The commensurability criterion for arithmeticity (after Margulis)

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These two talks will state and prove the famous theorem of Margulis that if the commensurator of a lattice in a semisimple Lie group is dense, then the lattice is arithmetic. The first talk will state the main results and provide outlines of their proofs. The second talk will fill in the proofs of key lemmas. Basic facts from ergodic theory (the study of measurable aspects of group actions) will play a crucial role.

$G = \mathrm{SL}(2,\mathbb{R})$ (= $\mathbf{G}(\mathbb{R})$ for $\mathbf{G}$ simply connected almost-simple / $\mathbb{R}$)

Question

What are the ***lattices*** in G?

discrete subgrp Γ, G/Γ has finite Haar measure

Example

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

Generalization

Embed $H := G \times \text{cpct} \hookrightarrow \mathrm{SL}(n,\mathbb{R})$ such that H is defined over $\mathbb{Q}$. (i.e., $\overline{H_{\mathbb{Q}}} = H$)

Then $H_{\mathbb{Z}}$ is a lattice in H. [Borel & Harish-Chandra, 1962]

Projection to G is an ***arithmetic*** lattice in G.

Defn. Γ_1 is ***commensurable*** to Γ_2 if $\dot\Gamma_1 = \dot\Gamma_2$

($\exists$ finite-index subgroups $\dot\Gamma_1, \dot\Gamma_2$)

Margulis Arithmeticity Theorem (1977, 1992)

$n \geq 3 \Rightarrow$ *every lattice in* $\mathrm{SL}(n,\mathbb{R})$ ($G \not\approx \mathrm{SO}(1,k), \mathrm{SU}(1,k)$)

is commensurable to an arithmetic lattice.

[*Margulis* (*Corlette, Schoen-Gromov*)]

Defn. $\Gamma_{\mathbb{Q}} := \mathrm{Comm}_G(\Gamma) := \{\, g \in G \mid g\Gamma g^{-1} \doteq \Gamma \,\}$.

Exercise

Γ arithmetic $\Rightarrow \mathrm{Comm}_G(\Gamma)$ is dense in G.

(*Hint:* $H_{\mathbb{Q}} \subseteq \mathrm{Comm}_H(H_{\mathbb{Z}})$.)

Γ arithmetic $\Rightarrow \Gamma_{\mathbb{Q}}$ is dense in G.

These two talks are about the **converse**:

Commensurability Criterion (Margulis 1977)

Γ *arithmetic* $\Leftrightarrow \Gamma_{\mathbb{Q}}$ *is dense in* G.

Corollary of:

Commensurator Superrigidity Thm (Margulis 1977)

Let $\rho\colon \Gamma_{\mathbb{Q}} \to \mathrm{SL}(k,\mathbf{R})$ (*with* $\mathbf{R} = \mathbb{R}$ *or* $\mathbb{Q}_p$)

- $\overline{\rho(\Gamma_{\mathbb{Q}})}$ *Zar-conn, simple*
- $\overline{\rho(\Gamma)}$ *not cpct*

$\Rightarrow \rho$ *extends to a continuous homo defined on* G.

$\mathbf{R} = \mathbb{Q}_p \Rightarrow$ extension is trivial $\Rightarrow \overline{\rho(\Gamma)}$ compact.

Why superrigidity implies arithmeticity

Let Γ be a lattice in $G = \mathrm{SL}(2,\mathbb{R})$.

We wish to embed $G \times \text{cpct} \hookrightarrow \mathrm{SL}(n,\mathbb{R})$,

so that $\Gamma \,\dot\subseteq\, \mathrm{SL}(n,\mathbb{Z})$, i.e., $\gamma_{i,j} \in \mathbb{Z}$.

Step 0. Use adjoint representation of G.

Step 1. $\gamma_{i,j}$ is algebraic

Suppose some $\gamma_{i,j}$ is transcendental.

Then $\exists$ field auto φ of $\mathbb{C}$ with $\varphi(\gamma_{i,j}) = ???$.

$\varphi \circ \mathrm{Ad}\colon \Gamma_{\mathbb{Q}} \to \mathrm{SL}(n,\mathbb{C})$ is a representation.

Superrigidity: extends to $\hat\varphi\colon G \to \mathrm{SL}(n,\mathbb{C})$.

There are uncountably many different φ's,

but G has only finitely many n-dim'l rep'ns

(up to change of basis). $\rightarrow\leftarrow$

Step 1. $\gamma_{i,j}$ is algebraic.

Step 2. $\gamma_{i,j} \in \mathbb{Q}$

Γ f.g., so $\{\gamma_{i,j}\}$ generates finite extension F of $\mathbb{Q}$.

"*algebraic number field*"

So $\Gamma \subseteq G_F$. Restriction of Scalars: $\Gamma \subseteq H_{\mathbb{Q}}$.

Lem. $\Gamma \subseteq G_F \Rightarrow \Gamma_{\mathbb{Q}} \subseteq G_F$ *if G is adjoint.*

Step 3. $\gamma_{i,j}$ has no denominator

Actually, show denominators are bounded.

(Then finite-index subgroup has no denoms.)

Γ f.g., so finitely many primes appear in denoms.

Suffices to show each prime occurs to bdd power.

This is *p-adic superrigidity* ($\mathbf{R} = \mathbb{Q}_p$).

We wish to embed $G \times \text{cpct} \hookrightarrow \mathrm{SL}(n,\mathbb{R})$,
so that $\Gamma \doteq \mathrm{SL}(n,\mathbb{Z})$.

We now know $\rho(\Gamma) \subseteq \mathrm{SL}(n,\mathbb{Z})$.

Step 4. ρ extends to embedding of $G \times$ cpct

$H = \mathrm{Res}_{F/\mathbb{Q}} G = H_1 \times H_2 \times \cdots \times H_k \times \text{cpct}$.
Superrigidity: $\pi_i \circ \rho$ extends to homo ρ_i on G,
so $G \to H_1 \times \cdots \times H_k$.
Image $\hat{G}$ is Zariski closed (up to finite index).
Then $\rho(\Gamma) \subseteq \hat{G} \times \text{cpct}$.

Proof of superrigidity (sketch)

Γ = lattice in $G = \mathrm{SL}(2,\mathbb{R})$. $\mathrm{Comm}_G(\Gamma) = \Gamma_{\mathbb{Q}}$ dense.
$\rho\colon \Gamma_{\mathbb{Q}} \to H$ homo, s.t. $\overline{\overline{\rho(\Gamma_{\mathbb{Q}})}} = H$ and $\overline{\rho(\Gamma)}$ not cpct.
Need to extend ρ to a continuous homo $\hat{\rho}\colon G \to H$.

$$P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset G \qquad \text{(minimal parabolic)}$$

Starting point

Furstenberg Lemma (probability / functional anal):
If Γ acts (by homeos) on cpct metric space Y, then $\exists$ Γ-equivariant meas'ble random $\psi\colon G/P \to Y$.
($\psi(x)$ is a probability distribution on Y)

Roughly, we choose $Y = H/L$ (and Γ acts via ρ).

$\exists$ Γ-equivariant meas'ble <u>random</u> $\psi\colon G/P \to H/L$.

"**Ergodicity vs. tameness**" (repeatedly) promotes ψ to:
well-defined $\Gamma_{\mathbb{Q}}$-equivariant $\psi'\colon G/P \to H/L'$.

Ergodicity vs. tameness one more time:
all G-translates of ψ' are in a single H-orbit:
$\dot{\forall} g \in G, \exists!\, h = h(g) \in H,\ \dot{\forall} x,$
$\psi'(gx) = h\,\psi'(x)$.
Uniqueness implies:

- $h\colon G \to H$ is a homo (meas'ble, so continuous)
- $h|_{\Gamma_{\mathbb{Q}}} = \rho$.

So h is the desired extension. □

Ergodicity vs. tameness

Eg. Let L = closed subgroup of G,
so $L \curvearrowright G$ by (right) translations.
G/L is very nice (manifold):
$\exists$ (measurable) $G/L \xrightarrow{1\text{-}1} \mathbb{R}^n$.

Defn. $H \curvearrowright Y$ is ***tame*** if $\exists$ (meas'ble) $Y/H \xrightarrow{1\text{-}1} \mathbb{R}^n$.

Example

Zar closed H acts (regularly) on variety
$\Rightarrow$ every H-orbit is locally closed
(open $\cap$ closed) [Andrei's book, p. 99]
$\Rightarrow$ action is tame. [exercise]

Eg. $\Gamma_{\mathbb{Q}}$ = <u>dense</u> subgroup of G.
$G/\Gamma_{\mathbb{Q}}$ is terrible: $\nexists$ (measurable) $G/\Gamma_{\mathbb{Q}} \xrightarrow{1\text{-}1} \mathbb{R}^n$.
Every $\Gamma_{\mathbb{Q}}$-inv't meas'ble func on $G/\Gamma_{\mathbb{Q}}$ is constant (a.e.)
i.e., the action is "**ergodic**."

Proof. G acts on $\{\psi\colon G \to [0,1]\}_{a.e.}$ by translation:
$(g * \psi)(x) = \psi(g^{-1}x)$.
For suitable topology, action is continuous. □

Exercise

$\Gamma \curvearrowright X$ ergodic, $H \curvearrowright Y$ tame, $\psi\colon X \to Y$ Γ-equi
$\Rightarrow \psi(X) \subseteq$ a single H-orbit (a.e.)

Can take this as the <u>definition</u> of "tame."

Ergodic: every meas'ble, inv't func is constant a.e.

Key example (Moore Ergodicity Theorem)

$\Gamma \curvearrowright G/P$ is ergodic. ($P \rightsquigarrow$ any closed, noncpct subgroup)

Proof.

Spse $\psi\colon G/P \to \mathbb{R}$ is Γ-invariant (and bdd). Lift to G.
$\tilde{\psi}\colon G \to \mathbb{R}$ is $(\Gamma \times P)$-invariant.
$\overline{\psi}\colon \Gamma\backslash G \to \mathbb{R}$ is P-invariant (and in L^2).
Let $u \in U := \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$, and $a = \begin{bmatrix} 1/2 & \\ & 2 \end{bmatrix}$.
$u * \overline{\psi} = u * a^n * \overline{\psi} = a^n * (u^{a^n}) * \overline{\psi} \doteq a^n * \overline{\psi} = \overline{\psi}$.
So $\overline{\psi}$ is U-invariant.
$\langle U, P\rangle = G$, so $\overline{\psi}$ is G-inv't, i.e., constant (a.e.). □

Part 2

Γ = lattice in $G = \mathrm{SL}(2,\mathbb{R})$. $\mathrm{Comm}_G(\Gamma) = \Gamma_{\mathbb{Q}}$ dense.

Cor (Margulis 1977). Γ *is arithmetic.*

Commensurator Superrigidity Thm (Margulis 1977)

Let $\rho\colon \Gamma_{\mathbb{Q}} \to \mathrm{SL}(k,\mathbf{R})$ *(with* $\mathbf{R} = \mathbb{R}$ *or* $\mathbb{Q}_p$*)*

- $H := \overline{\overline{\rho(\Gamma_{\mathbb{Q}})}}$ *Zar-conn, simple*
- $\overline{\rho(\Gamma)}$ *not cpct*

$\Rightarrow$ *ρ extends to a continuous homo defined on G.*

Proof uses "ergodicity vs. tameness."

Ergodicity vs. Tameness

Ergodic: every meas'ble, inv't func is constant a.e.

Examples

- $\Gamma_{\mathbb{Q}} \curvearrowright G$ ($\Gamma_{\mathbb{Q}}$ dense, $\mathrm{Stab}_G(\psi)$ closed)
- $\Gamma \curvearrowright G/P$ $\left(P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right)$
  $u * \psi = u * a^n * \psi = a^n * (u^{a^n}) * \psi \doteq a^n * \psi = \psi$
- $\Gamma \curvearrowright G/A$ $\left(A = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}\right)$
- $\Gamma \curvearrowright G/P \times G/P$ (*doubly ergodic* on G/P)
  - P transitive on G/P a.e. ("big cell"), Stab $= A$.
  - So G transitive (a.e.) on $G/P \times G/P \doteq G/A$.

$H \curvearrowright Y$ ***tame***: (Y is complete, separable metric space)

1. $h_n y \to y \Rightarrow [h_n] \to [e]$ in $H/\mathrm{Stab}_H(y)$.
2. For $y \in Y$, $H/\mathrm{Stab}_H(y) \to Hy$ is a homeo.
3. All orbits are locally closed (open $\cap$ closed)
   Eg. $H \curvearrowright$ variety or proper action.
4. Y/H is *countably separated.*
   $\exists\{E_n\}$ (Borel) that separates points of Y/H.
5. $\exists$ (meas'ble) $Y/H \xrightarrow{\text{1-1}} \mathbb{R}^n$.
   $\chi_{E_1} \times \chi_{E_2} \times \cdots$ maps Y/H to Cantor set.

Exercise

$\Gamma \curvearrowright X$ ergodic, $H \curvearrowright Y$ tame, $\psi\colon X \to Y$ Γ-equi
$\Rightarrow \psi(X) \subseteq$ a single H-orbit (a.e.)

Proof of superrigidity (outline)

Γ = lattice in $G = \mathrm{SL}(2,\mathbb{R})$. $\mathrm{Comm}_G(\Gamma) = \Gamma_{\mathbb{Q}}$ dense.
$\rho\colon \Gamma_{\mathbb{Q}} \to H$ homo, s.t. $\overline{\overline{\rho(\Gamma_{\mathbb{Q}})}} = H$ and $\overline{\rho(\Gamma)}$ not cpct.

Choose irred rep $\sigma\colon H \to \mathrm{SL}(n{+}1,\mathbf{R})$, so $H \curvearrowright \mathbb{P}\mathbf{R}^n$.

Furstenberg Lemma (P is amenable)

$\exists$ Γ-*equivariant* $\psi\colon G/P \to \mathrm{Meas}^1(\mathbb{P}\mathbf{R}^n)$.

Lemma

$H \curvearrowright \mathrm{Meas}^1(\mathbb{P}\mathbf{R}^n)$ *is tame.*

So $\psi(G/P) \subseteq H$-orbit: Can think of ψ as
a map to H/S, where $S = \mathrm{Stab}_H(\mu)$.
Let $\overline{\psi}\colon G/P \to H/\overline{\overline{S}}$.

Γ-equivariant $\overline{\psi}\colon G/P \to H/\overline{\overline{S}}$, where $S = \mathrm{Stab}_H(\mu)$.

Lemma (ergodicity vs. tameness)

S is not compact. (*because $\overline{\rho(\Gamma)}$ not compact*)

Lemma

$\overline{\overline{S}} \neq H$. (*stab of measure on $\mathbb{P}\mathbf{R}^n$ is $\approx$ Zariski closed*)

Lemma (ergodicity vs. tameness)

$\dot\Gamma$-equi map $G/P \to H/\overline{\overline{S}}$ is unique. (*slight exaggeration*)

For $\lambda \in \Gamma_{\mathbb{Q}}$, let ${}^{\lambda}\overline{\psi}(x) = \rho(\lambda)\,\overline{\psi}(\lambda^{-1}x)$.
Then ${}^{\lambda}\overline{\psi}$ is $\dot\Gamma$-equivariant.
So ${}^{\lambda}\overline{\psi} = \overline{\psi}$. I.e., $\overline{\psi}$ is $\Gamma_{\mathbb{Q}}$-equivariant.

$\overline{\psi}\colon G/P \to H/\overline{\overline{S}}$ is $\Gamma_{\mathbb{Q}}$-equivariant.

Define $\Psi_g(x) = \overline{\psi}(gx)$, so $\Psi\colon G \to \mathcal{F}(G/P, H/\overline{\overline{S}})_{a.e.}$
$\Psi_{\lambda g}(x) = \overline{\psi}(\lambda g x) = \rho(\lambda)\,\overline{\psi}(gx) = \rho(\lambda)\,\Psi_g(x)$.
So Ψ is $\Gamma_{\mathbb{Q}}$-equivariant.

Lemma

$H \curvearrowright \mathcal{F}(G/P, H/\overline{\overline{S}})_{a.e.}$ *is tame.* (*because $H/\overline{\overline{S}}$ is variety*)

All G-translates of $\overline{\psi}$ are in a single H-orbit:
$\dot\forall g \in G$, $\exists!\, h = h(g) \in H$, $\dot\forall x$, $\overline{\psi}(gx) = h\,\overline{\psi}(x)$.

Uniqueness implies:

- $h\colon G \to H$ is a homo (meas'ble, so continuous)
- $h|_{\Gamma_{\mathbb{Q}}} = \rho$.

So h is the desired extension. □

Lemma (ergodicity vs. tameness)

$\dot\Gamma$-equi map $\psi\colon G/P \to H/\overline{\overline{S}}$ is unique. *(exaggeration)*

Lemma (ergodicity vs. tameness)

$\dot\Gamma$-equi map $\psi\colon G/P \to H/L$

- *L Zariski closed*
- *$\dim L$ minimal*

$\Rightarrow$ $\overline{\psi}\colon G/P \to H/N_H(L^\circ)$ is unique.

Proof.

$\Psi\colon G/P \to H/L \times H/L$. Ergodicity vs. tameness:
$\Psi(G/P) \subseteq H(h_1L, h_2L)$, so $\psi_i(x) = h(x)\, h_i\, L$.
$h(\gamma x)\, h_1\, L = \rho(\gamma)\, h(x)\, h_1\, L$
$\Rightarrow h(\gamma x)^{-1}\, \rho(\gamma)\, h(x) \in {}^{h_1}L \cap {}^{h_2}L$.
So $\psi_i\colon G/P \to H/({}^{h_1}L \cap {}^{h_2}L)$ is $\dot\Gamma$-equi.
Minimality: $h_1^{-1}h_2 \in N_H(L^\circ)$. □

Lemma (Furstenberg (*P* is amenable))

$\exists$ Γ-equivariant $\psi\colon G/P \to \mathrm{Meas}^1(\mathbb{P}\mathbf{R}^n)$.

Proof.

$\{\,\Gamma$-equivariant $\psi\colon G \to \mathrm{Meas}^1(\mathbb{P}\mathbf{R}^n)\,\}_{a.e.}$.
Closed, convex subset of Banach space.
P acts by right translations.
Amenable (solvable) so must have a fixed point. □

Lemma (ergodicity vs. tameness)

S is not compact. *(because $\overline{\rho(\Gamma)}$ not compact)*

Proof.

$\Psi\colon\ G/P \times G/P \ \to\ H/S \times H/S$.
Ergodicity vs. tameness:
$\Psi(x_1, x_2) = (hy_1, hy_2)$, $\exists h = (x_1, x_2) \in H$
$\Rightarrow \psi(x_1) \in \mathrm{Stab}_H(y_2)\, y_1 = S y_1$
$\Rightarrow \rho(\Gamma) \subseteq$ compact   if S is compact. □

Lemma

$H \looparrowright \mathcal{F}(G/P, H/\overline{\overline{S}})_{a.e.}$ is tame. *(because $H/\overline{\overline{S}}$ is variety)*

Proof.

Spse $h_n\,\psi \to \psi$.
So (subseq) $h_n\psi(x) \to \psi(x)$ for a.e. $x \in G/P$.
$\mathrm{Stab}_H(\psi) = \{\, h \in H \mid h\,\psi(x) = \psi(x)$ for a.e. $x\,\}$
$= \{\, h \in H \mid h\,\psi(x_i) = \psi(x_i)$ for $i = 1, \ldots, k\,\}$
$= \mathrm{Stab}_H\big((y_1, \ldots, y_k)\big)$.
Since H is tame on variety $H/\overline{\overline{S}} \times \cdots \times H/\overline{\overline{S}}$
and $h_n \cdot (y_1, \ldots, y_k) \to (y_1, \ldots, y_k)$,
we have $[h_n] \to [e]$. □

Lemma

$\overline{\overline{S}} \neq H$. *(stab of measure on $\mathbb{P}\mathbf{R}^n$ is $\approx$ Zariski closed)*

Idea of proof.

Spse $\{a_n\}$ unbounded diagonal matrices in $\overline{\overline{S}}$.
E_n^+ = sum of eigenspaces of largest abs value,
E_n^- = sum of other eigenspaces.
$E_n^\pm \to E^\pm$ in Grasmannian.
In the limit, a_n contracts complement of E^- to E^+.
Since μ is a_n-inv't, it must be supported on $E^+ \cup E^-$.

Choose E of min dimension, such that $\mu(E) > 0$.
(So $\mu(E \cap gE) = 0$ unless $gE = E$.)
Then S-orbit of E is finite.
Stabilizer is a Zar-closed proper subgroup of H. □

Lemma

$H \looparrowright \mathrm{Meas}^1(\mathbb{P}\mathbf{R}^n)$ is tame.

Idea of proof

Spse $h_n\mu \to \mu$.
$h_nE_n^\pm \to E^\pm$ with μ supported on $E^+ \cup E^-$.
By induction on dim, $[h_n] \approx [e]$ on E^+ and E^-.

References

R. J. Zimmer:
Ergodic Theory and Semisimple Groups,
Birkhäuser, Basel, 1984, MR0776417.

N. A'Campo and M. Burger:
Réseaux arithmétiques et commensurateur
d'après G. A. Margulis.
Invent. Math. 116 (1994), no. 1-3, 1–25. MR1253187

G. A. Margulis:
Discrete Subgroups of Semisimple Lie Groups,
Springer, Berlin, 1991, MR1090825.