$.

6) ${}_n P_x$: $1 - {}_n q_x$

$$= \frac{l_{x+n}}{l_x}.$$

6) $_nL_x$: the no. of years lived, in the aggregate, by the cohort of l_0 persons between age x to $\overline{x+n}$.

$$\therefore {}_nL_x = \int^n l_{x+t}\,dt$$

Assuming that: L_x is a decreasing function of x i.e. deaths are uniformly distributed over the age interval x to $\overline{x+n}$.

For, $n=1$, we can write $l_{x+t} = l_x - t d_x$

$$\text{so, } \int_0^1 l_{x+t}\,dt = \int_0^1 (l_x - t d_x)\,dt$$

$$L_x = l_x - \frac{d_x}{2}$$

$$= \frac{l_x + l_{x+1}}{2} = l_{x+1/2}$$

7) T_x : The total future years lived by l_x persons who have attained age x.

$$\therefore T_x = L_x + L_{x+1} + \cdots\cdots$$

$$= \int_0^\infty L_{x+t}\,dt.$$

8) $_nm_x$: Age specific death rate for the age group x to $\overline{x+n}$ for the life table population.

$$\therefore {}_nm_x = \frac{{}_nd_x}{{}_nL_x}.$$

— It is used to smooth out irregularities.

9) e_x : Curtate expection of life.
Average no. of complete years lived by each of the l_x person often attaining age x.

10) $\mathring{e}_x$: Average no. of years lived by the l_x persons after attaining age x. It is called complete expectation of life.

$$\therefore \mathring{e}_x = \frac{T_x}{l_x} = \frac{\sum_{i=0}^{\infty} L_{x+i}}{l_x} \quad \Bigg| \quad \therefore e_x = \frac{\sum_{t=1}^{\infty} l_{x+t}}{l_x}$$

$$\boxed{\mathring{e}_x \approx e_x + \frac{1}{2}}$$

1) show that $\rightarrow$ $_nP_x = \prod_{i=0}^{n-1} P_{x+i}$

Ans: $\rightarrow$

$$_nP_x = \frac{l_{x+n}}{l_x}$$

$$= \frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{x+n-1}}{l_{x+n-2}} \cdots\cdots \frac{l_{x+1}}{l_x}$$

$$= \prod_{i=0}^{n-1} P_{x+i}$$

2) show that $\rightarrow$ $P_x = \frac{e_x}{1+e_{x+1}}$.

Ans: $\rightarrow$

$$e_x l_x = l_{x+1} + l_{x+2} + \cdots\cdots$$

$$e_{x+1} l_{x+1} = l_{x+2} + l_{x+3} + \cdots\cdots$$

$$e_x l_x - e_{x+1} l_{x+1} = l_{x+1}$$

$$\Rightarrow e_x l_x = l_{x+1}(1+e_{x+1})$$

$$\Rightarrow \frac{e_x}{1+e_{x+1}} = \frac{l_{x+1}}{l_x} = P_x .$$

(c.u) 3) $\boxed{\cancel{(\tfrac{1}{2} P_x + \tfrac{3}{2} P_x + \tfrac{5}{2} P_x + \cdots\cdots) = (P_x + 2P_x + 3P_x + \cdots)}}$

Show that $\rightarrow$ $\frac{m_x}{1+m_x} < q_x < m_x < \frac{q_x}{1-q_x}$.

Ans: $\rightarrow$

$$q_x = \frac{d_x}{l_x}$$

$$m_x = \frac{d_x}{L_x}$$

$$L_x = \int_0^1 l_{x+t}\, dt$$

$$l_{x+1} < L_x < l_x .$$

$$\Rightarrow \frac{1}{l_x} < \frac{1}{L_x} < \frac{1}{l_{x+1}}$$

$$\Rightarrow \frac{d_x}{l_x} < \frac{d_x}{L_x} < \frac{d_x}{l_{x+1}}$$

$$\therefore q_x < m_x < \frac{d_x}{l_x - d_x}$$

$$\Rightarrow q_x < m_x < \frac{q_x}{1-q_x}$$

Hence, the result.

$$m_x < \frac{q_x}{1-q_x}$$

$$m_x - m_x q_x < q_x$$

$$\Rightarrow q_x(1+m_x) > m_x$$

$$\Rightarrow q_x > \frac{m_x}{1+m_x} .$$

4) Find the expression of L_x.

Ans: →

$$L_x = \int_0^1 l_{x+t}\, dt$$

So, it lies between l_x and l_{x+1}.

We take, $L_x = f_x l_x + (1-f_x) l_{x+1}$ [convex combination]

where f_x is the weight function depending on x.

$$\frac{L_x}{d_x} = \frac{f_x l_x}{d_x} + \frac{(1-f_x) l_{x+1}}{d_x}$$

$$\Rightarrow \frac{1}{m_x} = \frac{f_x(l_x - l_{x+1})}{d_x} + \frac{l_{x+1}}{d_x}$$

$$\Rightarrow \frac{1}{m_x} = f_x + \frac{l_{x+1}}{d_x}$$

$$\Rightarrow \frac{1}{m_x} = f_x + \frac{l_x - d_x}{d_x}$$

$$\Rightarrow \frac{1}{m_x} = f_x + \frac{1}{q_x} - 1$$

$$\Rightarrow f_x = 1 + \frac{1}{m_x} - \frac{1}{q_x}$$

$$\therefore L_x = l_x\left(1 + \frac{1}{m_x} - \frac{1}{q_x}\right) + l_{x+1}\left(\frac{1}{q_x} - \frac{1}{m_x}\right)$$

***

FORCE OF MORTALITY → Since deaths occur at every year and at every fraction of time l_x is a continuous function of x. At age x, the rate of decreasing l_x is defined as $\lim_{t \to 0}\left(\frac{l_{x+t} - l_x}{t}\right) = -\frac{d}{dx}(l_x)$.

The force of mortality at age x is defined as the rate of instantaneous decreasing l_x to the value of l_x,

i.e. $$\mu_x = -\frac{1}{l_x}\cdot\frac{d}{dx}(l_x)$$

$$= -\frac{d}{dx}(\log l_x).$$

5) Prove that → $m_x \approx \mu_{x+\frac{1}{2}}$. (***)

Ans:→ $\frac{d}{dx}(L_x) = \frac{d}{dx}\left(\int_0^1 l_{x+t}\,dt\right)$

$= \int_0^1 \frac{d}{dx} l_{x+t}\,dt$ # | $\frac{d}{dx} l_{x+t} = \frac{d}{dt} l_{x+t}$

(supposing that the function l_{x+t} is sufficiently well-behaved) $= \int_0^1 \frac{d}{dt} l_{x+t}\,dt$ | $\frac{d}{dx} l_{x+t} = \frac{d}{d(x+t)} l_{x+t} \cdot \frac{d(x+t)}{dx}$

$= \int_0^1 d\,l_{x+t}$ | $= \frac{d}{d(x+t)} \cdot l_{x+t}$

$= l_{x+1} - l_x$

$= -d_x$

$$m_x = \frac{d_x}{L_x}$$

$$= -\frac{1}{L_x} \cdot \frac{d}{dx}(L_x)$$

$$= -\frac{1}{l_{x+1/2}} \cdot \frac{d}{dx}\left(l_{x+1/2}\right)$$

[∵ death are uniformly distributed]

$$= \mu_{x+1/2}$$

6) **Show that → $\frac{d}{dx}(\mathring{e}_x) = \mu_x \mathring{e}_x - 1$. / $\mu_x = \left(1 + \frac{d}{dx}\mathring{e}_x\right)/\mathring{e}_x$.

Ans:→

$$\mathring{e}_x = \frac{T_x}{l_x}$$

$$\frac{d}{dx}\mathring{e}_x = \frac{\frac{d}{dx}(T_x)\, l_x - T_x \frac{d}{dx}(l_x)}{l_x^2}$$

$$= \frac{1}{l_x} \cdot \frac{d}{dx}(T_x) - \frac{T_x}{l_x} \cdot \frac{1}{l_x} \cdot \frac{d}{dx}(l_x)$$

$$= \frac{1}{l_x} \cdot \frac{d}{dx}(T_x) + \mathring{e}_x \mu_x .$$

Now, $T_x = \int_x^\infty l_t\,dt$

$$\Rightarrow \frac{d}{dx}(T_x) = -l_x \quad [\because l_\infty = 0]$$

$$\therefore \frac{d}{dx}(\mathring{e}_x) = -1 + \mathring{e}_x \mu_x$$

7) Prove that the average age at death of those persons who die between age x and age y is

$$x + \{T_x - T_y - (y-x)l_y\}/(l_x - l_y).$$

Ans →

Average age at death of the persons

$$= x + \frac{\int_0^{y-x} t\, l_{x+t}\, \mu_{x+t}\, dt}{\int_0^{y-x} l_{x+t}\, \mu_{x+t}\, dt} \quad \text{---} (*)$$

Now, $\mu_{x+t} = \frac{1}{l_{x+t}} \cdot \left(-\frac{d}{dt} l_{x+t}\right)$

$$\Rightarrow l_{x+t}\, \mu_{x+t} = -\frac{d}{dt} l_{x+t}$$

$$\Rightarrow \int_0^{y-x} l_{x+t}\, \mu_{x+t}\, dt = -\int_0^{y-x} \left(\frac{d}{dt} l_{x+t}\right) dt$$

$$= \left[-l_{x+t}\right]_0^{y-x}$$

$$= -l_y + l_x \quad \text{---} (1)$$

& $\int_0^{y-x} t\, l_{x+t}\, \mu_{x+t}\, dt = \int_0^{y-x} t \left(-\frac{d}{dt} l_{x+t}\right) dt$

$$= \left[-t \cdot l_{x+t}\right]_0^{y-x} + \int_0^{y-x} l_{x+t}\, dt$$

$$= -(y-x)l_y + \int_x^{\infty} l_t\, dt - \int_y^{\infty} l_t\, dt$$

$$= T_x - T_y - (y-x)l_y \quad \text{---} (2)$$

∴ Required number is $= x + \frac{T_x - T_y - (y-x)l_y}{l_x - l_y}$

[Using (1) & (2) in (*)]

[Proved]

8) You are given that

$$1/\mu_x = (a_0 + a_1 x)(b_0 + b_1 x).$$

Find an expression for l_x.

Ans:
$$\mu_x = \frac{1}{(a_0+a_1x)(b_0+b_1x)}$$

$$\Rightarrow \mu_x = \frac{1}{a_1b_0 - b_1a_0}\left[\frac{a_1}{a_0+a_1x} - \frac{b_1}{b_0+b_1x}\right]$$

$$\Rightarrow -\frac{d}{dx}\ln l_x = \frac{1}{a_1b_0 - b_1a_0}\left[\frac{a_1}{a_0+a_1x} - \frac{b_1}{b_0+b_1x}\right]$$

$$\Rightarrow -\int d\ln l_x = \frac{1}{a_1b_0 - b_1a_0}\left[\int \frac{a_1}{a_0+a_1x}dx - \int \frac{b_1}{b_0+b_1x}dx\right] + \ln l.$$

$$= \frac{1}{a_1b_0 - b_1a_0}\left[\ln(a_0+a_1x) - \ln(b_0+b_1x)\right] + \ln l$$

$$= \ln c\left(\frac{a_0+a_1x}{b_0+b_1x}\right)^{\frac{1}{a_1b_0 - a_0b_1}}$$

$$\therefore -\ln l_x = \ln c\left(\frac{a_0+a_1x}{b_0+b_1x}\right)^{\frac{1}{a_1b_0 - a_0b_1}}$$

$$\Rightarrow l_x = c\cdot\left(\frac{a_0+a_1x}{b_0+b_1x}\right)^{\frac{1}{a_1b_0 - a_0b_1}}.$$

C.U.

9) Prove the following inequalities:

i) $q_x < m_x < \frac{q_x}{1-q_x}$, ii) $\frac{m_x}{1+m_x} < q_x < m_x$.

Ans: The central mortality rate (m_x) is given by

$$m_x = \frac{\text{Number of deaths within age-interval } (x, x+1)}{\text{Average number of persons living in the age-group}}$$

$$= \frac{d_x}{L_x}.$$

Note that, $L_x = \int_0^1 l_{x+t}\,dt$, since l_{x+t} decreases as t increases.

$$l_{x+1} < L_x < l_x.$$

Hence, $\frac{d_x}{l_x} < \frac{d_x}{L_x} < \frac{d_x}{l_{x+1}}$

$\Rightarrow \frac{d_x}{l_x} < \frac{d_x}{L_x} < \frac{d_x}{l_x - d_x} = \frac{\frac{d_x}{l_x}}{1-\frac{d_x}{l_x}}$

$\Rightarrow q_x < m_x < \frac{q_x}{1-q_x}$ ——— (i)

Again, $m_x < \frac{q_x}{1-q_x} \Rightarrow m_x - m_x \cdot q_x < q_x$

$\Rightarrow m_x < q_x(1+m_x)$

$\Rightarrow \frac{m_x}{1+m_x} < q_x$

Hence, $\frac{m_x}{1+m_x} < q_x < m_x$ ——— (ii)

(C.U) 10) On the life-table with $l_x = \frac{100-x}{190}$, $5 \le x \le 100$, work out —

(a) the chance that a child who has reached age 5 will live to 60.

(b) the chance that a man of 30 lives to age 80.

(c) the probability of dying within five years for a man aged 40.

(d) the average age at death of those dying between ages 40 and 45.

(e) the expectation of life at age 40.

(f) the chance that of three men aged 30, at least one survives to age 80.

Soln. → In life-table, ${}_nP_x$ is the probability that a person aged x will survive upto age $(x+n)$. Then by definition,

$${}_nP_x = \frac{\text{the number of persons living at age } (x+n)}{\text{the number of persons living at age } x} = \frac{l_{x+n}}{l_x},$$

(a) Required Probability $({}_{55}P_5) = \frac{l_{60}}{l_5} = \frac{\frac{100-60}{190}}{\frac{100-5}{190}} = \frac{40}{95}$.

(b) Required Probability $({}_{50}P_{30}) = \frac{l_{80}}{l_{30}} = \frac{(100-80)/190}{(100-30)/190} = \frac{2}{7}$,

(c) In life table, ${}_nq_x$ = the probability that a person of the cohort living at age x will die before reaching age $(x+n)$.

$$= \frac{\text{the number of persons dying in the age interval } (x, x+n)}{\text{the number of persons living at age } x}$$

$$= \frac{l_x - l_{x+n}}{l_x} = 1 - {}_np_x .$$

$$\text{Required probability} = 1 - \frac{l_{45}}{l_{40}} = 1 - \frac{(100-45)/190}{(100-40)/190}$$

$$= 1 - \frac{55}{60}$$

$$= \frac{1}{12} \text{ (Ans)}$$

(e) $$e^0_{40} = \frac{T_{40}}{l_{40}} = \frac{1}{l_{40}} \int_{40}^{100} l_x \, dx = \frac{1}{\frac{100-40}{190}} \int_{40}^{100} \frac{100-x}{190} \, dx$$

$$= \frac{1}{60} \int_{40}^{100} (100-x) \, dx$$

$$= \frac{1}{60} \left[- \frac{(100-x)^2}{2} \right]_{40}^{100}$$

$$= \frac{1}{60} \left\{ 0 + \frac{(60)^2}{2} \right\} = 30$$

(f) the probability that of three men aged 30 at least one survives to age 80 = 1 − P[all three men aged 30 die before reaching age 80]

$$= 1 - ({}_{50}q_{30})^3$$

$$= 1 - \left(\frac{l_{30} - l_{80}}{l_{30}} \right)^3$$

$$= 1 - \left(1 - \frac{l_{80}}{l_{30}} \right)^3 = 1 - \left(1 - \frac{2}{7} \right)^3$$

$$= 1 - \left(\frac{5}{7} \right)^3 \text{ [Ans]}$$

(d)

FORCE OF MORTALITY: This is a life table function which is most useful but frequently not printed in tables. At any age x, l_x be the number of persons of exact age x and $-\Delta l_x$ be the number of persons among them who die between age x and age $x+\Delta x$.

The rate of decrease in l_x at age x is

$$\lim_{\Delta x \to 0} -\frac{\Delta l_x}{\Delta x} = -\lim_{\Delta x \to 0} \frac{l_{x+\Delta x} - l_x}{\Delta x}$$

$$= -\frac{dl_x}{dx}.$$

The force of mortality at age x is defined as the ratio of instantaneous rate of decrease in l_x or instantaneous death rate at age x to the value of l_x and it is denoted by μ_x.

Hence, $\mu_x = -\frac{1}{l_x} \cdot \frac{dl_x}{dx}$

$$= -\frac{d}{dx}(\log_e l_x)$$

i) Integrating, we get $\int_0^x \mu_t dt = -\log_e l_x + c$

$$\Rightarrow l_x = e^c \cdot e^{-\int_0^x \mu_t dt}$$

putting $x=0$, $l_0 = e^c \cdot e^{-0} \Rightarrow e^c = l_0$.

Hence, $l_x = l_0 \cdot e^{-\int_0^x \mu_t dt}$

Note that, $${}_nP_x = \frac{l_{x+n}}{l_x} = \frac{l_0 e^{-\int_0^{x+n} \mu_t dt}}{l_0 e^{-\int_0^x \mu_t dt}} = e^{-\int_x^{x+n} \mu_t dt}$$

$$= e^{-\int_0^n \mu_t dt}$$

ii) Again, note that,

$$l_x \mu_x = -\frac{dl_x}{dx} \text{ and } \int_0^n \mu_{x+t} \cdot l_{x+t}\, dt$$

$$= \int_0^n -\left(\frac{d}{dt} l_{x+t}\right) dt$$

$$= \left[-l_{x+t}\right]_0^n$$

$$= l_x - l_{x+n}$$

Hence, $${}_nq_x = \frac{l_x - l_{x+n}}{l_x} = \int_0^n \frac{l_{x+t}}{l_x} \cdot \mu_{x+t}\, dt = \int_0^n {}_tP_x \cdot \mu_{x+t}\, dt$$

N.P. (C.U.)

1) Assume $\mu_{x+t} = \mu$, $0 \le t \le 1$

i) Prove that, ${}_tq_x = 1 - e^{-\mu t}$, $0 \le t \le 1$

ii) Prove that, ${}_{1-t}q_{x+t} = 1 - e^{-\mu(1-t)}$, $0 \le t \le 1$

iii) Prove that, $l_{x+t} = l_x \cdot e^{-\mu t}$, $0 \le t \le 1$.

Soln. →

iii) $\mu_{x+t} = \mu$

$$\Rightarrow -\frac{d}{dt}\left(\log_e l_{x+t}\right) = \mu, \quad 0 \le t \le 1.$$

Integrating, $$\int_0^t -\frac{d}{dz}\left(\log_e l_{x+z}\right)dz = \int_0^t \mu\, dz$$

$$\Rightarrow -\log_e l_{x+z}\Big|_0^t = \mu t$$

$$\Rightarrow \frac{l_{x+t}}{l_x} = e^{-\mu t}$$

i) $${}_tq_x = 1 - \frac{l_{x+t}}{l_x} = 1 - e^{-\mu t}$$

ii) $${}_{1-t}q_{x+t} = 1 - \frac{l_{x+1}}{l_{x+t}} = 1 - \frac{l_x e^{-\mu \cdot 1}}{l_x \cdot e^{-\mu \cdot t}}$$

$$= 1 - e^{-\mu(1-t)}$$

2) Define in symbols $\overline{x}_D$, the mean age at death in the stationary population. Prove that it is the same as the expectation of life e_0^0.

Soln. → Rate of decrease in l_x at age x is $-\frac{dl_x}{dx}$.

and $\int_0^\infty -\frac{dl_x}{dx}\cdot dx = -l_x\Big|_0^\infty = l_0 - l_\infty = l_0$. Hence, the function $\left(-\frac{dl_x}{dx}\right)/l_0 = \frac{1}{l_0}(l_x\mu_x)$ is the density of the population (life table) at age x. Hence $\overline{x}_D$, the mean age at death in the life table or stationary population, is

$$\int_0^\infty x\left(\frac{l_x\mu_x}{l_0}\right)dx = \frac{1}{l_0}\int_0^\infty x(-dl_x) = \frac{1}{l_0}\left\{\left[-xl_x\right]_0^\infty - \int_0^\infty 1.(-l_x)dx\right\}$$

$$= \frac{1}{l_0}\int_0^\infty l_x dx = \frac{T_0}{l_0} = e_0^0.$$

3) Let ${}_tq_x = t.q_x$, $0 \le t \le 1$, Show that $\mu_{x+t} = \dfrac{q_x}{1-t.q_x}$.

Soln. → Note that, ${}_tq_x = 1 - \dfrac{l_{x+t}}{l_x}$

$$= 1 - \frac{l_0 e^{-\int_0^{x+t} \mu_z dz}}{l_0 e^{-\int_0^x \mu_z dz}}$$

$$= 1 - e^{-\int_x^{x+t} \mu_z dz} \text{, since}$$

$$\mu_x = -\frac{d}{dx} \log_e l_x$$

$$\Rightarrow l_x = l_0 . e^{-\int_0^x \mu_t dt}$$

Hence, $e^{-\int_0^t \mu_{x+z} dz} = 1 - t.q_x$

$$\Rightarrow -\int_0^t \mu_{x+z} dz = \ln(1 - t.q_x)$$

Differentiating w.r.t.,

$$-\mu_{x+t} = \frac{1}{1-t q_x}(-q_x)$$

$$\therefore \Rightarrow \mu_{x+t} = \frac{q_x}{1-t.q_x}$$

4) Prove that if at age x and above, mortality is fixed at μ, then the expectation of life at age x is $\dfrac{1}{\mu}$.

Soln. → $\mathring{e}_x = \dfrac{\int_0^\infty l_{x+t} dt}{l_x} = \displaystyle\int_0^\infty \frac{l_0 e^{-\int_0^{x+t} \mu_z dz}}{l_0 e^{-\int_0^x \mu_z dz}} dt$

$$= \int_0^\infty e^{-\int_x^{x+t} \mu_z dz} dt \quad \left[\because l_x = l_0 e^{-\int_0^x \mu_z dz} \right]$$

$$= \int_0^\infty e^{-\int_0^t \mu_{x+z} dz} dt$$

$$= \int_0^\infty e^{-\int_0^t \mu dz} dt$$

$$= \int_0^\infty e^{-\mu t} dt = \frac{1}{\mu}.$$

"A life table is meaningful only when the population is stationary" — Comment. (***)

A life table is a life history, as it diminishes gradually by deaths, of a hypothetical group of $l_0 (=10^5)$ babies born at the same time. Subjected to the mortality conditions of a certain year, the cohort will be diminished through deaths at each age till finally all have died.

Certain assumptions are made while constructing a life table. They are —

i) there is no effect of migration on the cohort.

ii) The deaths occurring in between two consecutive birth days are uniformly distributed.

iii) the cohort experiences the age specific mortality in the population under consideration.

A life table is meaningful for a population if there are exactly l_0 births every year, these being distributed uniformly throughout the year and that the death rate at each age remains the same — same as that given by the q_x column of the life table. Further let there be no migration. If a population satisfies the above conditions, in long run the population becomes a stationary population. Therefore a life table is meaningful only when the population is stationary. For non-stationary population, a life-table is not meaningful.

Uses of Life Table :— The main objective of life tables is to gives a clear picture of the age distribution of mortality in a given population group, it has been used widely in a large number of spheres. Today life table is widely accepted as important basic material in demographic and public health studies. William Farr says life table as 'Biometer' of the population.

The main uses are the followings :—

i) For use by Actuaries in Insurance: Life tables are indispensible for the solution of all question concerning the duration of human life. Life tables form the basis for determining the rates of premiums to be paid by persons of different age groups, for various amount of life insurance. Life tables provide the acturial science with a sound foundation. Converting the insurance business from a mere gambling in human lives to the ability to offer well calculated safeguard in the event of death.

ii) For population Projection: Life tables are used by demographers to devise measures as "Net Reproduction Rate" (NRR) to study the rate of growth of population.

iii) Life tables are as well used by the government and the private establishments for determining the rates of retirement benefits to be given to its employees or for formulating various programmes for retired persons.

iv) Life tables are also used:

a) For making policies and programmes relating to public health, by the government and public administration.

b) To evaluate the impact of family welfare programmes on the population growth.

c) Life tables of two or more population groups may also be compared to determine relative mortality. The most familiar of such comparisons is in regard to e_0^0, the average longevity per member of a population.

***

NOTE:- "Life table is not suitable for popln. projection"

— Generally life table is based on the mortality rates experienced by the cohort (popln.). It does not take ~~[illegible]~~ into consideration the current fertility rate. Popln. projection tells us what the future popln. would be if a particular set of assumptions were to hold true. So, life table is not used for population projection.

Cohort & Current Life Tables:

We may distinguish two types of life tables according to the reference year of the table: the current or period life-table and the generation or cohort life-table. The first type is based on the experience over a short period of time, such as 1 year, 3 years or intercensal period, in which mortality has remained substantially the same. Commonly, the death statistics used for a current life-table relate to a period of 1 to 3 years, and the population data used relate to the middle of that period (usually close to the date of a census). This type of table, therefore, represents the combined mortality experience by age of the population in a particular short period of time (viewed cross-sectionally); it does not represent the mortality experience of an actual cohort. Instead, it assumes a hypothetical cohort that is subject to the age-specific death rates observed in the particular period. therefore, a current life table may be viewed as a snapshot of current mortality. It is an excellent summary description of mortality in a year or a short period.

The second type of life-table, the generation life table, is based on the mortality rates experienced by a particular birth cohort, e.g. all persons born in the year 1990. According to this type of table, the mortality experience of the persons in the cohort could be observed from their moment of birth, through each consecutive age in successive calendar years until all of them die. The generation life-table provides a longitudinal perspective in that in follows the mortality experience of an actual cohort. Obviously, data over a long period of years one needed to complete a single table, and it is not possible cohollly on the basis of actual data to construct generation tables for cohorts born in 20th century. this type of table is useful for projections of mortality, for studies of mortality trends, and for the measurement of fertility and reproductivity.

Remarks on "Expectation of Life" :–

1) The expectation of life in most developing countries is seen to increase between 0 l.b.d. and 3 or 4 l.b.d. and then decreases with increase in age – explain the phenomenon.

It has been observed that within the very early weeks of life there is considerable variation in mortality, the risk of death at the time of birth is maximum and then it gradually decreases as age increases. In developing country, due to lack of medical facilities or health care system, there is a large number of deaths in the age group (0,1) and the number of deaths gradually decreases in the successive age groups.

By definition, $e_x^0 = \frac{T_x}{l_x}$, is the expectation of life at age x.

Note, $\frac{e_1^0}{e_0^0} = \frac{T_1}{l_1} \times \frac{l_0}{T_0} = \left(\frac{l_1 + d_0}{l_0}\right) \Big/ \left(\frac{T_1 + L_0}{T_1}\right)$

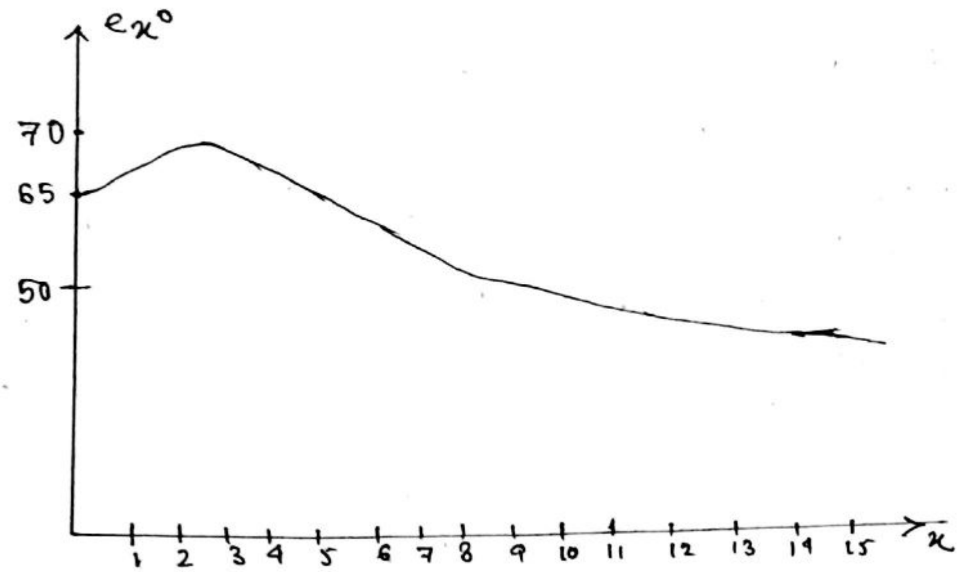
$$= \frac{1 + \frac{d_0}{l_1}}{1 + \frac{L_0}{T_1}}.$$

As d_0, the number of deaths in the age group (0,1), is large and $\frac{d_0}{l_1}$ is not a small fraction where as L_0, the number of years lived by the cohort l_0 in the age group (0,1), is small compare $T_1 = \sum_{x=1}^{\infty} L_x$ and $\frac{L_0}{T_1}$ is a small fraction.

Therefore, $\frac{d_0}{l_1} > \frac{L_0}{T_1} \Rightarrow 1 + \frac{d_0}{l_1} > 1 + \frac{L_0}{T_1}$

$$\Rightarrow e_1^0 > e_0^0$$

The similar argument is valid upto age 3 or 4 l.b.d, hence e_x^0 increases between $x = 0$ to $x = 3$ or 4.
For $\underline{x \geq 5}$, d_x gradually decreases except the last few age groups and $\frac{d_x}{l_{x+1}}$ is a small fraction where as L_x/T_{x+1} is not a small fraction, i.e. $\frac{d_x}{l_{x+1}}$ decreases but $\frac{L_x}{T_{x+1}}$ & decreases more slowly. Hence $e_{x+1}^0 / e_x^0 = \frac{1 + \frac{d_x}{l_{x+1}}}{1 + \frac{L_x}{T_{x+1}}} < 1 \Rightarrow e_x^0$ decreases as age $x (\geq 5)$ increases.



Graph of e_x^0 in developing countries.

2) The expectation of life at birth (e_0^0) is the life table function most frequently used as an index of the level of mortality. In fact $1/e_0^0$ is equivalent to CDR of the life table population. The life expectancy is a statistical measure used to determine the average life span of the population of a certain nation or area. Life expectancy is one of the factors in measuring the Human Development Index of each nation and it is also a factors in finding physical quality of life of an area. It is an indication of the overall heath of a country. Improvements in heath and welfare increase life expectancy. The higher the life expectancy, the better shape a country is in. High life expectancy, indicates low infant and chip mortality, an ageing population and a high quality of health care delivery.

PROBLEM OF LIFE TABLE :→

• **Question :→** In a certain family the grand father, mother and daughter are respectively 65, 42 and 16 years of age (all ages l.b.d.). Suppose the daughter intends to get married at 21. Assuming the population to be stationary find the following probabilities in terms of the life table functions

(i) there will be a wedding attended by both the father and the grandfather.

(ii) the grandfather will die before the wedding takes place.

(iii) the grandfather will be alive for the wedding but will die within 3 years of it.

Soln. →

i) A : the father will attend the marriage.
B : the grandfather will attend the marriage.

∴ Required probability $= P(A \cap B) = P(A)P(B)$

[since A & B are independent] $= \dfrac{l_{47}}{l_{42}} \times \dfrac{l_{70}}{l_{65}}$

ii) Required probability $= P(B^c) = 1 - P(B)$

$$= 1 - \frac{l_{70}}{l_{65}}.$$

iii) A : the grandfather will attend the wedding.
B : the " " die within 3 years of wedding.

∴ Required probability $= P(A \cap B)$

$$= P(A)P(B|A)$$

$$= \frac{l_{70}}{l_{65}} \times \left(\frac{l_{70} - l_{73}}{l_{70}}\right)$$

$$= \frac{l_{70} - l_{73}}{l_{65}}.$$

C.U.

Question:→ Suppose separate life tables for male and females are given (use 'm' and 'f' suffix for the two sets of life table functions). Suppose Mr. A of age 28 l.b.d. marries Mrs. B of age 24 l.b.d. Then find the probability that —

i) Mr. A will survive age 60.

ii) Mr. A and Mrs. B will enjoy at least 40 years of married life.

iii) Mr. A and Mrs. B will enjoy exactly 40 years of married life.

Soln.→ In life table, ${}_nP_x$ is the probability that a person aged x will survive upto age $(x+n)$.

Then,

$$ {}_nP_x = \frac{\text{the number of persons living at age } (x+n)}{\text{the number of persons living at age } x} $$

$$ = \frac{l_{x+n}}{l_x}. $$

i) Required Probability $= \dfrac{l^m_{60}}{l^m_{28}}$.

ii) Required Probability $= \dfrac{l^m_{68}}{l^m_{28}} \times \dfrac{l^f_{64}}{l^f_{24}}$.

iii) C : Mr. A will enjoy exactly 40 years of married life

D : Mrs. B " " " " " " " .

So, the required probability is → $P(C \cap D)$

$$ = \frac{l^m_{68} - l^m_{69}}{l^m_{28}} \times \frac{l^f_{64} - l^f_{65}}{l^f_{24}}. $$

MORBIDITY STATISTICS

Basic concepts & Definitions :

Morbidity = Sickness

1) The state of Health : According to the constitution of the WHO, "Health is a state of complete physical, mental and social well-being and not merely the absence of disease or infirmity."

A morbid condition is a departure from the normal healthy condition. Although deviations from this ideal situation may be regarded as morbid conditions, a better classification of the states of health in a population is the following:

a) free from any defect or disease.

b) having known congential or acquired defects, deseases which cause no current disability; e.g. visual defects, some orthopedic conditions.

c) with incipient disease usually known to the person affected; e.g. early tuberculosis or diabetes.

d) ill or sick ; this includes persons who believe or recognize that they are not well ; it also includes the cases of injury.

Until recently, classes (b) and (d) generally provide practically all of the observations for morbidity data or statistics.

2) Sources of statement of sickness or illness :

In compilation of morbidity data, the source of the statement regarding the illness will be on most occasions :

a) the individual affected or somebody speaking on his behalf.

b) a physician after a clinical examination.

c) the result of a diagonastic test.

3) Chronic illness : The chronic illness are becoming increasingly important in the total morbidity picture because

a) the acute infection diseases have been brought largely under control.

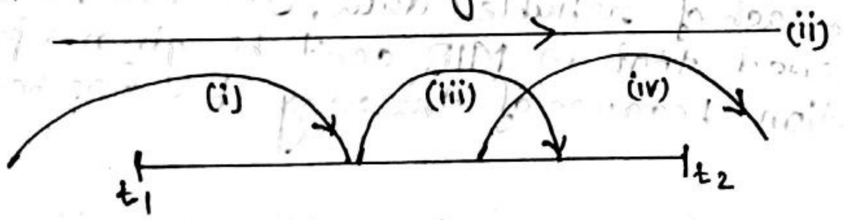
b) the no. of aged persons, among whom the chronic illness are most frequent is growing more rapidly than the rest of the population.

4) Classification of Morbidity Data :- According to the purpose to be served, morbidity data may be classified to show details for specific morbid conditions, the extent of disability involved, the duration of the condition and the kind and amount of medical care received.

Measures of Morbidity :- For many of the purposes served by morbidity statistics, the primary interest is in the number of cases or of persons involved. However, there are also many problems that require the use of rates to measure morbidity, such as comparisons among communities or the study of time-trends. In any particular situation, the rate to be used will accordingly depend upon the definition of illness, the manner in which the basic data have been collected and the specific problem on hand. There is thus great variety in the rates proposed to measure morbidity.

Since there is always an element of duration to illness, the observed cases during a specific period will fall into one of the following four categories :

(i) illness that began before the period but terminated during the period
(ii) " " " " " " and " after the period
(iii) " " " as well as terminated during the period, and
(iv) " " " during the period and terminated after the period.



We have also to distinguish between persons and illnesses, remembering that a person can have more than one illness within an interval and even at the same point of time. Measures of morbidity may therefore be derived from a count of illness and from a count of persons affected.

(C.U)

Morbidity incidence Rate : ~ The term "incidence" relates to the emergence of new cases of illness, and this rate is defined in terms of new cases of illness observed during a period, i.e. cases falling under categories (iii) & (iv) constitute the total no. of new cases of illness arising during the period of observation.

The morbidity incidence rate (the crude incidence Rate), MIR is given by

$$MIR = \frac{I}{P} \times 1000.$$

where, I = total number of new cases of illness in the given period in the given community,

and P = total population of the community during the period.

An MIR may either be a crude rate (when it relates to the whole population) or an age-specific rate (when it relates to a specific age-group). Again, an MIR may relate to a specific type of illness (or injury) rather than all kinds of illness.

Apart from the difficulty in computing an MIR for lack of reliable data, it should be remembered that an MIR can't be given a probability interpretation because of the way it has been defined.

(C.U)

Morbidity prevalence rate : ~ The term 'prevalence' relates to the number of existing cases of illness (a 'stock' concept) during the period of observation. The sum of categories (i) through (iv) makes up the total no. of cases of illness which exist in a community at any time during the period of observation. The rate is thus defined by

$$MPR = \frac{C}{P} \times 1000,$$

where, C = number of cases of illness observed to exist in the given community during the given period.

and P = total population or average number of persons living in the community during the period of observation.

The rate may be made age-specific for a particular kind of illness or a crude MPR.

Usually, an MPR relates to a short interval of time, such as a day or week, where as an MIR generally relates to a longer period. In cases of acute illness of short duration, like influenza and typhoid fever, the MPR would approximate the MIR, provided the period of observation is long enough.

(c.v)

Prevalence Vs. Incidence: — Prevalence is distinct from incidence. Prevalence is a measurement of all existing cases of illness within a particular period of time, whereas incidence is a measurement of the number of new cases of illness during a particular period of time. Incidence conveys information about the risk of contracting the disease, where as prevalence indicates how widespread the disease is.

To illustrate, a long term disease that was spread widely in a community in 2002 will have a high prevalence at a given point of 2003 (assuming it has a long duration) but it might have a low incidence rate during 2003 (i.e. lots of existing cases but not many new cases in that year). Conversely, a disease that is easily transmitted but has a short duration might spread widely during 2002 but is likely to have a low prevalence at a given point in 2003 (as many people develop the disease). As such, prevalence is a useful parameter when talking about long lasting diseases, such as HIV, but incidence is more useful when talking about diseases of short duration. In the case for acute illness of short duration, the prevalence rate approximates the incidence rate when the period of observation is sufficiently long.

(c.u)

*Example: "It is reported that in a country the MPR of Malaria in 2004 was 58 while the MIR was 72"– Comment on the report.

Comment: Malaria is an easily transmitted disease but it has a short duration. Malaria might spread widely during 2003 but is likely to have a low MPR at given point in 2004 due to its short duration but high incidence during 2004, as many people develop malaria. As a matter of fact, in a country the MPR of malaria in 2004 was 58 but the MIR was 72.

(c.u)

* Example: Suggest which of the two measures: MIR, MPR should be used to decide on the amount of medicine to be sent to a malaria affected area. Cite an example where the other rate can be useful.

⇒ Prevalence is a measurement of all existing cases of malaria within a particular period of time whereas incidence is a measurement of the number of new cases of malaria during a period of time. MIR conveys information about the risk of contracting the malaria, where MPR indicates how widespread the malaria is. Clearly the amount of medicine to be sent to a malaria affected area depends on how wide spread the malaria is in that area; that is depends on the Morbidity Prevalence rate for the area. In a single-visit sickness (malaria) surveys, the number of existing cases of malaria along with the number of persons living in the area during the period of observation, such as on the day of the survey, enumerators are collected and the MPR is computed – the medicine should be sent accordingly to the area.

From a single visit survey, we can also compute the MIR. As MIR conveys the information about the risk of contracting the malaria, the amount of medicine for malaria to be storied for the next year should depend on the computed value of MIR.

Case-fatality Rate :

$$CFR = \frac{\text{Number of deaths from cause } i \text{ in a given period}}{\{\text{No. of new cases of illness or injury for cause '}i\text{' during that period}\}} \times 1000$$

$$= \frac{D^i}{I^i} \times 1000$$

The case-fatality rate is intended to measure the risk of death from a specified condition among those suffering from it.

It is truly a probability rate, for those who have a specified disease are the ones truly exposed to the risk of dying of that disease. The case-fatality rate for 'Cancer', for instance, gives the probability that a person suffering from cancer in a given long period will die of that disease in that period. This rate can be used to study the rate of deaths among the cancer patients under a given treatment. This rate is of the greatest interest to clinicians.

USE OF MORBIDITY STATISTICS / RATES :

The purposes served by morbidity statistics are many : they are essential to public health agencies for the control of epidemics and communicable disease, and for the location, design and administration of public health and medical care facilities and services. Health agencies also rely on morbidity data for their operations in both the solicitation and disbursement of fund.

FERTILITY RATES :— Fertility means the actual birth performance as evidenced by the number of offspring. The level of fertility in a community may be measured by the rate at which a population adds to itself by births. In measurement of fertility we normally relate births to the population under consideration. Usually only the live births are taken into account, while still births are excluded. Here we take into account the legitimate births only.

Measurement of Fertility :—

1) CRUDE BIRTH RATE :— The simplest measure of fertility is defined as

$$CBR = \frac{B}{P} \times 1000,$$

where, B = total number of live births which occurred in the given region during the given period;

P = total population of the given region during the given period.

CBR provides an index of the rate at which total population during a calender year adds to itself.

"The CBR per year is 23.7 for India" means that at the rate of 23.7 births per 1000 population during the year adds to the total population.

MERIT :—

1) CBR is simple to interpret ; it gives the no. of live births occurring on the average per thousand people over the given region.

2) It is easy to compute as it requires only the total no. of live births and total population.

3) The CBR is satisfactory measure of fertility only when it is used for the same community in a short period of times or in comparing the birth rates of communities whose populations are known to be nearly, if not quite, equal in their age and sex composition and in martial condition.

DEMERIT :—

1) CBR is not a probability rate, because the whole population is not exposed to the risk of giving birth of babies. Only females and only those between reproductive ages are really liable to this risk. (15–49 years in India)

2) It is calculated without paying regard to the age and sex composition of the community, though these are the most important factors affecting fertility.

3) CBR is not an adequate measure of fertility, as it is not suitable for comparing the fertility levels of two regions for the same period or of the same region for two distant periods of time.

2) GENERAL FERTILITY RATE :— The level of CBR will be influenced by the sex-composition of the total population P. Thus, the CBR will be low if there is a small proportion of married females at the reproductive ages in the total population. To measure the level of fertility in a community more effectively, the following ratio have been used:

$$GFR = \frac{B}{\sum_{x=\omega_1}^{\omega_2} {}^fP_x} \times 1000$$

where, B = total no. of live births in the given region during the given period,

${}^fP_x$ = Number of females of age x l.b.d. in the given region during the given period; and

ω_1, ω_2 = lower and upper limits of the female reproductive period, generally in India the period is taken as 15 to 49 years.

GFR reflects the extent to which the female population in reproductive ages increases the existing population through live births. "The GFR = 116·6" show that 116·6 live births per 1000 women in the child-bearing ages have added to the existing population through births, during a calender year.

MERIT :—

1) GFR is a probability rate since the denominator $\sum_{x=\omega_1}^{\omega_2} f P_x$ consists of the entire female population which is exposed to the risk of producing children.

2) GFR takes into account the sex-distribution of the population in the child-bearing ages and also the age structure to a certain extent.

3) It shows how much the women in the child-bearing ages have added to the existing population through births. GFR is easy to compute and interpret.

DEMERIT :—

1) The GFR ignores that all women in the child-bearing age are not equally capable of giving births to babies. For instance, a woman of age 25 can't be compared with a woman of age 45 in this regard.

2) GFR also is not suitable for comparing the fertility of two communities as it does not take into account the age composition of the females within their reproductive period.

3) AGE SPECIFIC FERTILITY RATE :—

Although, in GFR, we have taken into account of the child bearing population as distinct from the total popln., we have not subdivided this new popln. according to age in order to take account of differential fertility at different ages within the group. In order to overcome the drawback of GFR and get a better idea of the fertility situation in a community, it is necessary to compute the fertility rates for different age-groups or ages in the reproductive age separately. The fertility rate so computed on the basis of the specification of age is called the Age-specific fertility rate (ASFR).

▨ The ASFR for the age group $[x, x+n-1)$, denoted by ${}_n i_x$, is given by the formula,

$$ {}_n i_x = \frac{{}_n B_x}{{}_n^f P_x} \times 1000 $$

where, ${}_n B_x$ = total number of live births to women of age x to $\overline{x+n-1}$ in the given region during the given period.

${}_n^f P_x$ = number of women of age x to $\overline{x+n-1}$ in the region during the given period.

In particular, if we take $n=1$, we get the so-called annual age-specific fertility rate, given by

$$ i_x = \frac{B_x}{{}^f P_x} \times 1000 $$

The ASFR ${}_n i_x$ gives the number of children born alive during a period, per thousand women of the age-group $[x, x+n-1)$.

<u>REMARK</u> :→ Fertility data for different countries show that usually specific fertility starts from a low point, rises to a peak somewhere between 20 and 29 years of age and after that declines steadily. The curve of i_x against x, called the fertility curve. <u>The fertility curve is positively skewed</u>. There are several factors which influence (i_x), $x = 15(1)49$ and thus the shape of the fertility curve. The important factors are: female age at marriage, incidence of widowhood among women and the extent of the adoption of birth control and family planning methods, etc. Since the fertility curve is positively skewed and highly peaked, in computation of ASFR, it is better to consider one year age group rather than five years for to put the women in common with other, in the chosen age group, of the child bearing capacity.

MERIT :—

1) ASFR is a probability rate.
2) ASFR takes into consideration the age composition of the females in the child bearing age-group.
3) ASFR is suitable for comparative studies.

DEMERIT :—

1) Computation of ASFR requires more elaborate data than are required for CBR or GFR.
2) The use of ASFR for comparing the fertility situations of two regions is not an easy job. Generally ASFR will be higher for certain age-groups and lower for remaining age groups in one region than in the other. Accordingly, it is difficult to say if the fertility is higher or loco in one region as compared to the other.

4) TOTAL FERTILITY RATE :—

Though ASFR's reflect the fertility experience of a community in a precise manner, these can't be readily used in comparing the fertility experience of two regions or of the same region at two different perio. For temporal & spatial comparison the ASFR's are combined into a single index or measure. A simple method is to add up the annual ASFR's and take the sum, is called the Total Fertility Rate (TFR), as an index of the overall fertility of the community.

Thus we have,

$$TFR = \sum_{x=\omega_1}^{\omega_2} i_x ,$$

where, $i_x = \frac{B_x}{{}^f P_x}$, B_x is the number of live births to women of age x l.b.d. and ${}^f P_x$ is the number of women of age x l.b.d. in the region during the given period.

NOTE :- 1) Here the multiplier 1000 is not considered.

2) The TFR is a hypothetical figure. This figure indicate the number of children that could be born per woman of the females in the reproductive period, were subjected to the observation of ASFR's and none died in that reproductive period.

3) " The TFR = 3·29 of a state in India", indicates that the average number 3·29 of children would be born alive by a woman in her entire reproductive period provided none of them died before reaching the end of the reproductive period and they are subjected to the observation of ASFR's of the given state.

▨ If we deal with quinquennial age group, i.e. n = 5 for each class there TFR can be approximated by

$$TFR = \sum_x n.({}_n i_x)$$

$$= \sum_x 5({}_5 i_x) = 5\left(\sum_x {}_5 i_x\right).$$

Here the sum is taken over all five-year age-groups in the reproductive period.

MERIT :— 1) It is the best measure of overall fertility over a community.
2) TFR is suitable for comparing overall fertility of two regions for the same period.

DEMERIT :— 1) Computation of TFR requires, like ASFR, elaborate data. 2) TFR is a hypothetical figure.

C.U. Distinguish between GFR and TFR :— ***

i) GFR reflects the extent to which the female population in reproductive ages increases the existing population through live births. The TFR estimates the number of children, a cohort of 1000 women would bear if they all went through their child bearing years exposed to the ASFR in effect for a particular time.

ii) The GFR is an age or sex specific birth rate while the total fertility rate is age or sex-adjusted birth rate. The TFR is an age-adjusted rate because it is based on the assumptions that there are the same number of women in the age group.

[C.U.] MEASUREMENT OF REPRODUCTION OR POPULATION GROWTH

- It is the study of population growth in a community over time on the basis of mortality and fertility only under the following assumptions:

i) the community is closed to migration (i.e. immigration and emegration)

ii) the current mortality and fertility situations will remain unchanged over time.

CRUDE RATE OF NATURAL INCREASE :—

The simplest measure of population growth is the crude rate of national increase, which is given by :

Crude Rate Of National Increase = CBR − CDR = 'r'

The CBR gives the proportion by which the population increases through births while the CDR represents the proportion by which it decreases through deaths. Hence, this rate shows the net gain or loss in the population size through births and deaths taken together.

For India in year 1986, CBR = 29·6 per 1000, CDR = 11·5 per 1000 ⇒ Crude Rate of national increase is 29·6−11·5 = 18·1, i.e. the population has registered a natural increase of 18·1 per 1000 of population during 1986.

The introduction of public health measures, such as better nutrition, greater access to medical system and improved sanitation. Although death rates declined in all age-groups the reduction among infants and children had — and will continue to have — the greatest impact on population growth. This is because they will soon be having children of their own. This situation, resulting in a rapid rate of population growth, is characteristic of many of the developing country.

VITAL INDEX :
It is an another indicator of population growth based on births and deaths taken together, is provided by R. Pearle's Vital Index, defined as follows :

$$\text{Vital index } (i) = \frac{\text{Number of births in the given period}}{\text{Number of deaths in the given period}} \times 100$$

$$= \frac{CBR}{CDR} \times 100$$

REMARK:

i) Both the indices suffer from the drawbacks of CDR and CBR and as such are not suitable for comparative studies.

ii) Both the indices merely give a measure whether births exceed deaths or not. It certainly fails to give us any idea about the trend in the population growth.

iii) $i = 100 \Leftrightarrow r = 0$ indicates stagnation in the population growth.

$i > 100 \Leftrightarrow r > 0$ indicates an increase in the popln.,

$i < 100 \Leftrightarrow r < 0$ implies the population is not holding its own.

(c.u)

GROSS REPRODUCTIVE RATE :
To get a proper measure of population growth, it is first of all necessary to take into account the age-sex composition of the population. Our concern being to measure population growth, it is also appropriate that we should consider female births alone, since it is mainly through females that a popln. increases. Our age-specific fertility rates will then be given by

$$ASFR = {}^{f}i_x = \frac{{}^{f}B_x}{{}^{f}P_x},$$

where, ${}^{f}B_x$ is the number of female births to women of age x during the given period in the given community. Summing these rates for all ages in the reproductive period, a measure of popln. growth, called the

Gross Reproduction rate (GRR), is obtained. Then

$$GRR = \sum_{x=\omega_1}^{\omega_2} {}^f i_x ,$$

Like the TFR, the GRR is a hypothetical figure. GRR (=1·59) indicates the number of (1·59) daughters would be born, on the average, to each of a group of females beginning life together, supposing none of them died before reaching the end of the child-bearing period, if they experienced throughout this period the current level of fertility as represented by ${}^f i_x$.

If the given fertility rates are for quinquennial age-groups, i.e.

$$ {}_5^f i_x = \frac{{}_5^f B_x}{{}_5^f P_x} ,$$

then the GRR will be approximately given by

$$GRR \simeq 5 \times \sum_x {}_5^f i_x ,$$

the sum being taken over all quinquennial age-groups in the reproductive period.

▨ The computation of GRR requires the availability of the following data i) the classification of the births according to the age of the mothers at the time of birth, ii) the sex of the new-born babies. Usually such data are not available. In that case, an approximate value of GRR may be obtained under the assumption that sex-ratio at birth remains more or less constant at all the ages of the women in the reproductive period.

Then, we have,

$$\text{Sex-ratio} = \frac{{}^f B_x}{{}^m B_x} = b \text{ (constant)}, \ \forall\ x = 15(1)49$$

$$\Rightarrow \frac{{}^f B_x}{{}^f B_x + {}^m B_x} = \frac{b}{b+1}$$

$$\Rightarrow \frac{{}^f B_x}{B_x} = c \text{ (say)}, \ \forall\ x = 15(1)49$$

Hence, $c = \frac{{}^fB_x}{B_x} = \frac{\sum_x {}^fB_x}{\sum_x B_x} = \frac{{}^fB}{B}$, where ${}^fB$ is the numbers of female births among the total number of births, denoted by B.

Hence, ${}^fB_x = \frac{{}^fB}{B} \times B_x$, finally, we obtain an estimate of GRR as

$$GRR = \sum_x {}^f\tilde{i}_x = \sum_x \frac{{}^fB_x}{{}^fP_x} = \frac{{}^fB}{B} \sum_x \frac{B_x}{{}^fP_x} = \frac{{}^fB}{B}\left\{\sum_x \tilde{i}_x\right\}$$
$$= \frac{{}^fB}{B} . TFR$$

$$\text{***} \quad \Rightarrow \boxed{GRR = \frac{{}^fB}{B}.(TFR)}$$

Interpretation of GRR :— Assume that a cohort of females is traced from birth to the end of their reproductive period and none of them die meanwhile. Assume further that as they pass through their reproductive period, they give birth to daughters on the basis of current ASFR, counting female births only. Then GRR is the ratio of the total number of their daughters to the original no. of females in the cohort.

GRR may be interpreted as the number of female babies born on an average to each of a female cohort during their reproductive period assuming that

i) none of them die during the said period,

ii) they experience the fertility situation of the community as given by the current female ASFRs.

GRR may also be interpreted as the ratio of female babies in two successive generations, assuming that the first generation experiences only fertility but not mortality.

$GRR > 1 \Rightarrow$ Female popln. size increases.

$GRR = 1 \Rightarrow$ " " " remains constant.

$GRR < 1 \Rightarrow$ " " " decreases.

GRR =1 ⇒ There is exact replacement and the population could remain constant in size irrespective of how high or low the death rate may be.

GRR >1 or <1 ⇒ then a group of female is expected to be replaced by a larger or a smaller number of females in the next generation, under the current level of fertility and mortality and the population shows a tendency to increase or decrease no matter how low or high the death rate may be.

In this sense, the GRR may be looked upon as an index of population growth.

☑ The main drawback of GRR is that it ignores the fact that some of the females who are assumed to ~~begin~~ their life together are very likely to die before the expiry of the reproductive period. This drawback is overcome in NRR. GRR take into account current fertility but ignores current mortality – principal drawback of GRR.

LIMITATIONS OF GRR :– The GRR fails to take account of the mortality of female child before they themselves become the same age as that of the mothers, they are supposed to replace and of mortality among mothers before the end of the child bearing period. As such GRR appears to present overestimates of population replacement and population growth; the population growth depends on the balance between fertility and mortality when the effect of migration is neglected.

NET REPRODUCTION RATE: (***)

(C.U) NRR is an improvement over GRR by considering the survivorship on the basis of the current mortality rates. NRR gives more appropriate measure of population growth can be obtained by combining both mortality and fertility situations.

To take into the factors of mortality in measuring population growth, let ${}^f_xP_0$ be the probability of survival from birth to age x, according to the currently applicable life table for the female population. Hence, the new measure of population growth is

$$R_0 = \sum_{x=\omega_1}^{\omega_2} {}^f i_x \cdot {}^f_xP_0$$

and is called the NRR.

For the life table for female based on current mortality of female, the quantities ${}^f_xP_0 = \frac{{}^f l_x}{{}^f l_0}$ are called survivorship factor or rate for females. Hence,

$$R_0 = \frac{1}{{}^f l_0} \sum_{x=\omega_1}^{\omega_2} {}^f i_x \times {}^f l_x .$$

▣ This is evident from the defn. that NRR is modified GRR with mortality situation built into it. The explanation of NRR follows the patern of the GRR, except that females are traced from birth to the end of their reproductive period on the basis of survivorship corresponding to current mortality. It should be recognized that R_0 is intended only as a measure of popln. growth corresponding to current mortality and fertility.

▣ To make our idea mathematically, construct a life table for females on the basis of the observed ASDRs for females, ${}^f m_x$. The values in the L_x column of the life table, denoted by ${}^f L_x$ in this case, give the mean size of the cohort of ${}^f l_0$ females in the age interval x to $\overline{x+1}$. Hence ${}^f i_x \times {}^f L_x$ gives the number of female children that would be born to the cohort at the age x l.b.d.

The sum $\sum_{x=\omega_1}^{\omega_2} {}^f\mathring{i}_x \times {}^fL_x$ is the total number of female children that are expected to be born to the ${}^fl_0$ females during their life-time. Then, the measure of population growth is $NRR = \frac{1}{{}^fl_0} \sum_{x=\omega_1}^{\omega_2} {}^f\mathring{i}_x \times {}^fL_x$.

▨ With quinquennial fertility rates ${}^f_5\mathring{i}_x$, an estimate of the NRR is obtained as

$$NRR = \frac{1}{{}^fl_0} \sum {}^f_5\mathring{i}_x \times {}^f_5L_x, \text{ where}$$

$${}^f_5L_x = {}^fL_x + {}^fL_{x+1} + \cdots + {}^fL_{x+4}.$$

The NRR is also a hypothetical figure;

c.u "NRR = 1·5" means that 1·5 female children would be born, on the average, per member of a group of females beginning life together, if they were subjected to the observed rates of mortality and fertility throughout their life time. Hence, NRR measures the extent to which a generation of girl babies survive to reproduce themselves as they pass through the child-bearing age group under the given fertility and mortality conditions throughout her

c.u life time.

➪ NRR ≤ GRR

$$NRR = \sum_{x=\omega_1}^{\omega_2} {}^f\mathring{i}_x \cdot {}^f_xp_0 = \sum_{x=\omega_1}^{\omega_2} {}^f\mathring{i}_x \cdot \left(\frac{{}^fl_x}{{}^fl_0}\right) \leq \sum_{x=\omega_1}^{\omega_2} {}^f\mathring{i}_x = GRR$$

Since the survivorship rate ${}^f_xp_0 = \frac{{}^fl_x}{{}^fl_0} \leq 1 \;\forall x$.

Hence the GRR be regarded as a limit above which the NRR can't be raised with fertility as it is, simple by reducing mortality. Hence for a group females beginning their life together,

c.u GRR = NRR iff ${}^f_xp_0 = 1$,

iff none of the new-born girls die before reaching the end of the reproductive period.

(i.v) (***)

NRR = 1 ⇒ If the current fertility rates and mortality rates prevail, then a group of new born girls will exactly replace itself in the next generation. Thus in this case the population has a tendency to remain more or less constant.

(***)

NRR > 1 or < 1 ⇒ In this case, a group of females is expected to be replaced by a larger (or, a smaller) number of females in the next generation under the given rates of fertility and mortality & the population will show a tendency to increase (or decrease).

In this sense, <u>the NRR may be looked upon as a good index of population growth.</u> [i.v]

⇨ <u>LIMITATIONS OF NRR :</u> —

(a) The community is assumed to be closed to migration.

(b) The current ~~[illegible]~~ ASDRs and ASFRS' are assumed to be constant over time.

(c) The reproduction rates are based on a hypothetical age-sex distribution of the population as given in the life table, whereas the reproductive capacity of a population depends on the actual age-sex composition of the population.

⇨ <u>NRR can not be used in population projection for the following reasons :</u> —

1) It assumes that current mortality and fertility rates prevail in future, an assumption which is not true.

2) It overlooks the factor of migration. The population in any region may be decreased through emigration or may increase as a result of immigration.

(c.u.)

Ex. (1). Show that — $\frac{TFR}{1000} > GRR > NRR$.

Soln.→ By defn, $TFR = \sum_{x=\omega_1}^{\omega_2} \hat{i}_x \times 1000$, with multiplier 1000.

$$= \sum_{x=\omega_1}^{\omega_2} \frac{B_x}{{}^f P_x} \times 1000.$$

$$GRR = \sum_{x=\omega_1}^{\omega_1} {}^f i_x = \sum_{x=\omega_1}^{\omega_2} \frac{{}^f B_x}{{}^f P_x} < \sum_{x=\omega_1}^{\omega_2} \frac{B_x}{{}^f P_x} = \sum_{x=\omega_1}^{\omega_2} \hat{i}_x = \frac{TFR}{1000}$$

Since, ${}^f B_x$ = the no. of female births to women of age x,
$\leq B_x$ = the no. of births to women of age x, $\forall_x$.

(c.u.) The part 'GRR > NRR' has been solved earlier.

Ex. (2). Interpret the result : NRR = 0·92. What would ultimately happen if NRR = 0·92?

Soln.→ "NRR = 0·92" implies "a group of 100 females is expected to be replaced by 92 females in the next generation under the given rates of fertility and mortality and the population will show a tendency to decrease".

Hence the popln. will ultimately decrease and will ultimately die out, unless the fertility and mortality change.

Ex. (3). If NRR = 0·91 and the sex-ratio at the birth is 6 : 5 in favour of males, in a community, then comment on the growth of the population.

Soln.→ NRR = 0·91 implies that a group of 1000 females is expected to replaced by 910 females in the next generation, under the given fertility and mortality rates. As the sex-ratio at birth is 6 : 5, then a group of 1000 females is expected to replaced by $\frac{6+5}{5} \times 910 = 2002$ children (males & females) in the next generation but popln. only increases/grows through females. As a group of females is expected to replaced by a less number of females in the next generation, the population will ultimately decrease.

POPULATION ESTIMATES & PROJECTIONS

Currently, most complete and reliable source of information on popln. of countries and communities is a Census. However, popln. changes constantly and sometimes quite rapidly, making census for every 10^{th} year, inadequate for most purposes. Population estimates are used by government official market research analysis, public and private planners.

To meet the need for up-to-date population figures, a wide variety of estimating techniques apply. The analytic techniques involving the use of vital statistics immigration and mathematical methods, are relatively inexpensive to apply and can be used to prepare estimates for past and future dates as well as for current dates.

■ TYPES OF POPULATION ESTIMATES :→ Estimates can be broadly divided into three types on the basis of their time reference and method of deviation. These types pose different methodological problems and are associated with different levels of reliability are :

i) Intercensal estimates, which relate to a date intermediate to two censuses and take the results of these censuses into account.

ii) Postcensal estimates, which relate to a past or current date following a census and take that census and possibly earlier censuses into account but not later censuses.

iii) Projections, which are 'conditional' estimates of popln. at future dates : showing what the future popln. would be if a particular set of assumptions were to hold true. C.U

■ Estimate, Projection & Forecast :→ Information about a present or past population, not based on a census or register is called an 'estimate'. Demographers typically refer to information about the future as either a 'projection' or a 'forecast'.

A projection may be defined as the numerical outcome of a particular set of assumptions regarding future population. It is a conditional calculation showing what the future popln. would be if a particular set of assumptions were to hold true. A projection does not attempt to predict whether the assumptions actually will hold true.

A forecast may be defined as the projection in which the assumptions are considered to yield a realistic picture of the probable future development of a population. As such, it represents a specific viewpoint regarding the validity of the underlying data and assumptions.

The quality of projection is determined by their internal validity, i.e. whether they accurately and consistently model the relation among vital statistics. On the other hand, the gauge of a forecast, is its external validity i.e. how well predictions correspond to subsequent events. Unlike forecasts, population projection can be made for the past as well as for the future.

Here we use the term estimate to refer to a present or past calculation and projection to refers to a future population regardless of their intended uses. We use the term forecast for particular projections when discussing their accuracy.

POPULATION ESTIMATES :→

(a) Component Method :→ Let P_0 and P_1 be the population counted at two successive censuses at time points '0' & '1'.

i) Inter-censal estimates :→ If $B^{(0-t)}$, $D^{(0-t)}$, $I^{(0-t)}$ and $E^{(0-t)}$ denote the number of births, the number of deaths, the total immigration and the total emigration occuring between time '0' and 't' $(0 < t < 1)$, then the inter-censal value of P_t is

$$\hat{P}_t = P_0 + \left\{ B^{(0-t)} - D^{(0-t)} \right\} + \left\{ I^{(0-t)} - E^{(0-t)} \right\} \quad \text{—(1)}$$

This will be an inter-censal estimate because the data are bounded to involve some errors. The difference (c) between the census value P_1 and the value of $\hat{P}_1$ as obtained by equation(1), is known as the 'error of closure', so that → $c = P_1 - \hat{P}_1$.

We may improve upon eqn. (1) by adding to (1) a fraction 't' of the error of closure, as follows —

$$\hat{P}_t = P_0 + \{B^{(0-t)} - D^{(0-t)}\} + \{I^{(0-t)} - E^{(0-t)}\} + t.c$$

ii) Post-censal estimates : — If $B^{(1-t)}$, $D^{(1-t)}$, $I^{(1-t)}$ and $E^{(1-t)}$ denote the numbers of births, the number of deaths, the total immigration and the total emigration occuring between time '1' and 't'. When $t > 1$, then we have the post-censal value of P_t as

$$\hat{P}_t = P_1 + \{B^{(1-t)} - D^{(1-t)}\} + \{I^{(1-t)} - E^{(1-t)}\}, \text{with } t > 1.$$

Here P_1 is the last census value. Unlike the case of an intercensal estimate, no adjustment of this value is possible. [C.U]

(b) MATHEMATICAL METHODS : —

i) Arithmetic Progression Method (A.P. Method) : — (***)

There is proper reason to believe that the population changes rather uniformly during the inter-censal or post-censal periods, a linear interpolation between succes'sive censuses will usually produce reasonably satisfactory results.

Then we assume linear or Arithmatic Progression (A.P.) growth for population, then we may write $P_t = a + bt$.

Taking $t = 0$ and $t = 1$, we then have $P_0 = a$ and $P_1 = a + b$, so that the estimate of 'a' and 'b' are $a = P_0$, $b = (P_1 - P_0)$.

The fitted equation is then, $P_t = P_0 + t(P_1 - P_0)$

or, $P_t = (1-t)P_0 + tP_1$ —— (2)

ii) Geometric Progression Method (G.P. Method) : (***)

A graph of historical series of census population may indicate that an assumption of geometrical growth during the inter-censal interval is more representative than that of linear growth; then we assume Geometrical Progression (G.P.) or exponential growth and we may write

$$P_t = ab^t$$

Taking $t=0$ and $t=1$, we have $P_0 = a$ and $P_1 = a.b$,

$$\Rightarrow a = P_0 \ , \ b = \frac{P_1}{P_0}.$$

the fitted equation is then $P_t = P_0 \left(\frac{P_1}{P_0}\right)^t$

$$= P_0^{(1-t)} . P_1^t \quad \text{——} (3)$$

N.P. → For inter-censal estimate at time t, $0 < t < 1$, we may use equn. (2) or equn. (3) with $0 < t < 1$.

For post-censal estimate at time t, $t > 1$, we may use equn. (2) or equn (3) with $t > 1$.

C.U

REMARK :→ Where the requisite data on birth, deaths and migration are available, it is preferable to make intercensal population estimates by the component method because it takes into account temporal factors that affect population change from one year to the next. Mathematical methods assume uniform yearly changes in the population; however, they avoid the problem of closure. Mathematical methods will be used in case of registration of deaths, births and migration are either unavailable or unsatisfactory.

Population Projections : —

1) By Mathematical Methods : — For population projections, the mathematical methods carry with it the assumption that the social and economic forces that have molded population growth in the past will continue into the future. Since projection are often made over long periods, the choice of the functional form has to be made with much greater care. [C.U]

(a) Logistic Curve : — Suppose a population has the size P at time 't' and the size $P + \Delta P$ at time $t + \Delta t$. The rate of increase of the population at time t is

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}.$$

We may consider the relative growth rate of P, which is

$$\frac{1}{P} \times \frac{dP}{dt} = \text{function of } 't'.$$

$$* \; \frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = r(t)$$
$$\Rightarrow \frac{d}{dt}(\ln P(t)) = r(t) \; *$$

***

i) Lotka Model : — It is assumed that

$$\frac{1}{P} \cdot \frac{dP}{dt} = r \text{ (constant)}$$

Solving this differential eqn. as follows —

$$\therefore \int \frac{dP}{P} = r \int dt$$

$$\Rightarrow \ln P = rt + a \quad [a = \text{integration constant}]$$

$$\text{or, } P = Ae^{rt}$$

where, A is some positive constant and r is the rate of increase.

For $r > 0$ as $t \to -\infty$; $P \to 0$ which is quite realistic because in the initial stage, any region may have zero population.

For $r > 0$ as $t \to \infty$; $P \to \infty$ implies an (geometric growth) exponential law, appears unrealistic, because for a region with limited means of sustenance, it is unthinkable that the population can increase without limits.

$\therefore$ Population projection by Geometric Progression is hardly feasible in most instances because, with positive r, it may produce impossibly large numbers in the long run.

Remark:→

⟨a⟩ A population count at time t will be denoted by P_t when discrete time values are being used. A rate of increase r in population can be expressed as

$$r^* = \frac{P_{t+1} - P_t}{P_t}$$

$$\Rightarrow P_{t+1} = P_t(1+r^*)$$

A population model reflects geometric growth if

$P_t = P_0(1+r^*)^t$ where the unit of time may be a month, a year, 5 years, etc, provided r^* is a fraction of increase per unit of time.

If the relative growth rate,

$\frac{1}{P_t} \cdot \frac{dP_t}{dt} = r$, then $P_t = P_0 \cdot e^{rt}$. Hence,

$P_0(1+r^*)^t = P_t = P_0 \cdot e^{rt} \Rightarrow r^* = e^r - 1$.

where, r^* is the rate of increase, compounded annually.

⟨b⟩ The relative growth rate 'r' in the exponential or geometric growth $P_t = P_0 e^{rt}$, is sometimes taken as the Crude rate of Natural Increase.

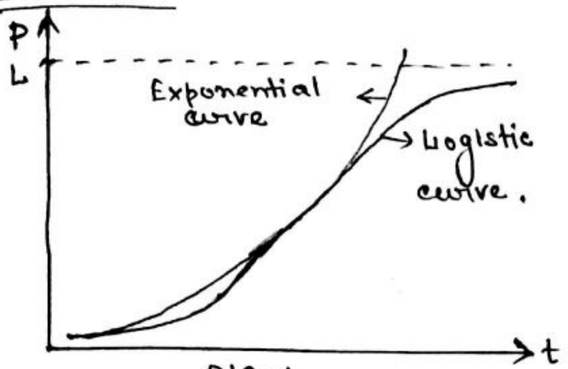
ii) Most populations are constrained by limitations on resources – even in the short run, and none is unconstrained forever. A very plausible assumption for a popl^n. that is growing in an area of fixed limits would be that the relative growth rate gradually decreases as t and P increase, one of the simplest forms of decreasing functions of P is $r(1-KP)$ where r and k are positive constants.

Verhulst Model:→ It is assumed that, relative growth rate is an decreasing function in time and population. In this case, the differential equation

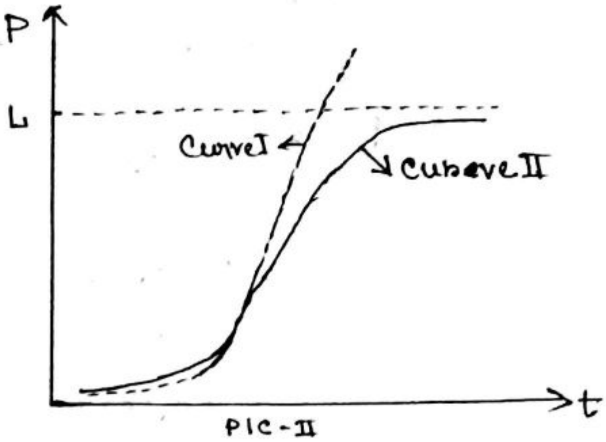
$$\frac{1}{P} \cdot \frac{dP}{dt} = r(1-KP).$$

This model is called the Logistic growth or Verhulst model.

Rationale :→



PIC-I



PIC-II

In Pic-II, the following figure shows two possible curves for growth of a population, the curve I following an exponential, the curve II constrained so that the popln. is always less than some number $1/k$. When the popln. size is small relative to $1/k$, the two patterns are virtually identical, i.e. the constraint does not make much difference. But for the 2nd curve, as P becomes a significant fraction of $\frac{1}{k}$. The curve begins to deviate from the curve-I and as P gets close to $\frac{1}{k}$, the growth rate drops to 0. Hence we may account for the growth rate in the model a factor $(1-kP)$ – which is close to 1 (has no effect) when P is much smaller than $1/k$ & which is close to 0 when P is close to $1/k$. The resulting model is

$\boxed{\text{c.u}}$

$$\frac{1}{P}\cdot\frac{dP}{dt} = r(1-kP) \quad ***$$

]] Derivation of logistic function :— A more realistic assumption is to consider the relative growth rate to be a decreasing function of t on P. W.L.G. let us consider,

$$\frac{1}{P}\cdot\frac{dP}{dt} = r(1-kP)$$

$$\Rightarrow \frac{dP}{P(1-kP)} = r\,dt$$

(U.C) $$\Rightarrow \frac{kP+1-kP}{P(1-kP)}\,dP' = r\,dt$$

$$\Rightarrow \int\left(\frac{1}{P}+\frac{k}{1-kP}\right)dP = r\int dt$$

$$\Rightarrow \ln P - \ln(1-kP) = rt + c$$

$$\Rightarrow \ln\frac{P}{1-kP} = rt + c. \Rightarrow \frac{P}{1-kP} = Ae^{rt}$$

$\therefore \frac{P}{1-kP} = Ae^{rt}$

$\Rightarrow P = Ae^{rt} - kPAe^{rt} \Rightarrow P(1+kAe^{rt}) = Ae^{rt}$

$$\Rightarrow P = \frac{Ae^{rt}}{1+kAe^{rt}}$$

$$= \frac{1}{k+\frac{1}{Ae^{rt}}}$$

$$= \frac{1}{k+\frac{1}{A}\cdot e^{-rt}}, \text{ where } A>0$$

$$= \frac{1/k}{1+\frac{1}{kA}e^{-rt}}$$

$$= \frac{L}{1+\frac{L}{A}e^{-rt}} \quad \left[\text{where } \frac{1}{k} = L\right] \quad (*)$$

Here also as $t \to -\infty$, $P \to 0$ which is quite realistic and as $t \to \infty$, $P \to \frac{1}{k}$. we denoting this upper limit to the popl^{n}. size by L for that region.

Now, suppose $P = \frac{L}{2}$ when $t = \beta$. Then

$$\frac{L}{2} = \frac{L}{1+\frac{L}{A}e^{-r\beta}}$$

$$\Rightarrow 1+\frac{L}{A}e^{-r\beta} = 2$$

$$\Rightarrow \frac{L}{A}e^{-r\beta} = 1$$

$$\Rightarrow e^{-r\beta} = \frac{A}{L}$$

$$\Rightarrow \frac{L}{A} = e^{r\beta}$$

Now subtituting the value of $\frac{L}{A}$ in $(*)$, we have

$$P = \frac{L}{1+e^{r\beta - rt}}$$

i.e. $\boxed{P = \frac{L}{1+e^{r(\beta-t)}}} \longrightarrow$ Logistic Law. (C.U)

which is the logistic function of t where L, r and β are model parameter & the corresponding curve is called Logistic Curve of the population figure in that region.

[C.U]

Properties of Logistic function :— (***)

(a) $$P = \frac{L}{1+e^{r(\beta-t)}} = \frac{L}{1+e^{-r(t-\beta)}}$$

(i) As $t \to \infty$; $P \to L$
(ii) As $t \to -\infty$; $P \to 0$
(iii) As $t = \beta$; $P = \frac{L}{2}$

(b) $$\frac{1}{P}\cdot\frac{dP}{dt} = r(1-KP)$$

$$\Rightarrow \frac{dP}{dt} = rP\left(1-\frac{P}{L}\right) \quad \left[\because K = \frac{1}{L}\right]$$

where L is the upper limit to the population size.
When, $t \to -\infty$, $P \to 0$ and $\frac{dP}{dt} \to 0$,
i.e. growth rates start from 0.
Also when, $t \to \infty$, $P \to L$ and $\frac{dP}{dt} \to 0$,
i.e. growth rate drops to '0'.
Hence, the logistic curve has two asymptotes at $P = 0$ and $P = L$.

$$\left[\frac{dP}{dt} = rP\left(1-\frac{P}{L}\right) \; ; \; \frac{dP}{dt} = 0 \text{ at } P=0, P=L\right]$$

(c) since $r, P, \left(1-\frac{P}{L}\right)$ are all positive quantities, $\frac{dP}{dt}$ is also positive, so that P is, according to the logistic law, continuously increasing with t.

(d) Again, $$\frac{dP}{dt} = rP\left(1-\frac{P}{L}\right)$$

$$\frac{d^2P}{dt^2} = r\cdot\frac{dP}{dt}\left(1-\frac{P}{L}\right) + rP\left(-\frac{1}{L}\right)\frac{dP}{dt}$$

$$= r\frac{dP}{dt}\left(1-\frac{2P}{L}\right)$$, $\frac{dP}{dt}$ is always +ve

$$\therefore \quad \frac{d^2P}{dt^2} \gtrless 0$$

$$\Rightarrow \left(1-\frac{2P}{L}\right) \gtrless 0$$

$$\Rightarrow \frac{2P}{L} \lessgtr 1$$

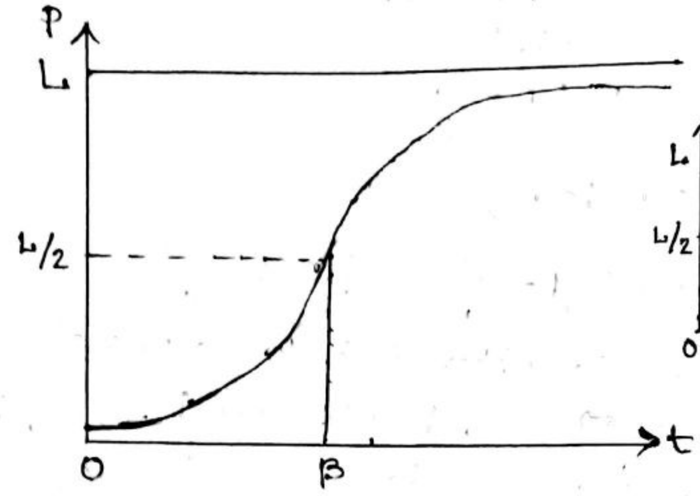
$$\Rightarrow P \lessgtr \frac{L}{2}$$

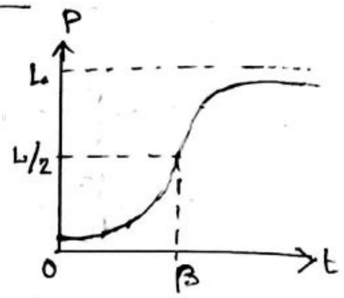
$$\Rightarrow t \lessgtr \beta$$

i.e. $\frac{d^2P}{dt^2} \gtrless 0$
iff $P \lessgtr \frac{L}{2}$
iff $t \lessgtr \beta$.

Hence, the curve has a point of inflection at $t=\beta$ and is concave upwards for $t<\beta$ and convex upwards for $t>\beta$.

(c) The curve is shaped like an elongated S.





(b)

FITTING OF LOGISTIC CURVE :

To fit a logistic curve to a set of data, we have to estimate the constants L, r and β from the observed popln. figures. Let P_t denotes the population at time 't' and the population figures are given for N equidistant points of time, say, $t=0,1,2,\ldots,N-1$.

(A) Method of 3 selected points : ***

$T=\{t: 0, 2n\}$

Selected points are $0, n$ & $2n$.

$$\frac{1}{P_0}=\frac{1}{L}+\frac{e^{r\beta}}{L}$$

$$\frac{1}{P_n}=\frac{1}{L}+\frac{e^{r(\beta-n)}}{L}$$

$$\frac{1}{P_{2n}}=\frac{1}{L}+\frac{e^{r(\beta-2n)}}{L}$$

Now, writing $d_1 = \frac{1}{P_0} - \frac{1}{P_n}$

$$\therefore \quad d_1 = \frac{1}{L} e^{r\beta}\left(1 - e^{-rn}\right)$$

and $d_2 = \frac{1}{P_n} - \frac{1}{P_{2n}}$

$$\therefore \quad d_2 = \frac{1}{L} e^{r\beta - rn}\left(1 - e^{-rn}\right)$$

whence $\frac{d_1}{d_2} = e^{rn}$

or, $\boxed{\hat{r} = \frac{1}{n} \ln \frac{d_1}{d_2}}$ ——— (*)

Further,
$$\frac{d_1^2}{d_1 - d_2} = \frac{\frac{1}{L^2} e^{2r\beta}\left(1 - e^{-rn}\right)^2}{\frac{1}{L} e^{r\beta}\left(1 - e^{-rn}\right)\left[1 - e^{-rn}\right]}$$

$$= \frac{e^{r\beta}}{L}$$

$$= \left(\frac{1}{P_0} - \frac{1}{L}\right)$$

$\left[\because \frac{e^{r\beta}}{L} = \frac{1}{P_0} - \frac{1}{L}\right.$ from the first equation. $\left.\right]$

$$\therefore \quad \boxed{\frac{1}{\hat{L}} = \frac{1}{P_0} - \frac{d_1^2}{d_1 - d_2}} \text{ ——— (**)}$$

We estimate $\hat{r}$ and $\hat{L}$ from eqn. (*) and (**).

Now, $e^{r\beta} = \frac{L}{P_0} - 1$

or, $\beta = \frac{1}{r} \ln\left(\frac{L}{P_0} - 1\right)$

$$\therefore \quad \boxed{\hat{\beta} = \frac{1}{\hat{r}} \ln\left(\frac{\hat{L}}{P_0} - 1\right)} \text{ ——— (***)}$$

By 'method of three selected points', we can estimate the constants L, r and β from the observed popl^n. figures..

[c,v]

(B) Method of Rhodes : ***

This is the best method of fitting the logistic curve. Considering the popl^n. according to the logistic law for the time point $t = i-1$ and $t = i$.

$$\frac{1}{P_{i-1}} = \frac{1}{L} + \frac{e^{r(\beta - i+1)}}{L} \quad \text{—(1)}$$

and $$\frac{1}{P_i} = \frac{1}{L} + \frac{e^{r(\beta - i)}}{L},$$

From (1),

$$\left(\frac{1}{P_{i-1}} - \frac{1}{L}\right) = \frac{e^{r(\beta - i+1)}}{L}$$

or, $$\left(\frac{1}{P_{i-1}} - \frac{1}{L}\right) e^{-r} = \frac{e^{r(\beta - i)}}{L}$$

so that, $$\frac{1}{P_i} = \frac{1}{L} + e^{-r}\left(\frac{1}{P_{i-1}} - \frac{1}{L}\right)$$

$$\Rightarrow \frac{1}{P_i} = \frac{1-e^{-r}}{L} + \frac{e^{-r}}{P_{i-1}}$$

This relationship may be put in the form

$$y_i = A + Bx_i,$$

where, $y_i = \frac{1}{P_i}$, $x_i = \frac{1}{P_{i-1}}$, $A = \frac{1-e^{-r}}{L}$, $B = e^{-r}$.

Thus the two variables x and y should be exactly linearly related if the popl^n. precisely follows the logistic law. The problem is to estimate the constants A and B, assuming that the deviations of the points (x_i, y_i) from an exact linear relationship arise from errors in both x_i and y_i

The proper estimates of A and B are taken to be

$$b = \sqrt{\sum_{i=1}^{N-1}(y_i-\bar{y})^2 \Big/ \sum_{i=1}^{N-1}(x_i-\bar{x})^2}$$

and $\quad a = \bar{y} - b\bar{x}$,

where $\quad \bar{x} = \sum_{i=1}^{N-1} x_i/(N-1) \quad ; \quad \bar{y} = \sum_{i=1}^{N-1} y_i/(N-1)$

$$= \bar{x} + \left[\frac{1}{P_{N-1}} - \frac{1}{P_0}\right]$$

So, $\hat{r} = -\ln b$,

$$\hat{L} = \frac{1-e^{-\hat{r}}}{a} = \frac{1-e^{\ln b}}{a} .$$

$$\therefore \hat{L} = \frac{1-b}{a} .$$

Now, we know from the logistic law —

$$P_i = \frac{L}{1+e^{r(\beta-i)}}$$

$$\Rightarrow e^{r(\beta-i)} = \frac{L}{P_i} - 1$$

$$\Rightarrow \beta = \frac{1}{r}\ln\left(\frac{L}{P_i}-1\right) +$$

$$P_t = \frac{L}{1+e^{r(\beta-t)}}$$

$$e^{r(\beta-t)} = \frac{L}{P_t} - 1$$

$$\therefore \beta = t + \frac{1}{r}\ln\left(\frac{L}{P_t}-1\right) \quad \forall \ t = 0(1)N-1 .$$

$$\Rightarrow \sum_{t=0}^{N-1}\beta = \sum_{t=0}^{N-1} t + \frac{1}{\hat{r}}\sum_{t=0}^{N-1}\ln\left(\frac{\hat{L}}{P_t}-1\right)$$

$$\therefore \hat{\beta} = \frac{N-1}{2} + \frac{1}{N\hat{r}}\sum_{t=0}^{N-1}\ln\left(\frac{\hat{L}}{P_t}-1\right) .$$

(c.v.)

(C) Fisher's Method :- *** We start from the differential equation

$$\frac{1}{P_t} \cdot \frac{dP_t}{dt} = r\left(1 - \frac{P_t}{L}\right)$$

i.e. $$P_t = \frac{L}{1+e^{r(\beta - t)}}$$

Let us denote,

$$Z_t = \frac{1}{P_t} \cdot \frac{dP_t}{dt} \text{ and } Y_t = P_t$$

thus the differential eqn. will be

$$Z_t = \left(r - \frac{r}{L} \cdot Y_t\right)$$

If we estimate Z_t, then the estimated values of Z_t can be regressed Y_t (through a linear regression) which provides the estimation of r and L. Upto this technique remain the same however the method of estimation of r and L by Fisher's Technique. Fisher expected Z_t i.e. $\frac{d}{dt} \log P_t$ by Stirling's central difference formula on the assumption that the third order or higher order differences are neglected.

$$\frac{d}{dt} \ln P_t \simeq \frac{1}{2}\left(\Delta \ln P_t + \Delta \ln P_{t-1}\right)$$

$$= \frac{1}{2}\left[\ln P_{t+1} - \cancel{\ln P_t} + \cancel{\ln P_t} - \ln P_{t-1}\right]$$

$$= \frac{1}{2} \ln \frac{P_{t+1}}{P_{t-1}}.$$

Thus Z_t being estimated, one can estimate r and L by regression equation.

$$Z_t = a + bY_t$$

where, $a = r$, $b = -\frac{r}{L}$

It is to be noted that here

$$Z_t = A + B\ Y_t \quad , \quad t = 0(1)N-1$$

$$\therefore \hat{B} = \frac{S_{ZY}}{S_{YY}} = \frac{\sum_{t=1}^{N-1}(Z_t - \bar{Z})(Y_t - \bar{Y})}{\sum_{t=1}^{N-1}(Y_t - \bar{Y})^2},$$

where, $\bar{Y} = \frac{1}{N-1}\sum_{t=0}^{N-1} Y_t$

$$\bar{Z} = \frac{1}{N-1}\sum_{t=0}^{N-1} Z_t$$

And finally, $A = \bar{Z} - \hat{B}\bar{Y}$

$$= \hat{r}$$

$$B = -\frac{r}{L}$$

i.e. $\hat{L} = -\frac{\hat{r}}{\hat{B}}$ & $\hat{r} = A$.

It may be noted the method of Fisher does not consider seperately the estimation of the parameter β. However, estimation of r and L can be done by Fisher's above method. But one can estimate β by Rhode's method.

c.v **Example (1).**

(a) Suppose a population numbering 1,00,000 growing geometrically at annual rate $r = 0\cdot 02$. If r is compounded annoually, determine the size of the popl^n. at the end of 10 yrs.

(b) If another population also of 1,00,000 is growing arithmetically with an increment of $0\cdot 02$ of its initial number per year, how does its size at the end of 10 years compare with that in (a).

(c) At the end of 100 years, what is the difference?

Sol^n. →

(a) $P_{10} = P_0 (1+r)^{10} = 1,00,000\ (1+0\cdot 2)^{10} = 121,899$

(b) $P_{10} = P_0 (1+10.r) = 100000\ (1+10\times 0\cdot 02) = 120,000$

(c) $D = P_0 \{(1+r)^{100} - 1+100.r\} = 100000 \{(1\cdot 02)^{100} - 3\}$

c.v **Example (2).** A popl^n growing exponentially stood at 10,00,000 in 1930 and at 30,00,000 in 1970.

(a) What is its rate of increase?

(b) What is its doubling time?

(c) How long would it take to multiply by 9?

Sol^n. →

(a) $P_t = P_0 e^{rt}$

$\therefore\ 30,00,000 = 10,00,000 \times e^{r.40}$

$\Rightarrow r = \frac{1}{40}.\ln 3$.

(b) Let $P_{t*} = 2P_0$, t^* is the doubling time.

$\Rightarrow P_0 e^{r.t^*} = 2P_0$

$\Rightarrow rt^* = \ln 2$

$\Rightarrow t^* = 40.\frac{\ln 2}{\ln 3}$.

(c) Let, $P_{t_1} = 9P_0$. Then $P_0 e^{r.t_1} = 9P_0$

$\Rightarrow r.t_1 = \ln 9$

$\Rightarrow t_1 = \frac{2\ln 3}{\frac{1}{40}\ln 3} = 80$ years.

Example (3). The Official United States Population estimate for mid-1965 was 194,303,000; for mid-1970 was 204,879,000. Extrapolate to 1975 assuming
(a) fixed absolute increase i.e. Arithmatic progression.
(b) fixed ratio of increase i.e. Geometric Progression.

Soln.→ (a) Let $P_t = a + bt$, assuming A.P. Method and considering mid-1965 as the origin.

Here, $P_0 = 194,303,000$ and $P_5 = 204,879,000$.

$\therefore a = 194,303,000$

$\therefore a + b.5 = 204,879,000$

$\Rightarrow 5b = 10,576,000$

Hence, estimate of P_{10}, the popln. at 1975, is

$$P_{10} = a + b.10$$
$$= 194,303,000 + 21,152,000$$
$$= 215,455,000$$

(b) Similarly, by G.P method, $P_t = ab^t$

$$a = P_0, \quad P_5 = ab^5$$
$$\Rightarrow b^5 = \frac{P_5}{P_0}$$

Hence, $P_{10} = a.b^{10} = P_0\left(\frac{P_5}{P_0}\right)^2 = 216,031,000$.

C.U.

Example (4). Consider a logistic population model:

$$P(t) = [A + Be^{-ut}]^{-1}, t \geq 0,$$

Verify that $P(t)$ satisfies the differential equation

$$\frac{d}{dt}P(t) = uP(t) - uA\{P(t)\}^2$$

Find the co-ordinates of the point of inflection for $P(t)$.

Soln.→ $P(t) = [A + Be^{-ut}]^{-1}$

Differentiating w.r.t. 't', we get —

$$\frac{dP(t)}{dt} = (-1)[A + Be^{-ut}]^{-2}.(-uB)\ e^{-ut},$$
$$= uBe^{-ut}.\{P(t)\}^2$$
$$= u[\{P(t)\}^{-1} - A]\{P(t)\}^2$$
$$= uP(t) - uA\{P(t)\}^2$$

Differentiating again w.r.t. 't',

$$\frac{d^2P(t)}{dt^2} = uP'(t) - uA.2P(t).P'(t)$$
$$= uP'(t)\{1-2AP(t)\}$$

for point of inflexion,

$$\frac{d^2P(t)}{dt^2} = 0$$

$$\Rightarrow P(t) = \frac{1}{2A}$$

$$\Rightarrow \frac{1}{A+Be^{-ut}} = \frac{1}{2A}$$

$$\Rightarrow e^{-ut} = \frac{A}{B}$$

$$\Rightarrow t = -\frac{1}{u}\ln\left(\frac{A}{B}\right).$$

(c.u)

Example(5). A certain country had a population on July 1, 1980 of 15,000,000, the births and deaths during 1980 were 7,50,000 and 3,00,000, respectively.

(a) What is the annual crude rate of increase?

(b) Use an exponential model to project the population to 2030 (July 1).

Soln. → (a) $r = CRI = CBR - CDR = \frac{750,000}{15,000,000} - \frac{300,000}{15,000,000}$

$$= 0.03.$$

(b) $P(t) = a.b^t$, where the origin of 't' is 1980 (July 1).

$\therefore a = P(0) = 15,000,000$

Note that, $b = 1 + r$

$$= 1 + 0.03 = 1.03$$

Hence, $P(t) = 15,000,000 \times (1.03)^{50} = 67,225,000.$

Example (6). If the annual births at time t_1 were n_1 and at time t_2 were n_2, what's the total number of births occur between t_1 and t_2 → (a) On the assumption of straight line or arithmetic increase, (b) On the assumption of exponential or geometrical increase?

Soln. → (a) $P_t = a + bt$

Here, $n_1 = P_{t_1} = a + bt_1$
$n_2 = P_{t_2} = a + bt_2$ $\Rightarrow b = \frac{n_2 - n_1}{t_2 - t_1}.$

(b) $P_t = ab^t$.

Here $n_1 = ab^{t_1}$, $n_2 = ab^{t_2}$ $\Rightarrow b^{t_2 - t_1} = n_2 - n_1$

$\Rightarrow b = (n_2 - n_1)^{\frac{1}{t_2 - t_1}}$.

Example (7). The rate of increase of a population at time 't' is $r(t) = 0.01 + 0.0001.t^2$. If the population totals 1,000,000 at time zero, what is it at time 30?

Soln → $\frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = r(t)$

$$\Rightarrow \frac{d}{dt}\{\ln P(t)\} = r(t)$$

On integration, $\int_0^t d\{\ln P(x)\} = \int_0^t r(z)dz$

$$\Rightarrow \ln\left\{\frac{P(t)}{P(0)}\right\} = \int_0^t \{0.01 + 0.0001 \times z^2\} dz$$

$$\Rightarrow P(t) = P(0) \times e^{\int_0^t \{0.01 + 0.0001 z^2\} dz}$$

$$\Rightarrow P(t) = P(0).e^{0.01t + 0.0001.t^3/3}$$

$$\therefore P(30) = 10^6.e^{0.3 + 0.9} = 10^6.e^{1.2}.$$

(c.u.) **Example (8).** A population grows as $P(t) = \left(\frac{\alpha}{t_0 - t}\right)^2$. What is its rate of increase at time t? What does this say about the birthrate as $t \to t_0$?

Soln → $$\frac{dP(t)}{dt} = 2\alpha^2 (t_0 - t)^{-3}$$

$$= \frac{2\alpha^2}{(t_0 - t)^3}$$

$$= \left(\frac{\alpha}{t_0 - t}\right)^2 \left(\frac{2}{t_0 - t}\right)$$

$$\Rightarrow r(t) = \frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = \frac{2}{t_0 - t}.$$

Note that, $r(t) \longrightarrow +\infty$ as $t \to t_0$.

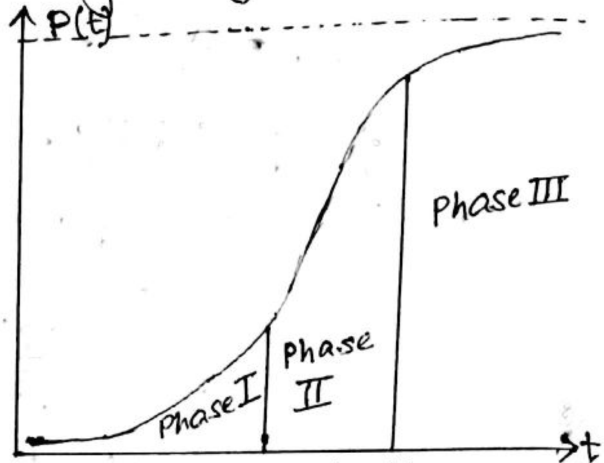
$r(t) \longrightarrow +\infty$ as $t \to t_0$.

$\Rightarrow$ birth rate $\longrightarrow +\infty$ as $t \to t_0$.

The Type of the population data that a Logistic curve fits well into.

The shape of logistic curve is characterized by three phases : Phase I – shows little increase
Phase II – Shows rapid climb in upward direction
Phase III – shows slowing down of change to reach a stationary stage.

The population growths of many developed countries follow the more or less well defined phases as described below :



Phase I / Pre-transitional Phase : → During this phase, both CBR and CDR remain high and nearly equal ; the Rate of natural increase is negligible and so is the growth of the population.

Phase II / Transitional Phase : → During this phase, CBR remains on the high side or decline slowly, while the death rates are decline more sharply due to improvement in health care and medical system. As a result, the rates of natural increase are high and the population grows fast.

Phase III / Post-transitional Phase : → During this phase, CBR and CDR decline to more or less to comparable levels due to improvement in the social, cultural and medical system. the growth of the population is very low or near zero.

If the population growth of a country follow the above pattern and the growth can be explained or analysed by the law described by the Logistic curve.

Demographic Scenario of India & Logistic Law :-

The shape of the Logistic Law is characterised by three phases :

Phase I : It shows little change and this is characterized by high CBR and CDR ; and are nearly equal. Crude rate of increase is low.

Phase II : It shows rapid upward climb and this is characterized by high CBR and rapidly declining CDR. The crude rate of natural increase is high.

Phase III : Here the population growth is very slow and this is due low CBR and CDR to comparable levels. The crude rate of natural increase is low.

The population may be said to have in Phase I upto 1930-40. The rates of natural increase are more than 2% since 1951. India may be said to have in Phase II in the middle of the 19th century. India is still at this phase, it would appear, with the rate of natural increase still on the high side and with little prospect of the same coming down. the phase III is still a long way off for the indian population. The Indian popln. data does not follow the logistic law. Hence, the logistic curve is not appropriate for India or other developing countries.