Weighted Utility Theory with Incomplete Preferences

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Abstract

This paper axiomatizes the representations of weighted utility theory with incomplete preferences. These include the general multiple weighted utility representation as well as special cases of multiple utilities or multiple weights only.

Keywords: Incomplete preferences, weighted utility theory, multiple weighted expected utility representation

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1 Introduction

1.1 Motivation and Literature Review

Two of the assumptions of expected utility theory seem less satisfactory than the others, those of completeness and independence. Completeness requires that decision makers are able to compare and express clear preferences between any two risky prospects, while independence requires that decision makers rank prospects only by their distinct characteristics, disregarding their common aspects.

That the completeness axiom may be too demanding was recognized from the outset by von Neumann and Morgenstern (1947) who say that "It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable." Aumann (1962), who was the first to study expected utility theory without the completeness axiom, claims that "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint." Later studies by Shapley and Baucelles (1998), Dubra, Maccheronni and Ok (2004) and, most recently, Galaabaatar and Karni (2012) all conclude that the departure from completeness axiom leads to expected multi-utility representations.

Experimental evidence, such as the Allais paradox, motivated developments in the 1980s of theories of decision making under risk that depart from the independence axiom. These theories include Quiggin's (1982) anticipated utility theory, Chew and MacCrimmon's (1979) weighted utility theory, Yaari's (1987) dual theory, Dekel's (1986) implicit weighted utility, and Gul's (1991) theory of disappointment aversion.[1]

Thus far, the only works that simultaneously depart from both the completeness and independence axioms are Maccheroni (2004), Safra (2014) and Zhou (2014). Maccheroni (2004) showed that without the completeness axiom the representation of Yaari's dual theory entails the existence of a set of probability transformation functions such that one risky prospect is preferred over another if and only if its rank-dependent expected value is larger according to every probability transformation function in that set. Safra (2014) studied a general model of decision making under risk that has the betweenness property.[2] Safra showed that without completeness, the representation entails the existence of a set of continuous functionals displaying betweenness such that one risky prospect is preferred over another if and only if it is assigned a higher value by every element in this set. Weighted utility theory, the subject of this work, also displays the betweenness property but is more structured and therefore calls for a different analysis.

The objective of this paper is to study weighted utility theory without the completeness axiom. Introduced by Chew and MacCrimmon (1979) and Chew (1983, 1989), weighted utility theory is based on a natural weakening of the independence axiom to a ratio substitution property, allowing the outcomes to hold different degrees of salience for the decision maker, captured in the representation by the namesake weight function. Incompleteness in weighted utility theory may thus be the result not only of indecisive tastes, captured by a set of utility functions that rank the outcomes differently, but also of conflicting perceptions of the alternatives presented, captured by a set of weight functions that represent different transformations of the probabilities, or some combination of both. We begin by analyzing the general multiple weighted expected utility model, and follow with the two special cases of multiple utilities paired with a single weight function, or a single utility paired with multiple weights.

1.2 An Informal Review

To set the stage and develop some intuition, we begin with an informal review. Let $X = \{x_1, \ldots, x_n\}$ be the set of outcomes, and denote the set of lotteries over X by $\Delta(X) = \{p \in \mathbb{R}^n_+ : \sum_{x \in X} p(x) = 1\}$.[3] Denote by δ_x the degenerate lottery that assigns $x \in X$ unit probability mass. Let $\succ$ be a strict preference relation over $\Delta(X)$, that is, an irreflexive and transitive binary relation which may or may not be negatively transitive. If $\succ$ violates the independence axiom and instead satisfies only the weaker

[1] See Karni and Schmeidler (1991) for a review of this literature.

[2] Models of decision making under risk with the betweenness property include Chew (1983), Dekel (1986) and Gul (1991).

[3] For each $p, q \in \Delta(X)$ and $\alpha \in [0, 1]$, define $\alpha p + (1 - \alpha)q \in \Delta(X)$ by $(\alpha p + (1 - \alpha)q)(x) = \alpha p(x) + (1 - \alpha)q(x)$ for all $x \in X$. Then $\Delta(X)$ is a convex subset of the linear space $\mathbb{R}^n$.

substitution axiom of Chew (1989), then there exists a utility function u and a nonnegative weight function w mapping X to $\mathbb{R}$, such that, for all $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}.$$

For example, if $n = 3$ and $\delta_{x_3} \succ \delta_{x_2} \succ \delta_{x_1}$, the indifference map induced by (**??**) is depicted in Figure 1 below. The indifference curves all emanate from a *source point* o lying outside the simplex.

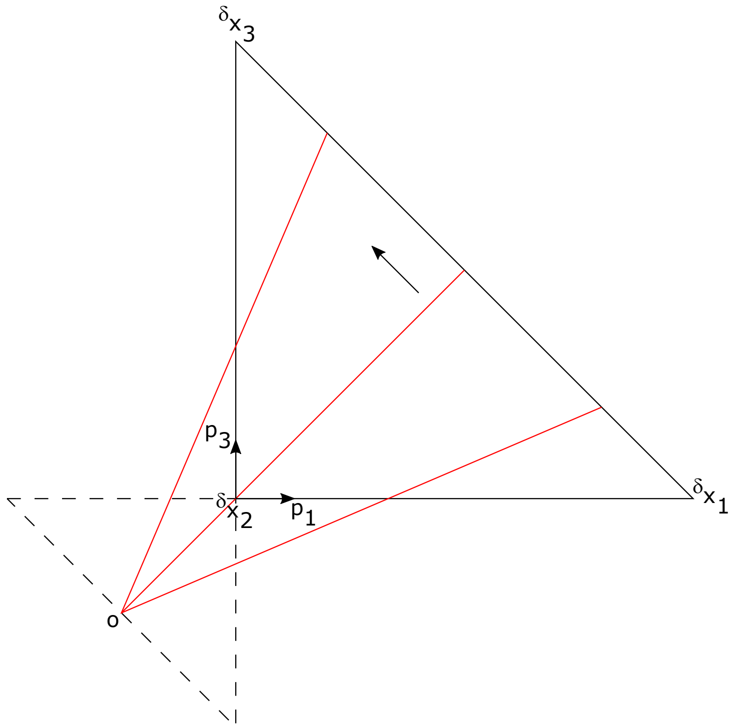

Figure 1: Weighted Utility

Figure 1 depicts a decision maker that attaches greater weight to the extreme outcomes x_1 and x_3 than to the median outcome x_2, indicating that the former have more influence on his evaluation of any particular lottery than their probability would justify. The degree of risk aversion would vary across the simplex and thus the decision maker would exhibit Allais-type behavior, being willing to take risks when he feels he has nothing to lose that he would otherwise avoid if his alternatives were more attractive. The extent of this distortion depends on the proximity of the source point o to the simplex and as it is moved farther away from the diagram, approaches the parallel indifference map of expected utility.

Now suppose that $\succ$ is also incomplete. As in multiple expected utility models with independence such as those of Dubra, Maccheroni, and Ok (2004), and Galaabaatar and Karni (2012), the preference relation cannot be meaningfully characterized with indifference curves, as two lotteries that are not strictly comparable are not necessarily equivalent. Consider the lottery p in Figure 2 below, let $B(p) = \{r \in \Delta(X) : r \succ p\}$ and $W(p) = \{r \in \Delta(X) : r \prec p\}$ respectively denote the upper and lower contour sets of p, and observe that they are demarcated by rays emanating from a pair of distinct source points o^1 and o^2. Unlike in classic weighted utility theory, these rays are not indifference curves, but indicate only that no two lotteries lying on a single ray are strictly comparable, a relation which is not transitive and hence not an equivalence relation.

Nevertheless these *incomparability curves* do inherit many of the properties of indifference curves from the weighted utility setup. As each set of such curves converges at a source point, each in turn has a weighted linear utility representation as in (**??**), with the two sources o^1 and o^2 respectively corresponding to utility and weight pairs (u^1, w) and (u^2, w). As the diagram indicates, for any lottery to be strictly

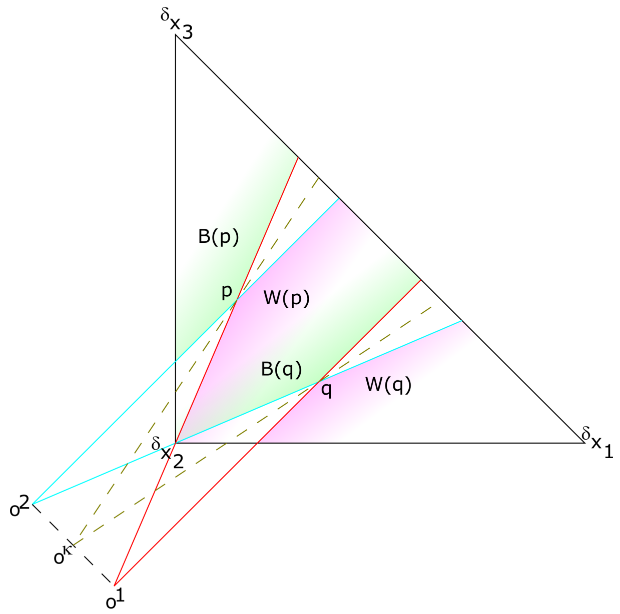

Figure 2: Multiple Utilities

preferred to p it must lie above both of the incomparability curves intersecting p, and thus the preference relation has a multiple weighted expected utility representation, with a set of utilities $\mathcal{U} = \{u^1, u^2\}$.

$$p \succ q \iff \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)}, \; \forall\, u \in \mathcal{U}. \tag{1}$$

Two other aspects of this setup are noteworthy. Firstly, every point on the line segment connecting o^1 and o^2 also projects a set of incomparability curves, always lying between the rays projected by the two endpoints. Any such point o^κ would thus also be a source point and correspond to some utility u^κ, which could be included within $\mathcal{U}$ without altering the preference relation it represents. The location of o^κ between o^1 and o^2 implies that u^κ would be some convex combination of u^1 and u^2, and thus could not contradict any ordering jointly established by these utilities. This leads us to conclude that, just as in multiple expected utility models with independence, the representation will only be unique up to some closed convex hull, though as the utilities here are not linear we will need to adopt a slightly different approach to establish this result.

Secondly, the line segment connecting the source points o^1 and o^2 is parallel to that connecting the best and worst outcomes δ_{x_3} and δ_{x_1}. Hence these sources are equidistant from the simplex and represent different utilities paired with the same weight function. As the incomparability curves projected from o^1 are everywhere steeper than those projected from o^2, u^1 is uniformly more risk averse than u^2. This naturally leads us to consider the dual case, where a single utility function might be paired with multiple weight functions.

Figure 3(a) depicts such a case, where there are a pair of source points o^1 and o^2 corresponding to utility-weight pairs (u, w^1) and (u, w^2). Here the utility functions are identical, as the incomparability curves drawn from both sources through δ_{x_2} coincide and thus rank the median outcome identically, but as o^2 is closer to the simplex, w^2 represents a greater deviation from the uniform weights of expected utility theory. Figure 3(b) depicts a similar case, where there are again two sources o^1 and o^2, and two corresponding utility-weight pairs (u, w) and (u, w^2), but here o^2 is located on the other side of the simplex. This produces incomparability curves that fan in rather than out, and indicating that x_2 is weighted more heavily than the extreme outcomes, rather than less. The preferences depicted in either incomparability map would have a representation consisting of the single utility u and multiple weights

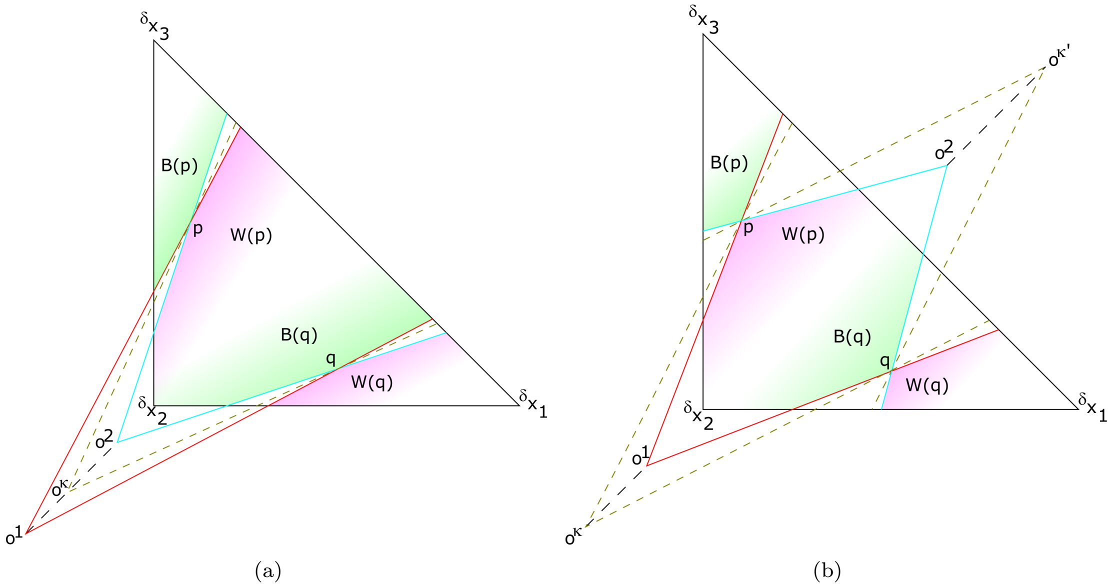

Figure 3: Multiple Weights

$\mathcal{W} = \{w^1, w^2\}$.

$$p \succ q \iff \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}, \ \forall\, w \in \mathcal{W}. \tag{2}$$

Note that in the multiple weight case depicted in Figure 3(a), analogously to the multiple utility case depicted in Figure 2, we may include in $\mathcal{W}$ the weight function w^κ corresponding to any point o^κ on the line segment connecting o^1 and o^2 without altering the preferences. However, attempting the same in Figure 3(b) would be invalid, as it would produce source points lying within the simplex. In this case, we can instead include any source points lying on the line defined by o^1 and o^2 but not on the segment connecting them, effectively connecting o^1 to o^2 through the point at infinity, as any of these would produce incomparability curves that lie between those projected from the endpoints and hence their inclusion would not alter the representation.

Finally, we consider the general case that incorporates both multiple utilities and multiple weights, as depicted in Figure 4. Here the four source points $\Omega = \{o^{11}, o^{12}, o^{21}, o^{22}\}$ correspond to pairs of utility and weight functions $\mathcal{V} = \{(u^1, w^1), (u^1, w^2), (u^2, w^1), (u^2, w^2)\}$ and the preferences depicted have the representation

$$p \succ q \iff \frac{\sum_{x \in X} p(x)w(x)u(x)}{\sum_{x \in X} p(x)w(x)} > \frac{\sum_{x \in X} q(x)w(x)u(x)}{\sum_{x \in X} q(x)w(x)}, \ \forall\, (u, w) \in \mathcal{V}. \tag{3}$$

Here the set of utility-weight pairs is separable, as $\mathcal{V} = \mathcal{U} \times \mathcal{W} = \{u^1, u^2\} \times \{w^1, w^2\}$, though this need not be the case generally. Any point lying in the convex hull of Ω would map to a utility-weight pair that could be included in $\mathcal{V}$ without altering the preferences represented. Therefore, this representation admits any of the models considered so far as special cases, with the single utility or single weight cases in (1) and (2) if respectively $\mathcal{U}$ or $\mathcal{W}$ are singletons, weighted utility if $\mathcal{V}$ is a singleton, multiple expected utility if every element of $\mathcal{W}$ is a constant function, and finally expected utility if all of these hold.

The next section introduces the basic model. Section 3 details the general multiple weighted expected utility model, with the special cases of a single utility or single weight covered in section 4. Concluding remarks appear in section 5 and the proofs are collected in section 6.

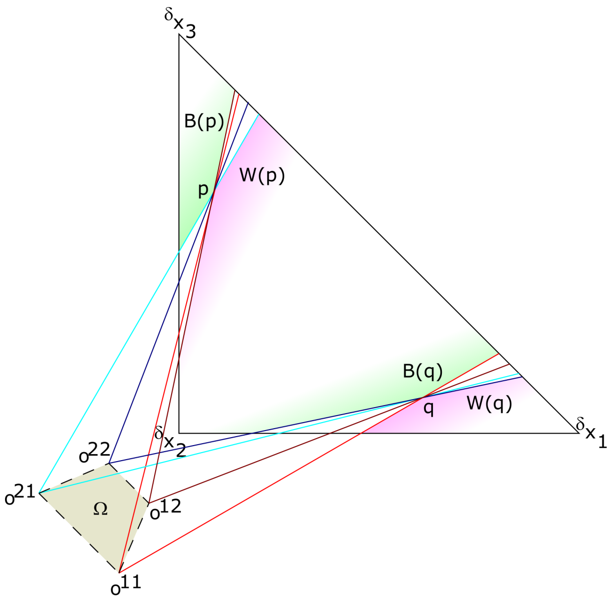

Figure 4: Multiple Weighted Expected Utility

2 Analytical Framework

2.1 Preference Structure

Let $X = \{x_1, \dots, x_n\}$ be a set of outcomes and $\Delta(X) = \{p \in \mathbb{R}^n_+ : \sum_{x \in X} p(x) = 1\}$ the set of lotteries over X. Let $\succ$ be a binary relation on $\Delta(X)$, which we refer to as a *strict preference relation*. The set $\Delta(X)$ is said to be $\succ$*-bounded* if there are best and worst outcomes $\overline{x}, \underline{x} \in X$ such that $\delta_{\overline{x}} \succ \delta_x \succ \delta_{\underline{x}}$, for all $x \in X \setminus \{\overline{x}, \underline{x}\}$, which we assume throughout.[4] Number the elements in X in nondecreasing order of preference, so that $\underline{x} = x_1$ and $\overline{x} = x_n$.

If the strict preference relation $\succ$ is negatively transitive, then its negation $\neg(p \succ q)$ defines the complete and transitive weak preference relation $p \preccurlyeq q$. The multiplicity of the utility representation thus depends on this assumption being violated, so that defining the *incomparability relation* $p \asymp q$ as the conjunction of $\neg(p \succ q)$ and $\neg(p \prec q)$, we obtain a relation that is not necessarily transitive and thus not necessarily an equivalence relation, unlike the indifference relation this would normally define under completeness.[5] Intuitively, the inability to rank a pair of alternatives does not necessarily mean that the decision maker considers them to be identical, but rather may imply that he evaluates them by multiple criteria that disagree on their ranking.

We assume throughout that $\succ$ is a continuous strict partial order.

(A.1) (Strict Partial Order) The preference relation $\succ$ is irreflexive and transitive.

(A.2) (Continuity) For all $p, q, r \in \Delta(X)$ if $p \succ q$ then there is $\underline{\alpha} \in (0, 1)$ such that $\alpha p + (1 - \alpha) r \succ q$ for all $\alpha > \underline{\alpha}$ and if $q \succ r$, there is $\overline{\alpha} \in (0, 1)$ such that $q \succ \alpha p + (1 - \alpha) r$ for all $\alpha < \overline{\alpha}$.

While this continuity axiom is not standard, it has the advantage of implying both the Archimedean and betweenness properties, which are standard in a range of models including expected, weighted, and implicit weighted utility theory.[6] The Archimedean property implies that no lotteries in $\Delta(X)$ are infinitely superior or inferior to any other, while betweenness asserts that a probability mixture of two lotteries must rank between them.

[4] Generally speaking, $\Delta(X)$ is $\succ$-bounded if there are $\overline{p}, \underline{p} \in \Delta(X)$ such that $\overline{p} \succ p \succ \underline{p}$ for all $p \in \Delta(X) \setminus \{\overline{p}, \underline{p}\}$. However, anticipating the monotonicity of the strict preference relation described below, there is no essential loss in our definition.

[5] We may still define weak preference and indifference relations that have the usual properties by following Galaabaatar and Karni (2013) and letting $p \succcurlyeq q$ if $r \succ p$ implies $r \succ q$, and $p \sim q$ if $p \succcurlyeq q$ and $p \preccurlyeq q$.

[6] See Dekel (1986) for an example and Chew (1989) for a review of this class of models.

(Archimedean) For all $p, q, r \in \Delta(X)$ such that $p \succ q \succ r$ there are $\alpha, \alpha' \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \alpha' p + (1-\alpha')r$.

(Betweenness) For all $p, r \in \Delta(X)$ and $\alpha \in (0,1)$, $p \succ r$ implies $p \succ \alpha p + (1-\alpha)r \succ r$.

Proposition 1 Continuity implies the Archimedean and betweenness properties.

For every $\alpha \in [0,1]$, let $\zeta_\alpha \equiv \alpha\delta_{x_n} + (1-\alpha)\delta_{x_1}$. For every $p \in \Delta(X)$, let $A(p) = \{\alpha \in [0,1] : p \asymp \zeta_\alpha\}$ denote the range of utility values assigned to p, measured along the line connecting the best and worst outcomes. The following proposition establishes that each of these utility ranges is a closed interval.

Proposition 2 For all $p \in \Delta(X)$, there are $\underline{\alpha}, \overline{\alpha} \in [0,1]$ such that $A(p) = [\underline{\alpha}, \overline{\alpha}]$.

In the standard expected utility and multiple expected utility models, applying the independence axiom at this step produces the desired utility representation.

2.2 Partial Substitution

At the core of weighted utility theory is the weak substitution axiom that replaces the independence axiom,[7] which can be equivalently expressed as a ratio substitution property.

(Weak Substitution) For all $p, q \in \Delta(X)$, $p \sim q$ if and only if for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that $\beta p + (1-\beta)r \sim \gamma q + (1-\gamma)r$ for all $r \in \Delta(X)$.

(Ratio Substitution) For all $p, q \in \Delta(X)$, $p \sim q$ if and only if there is $\tau > 0$ such that for every $\beta \in (0,1)$, $\beta p + (1-\beta)r \sim \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$ for all $r \in \Delta(X)$.

That these are equivalent can be shown by setting $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$, and interpreting this odds ratio as the weight of p relative to that of q. If $\succ$ is complete, a weighted linear utility function can thus be obtained by finding, for each $x_i \in X$, the unique α_i such that $\delta_{x_i} \sim \zeta_{\alpha_i}$, and τ_i satisfying ratio substitution between these two lotteries, and for any $p \in \Delta(X)$ repeatedly applying weak substitution to obtain

$$p \equiv \sum_{i=1}^n p_i \delta_{x_i} \sim \frac{p_1\tau_1\zeta_{\alpha_1} + \sum_{i=2}^n p_i\delta_{x_i}}{p_1\tau_1 + \sum_{i=2}^n p_i} \sim \cdots \sim \frac{\sum_{i=1}^n p_i\tau_i\zeta_{\alpha_i}}{\sum_{i=1}^n p_i\tau_i} = \zeta_{\frac{\sum_{i=1}^n p_i\tau_i\alpha_i}{\sum_{i=1}^n p_i\tau_i}} \equiv \zeta_{\alpha_p}. \tag{4}$$

By betweenness, the above implies that for any $p, q \in \Delta(X)$, $p \succ q \Leftrightarrow \alpha_p > \alpha_q$, so that we obtain a weighted utility representation by setting $u(x_i) = \alpha_i$ and $w(x_i) = \tau_i$ for $i = 1, \ldots, n$. The critical step in this construction lies in exploiting the transitivity of the indifference relation $\sim$.

For preferences $\succ$ that are not necessarily complete, we consider a modification that replaces the indifference relation $\sim$ with the incomparability relation $\asymp$ defined earlier.

(A.3) (Partial Substitution) For all $p, q \in \Delta(X)$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ there is $\gamma \in (0,1)$ such that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$ for all $r \in \Delta(X)$.

The next lemma establishes the analogous ratio substitution property in our setup.

Lemma 1 If $\succ$ satisfies (A.1)-(A.3), then for all $p, q \in \Delta(X)$, $p \asymp q$ if and only if there is $\tau > 0$ such that for every $\beta \in (0,1)$, $\beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}$ for all $r \in \Delta(X)$.

For any pair of incomparable lotteries, define the set of substitution odds ratios as

$$T(p,q) = \left\{\tau > 0 : \beta p + (1-\beta)r \asymp \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)},\ \forall\, \beta \in (0,1), r \in \Delta(X)\right\}.$$

Note that by (A.3), $T(p,q) \neq \varnothing$ if and only if $p \asymp q$. Under completeness, weak substitution implies that for every $\beta \in (0,1)$ we have a unique $\gamma \in (0,1)$, which can be seen by picking any $r \succ q$ and applying betweenness, and therefore that the odds ratio τ must be unique as well. This is not the case here however, as $\asymp$ is intransitive and hence $T(p,q)$ is not necessarily a singleton, implying that lotteries may have a range of weights in addition to a range of utility values.

[7] See Chew (1989).

Proposition 3 For all $p, q \in \Delta(X)$ such that $p \asymp q$, there are $\underline{\tau}, \overline{\tau} > 0$ such that $T(p,q) = [\underline{\tau}, \overline{\tau}]$.

We can now attempt to replicate the construction of the utility representation as in (4). For every $i = 1, \dots, n$, consider picking some $\alpha_i \in A(\delta_{x_i})$ and $\tau_i \in T(\delta_{x_i}, \zeta_{\alpha_i})$, and then repeatedly applying partial substitution to yield

$$p \equiv \sum_{i=1}^{n} p_i \delta_{x_i} \asymp \frac{p_1 \tau_1 \zeta_{\alpha_1} + \sum_{i=2}^{n} p_i \delta_{x_i}}{p_1 \tau_1 + \sum_{i=2}^{n} p_i} \asymp \cdots \asymp \frac{\sum_{i=1}^{n} p_i \tau_i \zeta_{\alpha_i}}{\sum_{i=1}^{n} p_i \tau_i} = \zeta_{\frac{\sum_{i=1}^{n} p_i \tau_i \alpha_i}{\sum_{i=1}^{n} p_i \tau_i}} \equiv \zeta_{\alpha_p}. \tag{5}$$

However, as $\asymp$ is intransitive, (5) does not necessarily imply that $p \asymp \zeta_{\alpha_p}$. Intuitively, if $\succ$ is complete, then every α_i and τ_i is unique, so that we can obtain for any p the unique α_p by simply taking the weighted convex combination as in (4). Under incompleteness, while we know that each $x_i \in X$ has utility range $A(\delta_{x_i})$, if we arbitrarily select $\{\alpha_1, \dots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i})$, these values need not be assigned by the same utility function, and hence the α_p produced by (5) need not belong to $A(p)$. It is the converse, that every $\alpha_p \in A(p)$ can be constructed from some $\{\alpha_1, \dots, \alpha_n\} \in \prod_{i=1}^{n} A(\delta_{x_i})$, which we need to show to ensure that preference relation can be represented by a set of weighted linear utility functions.

2.3 Source Space

As discussed in the introduction, a preference relation with a multiple weighted expected utility representation can be visualized as a simplex of lotteries with incomparability curves projected from a set of source points lying outside the simplex. Suppose we have $p, q \in \Delta(X)$ such that $p \asymp q$, and some $\tau \in T(p,q)$. By definition, for every $\beta \in (0,1)$ and $r \in \Delta(X)$, the line defined by $\beta p + (1-\beta) r$ and $\frac{\beta \tau q + (1-\beta) r}{\beta \tau + (1-\beta)}$ is an incomparability curve. All of these curves converge at some source point o, and as its location can depend on neither β nor r, we have that

$$o = \frac{1}{\beta(1-\tau)} [\beta p + (1-\beta) r] - \frac{\beta \tau + (1-\beta)}{\beta(1-\tau)} \left[\frac{\beta \tau q + (1-\beta) r}{\beta \tau + (1-\beta)} \right] = \frac{p - \tau q}{1-\tau}. \tag{6}$$

Define the source space Ω as the collection of all such source points.

$$\Omega = \left\{ o = \frac{p - \tau q}{1 - \tau} : p \asymp q, \tau \in T(p,q) \right\}$$

The following proposition asserts that Ω fully characterizes the incomparability relation $\asymp$ and, consequently, the preference relation $\succ$ as well. It states that any line connecting two lotteries is an incomparability curve if and only if it is projected from some source point $o \in \Omega$.

Proposition 4 For every $p, q \in \Delta(X)$, $p \asymp q$ if and only if there is $\tau \in T(p,q)$ such that $o = \frac{p - \tau q}{1-\tau} \in \Omega$.

For each p define $\Phi(p) = \{(\alpha_p, \tau_p) : \alpha_p \in A(p), \tau_p \in T(p, \zeta_{\alpha_p})\}$ as the collection of associated utility and weight pairs, each defining a source point $o_p = \frac{p - \tau_p \zeta_{\alpha_p}}{1 - \tau_p} \in \Omega$.

A utility function over X is given by a collection of utility weight pairs $\{(\alpha_i, \tau_i)\}_{i=1}^{n}$ corresponding to each of the degenerate lotteries $\{\delta_{x_i}\}_{i=1}^{n}$ such that $(\alpha_p, \tau_p) = \left(\frac{\sum_{i=1}^{n} p_i \tau_i \alpha_i}{\sum_{i=1}^{n} p_i \tau_i}, \sum_{i=1}^{n} p_i \tau_i \right) \in \Phi(p)$ for any lottery $p \in \Delta(X)$. Define the set of all such collections as

$$\Psi = \left\{ \{(\alpha_i, \tau_i)\}_{i=1}^{n} : (\alpha_p, \tau_p) = \left(\frac{\sum_{i=1}^{n} p_i \tau_i \alpha_i}{\sum_{i=1}^{n} p_i \tau_i}, \sum_{i=1}^{n} p_i \tau_i \right) \in \Phi(p), \ \forall p \in \Delta(X) \right\}.$$

Every $\psi = \{(\alpha_i^{\psi}, \tau_i^{\psi})\}_{i=1}^{n} \in \Psi$ defines a weighted linear utility function, and letting $o_i^{\psi} = \frac{\delta_{x_i} - \tau_i^{\psi} \zeta_{\alpha_i^{\psi}}}{1 - \tau_i^{\psi}} \in \Omega$ denote the source point that ψ associates with outcome x_i, we have for every $p \in \Delta(X)$ that

$$o_p^{\psi} = \frac{p - \tau_p^{\psi} \zeta_{\alpha_p^{\psi}}}{1 - \tau_p^{\psi}} = \frac{\sum_{i=1}^{n} p_i (1 - \tau_i^{\psi}) o_i^{\psi}}{\sum_{i=1}^{n} p_i (1 - \tau_i^{\psi})} \in \Omega. \tag{7}$$

That is, the source point associated with any lottery p is also a weighted convex combination of elements of $\{o_i^\psi\}_{i=1}^n$.[8] Collecting these points forms a convex subset of the source space, $O^\psi = \{o_p^\psi\}_{p\in\Delta(X)} \subseteq \Omega$ which characterizes a function pair (u^ψ, w^ψ). Therefore, to establish the representation theorem, we need to show that the collection of these subsets covers the source space $\bigcup_{\psi\in\Psi} O^\psi = \Omega$, so that the collection of function pairs Ψ defines fully characterizes the incomparability map and by extension the preference relation $\succ$ itself.

3 Representation

3.1 Existence

Before presenting the main theorem, we first establish some preliminary results. Suppose we start with any single lottery p, then we can find some other lottery q to which it is incomparable $p \asymp q$. Applying the partial substitution axiom, this relation implies the existence of a set of incomparability curves converging at a source point o, which is collinear with p and q. Lemma 2 asserts that we can find a third lottery r that is incomparable to both p and q, as well as any lottery on the incomparability curve they define, so that the three lotteries together define an incomparability plane.

Lemma 2 If $\succ$ satisfies (A.1)-(A.3), then for all $p, q \in \Delta(X)$, the following statements are equivalent:

1. [(i)]
2. $p \asymp q$.
3. There exists $r \in \Delta(X)$ such that $\lambda p + (1-\lambda)q \asymp r$ for all $\lambda \in [0,1]$.
4. There exists $r \in \Delta(X)$ and $\tau_p, \tau_q > 0$ such that for all $\lambda \in [0,1]$ and $\beta \in (0,1)$, $\beta[\lambda p + (1-\lambda)q] + (1-\beta)s \asymp \frac{\beta[\lambda\tau_p+(1-\lambda)\tau_q]r+(1-\beta)s}{\beta[\lambda\tau_p+(1-\lambda)\tau_q]+(1-\beta)}$ for all $s \in \Delta(X)$.
5. There exists $r \in \Delta(X)$ such that $p' \asymp q'$ for all $p', q' \in \Delta(\{p, q, r\})$.

By Lemma 2, every source point that projects a set of incomparability curves lies on a source line, on which every point is itself a source, which in turn projects a set of incomparability planes. The natural next step is to generalize this property, allowing us to construct a set of source points that will fully characterize a utility function.

Any collection $P \subseteq \Delta(X)$ of lotteries constitutes an *incomparability set* if $p \asymp q$ for any $p, q \in \Delta(P)$,[9] thus forming the natural higher dimensional analogue to the incomparability curves and incomparability planes encountered so far. As $\delta_{x_n} \succ \delta_{x_1}$ implies $\succ$ is non-empty, $\Delta(P) \subsetneq \Delta(X)$ and an incomparability set is at most of dimension $n-2$. The following lemma shows that, starting from any pair of incomparable lotteries, we can build up to such a maximal set.

Lemma 3 If $\succ$ satisfies (A.1)-(A.3), then for all $P \subseteq \Delta(X)$, the following statements are equivalent:

1. [(i)]
2. $p \asymp q$ for all $p, q \in \Delta(P)$ and $\dim P < n-2$.
3. There exists $r \in \Delta(X) \setminus \Delta(P)$ such that $p \asymp r$ for every $p \in \Delta(P)$

[8] By definition of ψ, we have that

$$o_p^\psi = \frac{p - \tau_p^\psi \zeta_{\alpha_p^\psi}}{1-\tau_p^\psi} = \frac{\sum_{i=1}^n p_i\delta_{x_i} - (\sum_{i=1}^n p_i\tau_i^\psi)\zeta_{\frac{\sum_{i=1}^n p_i\tau_i^\psi\alpha_i^\psi}{\sum_{i=1}^n p_i\tau_i^\psi}}}{1-(\sum_{i=1}^n p_i\tau_i^\psi)} = \frac{\sum_{i=1}^n p_i\delta_{x_i} - (\sum_{i=1}^n p_i\tau_i^\psi)\left(\frac{\sum_{i=1}^n p_i\tau_i^\psi\zeta_{\alpha_i^\psi}}{\sum_{i=1}^n p_i\tau_i^\psi}\right)}{1-(\sum_{i=1}^n p_i\tau_i^\psi)}$$

$$= \frac{\sum_{i=1}^n p_i(\delta_{x_i} - \tau_i^\psi\zeta_{\alpha_i^\psi})}{\sum_{i=1}^n p_i(1-\tau_i^\psi)} = \frac{\sum_{i=1}^n p_i(1-\tau_i^\psi)\left(\frac{\delta_{x_i}-\tau_i^\psi\zeta_{\alpha_i^\psi}}{1-\tau_i^\psi}\right)}{\sum_{i=1}^n p_i(1-\tau_i^\psi)} = \frac{\sum_{i=1}^n p_i(1-\tau_i^\psi)o_i^\psi}{\sum_{i=1}^n p_i(1-\tau_i^\psi)} \in \Omega.$$

[9] We slightly abuse notation here to define $\Delta(P) = \{q = \sum_{p\in P}\pi_p p : \{\pi_p\}_{p\in P} \subseteq \mathbb{R}_{\geq 0}\}$, that is, the set of reduced compound lotteries over P.

4. There exists $r \in \Delta(X) \setminus \Delta(P)$ and $\{\tau_p\}_{p\in P} \subseteq \mathbb{R}_{>0}$ such that for all $\{\pi_p\}_{p\in P} \subseteq \mathbb{R}_+$ such that $\sum_{p\in P} \pi_p = 1$ and $\beta \in (0,1)$, $\beta\left(\sum_{p\in P} \pi_p p\right) + (1-\beta)s \asymp \frac{\beta\left(\sum_{p\in P} \pi_p \tau_p\right)r+(1-\beta)s}{\beta\tau_q+(1-\beta)}$ for all $s \in \Delta(X)$.

5. There exists $r \in \Delta(X) \setminus \Delta(P)$ such that $p' \asymp q'$ for all $p', q' \in \Delta(P \cup \{r\})$.

Starting with any pair of incomparable lotteries, we can repeatedly apply Lemma 3 to add to it until we obtain a maximal incomparability set P. The following lemma shows that every such P maps to some utility function generated by a $\psi \in \Psi$.

Lemma 4 If $\succ$ satisfies (A.1)-(A.3), then for all $p, q \in \Delta(X)$, $p \asymp q$ if and only if there is $\psi \in \Psi$ such that $\frac{\sum_{i=1}^n p_i \tau_i^\psi \alpha_i^\psi}{\sum_{i=1}^n p_i \tau_i^\psi} = \frac{\sum_{i=1}^n q_i \tau_i^\psi \alpha_i^\psi}{\sum_{i=1}^n q_i \tau_i^\psi}$.

Note that Lemma 4 also implicitly establishes $\Psi \neq \varnothing$ as long as the incomparability relation is itself nonempty. We are now ready to present the main representation theorem.

Theorem 1 Let $\succ$ be a binary relation over $\Delta(X)$, then $\Delta(X)$ satisfies (A.1)-(A.3) if and only if there is a set $\mathcal{V}$ of utility $u : X \mapsto \mathbb{R}$ and weight $w : X \mapsto \mathbb{R}_{>0}$ function pairs (u, w) such that for every $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)},\ \forall\,(u,w) \in \mathcal{V}.$$

3.2 Uniqueness

Having established the existence of a utility representation, we now turn our attention to the question of uniqueness. As our model lies at the convergence of weighted utility and multiple utility theory, our uniqueness result naturally must incorporate elements of the uniqueness results found in both. From weighted utility models, we know that taking an affine transformation of a utility function u will not preserve its weighted linearity, and we must instead apply a rational affine transformation to both u and the associated weight function w jointly.

Proposition 5 For utility and weight function (u, w) and constants a, b, c, d such that $ad > bc$, define the rational affine transformation $(\tilde{u}, \tilde{w}) = \left(\frac{au+b}{cu+d}, w[cu + d]\right)$. Then for every $p, q \in \Delta(X)$,

$$\frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)} \Longleftrightarrow \frac{\sum_{x\in X} p(x)\tilde{w}(x)\tilde{u}(x)}{\sum_{x\in X} p(x)\tilde{w}(x)} > \frac{\sum_{x\in X} q(x)\tilde{w}(x)\tilde{u}(x)}{\sum_{x\in X} q(x)\tilde{w}(x)}.$$

The utility functions we construct from Ψ are normalized, with $(u^\psi(x_1), w^\psi(x_1)) = (0, 1)$ for the worst outcome and $(u^\psi(x_n), w^\psi(x_n)) = (1, 1)$ for the best, for every $\psi \in \Psi$. The following proposition shows that any function pair (u, w) has a rational affine transformation $(\hat{u}, \hat{w})$ that is similarly normalized, which in turn maps to some collection $\psi \in \Psi$.

Proposition 6 Let the collection $\mathcal{V}$ represent $\succ$. Then for every $(u, w) \in \mathcal{V}$ there is a normalized rational affine transformation $(\hat{u}, \hat{w})$ such that $\{(\hat{u}(x_i), \hat{w}(x_i))\}_{i=1}^n \in \Psi$.

For any $\mathcal{V}$, define the normalized set as $\hat{\mathcal{V}} = \{(\hat{u}, \hat{w}) : (u, w) \in \mathcal{V}\}$. In multiple expected utility models with independence, if we have a set $\hat{\mathcal{U}}$ of normalized utilities, some elements may merely be convex combinations of others and may be excluded without altering the preferences that $\hat{\mathcal{U}}$ represents. A similar idea exists here, though we cannot simply take convex combinations of weighted linear functions that simultaneously preserve weighted linearity while maintaining the preference ordering.[10] Rather than trying to define some convex closure of $\hat{\mathcal{V}}$ directly, we instead narrow our focus to a neighborhood around some $p \in \Delta(X)$. For every $(u, w) \in \hat{\mathcal{V}}$, define the local utility function $u_p : X \mapsto \mathbb{R}$ by setting

$$u_p = wu + (1-w)\left[\frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)}\right] \equiv wu + (1-w)\bar{u}_p. \tag{8}$$

[10] Given $(u^1, w^1), (u^2, w^2) \in \hat{\mathcal{V}}$ and $\kappa \in (0, 1)$, we could take a direct convex combination by setting $u^\kappa = \kappa u^1 + (1-\kappa)u^2$ and $w^\kappa = \kappa w^1 + (1-\kappa)w^2$, but unless $w^1 = w^2$, this would not preserve weighted linearity. Alternatively, we could take the weighted convex combination and set $u^\kappa = \frac{\kappa w^1 u^1 + (1-\kappa)w^2 u^2}{\kappa w^1 + (1-\kappa)w^2}$ and $w^\kappa = \kappa w^1 + (1-\kappa)w^2$. This however would not necessarily preserve the ordering of lotteries, as $u^k(x_i) > u^k(x_j)$ for $k = 1, 2$ does not imply $u^\kappa(x_i) > u^\kappa(x_j)$.

As u_p gives a normalized linear approximation[11] of (u, w) around p, collecting all such u_p gives a set $\hat{\mathcal{U}}_p = \{u_p = wu + (1-w)\bar{u}_p : (u, w) \in \hat{\mathcal{V}}\}$ of local utility functions that form a multiple expected utility representation that approximates $\succ$ around p. Furthermore, as in standard multiple utility models, we may include any local utility in the convex hull $\langle \hat{\mathcal{U}}_p \rangle = \{u_p^\pi = \sum_{u_p^k \in \mathcal{U}_p} \pi^k u_p^k : \sum_{u_p^k \in \mathcal{U}_p} \pi^k = 1\}$ without altering the preferences.

Proposition 7 For every $p \in \Delta(X)$ there is a convex set $\langle \hat{\mathcal{U}}_p \rangle$ of utilities $u_p : X \mapsto \mathbb{R}$ such that for every $q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \sum_{x \in X} p(x) u_p(x) > \sum_{x \in X} q(x) u_p(x), \; \forall \, u_p \in \langle \hat{\mathcal{U}}_p \rangle.$$

The transformation in (8) is many to one, as around every p, any pair $(u, w) \in \hat{\mathcal{V}}$ maps to local utility $u_p \in \hat{\mathcal{U}}_p$, but it is not necessarily the case that $wu + (1-w)\bar{u}_p \in \hat{\mathcal{U}}_p$ implies $(u, w) \in \hat{\mathcal{V}}$. By Proposition 7, any u_p in the convex hull $\langle \hat{\mathcal{U}}_p \rangle$ is also consistent with $\succ$ in a neighborhood around p, which leads us to define the set of pairs (u, w) everywhere consistent with $\succ$ as

$$\langle \hat{\mathcal{V}} \rangle = \{(u, w) : u_p = wu + (1-w)\bar{u}_p \in \langle \hat{\mathcal{U}}_p \rangle, \forall \, p\}.$$

The set $\langle \hat{\mathcal{V}} \rangle$ is the maximal set of normalized utility and weight pairs that agrees with the ordering of lotteries prescribed by $\succ$. The uniqueness theorem presented below asserts that two utility representations are identical if and only if they generate the same such maximal normalized sets.

Theorem 2 For $j = 1, 2$, let $\succ^j$ be a binary relation over $\Delta(X)$ that has a multiple weighted expected utility representation by a set $\mathcal{V}^j$ of utility $u : X \mapsto \mathbb{R}$ and weight $w : X \mapsto \mathbb{R}_{>0}$ function pairs (u, w). The preferences are identical $\succ^1 = \succ^2$ if and only if $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle$.

4 Special Cases

Weighted utility theory with incomplete preferences admits incompleteness arising either from conflicting perceptions, represented by multiple weight functions, or from indecisive tastes, represented by multiple utility functions. Thus the general framework we have devised admits a pair of special cases, those of multiple utilities paired with a single weight function $\mathcal{V} = \mathcal{U} \times \{w\}$ or a single utility paired with multiple weights $\mathcal{V} = \{u\} \times \mathcal{W}$.

In non-expected utility, local risk attitudes are captured by the local utility functions and global risk attitude depends on the variations of the local risk attitudes, as in Machina (1982). In weighted utility, the utility and weight functions play distinct roles, with the shape of the utility function capturing the decision maker's risk attitude while the weight function captures the nature and degree of the variation in local attitudes. Specifically, the weight function reflects the extent to which the indifference map exhibits the fanning in or fanning out structure described by Machina (1982).[12]

A decision maker with multiple utility functions and a single weight function has incomplete preferences solely due to his indecisive risk attitude, and has no more difficulty evaluating a lottery than he would evaluating each of its possible outcomes. On the other hand, a decision maker with a single utility function and multiple weight functions is sure of his risk attitude, but is indecisive when comparing lotteries because he is unsure of how to perceive randomness, and thus cannot always rate lotteries properly even if he knows how would rank their components.

[11] Since $w(x_1) = w(x_n) = 1$, we have that $u_p(x_1) = u(x_1) = 0$ and $u_p(x_n) = u(x_n) = 1$. Furthermore,

$$\sum_{x \in X} p(x) u_p(x) = \sum_{x \in X} p(x) w(x) u(x) + \left[1 - \sum_{x \in X} p(x) w(x)\right] \left[\frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)}\right] = \frac{\sum_{x \in X} p(x) w(x) u(x)}{\sum_{x \in X} p(x) w(x)}.$$

[12] Fanning out reflects a decision maker who underweights the median outcome relative to the extremes, corresponding to monotonically increasing local risk aversion with respect to first order stochastically dominating shifts, whereas fanning in reflects an overweight of the median outcome and hence decreasing local risk aversion.

4.1 Multiple Utilities

A decision maker whose preferences are represented by multiple utilities paired with a single weight function is indecisive about the valuation of each of the outcomes in X, but is confident of how much importance to attach to these outcomes when evaluating any lottery $p \in \Delta(X)$. For example, the decision maker may have several utilities exhibiting varying degrees of risk aversion, but is sure of how much attention he should pay to each of the possible payoffs. To ensure that a preference relation $\succ$ has such a representation, we adopt a stronger variant of the partial substitution axiom.[13]

(A.4) **(Unique Substitutability)** For all $p, q \in \Delta(X)$, $p \asymp q$ if and only if for every $\beta \in (0,1)$ and $\gamma \in (0,1)$, either $\gamma q + (1-\gamma)r \succ \beta p + (1-\beta)r$ or $\beta p + (1-\beta)r \succ \gamma' q + (1-\gamma')r$ for all $\gamma' < \gamma$ and $r \in \Delta(X)$.

Axioms (A.1), (A.2), and (A.3) jointly with (A.4) imply that, for every $p \asymp q$, there must be a unique substitution ratio $T(p,q) = \{\tau_{p,q}\}$. We prove this claim as part of the proof of the following lemma which shows that for every p we can pair a unique weight τ_p with any of the utility values $\alpha_p \in A(p)$.

Lemma 5 If $\succ$ satisfies (A.1), (A.2), (A.3) and (A.4), then for all $p \in \Delta(X)$ there is $\tau_p > 0$ such that $\Phi(p) = A(p) \times \{\tau_p\}$.

This result leads directly into the following representation theorem.

Theorem 3 Let $\succ$ be a binary relation over $\Delta(X)$, then $\Delta(X)$ satisfies (A.1),(A.2),(A.4) if and only if there is a set $\mathcal{U}$ of utility functions $u : X \mapsto \mathbb{R}$ and a weight function $w : X \mapsto \mathbb{R}_{>0}$ such that for every $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)},\ \forall\, u \in \mathcal{U}.$$

It directly follows that a multiple utility, single weight representation is equivalent to applying a one-time transformation to the entire probability space and constructing a multiple expected utility representation over the transformed probability space. For any p, define the transformed lottery p^w such that $p^w(x) = \frac{p(x)w(x)}{\sum_{x\in X} p(x)w(x)}$ for every $x \in X$, and define the relation $\succ^w$ such that $p^w \succ^w q^w$ if and only if $p \succ q$. Then it immediately follows that $\succ^w$ has a multiple expected utility representation as

$$p^w \succ^w q^w \Longleftrightarrow \sum_{x\in X} p^w(x)u(x) > \sum_{x\in X} q^w(x)u(x),\ \forall\, u \in \mathcal{U}.$$

While this transformation may appear to indicate that we could apply the uniqueness results from multiple expected utility to $\mathcal{U}$, by Theorem 2 there is in fact a broader set of equivalent representations. For example, suppose we had $\mathcal{U} = \{u^1, u^2\}$ and set $\tilde{u}^j = \frac{au^j+b}{cu^j+d}$ and $\tilde{w}^j = w[cu^j + d]$ for $j = 1,2$. Then $\tilde{\mathcal{V}} = \{(\tilde{u}^1, \tilde{w}^1), (\tilde{u}^2, \tilde{w}^2)\}$ would represent the same preferences as $\mathcal{U} \times \{w\}$, even though the former has multiple weight functions while the latter has only one.

4.2 Multiple Weights

A decision maker whose preferences are represented by a single utility function paired with multiple weight functions is confident of how he would evaluate all of the outcomes in X, but is indecisive over how much importance each of these outcomes carries when evaluating a lottery $p \in \Delta(X)$. Such a decision maker would be sure of his tastes, but when trying to compare alternative lotteries is unable to determine what aspects to focus on and attach more weight to. To ensure that a preference relation has such a representation, we impose the following axiom.

(A.5) **(Unique Solvability)** For all $x \in X$, and $\alpha \in [0,1]$ either $\zeta_\alpha \succ \delta_x$ or $\delta_x \succ \zeta_{\alpha'}$, for all $\alpha' < \alpha$.

Axioms (A.1), (A.2), and (A.3) jointly with (A.5) implie that every degenerate lottery has only a single utility value. We prove this claim as part of the proof of Theorem 4, below. Consequently, $A(\delta_{x_i}) = \{\alpha_i\}$

[13] Axioms (A.4) and (A.5) below are based on an idea first advanced by Galaabaatar and Karni (2013) in the form of an axiom dubed complete beliefs.

for every $i = 1, \ldots, n$, but may take multiple weight values so that $T(\delta_{x_i}, \zeta_{\alpha_i})$ need not be a singleton, and hence non-degenerate lotteries $p \in \Delta(X) \setminus X$ may still have multiple utility values. Imposing this assumption leads to a single utility, multiple weight representation.

Theorem 4 Let $\succ$ be a binary relation over $\Delta(X)$, then $\Delta(X)$ satisfies (A.1)-(A.3), (A.5) if and only if there is a utility function $u : X \mapsto \mathbb{R}$ and a set $\mathcal{W}$ of weight functions $w : X \mapsto \mathbb{R}_{>0}$ such that for every $p, q \in \Delta(X)$,

$$p \succ q \Longleftrightarrow \frac{\sum_{x\in X} p(x)w(x)u(x)}{\sum_{x\in X} p(x)w(x)} > \frac{\sum_{x\in X} q(x)w(x)u(x)}{\sum_{x\in X} q(x)w(x)},\ \forall\, w \in \mathcal{W}.$$

Unlike in the multiple utility, single weight case, there is no simple transformation here that we can apply to produce a more familiar multiple utility representation. In this case, the decision maker is unsure of how to perceive randomness, as each of his weight functions distort his focus differently. While he is able to rank all of the outcomes, his ability to evaluate lotteries is compromised by his inability to determine which components he should be paying attention to.

4.3 Behavioral Manifestations

To illustrate the difference between the two sources of indecisiveness, recall that the main empirical manifestation of incompleteness is *inertia.* Given an alternative, a, in some choice set, there is a range of non-comparable alternatives that will not be accepted if they were offered in exchange for a. In other words, given the default alternative, a, the decision makers' behavior may be described by the maxim "when in doubt do nothing."

In weighted utility theory with incomplete preferences, the nature of inertia depends on the source of indecisiveness. Specifically, if his indecisiveness is due to risk attitude then the decision maker displays inertia everywhere. By contrast, if the source of his indecisiveness is perceived randomness then the decision maker displays inertia everywhere except at degenerate lotteries δ_x. These are testable implications. For example, the subject in an experiment may receive δ_x by default and be offered the opportunity to trade it for some lottery $\{\zeta_\alpha$. Using standard experimental methods it is possible to verify if the subject switches at one point, thus indicating indecisiveness due to perceived randomness, or choose to hold on to δ_x over a range ζ_α indicating indecisiveness due to risk attitudes.

The following example illustrates another property that distinguishes the two special case. Let $X = \{x_1, x_2, x_3\}$, and suppose that $\delta_{x_3} \succ \delta_{x_2} \succ \delta_{x_1}$. Let $p = \beta\delta_{x_3} + (1-\beta)\,\delta_{x_2}$ and $p' = \beta\delta_{x_1} + (1-\beta)\,\delta_{x_2}$ for some $\beta \in (0,1)$. Define $\bar{\alpha}_2 = \inf\{\alpha \mid \zeta_\alpha \succ \delta_{x_2}\}$ and $\underline{\alpha}_2 = \sup\{\alpha \mid \delta_{x_2} \succ \zeta_a\}$. Then we have the following implications:[14]

If the source of indecisiveness is perceived randomness then $\bar{\alpha}_2 = \underline{\alpha}_2 := \hat{\alpha}$. Moreover,

$$\inf\{\gamma \mid \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\hat{\alpha}} \succ p\} = \inf\{\gamma \mid \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\hat{\alpha}} \succ p'\}$$

and

$$\sup\{\gamma \mid p \succ \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\hat{\alpha}}\} = \sup\{\gamma \mid p' \succ \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\hat{\alpha}}\}.$$

If the source of indecisiveness is risk attitude, then $\bar{\alpha}_2 > \underline{\alpha}_2$. Moreover,

$$\inf\{\gamma \mid \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\bar{\alpha}_2} \succ p\} = \sup\{\gamma \mid p' \succ \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\underline{\alpha}_2}\},$$

and

$$\sup\{\gamma \mid p \succ \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\bar{\alpha}_2}\} = \inf\{\gamma \mid \gamma\delta_{x_3} + (1-\gamma)\,\zeta_{\underline{\alpha}_2} \succ p'\}.$$

5 Concluding Remarks

In this paper, we considered a model of decision making under risk for preferences that satisfy neither independence nor completeness. We obtain a utility representation by the agreement of a set of utilities, as in multiple utility theory, each of which is weighted linear in the probabilities, as in weighted utility theory,

[14]The validity of these observations is immediate upon eyeballing Figures 2 and 3.

thus linking these separate strands in the literature under a unified framework. This representation further admits a variety of additional cases with distinct interpretations, as incomplete preferences may be due to ambivalent risk attitudes or incognizance of the relative salience of the possible outcomes. By directly imposing additional axioms that eliminate either of these possibilities, we obtain special cases where the multiplicity in the representation is restricted to either the utility or weight functions alone. The general framework we have devised thus serves as a useful foundation for studying decision making under risk from a variety of different perspectives.

6 Proofs

6.1 Proofs of Propositions

6.1.1 Proof of Proposition 1

To show that (A.2) implies the Archimedean property, let $p, q, r \in \Delta(X)$ such that $p \succ q \succ r$. Then, by (A.2) there are $\underline{\alpha}, \overline{\alpha} \in (0,1)$ such that if we pick $\alpha, \alpha' \in (0,1)$ such that $\alpha > \underline{\alpha}$ and $\alpha' < \overline{\alpha}$, $\alpha p + (1-\alpha) r \succ q \succ \alpha' p + (1-\alpha') r$.

To show that (A.2) implies betweenness, let $p, r \in \Delta(X)$ and $p \succ r$. Let $q = r$, then since $p \succ q$, by repeated application of (A.2) we have $\alpha p + (1-\alpha) r \succ r$, for all $\alpha \in (0,1)$. Now let $q = p$, then since $q \succ r$, by repeated application of (A.2) we have $p \succ \alpha p + (1-\alpha) r$, for all $\alpha \in (0,1)$. □

6.1.2 Proof of Proposition 2

Fix $p \in \Delta(X)$. Define $\overline{\alpha} = \inf\{\alpha : p \prec \zeta_\alpha\}$. Suppose for $\alpha' \leq \overline{\alpha}$ we have $p \prec \zeta_{\alpha'}$, then since by (A.2) there is $\beta \in (0,1)$ such that $p \prec \beta \zeta_{\alpha'} + (1-\beta)\delta_{\underline{x}} = \zeta_{\alpha''}$, but since $\alpha'' = \beta\alpha' < \overline{\alpha}$, this contradicts the definition of $\overline{\alpha}$. Now suppose that for $\alpha' > \overline{\alpha}$ we have $\neg(p \prec \zeta_{\alpha'})$, then for all $\alpha'' < \alpha'$ we have $\neg(p \prec \zeta_{\alpha''})$ or else $p \prec \zeta_{\alpha''} \prec \zeta_{\alpha'}$, which implies that $\alpha' \leq \inf\{\alpha : p \prec \zeta_\alpha\} = \overline{\alpha}$, a contradiction. Thus $\neg(p \prec \zeta_{\alpha'})$ if and only if $\alpha' \leq \overline{\alpha}$. Now define $\underline{\alpha} = \sup\{\alpha : p \succ \zeta_\alpha\}$, by a similar argument $\neg(p \succ \zeta_{\alpha'})$ if and only if $\alpha' \geq \underline{\alpha}$. Therefore, we have that $p \asymp \zeta_{\alpha'}$ if and only if $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$. □

6.1.3 Proof of Proposition 3

Fix $p, q \in \Delta(X)$ such that $p \asymp q$. For every $r \in \Delta(X)$, define

$$T^L(p,q,r) = \left\{\tau > 0 : \exists \beta,\ \beta p + (1-\beta) r \succ \frac{\beta\tau q + (1-\beta) r}{\beta\tau + (1-\beta)}\right\},$$
$$T^R(p,q,r) = \left\{\tau > 0 : \exists \beta,\ \beta p + (1-\beta) r \prec \frac{\beta\tau q + (1-\beta) r}{\beta\tau + (1-\beta)}\right\},$$
$$T(p,q,r) = \left\{\tau > 0 : \forall \beta,\ \beta p + (1-\beta) r \asymp \frac{\beta\tau q + (1-\beta) r}{\beta\tau + (1-\beta)}\right\}.$$

Let $R = \{r \in \Delta(X) : \neg(r \asymp p) \vee \neg(r \asymp q)\}$ denote the set of all lotteries that are comparable with either p or q. We will establish that for every $r \in R$, $T(p,q,r)$ is a closed interval $[\underline{\tau}_r, \overline{\tau}_r]$, and for every $r \notin R$, there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$. Taken together these will allow us to conclude that $T(p,q)$ is given by the intersection of closed intervals and is hence itself a closed interval.

Suppose $r \in R$. Then if $r \succ q$, define $\overline{\tau}_r = \inf T^L(p,q,r)$ and $\underline{\tau}_r = \sup T^R(p,q,r)$.[15] Suppose that for $\tau' \leq \overline{\tau}_r$ we have $\tau' \in T^L(p,q,r)$, then there is $\tau'' \in T^L(p,q,r)$ such that $\tau'' < \tau' \leq \overline{\tau}_r$, contradicting the definition of $\overline{\tau}_r$.[16] Now suppose that for $\tau' > \overline{\tau}_r$ we have $\tau' \notin T^L(p,q,r)$, then we must have

[15] Let $\overline{\tau}_r = \infty$ if $T^L(p,q,r) = \varnothing$ and $\underline{\tau}_r = 0$ if $T^R(p,q,r) = \varnothing$.

[16] If $\tau' \in T^L(p,q,r)$ then there is $\beta \in (0,1)$ such that $\beta p + (1-\beta) r \succ \frac{\beta\tau' q + (1-\beta) r}{\beta\tau' + (1-\beta)}$. This implies by (A.2) that there is $\lambda \in (0,1)$ such that

$$\beta p + (1-\beta) r \succ \frac{\lambda[\beta\tau' + (1-\beta)]\left[\frac{\beta\tau' q + (1-\beta) r}{\beta\tau' + (1-\beta)}\right] + (1-\lambda) r}{\lambda[\beta\tau' + (1-\beta)] + (1-\lambda)} = \frac{\lambda\beta\tau' q + (1-\lambda\beta) r}{\lambda\beta\tau' + (1-\lambda\beta)}.$$

Letting $\tau'' = \frac{\lambda\tau'\beta/(1-\lambda\beta)}{\beta/(1-\beta)} < \tau'$ completes the argument.

$\tau'' \notin T^L(p,q,r)$ for every $\tau'' < \tau'$[17], and therefore $\tau' \leq \inf T^L(p,q,r) = \overline{\tau}_r$, a contradiction. Thus $\tau' \in T^L(p,q,r)$ if and only if $\tau' > \overline{\tau}_r$, and by a similar argument $\tau' \in T^R(p,q,r)$ if and only if $\tau' < \underline{\tau}_r$. This implies that $\tau' \in T(p,q,r)$ if and only if $\tau' \in [\underline{\tau}_r, \overline{\tau}_r]$.

If $r \prec q$, then we can define $\overline{\tau}_r = \inf T^R(p,q,r)$ and $\underline{\tau}_r = \sup T^L(p,q,r)$[18] and apply a similar argument to the above. If $r \asymp q$ and either $r \succ p$ or $r \prec p$, we can again repeat the argument above by switching p and q and noting that $\tau' \in T(p,q,r)$ if and only if $\frac{1}{\tau'} \in T(q,p,r)$. Thus for every $r \in R$, $T(p,q,r)$ is a closed interval $[\underline{\tau}_r, \overline{\tau}_r]$.

Now suppose $r \notin R$. Then if there is $r' \in \Delta(\{p,q,r\})$ such that $r' \succ q$, there are $\lambda, \alpha \in [0,1]$ such that $r' = \lambda[\alpha p + (1-\alpha)r] + (1-\lambda)q \succ q$, so that by betweenness we have $s = \alpha p + (1-\alpha)r \succ q$. This implies that there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$.[19] A similar result follows if we have $s \in \Delta(\{p,q,r\})$ such that $s \prec q$. Likewise, if there is $s \in \Delta(\{p,q,r\})$ such that $s \succ p$ or $s \prec p$, we repeat the argument again noting that $\tau \in T(p,q,r)$ if and only if $\frac{1}{\tau} \in T(q,p,r)$, so that $T(p,q,r) \supseteq T(p,q,s)$ if and only if $T(q,p,r) \supseteq T(q,p,s)$.

Now suppose that for $s \in \Delta(\{p,q,r\})$ we have $s \asymp p$ and $s \asymp q$. If for some $\theta \in (0,1)$ we have $s \succ \theta p + (1-\theta)q$, then by betweenness $p \asymp \theta p + (1-\theta)q$ and by the argument above, $T(p, \theta p + (1-\theta)q, r) \supseteq T(p, \theta p + (1-\theta)q, s)$, which in turn implies $T(p,q,r) \supseteq T(p,q,s)$.[20] A similar result follows if for some $\theta \in (0,1)$ we have $s \prec \theta p + (1-\theta)q$. Finally, if for all $s \in \Delta(\{p,q,r\})$ and $\theta \in (0,1)$ we have $s \asymp \theta p + (1-\theta)q$, then $T(p,q,r) = \mathbb{R}$,[21] so that $T(p,q,r) \supseteq T(p,q,s)$ for all $s \in R$.

By definition we have that $T(p,q) = \bigcap_{r \in \Delta(X)} T(p,q,r)$. Note that $T(p,q)$ is bounded if any $T(p,q,r)$ is bounded, which we can establish by setting $r = \delta_{\overline{x}}$.[22] Since for every $r \notin R$ there is $s \in R$ such that $T(p,q,r) \supseteq T(p,q,s)$, we have that $T(p,q) = \bigcap_{r \in R} T(p,q,r) = \bigcap_{r \in R} [\underline{\tau}_r, \overline{\tau}_r]$. Letting $\underline{\tau} = \sup_{r \in R} \underline{\tau}_r$ and $\overline{\tau} = \inf_{r \in R} \overline{\tau}_r$, we have that $T(p,q) = [\underline{\tau}, \overline{\tau}]$. □

6.1.4 Proof of Proposition 4

Necessity is immediate from Lemma 1 and the definition of Ω. To prove sufficiency suppose that for $p,q \in \Delta(X)$ there is $\tau > 0$ such that $o = \frac{p - \tau q}{1-\tau} \in \Omega$, then there are $p', q' \in \Delta(X)$ such that $p' \asymp q'$ and

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[17] Otherwise if some $\tau'' \in T^L(p,q,r)$, then for some $\beta \in (0,1)$ we have $\beta p + (1-\beta)r \succ \frac{\beta\tau'' q + (1-\beta)r}{\beta\tau'' + (1-\beta)} \succ \frac{\beta\tau' q + (1-\beta)r}{\beta\tau' + (1-\beta)}$, implying $\tau' \in T^L(p,q,r)$ as well.

[18] As before, let $\overline{\tau}_r = \infty$ if $T^R(p,q,r) = \varnothing$ and $\underline{\tau}_r = 0$ if $T^L(p,q,r) = \varnothing$.

[19] Pick $\tau \in T(p,q,s)$ and for any $\beta \in (0,1)$, let $\beta' = \beta + (1-\beta)\alpha$ so that $p' = \beta p + (1-\beta)s = \beta' p + (1-\beta')r$ and let $q' = \frac{\beta'\tau q + (1-\beta')r}{\beta'\tau + (1-\beta')}$. Then we have that

$$p' \asymp \frac{\beta\tau q + (1-\beta)s}{\beta\tau + (1-\beta)} = \frac{\beta\tau q + (1-\beta)\alpha p + (1-\beta)(1-\alpha)r}{\beta\tau + (1-\beta)} = \frac{(1-\beta)\alpha p' + \beta[\beta'\tau + (1-\beta')]q'}{(1-\beta)\alpha + \beta[\beta'\tau + (1-\beta')]}.$$

By betweenness, the above implies that $p' \asymp q'$, and taking the odds ratio gives us $\tau \in T(p,q,r)$.

[20] We can show that for any $t \in \Delta(X)$ there is a one to one mapping from $T(p,q,t)$ to $T(p, \theta p + (1-\theta)q, t)$ by noting that, again by betweenness,

$$\beta p + (1-\beta)t \asymp \frac{\theta\tau[\beta p + (1-\beta)t] + (1-\theta)[\beta\tau + (1-\beta)]\left[\frac{\beta\tau q + (1-\beta)t}{\beta\tau + (1-\beta)}\right]}{\theta\tau + (1-\theta)[\beta\tau + (1-\beta)]} = \frac{\beta\tau[\theta p + (1-\theta)q] + (1-\beta)[\theta\tau + (1-\theta)]t}{\beta\tau + (1-\beta)[\theta\tau + (1-\theta)]}.$$

Taking the odds ratio of the above, we conclude that $\tau \in T(p,q,t)$ if and only if $\frac{\tau}{\theta\tau + (1-\theta)} \in T(p, \theta p + (1-\theta)q, t)$. Thus $T(p, \theta p + (1-\theta)q, r) \supseteq T(p, \theta p + (1-\theta)q, s)$ if and only if $T(p,q,r) \supseteq T(p,q,s)$.

[21] Pick $\beta \in (0,1)$ and let $s = \beta p + (1-\beta)r$, and for any $\tau > 0$ let $\theta = \frac{1}{1-\tau}$. Then by assumption $\beta p + (1-\beta)r \asymp \frac{p - \tau q}{1-\tau}$ and invoking betweenness yet again we have that

$$\beta p + (1-\beta)r \asymp \frac{[\beta p + (1-\beta)r] - \beta(1-\tau)\left[\frac{p-\tau q}{1-\tau}\right]}{1 - \beta(1-\tau)} = \frac{\beta\tau q + (1-\beta)r}{\beta\tau + (1-\beta)}.$$

This implies $\tau \in T(p,q,r)$ for every $\tau > 0$.

[22] Since for any $\overline{\beta} \in (0,1)$ we have $r \succ \overline{\beta}p + (1-\overline{\beta})r$, by (A.2) there is $\underline{\gamma} \in (0,1)$ such that $\underline{\gamma}q + (1-\underline{\gamma})r \succ \overline{\beta}p + (1-\overline{\beta})r$, implying $\underline{\tau}_r > \frac{\underline{\gamma}/(1-\underline{\gamma})}{\overline{\beta}/(1-\overline{\beta})}$. Likewise for any $\overline{\gamma} \in (0,1)$ we have $r \succ \overline{\gamma}q + (1-\overline{\gamma})r$, so there is $\underline{\beta} \in (0,1)$ such that $\underline{\beta}p + (1-\underline{\beta})r \succ \overline{\gamma}q + (1-\overline{\gamma})r$ so that $\overline{\tau}_r < \frac{\overline{\gamma}/(1-\overline{\gamma})}{\underline{\beta}/(1-\underline{\beta})}$.

$\tau' \in T(p', q')$ such that

$$o = \frac{p - \tau q}{1 - \tau} = \frac{p' - \tau' q'}{1 - \tau'}.$$

Rearranging, this implies that

$$\begin{aligned}
r' &\equiv \frac{(1-\tau')p - (1-\tau)p'}{\tau - \tau'} = \frac{(1-\tau')\tau q - (1-\tau)\tau' q'}{\tau - \tau'}, \\
p &= \frac{1-\tau}{1-\tau'} p' + \frac{\tau - \tau'}{1-\tau'} r' \equiv \beta p' + (1-\beta) r', \\
q &= \frac{(1-\tau)\tau'}{(1-\tau')\tau} q' + \frac{\tau - \tau'}{(1-\tau')\tau} r' \equiv \gamma q' + (1-\gamma) r'.
\end{aligned}$$

Since $p' \asymp q'$ by assumption and $\tau' = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} \in T(p', q')$, we have that

$$p = \beta p' + (1-\beta) r' \asymp \frac{\beta \tau' q' + (1-\beta) r'}{\beta \tau' + (1-\beta)} = q.$$

□

6.1.5 Proof of Proposition 5

Define $v = wu$, so that for $p \in \Delta(X)$ we may write

$$\begin{aligned}
\begin{bmatrix} V(p) \\ W(p) \end{bmatrix} &= \begin{bmatrix} v(x_1) & \cdots & v(x_n) \\ w(x_1) & \cdots & w(x_n) \end{bmatrix} \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{bmatrix} = \mathbf{V}\mathbf{p}, \\
U(p) &= \frac{\sum_{i=1}^n p(x_i) v(x_i)}{\sum_{i=1}^n p(x_i) w(x_i)} = \frac{V(p)}{W(p)}.
\end{aligned}$$

For $p, q \in \Delta(X)$, we have that $U(p) > U(q)$ if and only if

$$W(p)W(q)[U(p) - U(q)] = V(p)W(q) - V(q)W(p) = \begin{vmatrix} V(p) & V(q) \\ W(p) & W(q) \end{vmatrix} = \begin{vmatrix} \mathbf{Vp} & \mathbf{Vq} \end{vmatrix} = |\mathbf{VP}| > 0.$$

Now consider a positive affine transformation

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{v}(x_1) & \cdots & \tilde{v}(x_n) \\ \tilde{w}(x_1) & \cdots & \tilde{w}(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v(x_1) & \cdots & v(x_n) \\ w(x_1) & \cdots & w(x_n) \end{bmatrix} = \mathbf{AV}.$$

This implies that $\tilde{U}(p) > \tilde{U}(q)$ if and only if $|\tilde{\mathbf{V}}\mathbf{P}| = |\mathbf{A}||\mathbf{VP}| > 0$, so that the ranking of lotteries is unchanged as long as $|\mathbf{A}| > 0$, or $ad - bc > 0$. □

6.1.6 Proof of Proposition 6

Let $\mathcal{V}$ represent $\succ$, pick any pair $(u, w) \in \mathcal{V}$, and let $v = wu$. We begin by showing that there exists a normalized function pair $(\hat{u}, \hat{w})$ for which (u, w) is a rational affine transformation, so that there are a, b, c, d such that

$$\begin{bmatrix} v(x_1) & v(x_n) \\ w(x_1) & w(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \hat{v}(x_1) & \hat{v}(x_n) \\ \hat{w}(x_1) & \hat{w}(x_n) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solving for these constants, we see that we indeed have a positive rational affine transformation as long as u ranks the best element x_n above the worst x_1, since

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} v(x_1) & v(x_n) \\ w(x_1) & w(x_n) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} v(x_n) - v(x_1) & v(x_1) \\ w(x_n) - w(x_1) & w(x_1) \end{bmatrix}, \\
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)] = w(x_1)w(x_n)[u(x_n) - u(x_1)] > 0.
\end{aligned}$$

Inverting this matrix, we transform (u, w) back to the normalized $(\hat{u}, \hat{w})$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} w(x_1) & -v(x_1) \\ -[w(x_n) - w(x_1)] & v(x_n) - v(x_1) \end{bmatrix}}{w(x_1)[v(x_n) - v(x_1)] - v(x_1)[w(x_n) - w(x_1)]} = \frac{\begin{bmatrix} w(x_1) & -w(x_1)u(x_1) \\ -[w(x_n) - w(x_1)] & w(x_n)u(x_n) - w(x_1)u(x_1) \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}.$$

For any $x \in X$, we have that

$$\begin{aligned} \begin{bmatrix} \hat{v}(x) \\ \hat{w}(x) \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} v(x) \\ w(x) \end{bmatrix} = \frac{\begin{bmatrix} w(x_1) & -w(x_1)u(x_1) \\ -[w(x_n) - w(x_1)] & w(x_n)u(x_n) - w(x_1)u(x_1) \end{bmatrix} \begin{bmatrix} w(x)u(x) \\ w(x) \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]} \\ &= \frac{\begin{bmatrix} w(x_1)w(x)[u(x) - u(x_1)] \\ w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)] \end{bmatrix}}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}. \end{aligned}$$

This gives us the utility and weight functions

$$\begin{aligned} \hat{u}(x) &= \frac{\hat{v}(x)}{\hat{w}(x)} = \frac{w(x_1)w(x)[u(x) - u(x_1)]}{w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)]}, \\ \hat{w}(x) &= \frac{w(x_n)w(x)[u(x_n) - u(x)] + w(x_1)w(x)[u(x) - u(x_1)]}{w(x_1)w(x_n)[u(x_n) - u(x_1)]}. \end{aligned}$$

It is easily verified that $(\hat{u}(x_1), \hat{w}(x_1)) = (0, 1)$ and $(\hat{u}(x_n), \hat{w}(x_n)) = (1, 1)$. Now for every $p \in \Delta(X)$, define

$$\hat{U}(p) = \frac{\sum_{i=1}^n p(x_i)\hat{w}(x_i)\hat{u}(x_i)}{\sum_{i=1}^n p(x_i)\hat{w}(x_i)} = \frac{\sum_{i=1}^n p_i \tau_i \alpha_i}{\sum_{i=1}^n p_i \tau_i} = \alpha_p, \qquad \hat{W}(p) = \sum_{i=1}^n p(x_i)\hat{w}(x_i) = \sum_{i=1}^n p_i \tau_i = \tau_p.$$

Since the utility function is normalized by assumption, we have that for any α that

$$\hat{U}(\zeta_\alpha) = \frac{\alpha \hat{w}(x_n)\hat{u}(x_n) + (1 - \alpha)\hat{w}(x_1)\hat{u}(x_1)}{\alpha \hat{w}(x_n) + (1 - \alpha)\hat{w}(x_1)} = \alpha, \qquad \hat{W}(\zeta_\alpha) = \alpha \hat{w}(x_n) + (1 - \alpha)\hat{w}(x_1) = 1.$$

We have for every $\beta \in (0, 1)$ and $r \in \Delta(X)$ that

$$\begin{aligned} \hat{U}(\beta p + (1 - \beta)r) &= \frac{\beta \hat{W}(p)\hat{U}(p) + (1 - \beta)\hat{W}(r)\hat{U}(r)}{\beta \hat{W}(p) + (1 - \beta)\hat{W}(r)} \\ &= \frac{\beta \tau_p \hat{W}(\zeta_{\alpha_p})\hat{U}(\zeta_{\alpha_p}) + (1 - \beta)\hat{W}(r)\hat{U}(r)}{\beta \tau_p \hat{W}(\zeta_{\alpha_p}) + (1 - \beta)\hat{W}(r)} = \hat{U}\left(\frac{\beta \tau_p \zeta_{\alpha_p} + (1 - \beta)r}{\beta \tau_p + (1 - \beta)r} \right). \end{aligned}$$

This implies that every $\beta p + (1 - \beta)r \asymp \frac{\beta \tau_p \zeta_{\alpha_p} + (1-\beta)r}{\beta \tau_p + (1-\beta)}$, so that $(\alpha_p, \tau_p) \in \Phi(p)$ for every $p \in \Delta(X)$ and hence $\{(\hat{u}(x_i), \hat{w}(x_i))\}_{i=1}^n = \{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$. □

6.1.7 Proof of Proposition 7

Let $\hat{\mathcal{V}}$ be a normalized set of utilities that represents $\succ$, and for every $(u^k, w^k) \in \mathcal{V}$, define

$$U^k(p) = \frac{\sum_{x \in X} p(x)w(x)u^k(x)}{\sum_{x \in X} p(x)w^k(x)}, \qquad W^k(p) = \sum_{x \in X} p(x)w^k(x).$$

Around any lottery $p \in \Delta(X)$, let $\bar{u}_p^k = U^k(p)$ and $\hat{\mathcal{U}}_p = \{u_p^k = w^k u^k + (1 - w^k)\bar{u}_p^k : (u^k, w^k) \in \mathcal{V}\}$ be the set of normalized local utilities. For every $q \in \Delta(X)$, let

$$U_p^k(q) = \sum_{x \in X} q(x)u_p^k(x) = \sum_{x \in X} q(x)[w^k(x)u^k(x) + [1 - w^k(x)]U^k(p)] = W^k(q)U^k(q) + [1 - W^k(q)]U^k(p).$$

Since $U_p^k(p) = U^k(p)$, we have $U_p^k(p) - U_p^k(q) = W^k(q)[U^k(p) - U^k(q)]$. Thus $p \succ q$ if and only if $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \hat{\mathcal{V}}$, which in turn holds if and only if $U_p^k(p) > U_p^k(q)$ for every $u_p^k \in \hat{\mathcal{U}}_p$. Denote the convex hull of $\hat{\mathcal{U}}_p$ by $\langle \hat{\mathcal{U}}_p \rangle = \{u_p^\pi = \sum_{u_p^k \in \mathcal{U}_p} \pi^k u_p^k : \sum_{u_p^k \in \mathcal{U}_p} \pi^k = 1\}$, then

$$U_p^k(p) > U_p^k(q),\ \forall\, u_p^k \in \mathcal{U}_p \iff U_p^\pi(p) = \sum_{u_p^k \in \mathcal{U}_p} \pi^k U_p^k(p) > \sum_{u_p^k \in \mathcal{U}_p} \pi^k U_p^k(q) = U_p^\pi(q),\ \forall\, u_p^\pi \in \langle \hat{\mathcal{U}}_p \rangle.$$

Therefore, $p \succ q$ if and only if every $U_p^\pi(p) > U_p^\pi(q)$, completing the proof. □

6.2 Proofs of Lemmas

6.2.1 Proof of Lemma 1

Fix $p, q \in \Delta(X)$ such that $p \asymp q$. Fix $r \in \Delta(X)$ and pick $\beta, \gamma \in (0,1)$ such that $\frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} > 0$ and that satisfy partial substitution so that $s \equiv \beta p + (1-\beta) r \asymp \gamma q + (1-\gamma) r \equiv t$. Now pick $\beta', \gamma' \in (0,1)$ such that $\tau \equiv \frac{\gamma'/(1-\gamma')}{\beta'/(1-\beta')} = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$, proving the proposition requires showing that $u \equiv \beta' p + (1-\beta)' r \asymp \gamma' q + (1-\gamma') r \equiv v$.

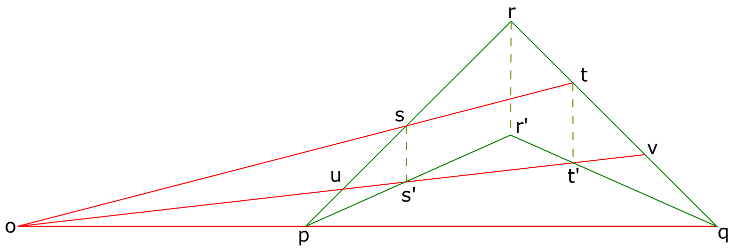

Figure 5: Proof of Lemma 1

As depicted in Figure 5, the extensions of the lines st and uv intersect at some source point lying outside of the simplex on the extended line pq, located at $o = \frac{p - \tau q}{1-\tau}$. Draw parallel lines from s and r such that the line from s intersects uv at some point s', and extending ps' intersects the line from r at some r', and let t' denote the intersection of uv and qr'. By Desargues' theorem, the triangles rst and $r's't'$ are perspective from the line opq, and hence the lines rr', ss', and tt' are parallel. This implies that $s' = \beta p + (1-\beta) r'$ and $t' = \gamma q + (1-\gamma) r'$, and hence by weak substitution that $s' \asymp t'$. Since both s' and t' lie on uv, we have by betweenness that $u \asymp v$ as well, completing the proof. □

6.2.2 Proof of Lemma 2

$(i) \Rightarrow (ii)$ Pick $p, q \in \Delta(X)$ and suppose that $p \asymp q$, and pick $\overline{r}, \underline{r} \in \Delta(X)$ such that $\overline{r} \succ p, q \succ \underline{r}$. For every λ, let $A_\lambda = \{\alpha \in [0,1] : \lambda p + (1-\lambda) q \asymp \alpha \overline{r} + (1-\alpha) \underline{r}\}$. By Proposition 2 we have that each A_λ is a closed interval $[\underline{\alpha}_\lambda, \overline{\alpha}_\lambda]$. If $A^* = \bigcap_\lambda A_\lambda = \varnothing$, then there are $\lambda_1, \lambda_2 \in [0,1]$ and $\alpha' \in (0,1)$ such that $\underline{\alpha}_{\lambda_1} > \alpha' > \overline{\alpha}_{\lambda_2}$, so that $\lambda_1 p + (1-\lambda_1) q \succ \zeta_{\alpha'} \succ \lambda_2 p + (1-\lambda_2) q$. By betweenness, if $\lambda_1 > \lambda_2$ then $p \succ q$, and if $\lambda_1 < \lambda_2$ then $p \prec q$. As either would contradict $p \asymp q$, we must have that $A^* \neq \varnothing$, so letting $r = \alpha \overline{r} + (1-\alpha) \underline{r}$ for any $\alpha \in A^*$ establishes the result.

$(ii) \Rightarrow (iii)$ Pick $p, q \in \Delta(X)$ such that $p \asymp q$ and pick any $\tau^* \in T(p,q)$, so that there is a source point $o^* = \frac{p - \tau^* q}{1 - \tau^*} \in \Omega$. Pick $\overline{r}, \underline{r} \in \Delta(X)$ such that $\overline{r} \succ p, q \succ \underline{r}$, then by the above, there is $r = \alpha \overline{r} + (1-\alpha) \underline{r} \in \Delta(X)$ such that every $\lambda p + (1-\lambda) q \asymp r$. We need to show that there are $\tau_p, \tau_q > 0$ such that every $\lambda \tau_p + (1-\lambda) \tau_q \in T(\lambda p + (1-\lambda) q)$.

For every $\lambda \in [0,1]$, let $p_\lambda = \lambda p + (1-\lambda) q$ and $\tau_\lambda^* = \lambda \tau^* + (1-\lambda)$, then since $\tau^* \in T(p,q)$ we have that

$$o^* = \frac{p - \tau^* q}{1 - \tau^*} = \frac{[\lambda p + (1-\lambda) q] - [\lambda \tau^* + (1-\lambda)] q}{1 - [\lambda \tau^* + (1-\lambda)]} = \frac{p_\lambda - \tau_\lambda^* q}{1 - \tau_\lambda^*} \in \Omega.$$

This implies by Proposition 4 that $\tau_\lambda^* \in T(p_\lambda, q)$. We now claim that there is some $\tau_q \in T(q, r)$ such that for every $\lambda \in [0,1]$, $\tau_\lambda^* \tau_q \in T(p_\lambda, r)$. Suppose not, then since by Proposition 3 the weight ranges are closed intervals $T(p_\lambda, r) = [\underline{\tau}_\lambda, \overline{\tau}_\lambda]$ and $T(q,r) = [\underline{\tau}_q, \overline{\tau}_q]$, there is $\lambda \in [0,1]$ such that $\tau^*_\lambda > \frac{\overline{\tau}_\lambda}{\underline{\tau}_q}$ or $\tau^*_\lambda < \frac{\underline{\tau}_\lambda}{\overline{\tau}_q}$. Assume the former without loss of generality, then there is $\tau'_q < \underline{\tau}_q$ and $\tau'_\lambda > \overline{\tau}_\lambda$ such that $\tau^*_\lambda = \frac{\tau'_\lambda}{\tau'_q}$. Let $s = \overline{r} \succ r$, then we have that since $\tau'_\lambda > \overline{\tau}_\lambda$, $o'_\lambda = \frac{p_\lambda - \tau'_\lambda r}{1-\tau'_\lambda} \notin \Omega$, so that for some $\beta \in (0,1)$,

$$p'_\lambda = \beta p_\lambda + (1-\beta)s \succ \frac{\beta\tau'_\lambda r + (1-\beta)s}{\beta\tau'_\lambda + (1-\beta)} \equiv r'.$$

Now let $\gamma = \frac{\beta\tau^*_\lambda}{\beta\tau^*_\lambda + (1-\beta)}$ so that $\tau^*_\lambda = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)}$. Since $\tau'_q < \underline{\tau}_q$, $o'_q = \frac{q-\tau'_q r}{1-\tau'_q} \notin \Omega$ so that

$$q' = \gamma q + (1-\gamma)s \prec \frac{\gamma\tau'_q r + (1-\gamma)s}{\gamma\tau'_q + (1-\gamma)} \equiv r''$$

Since by construction $\tau'_\lambda = \tau^*_\lambda\tau'_q$, taking the above together we have that $p'_\lambda \succ r' = r'' \succ q'$, but as $\tau^*_\lambda \in T(p_\lambda, q)$ implies $p'_\lambda \asymp q'$, this is a contradiction. Thus we must have that $\tau^*_\lambda \leq \frac{\overline{\tau}_\lambda}{\underline{\tau}_q}$, and a similar argument shows $\tau^*_\lambda \geq \frac{\underline{\tau}_\lambda}{\overline{\tau}_q}$. Therefore, there is $\tau_q \in [\underline{\tau}_q, \overline{\tau}_q]$ such that $\tau^*_\lambda\tau_q \in T(p_\lambda, r)$ for every λ.

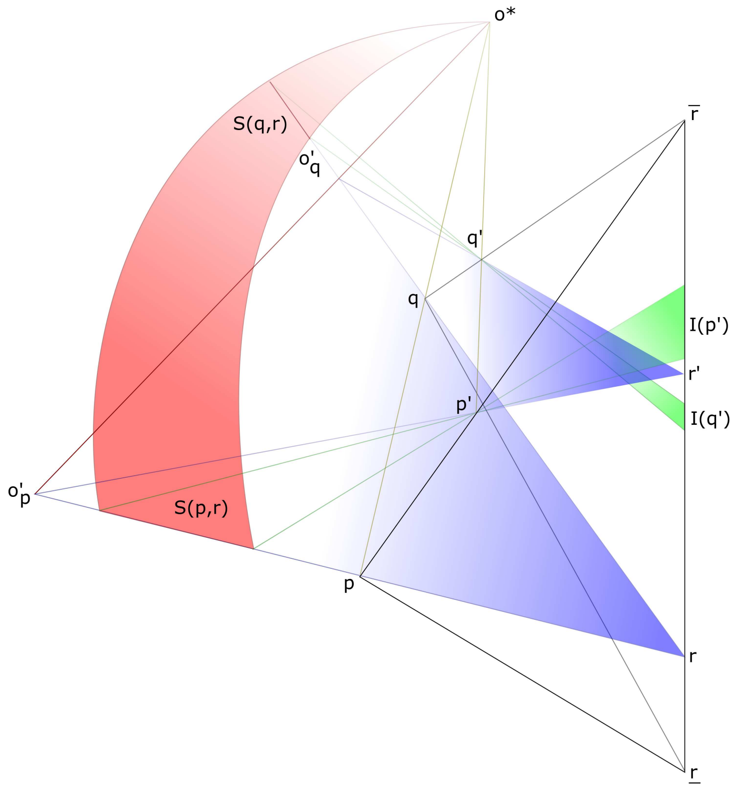

Figure 6: Proof of Lemma 2 $[(ii) \Rightarrow (iii)]$

This argument is illustrated in Figure 6, showing the contradiction when the claim is violated for $\lambda = 1$, so that $\tau^*\tau_q \notin T(p,r)$.[23] Letting $\tau_p = \tau^*\tau_q \in T(p,r)$, this implies $[\lambda\tau^* + (1-\lambda)]\tau_q = \lambda\tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda)q, r)$ for every λ, completing the proof.

[23]For every $p, q \in \Delta(X)$ let $S(p,q) = \{o = \frac{p-\tau q}{1-\tau} : \tau \in T(p,q)\}$ the range of source points on the line defined by p and q. For any $p \in \Delta(X)$, let $I(p) = \{r' = \alpha\overline{r} + (1-\alpha)\underline{r} \asymp p : \alpha \in [0,1]\}$ the range of mixtures of $\overline{r}$ and $\underline{r}$ to which p is incomparable, then $r \in I(p)$ and $\tau \in T(p,r)$ implies that $r' = \frac{\beta\tau r + (1-\beta)\overline{r}}{\beta\tau + (1-\beta)} \in I(\beta p + (1-\beta)\overline{r})$. As shown in Figure 6, if $o^* \in S(p,q)$, then we must be able to draw a line from it that intersects both $S(p,r)$ and $S(q,r)$, or else there are $o'_p, o'_q \notin \Omega$ that indicate $I(p')$ and $I(q')$ are disjoint, so that $p' \succ q'$ which in turn would imply $o^* \notin \Omega$.

$(iii) \Rightarrow (iv)$ Pick $p, q \in \Delta(X)$ such that $p \asymp q$ and suppose there is $r \in \Delta(X)$ and $\tau_p, \tau_q > 0$ such that every $\lambda\tau_p + (1-\lambda)\tau_q \in T(\lambda p + (1-\lambda)q, r)$. This defines a line of source points

$$O(p,q) = \left\{ o_\lambda = \frac{\lambda p + (1-\lambda)q - [\lambda\tau_p + (1-\lambda)\tau_q]r}{1 - [\lambda\tau_p + (1-\lambda)\tau_q]} : \lambda \in [0,1] \right\} \subseteq \Omega.$$

Now pick $p', q' \in \Delta(\{p,q,r\})$, then there are $\tau'_p, \tau'_q > 0$ such that we can define source points $o'_p = \frac{p' - \tau'_p r}{1-\tau'_p}$ and $o'_q = \frac{q'-\tau'_q r}{1-\tau'_q}$.[24] Now let $\tau'_t = \frac{\tau'_p}{\tau'_q}$ and $\lambda'_t = \frac{1}{1-\tau'_t}$, then we have that

$$o'_t = \frac{\lambda'_t(1-\tau'_p)o'_p + (1-\lambda'_t)(1-\tau'_q)o'_q}{\lambda'_t(1-\tau'_p) + (1-\lambda'_t)(1-\tau'_q)} = \frac{(p' - \tau'_p r) - \frac{\tau'_p}{\tau'_q}(q' - \tau'_q r)}{(1-\tau'_p) - \frac{\tau'_p}{\tau'_q}(1-\tau'_q)} = \frac{p' - \tau'_t q'}{1-\tau'_t} \in \Omega.$$

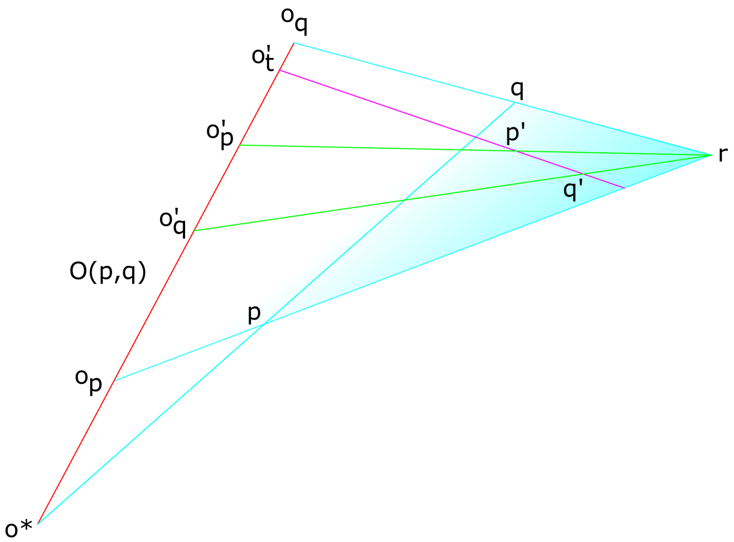

Figure 7: Proof of Lemma 2 [$(iii) \Rightarrow (iv)$]

As Figure 7 shows, for any $p', q' \in \Delta(\{p,q,r\})$ we can draw a line connecting these two lotteries that intersects the source line $O(p,q)$ at some o'_t. By Proposition 4 this implies that $p' \asymp q'$ and $\tau'_t \in T(p', q')$.

$(iv) \Rightarrow (i)$ This is immediate.

As $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$, the four statements are equivalent. $\square$

6.2.3 Proof of Lemma 3

The steps in this proof follow much of the same logic as the proof of Lemma 2.

$(i) \Rightarrow (ii)$ Suppose that P is an incomparability set, then if $\dim P < n-2$ we can pick $\overline{r}, \underline{r} \in \Delta(X) \setminus \Delta(P)$ such that $\overline{r} \succ p \succ \underline{r}$ for every $p \in \Delta(P)$. For every $p \in \Delta(P)$ let $A_p = \{\alpha : p \asymp \alpha\overline{r} + (1-\alpha)\underline{r}\} = [\underline{\alpha}_p, \overline{\alpha}_p]$. If $A^* = \bigcap_{p\in\Delta(P)} A_p = \varnothing$, then there are $p_1, p_2 \in \Delta(P)$ and α' such that $\underline{\alpha}_{p_1} > \alpha' > \overline{\alpha}_{p_2}$. But this implies that $p_1 \succ p_2$, a contradiction. Picking any $\alpha \in A^* \neq \varnothing$ and letting $r = \alpha\overline{r} + (1-\alpha)\underline{r}$ establishes the result.

[24] Let $p' \equiv \mu_p p + \mu_q q + (1-\mu_p-\mu_q)r$ and $q' \equiv \nu_p p + \nu_q q + (1-\nu_p-\nu_q)r$ and $\tau'_p = \mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)$ and $\tau'_q = \nu_p\tau_p + \nu_q\tau_q + (1-\nu_p-\nu_q)$. Then we have that

$$o'_p = \frac{p' - \tau'_p r}{1-\tau'_p} = \frac{[\mu_p p + \mu_q q + (1-\mu_p-\mu_q)r] - [\mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)]r}{1 - [\mu_p\tau_p + \mu_q\tau_q + (1-\mu_p-\mu_q)]} = \frac{\left[\frac{\mu_p p + \mu_q q}{\mu_p+\mu_q}\right] - \left[\frac{\mu_p\tau_p + \mu_q\tau_q}{\mu_p+\mu_q}\right]r}{1 - \left[\frac{\mu_p\tau_p+\mu_q\tau_q}{\mu_p+\mu_q}\right]}.$$

Letting $\lambda'_p = \frac{\mu_p}{\mu_p+\mu_q}$ shows that $o'_p \in O(p,q) \subseteq \Omega$, and by a similar argument $o'_q \in \Omega$.

(ii) ⇒ *(iii)* Fix $r \in \Delta(X) \setminus \Delta(P)$ satisfying (ii). Let $P_k = \{p_1, \ldots, p_k\} \subseteq P$ be a basis for $\Delta(P)$, then it will be sufficient to show that there are $\{\tau_1, \ldots, \tau_k\} \in \mathbb{R}^k_{>0}$ such that for every $q = \sum_{j=1}^k \pi_j p_j \in \Delta(P_k)$, we have that $\tau_q = \sum_{j=1}^k \pi_j \tau_j \in T(q, r)$.

Claim For any $\ell \leq k$, let $P_\ell = \{p_1, \ldots, p_\ell\}$, then there is $\{\tau_1, \ldots, \tau_\ell\} \in \mathbb{R}^\ell_{>0}$ such that for every $q = \sum_{j=1}^\ell \pi_j p_j \in \Delta(P_\ell)$, we have that $\tau_q = \sum_{j=1}^\ell \pi_j \tau_j \in T(q, r)$.

Proof We establish this property by induction. For the base case, if $\ell = 1$ then the simplex is a singleton $\Delta(\{p_1\}) = \{p_1\}$ so the property is trivially satisfied by picking any $\tau_1 \in T(p_1, r)$. For the inductive step, suppose that we have such $\{\tau_1, \ldots, \tau_\ell\} \in \mathbb{R}^\ell_{>0}$ and let $\tau_q = \sum_{j=1}^\ell \pi_j \tau_j$ for every $q \in \Delta(P_\ell)$. By Lemma 2, for every $q_1, q_2 \in \Delta(P_\ell)$ we have that $q_1 \asymp q_2$ and furthermore that $\frac{\tau_{q_1}}{\tau_{q_2}} \in T(q_1, q_2)$.

Pick any $p \in P \setminus \Delta(P_\ell)$, then since P is an incomparability set $p \asymp q$ for any $q \in \Delta(P_\ell)$, by Lemma 2 we have that

$$Z_p^q = \{\tau_p \in T(p, r) : \lambda \tau_p + (1 - \lambda)\tau_q \in T(\lambda p + (1 - \lambda)q, r), \ \forall \lambda \in [0, 1]\} = [\underline{\tau}_p^q, \overline{\tau}_p^q] \neq \varnothing.$$

By Lemma 2, $\tau_p \in Z_p^q$ implies that $\frac{\tau_p}{\tau_q} \in T(p, q)$. Suppose that $Z_p^* = \bigcap_{q \in \Delta(P_\ell)} Z_p^q = \varnothing$, then there are $q_1, q_2 \in \Delta(P_\ell)$ and τ_p' such that $\underline{\tau}_p^{q_1} > \tau_p' > \overline{\tau}_p^{q_2}$. This in turn implies that $\frac{\tau_p'}{\tau_{q_1}} < \inf T(p, q_1)$ and $\frac{\tau_p'}{\tau_{q_2}} > \sup T(p, q_2)$. Letting $s = \overline{r} \succ p$, for $\beta \in (0, 1)$ we have that

$$\beta q_1 + (1 - \beta)s \succ \frac{\beta \left(\frac{\tau_{q_1}}{\tau_p'}\right) p + (1 - \beta)s}{\beta \left(\frac{\tau_{q_1}}{\tau_p'}\right) + (1 - \beta)} \succ \frac{\beta \left(\frac{\tau_{q_1}}{\tau_{q_2}}\right) q_2 + (1 - \beta)s}{\beta \left(\frac{\tau_{q_1}}{\tau_{q_2}}\right) + (1 - \beta)}$$

This contradicts $\frac{\tau_{q_1}}{\tau_{q_2}} \in T(q_1, q_2)$, and hence $Z_p^* \neq \varnothing$. Let $p = p_{\ell+1}$ and pick any $\tau_{\ell+1} \in Z_{p_{\ell+1}}^*$, then we have that the set $\{\tau_1, \ldots, \tau_{\ell+1}\} \in \mathbb{R}^{\ell+1}_{>0}$ has the desired property.[25] This completes the proof of the claim. □

Returning to the proof of the lemma, set $\ell = k$ and choose $\{\tau_1, \ldots, \tau_k\} \in \mathbb{R}^k_{>0}$ that satisfies the claim. Then for every $p \equiv \sum_{j=1}^k \pi_j p_j \in P$, letting $\tau_p = \sum_{j=1}^k \pi_j \tau_j$ establishes the result.

(iii) ⇒ *(iv)* Fix $r \in \Delta(X) \setminus \Delta(P)$ and $\{\tau_p\}_{p \in P} \subseteq \mathbb{R}_{>0}$ satisfying (iii). Pick $q_1, q_2 \in \Delta(P \cup \{r\})$. Then for $i = 1, 2$, $q_i = \theta_i r + (1 - \theta_i) q_i'$ for some $q_i' \equiv \sum_{p \in \Delta(P)} \pi_{p,i} p \in P$. Then letting $\tau_i' = \sum_{p \in P} \pi_{p,i} \tau_p \in T(q_i', r)$ and $\tau_i = \theta_i + (1 - \theta_i)\tau_i'$, there is a source point at $o_i = \frac{q_i' - \tau_i' r}{1 - \tau_i'} = \frac{q_i - \tau_i r}{1 - \tau_i} \in \Omega$. By construction, we have that $\lambda \tau_1' + (1 - \lambda)\tau_2' \in T(\lambda q_1' + (1 - \lambda) q_2', r)$. Thus, letting $\tau^* = \frac{\tau_1}{\tau_2}$ and $\lambda^* = \frac{1}{1 - \tau^*}$ we have that

$$o^* = \frac{\lambda^*(1 - \tau_1)o_1 + (1 - \lambda^*)(1 - \tau_2)o_2}{\lambda^*(1 - \tau_1) + (1 - \lambda^*)(1 - \tau_2)} = \frac{(q_1 - \tau_1 r) - \frac{\tau_1}{\tau_2}(q_2 - \tau_2 r)}{(1 - \tau_1) - \frac{\tau_1}{\tau_2}(1 - \tau_2)} = \frac{q_1 - \tau^* q_2}{1 - \tau^*} \in \Omega$$

By Proposition 4, this implies that $q_1 \asymp q_2$ and $\tau^* \in T(q_1, q_2)$.

(iv) ⇒ *(i)* This is immediate.

As *(i)* ⇒ *(ii)* ⇒ *(iii)* ⇒ *(iv)* ⇒ *(i)*, the four statements are equivalent. □

6.2.4 Proof of Lemma 4

Fix $p, q \in \Delta(X)$. By repeated application of Lemma 3, we have that $p \asymp q$ if and only if they both belong to the some maximal incomparability set $P \subseteq \Delta(X)$ with $\dim P = n - 2$. Note that since $\delta_{x_n} \succ \delta_{x_1}$, by betweenness there is a unique α such that $\zeta_\alpha \in P$. Hence for every $i = 2, \ldots, n - 1$ there exists some $p_i = \lambda_i \delta_{x_i} + (1 - \lambda_i)\zeta_{\theta_i} \in \Delta(P) \cap \Delta(\{x_i, x_1, x_n\})$. By Lemma 3, this implies that there exist

[25] For any $q = \sum_{j=1}^{\ell+1} \pi_j p_j$, let $\lambda = \pi_j$ and $q' = \frac{\sum_{j=1}^{\ell} \pi_j p_j}{\sum_{j=1}^{\ell} \pi_j}$. Then since $\tau_{\ell+1} \in Z_{p_{\ell+1}}^* \subseteq Z_{p_{\ell+1}}^{q'}$, we have that $\sum_{j=1}^{\ell+1} \pi_j \tau_j = \lambda \tau_{\ell+1} + (1 - \lambda)\tau_{q'} \in T(\lambda p_{\ell+1} + (1 - \lambda) q', r) = T(q, r)$.

$\{\tau_2', \ldots, \tau_{n-1}'\}$ such that every $\tau_i' \in T(p_i, \zeta_\alpha)$ and hence $(\alpha, \tau_i') \in \Phi(p_i)$. Letting $\tau_i = \frac{1}{\lambda_i}\tau_i' + \left(1 - \frac{1}{\lambda_i}\right)$ and $\alpha_i = \frac{\tau_i'\alpha - (1-\lambda_i)\theta_i}{\tau_i' - (1-\lambda_i)} = \frac{[\lambda_i\tau_i + (1-\lambda_i)]\alpha - (1-\lambda_i)\theta_i}{\lambda_i\tau_i}$, then we have that

$$o_i = \frac{\delta_{x_i} - \tau_i\zeta_{\alpha_i}}{1-\tau_i} = \frac{\lambda_i\delta_{x_i} + (1-\lambda_i)\zeta_{\theta_i} - [\lambda_i\tau_i + (1-\lambda_i)]\zeta_\alpha}{1 - [\lambda_i\tau_i + (1-\lambda_i)]} = \frac{p_i - \tau_i'\zeta_\alpha}{1-\tau_i'} \in \Omega.$$

This implies that every $(\alpha_i, \tau_i) \in \Phi(\delta_{x_i})$.

For every $p' \equiv \sum_{i=1}^n \pi_i\delta_{x_i} \in \Delta(X)$ there is $q' = \sum_{i=2}^{n-1} \pi_i' p_i \in \Delta(P)$ such that $p' = \lambda' q' + (1-\lambda')\zeta_{\theta'}$. This implies that

$$(\alpha, \tau_q') = \left(\frac{\sum_{i=2}^{n-1}\pi_i'\tau_i'\alpha}{\sum_{i=2}^{n-1}\pi_i'\tau_i'}, \sum_{i=2}^{n-1}\pi_i'\tau_i'\right) = \left(\frac{\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i\alpha_i + (1-\lambda_i)\theta_i]}{\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)]}, \sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)]\right) \in \Phi(q').$$

Letting $\tau_p' = \lambda'\tau_q' + (1-\lambda')$ and $\alpha_p' = \frac{\lambda'\tau_q'\alpha + (1-\lambda')\theta'}{\lambda'\tau_q' + (1-\lambda')}$ we have

$$o_p' = \frac{p' - \tau_p'\zeta_{\alpha_p'}}{1-\tau_p'} = \frac{[\lambda' q' + (1-\lambda')\zeta_{\theta'}] - [\lambda'\tau_q'\zeta_\alpha + (1-\lambda')\zeta_{\theta'}]}{1 - [\lambda'\tau_q' + (1-\lambda')]} = \frac{q' - \tau_q'\zeta_\alpha}{1-\tau_q'} \in \Omega.$$

This implies $(\alpha_p', \tau_p') \in \Phi(p')$. Furthermore, letting $(\alpha_1, \tau_1) = (0, 1)$ and $(\alpha_n, \tau_n) = (1, 1)$, we have that

$$\begin{aligned}
p' &= \lambda' \sum_{i=2}^{n-1}\pi_i'[\lambda_i\delta_{x_i} + (1-\lambda_i)\zeta_{\theta_i}] + (1-\lambda')\zeta_{\theta'} \equiv \sum_{i=1}^n \pi_i\delta_{x_i}, \\
(\alpha_p', \tau_p') &= \left(\frac{\lambda'\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i\alpha_i + (1-\lambda_i)\theta_i] + (1-\lambda')\theta'}{\lambda'\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)] + (1-\lambda')}, \lambda'\sum_{i=2}^{n-1}\pi_i'[\lambda_i\tau_i + (1-\lambda_i)] + (1-\lambda')\right) \\
&= \left(\frac{\sum_{i=1}^n \pi_i\tau_i\alpha_i}{\sum_{i=1}^n \pi_i\tau_i}, \sum_{i=1}^n \pi_i\tau_i\right) \in \Phi(p').
\end{aligned}$$

Since the above holds for any $p \in \Delta(X)$, the collection $\{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$.

This construction is shown in Figure 8.[26] Returning to the proof, we have that for every $p, q \in \Delta(X)$ that $p \asymp q$ if and only if they lie on some maximal incomparability set P, which defines $\{(\alpha_i, \tau_i)\}_{i=1}^n \in \Psi$. Since there is a unique $\alpha \in [0, 1]$ for which $p, q, \zeta_\alpha \in P$, we must have $\alpha_p = \alpha_q = \alpha$, which completes the proof. ☐

6.2.5 Proof of Lemma 5

We begin by proving the following claim

Claim Given (A.1), (A.2), and (A.3), axiom (A.4) implies that for all $p, q \in \Delta(X)$, $p \asymp q$ if and only if for every $\beta \in (0, 1)$ there is a unique $\gamma \in (0, 1)$ such that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$ for all $r \in \Delta(X)$.

Proof Suppose that there are $p, q \in \Delta(X)$, $p \asymp q$ and $\beta \in (0, 1)$ and $\gamma, \gamma' \in (0, 1)$, $\gamma > \gamma'$, such that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$ and $\beta p + (1-\beta)r \asymp \gamma' q + (1-\gamma')r$ for some $r \in \Delta(X)$. Let $\bar\gamma = \sup\{\gamma \in (0, 1) \mid \beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r\}$. That such $\bar\gamma$ exists follows from the fact that the set is bounded and nonempty. By definition, $\bar\gamma \geq \gamma > \gamma'$. Moreover, since the incomparable sets are closed, $\beta p + (1-\beta)r \asymp \bar\gamma q + (1-\bar\gamma)r$. By definition, for all $\gamma \in (\bar\gamma, 1)$, $\gamma q + (1-\gamma)r \succ \beta p + (1-\beta)r$. By (A.4) since $\gamma' \in (0, \bar\gamma]$, $\beta p + (1-\beta)r \succ \gamma' q + (1-\gamma')r$ for all $r \in \Delta(X)$. This contradicts $\beta p + (1-\beta)r \asymp \gamma' q + (1-\gamma')r$ for some $r \in \Delta(X)$. Hence, γ is unique.

[26] The existence of a maximal incomparability set P implies that $\Delta(P)$ crosses every triangle $\Delta(\{x_1, x_i, x_n\})$, formed by the best and worst outcomes along with some third outcome $x_i \in X$, at some p_i'. Furthermore, as betweenness implies that we may have at most one $\zeta_\alpha \in \Delta(P)$, every $p_i \asymp \zeta_\alpha$, so that we may find a source point o_i in the usual manner. Drawing a line from o_i through δ_{x_i} allows us to find the utility weight pair (α_i, τ_i) for x_i. By the result of Lemma 3, every point on the line connecting two source points is itself a source point o_p', and drawing this through any lottery $p' \in \Delta(X)$ produces the pair (α_p', τ_p') which is in turn a linear combination of the (α_i, τ_i).

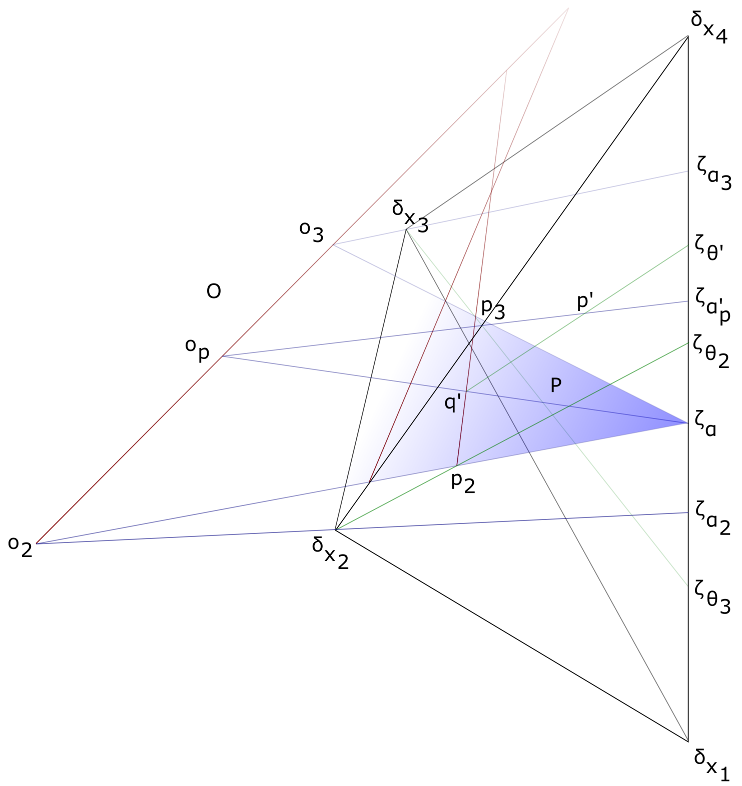

Figure 8: Proof of Lemma 4

Suppose there are $p, q \in \Delta(X)$, $\beta \in (0,1)$ and $\gamma, \gamma' \in (0,1)$, $\gamma > \gamma'$, and $(\gamma q + (1-\gamma)r \asymp \beta p + (1-\beta)r$ or $\beta p + (1-\beta)r \asymp \gamma' q + (1-\gamma')r)$, for some $r \in \Delta(X)$. Then, it is no true that $\gamma q + (1-\gamma)r \succ \beta p + (1-\beta)r$ or $\beta p + (1-\beta)r \succ \gamma' q + (1-\gamma')r$ for all $\gamma' < \gamma$ and $r \in \Delta(X)$. Thus, by $p \asymp q$. $\triangle$

To prove the lemma, fix $p \in \Delta(X)$ and $(\alpha^1, \tau^1), (\alpha^2, \tau^2) \in \Phi(p)$. If $\tau^1 \neq \tau^2$, let

$$\begin{aligned}
\beta^* &= \frac{\tau^2\alpha^2 - \tau^1\alpha^1}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1}, \\
\alpha^* &= \frac{\beta^*\tau^1\alpha^1}{\beta^*\tau^1 + (1-\beta^*)} = \frac{\beta^*\tau^2\alpha^2}{\beta^*\tau^2 + (1-\beta^*)} = \frac{\tau^2\alpha^2 - \tau^1\alpha^1}{\tau^2 - \tau^1}, \\
\tau^{1*} &= \beta^*\tau^1 + (1-\beta^*) = \frac{(\tau^2 - \tau^1)\tau^1\alpha^1}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1}, \\
\tau^{2*} &= \beta^*\tau^2 + (1-\beta^*) = \frac{(\tau^2 - \tau^1)\tau^2\alpha^2}{(1-\tau^1)\tau^2\alpha^2 - (1-\tau^2)\tau^1\alpha^1}.
\end{aligned}$$

Then for $j = 1, 2$, we have

$$o^j = \frac{p - \tau^j\zeta_{\alpha^j}}{1-\tau^j} = \frac{[\beta^* p + (1-\beta^*)\delta_{x_1}] - [\beta^*\tau^j\zeta_{\alpha^j} + (1-\beta^*)\delta_{x_1}]}{1 - [\beta^*\tau^j + (1-\beta^*)]} = \frac{[\beta^* p + (1-\beta^*)\delta_{x_1}] - \tau^{j*}\zeta_{\alpha^*}}{1-\tau^{j*}} \in \Omega.$$

This implies that $(\alpha^*, \tau^{j*}) \in \Phi(\beta^* p + (1-\beta^*)\delta_{x_1})$ for $j = 1, 2$, but as $\tau^1 \neq \tau^2$ implies $\tau^{1*} \neq \tau^{2*}$, this would violate (A.4), so we must have $\tau^1 = \tau^2$. Hence, there is a unique τ_p such that $\alpha_p \in A(p)$ implies $(\alpha_p, \tau_p) = \Phi(p) = A(p) \times \{\tau_p\}$. $\square$

6.3 Proof of Theorem 1

(Necessity) Suppose that there is such a $\mathcal{V}$ that represents $\succ$. Then for any $(u^k, w^k) \in \mathcal{V}$, define

$$U^k(p) = \frac{\sum_{x\in X} p(x)w^k(x)u^k(x)}{\sum_{x\in X} p(x)w^k(x)}, \qquad W^k(p) = \sum_{x\in X} p(x)w^k(x).$$

Hence $p \succ q$ if and only if $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. It is easily verified that U^k is weighted linear.[27] To show that $\succ$ satisfies (A.1), note that for every $p \in \Delta(X)$, $\neg(U^k(p) > U^k(p))$, so $\succ$ is irreflexive, and that for every $p, q, r \in \Delta(X)$, $U^k(p) > U^k(q) > U^k(r)$ implies $U^k(p) > U^k(r)$, so $\succ$ is transitive.

To show that $\succ$ satisfies (A.2), pick $p, q, r \in \Delta(X)$ such that $p \succ q$, then $U^k(p) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$. If so we can define $\alpha^k \in (0, 1)$ such that

$$U^k(\alpha^k p + (1-\alpha^k)r) = \frac{\alpha^k W^k(p)U^k(p) + (1-\alpha^k)W^k(r)U^k(r)}{\alpha^k W^k(p) + (1-\alpha^k)W^k(r)} = U^k(q),$$
$$\alpha^k = \frac{W^k(r)[U^k(q) - U^k(r)]}{W^k(p)[U^k(p) - U^k(q)] + W^k(r)[U^k(q) - U^k(r)]}.$$

If $U^k(q) > U^k(r)$, let $\underline{\alpha}^k = \alpha^k$, otherwise let $\underline{\alpha}^k = 0$, and let $\underline{\alpha} = \inf_{(u^k,w^k)\in\mathcal{V}} \underline{\alpha}^k$. Then for any $\alpha > \underline{\alpha}$ we have that $U^k(\alpha p + (1-\alpha)r)) > U^k(q)$ for every $(u^k, w^k) \in \mathcal{V}$ and hence $\alpha p + (1-\alpha)r \succ q$. By a similar argument, there is $\overline{\alpha} \in (0, 1)$ such that $\alpha < \overline{\alpha}$ implies $q \succ \alpha p + (1-\alpha)r$.

To show that $\succ$ satisfies (A.3), pick $p, q \in \Delta(X)$ such that $p \asymp q$. This implies that there are $(u^1, w^1), (u^2, w^2) \in \mathcal{V}$ such that $U^1(p) \geq U^1(q)$ and $U^2(p) \leq U^2(q)$. Define

$$U^\kappa(p) = \frac{\kappa W^1(p)U^1(p) + (1-\kappa)W^2(p)U^2(p)}{\kappa W^1(p) + (1-\kappa)W^2(p)}, \qquad W^\kappa(p) = \kappa W^1(p) + (1-\kappa)W^2(p)$$

Then there is some $\kappa \in [0, 1]$ such that $U^\kappa(p) = U^\kappa(q)$. For every $\beta \in (0, 1)$, fix $\gamma \in (0, 1)$ such that the odds ratio $\frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{W^\kappa(p)}{W^\kappa(q)}$, so for every $r \in \Delta(X)$ we have

$$\begin{aligned} U^\kappa(\beta p + (1-\beta)r) &= \frac{\beta W^\kappa(p)U^\kappa(p) + (1-\beta)W^\kappa(r)U^\kappa(r)}{\beta W^\kappa(p) + (1-\beta)W^\kappa(r)} = \frac{\beta \frac{W^\kappa(p)}{W^\kappa(q)} W^\kappa(q)U^\kappa(q) + (1-\beta)W^\kappa(r)U^\kappa(r)}{\beta \frac{W^\kappa(p)}{W^\kappa(q)} W^\kappa(q) + (1-\beta)W^\kappa(r)} \\ &= \frac{\gamma W^\kappa(q)U^\kappa(q) + (1-\gamma)W^\kappa(r)U^\kappa(r)}{\gamma W^\kappa(q) + (1-\gamma)W^\kappa(r)} = U^\kappa(\gamma q + (1-\gamma)r). \end{aligned}$$

Thus we can have neither that $\beta p + (1-\beta)r \succ \gamma q + (1-\gamma)r$ nor $\beta p + (1-\beta)r \prec \gamma q + (1-\gamma)r$, implying that $\beta p + (1-\beta)r \asymp \gamma q + (1-\gamma)r$.

Thus $\succ$ satisfies (A.1)-(A.3) if it has a multiple weighted expected utility representation.

(Sufficiency) Suppose $\succ$ satisfies (A.1)-(A.3). Then for every $\psi \in \Psi$ we can construct utility and weight functions by letting $u^\psi(x_i) = \alpha_i$ and $w^\psi(x_i) = \tau_i$ for $i = 1, \ldots, n$. For every $p \in \Delta(X)$, set

$$U^\psi(p) = \alpha_p = \frac{\sum_{i=1}^n p_i\tau_i\alpha_i}{\sum_{i=1}^n p_i\tau_i} = \frac{\sum_{i=1}^n p(x_i)w(x_i)u(x_i)}{\sum_{i=1}^n p(x_i)w(x_i)},$$
$$W^\psi(p) = \tau_p = \sum_{i=1}^n p_i\tau_i = \sum_{i=1}^n p(x_i)w(x_i).$$

[27] For every $p, q \in \Delta(X)$ and $\lambda \in (0, 1)$, we have that

$$\begin{aligned} U^k(\lambda p + (1-\lambda)q) &= \frac{\sum_{x\in X}(\lambda p + (1-\lambda)q)(x)w^k(x)u^k(x)}{\sum_{x\in X}(\lambda p + (1-\lambda)q)(x)w^k(x)} = \frac{\lambda\left[\sum_{x\in X} p(x)w^k(x)u^k(x)\right] + (1-\lambda)\left[\sum_{x\in X} q(x)w^k(x)u^k(x)\right]}{\lambda\left[\sum_{x\in X} p(x)w^k(x)\right] + (1-\lambda)\left[\sum_{x\in X} q(x)w^k(x)\right]} \\ &= \frac{\lambda W^k(p)U^k(p) + (1-\lambda)W^k(q)U^k(q)}{\lambda W^k(p) + (1-\lambda)W^k(q)}. \end{aligned}$$

Suppose $p \succ q$, then by Lemma 4 we have that for every $\psi \in \Psi$, $U^\psi(p) \neq U^\psi(q)$. Suppose that for some $\psi \in \Psi$ we have $U^\psi(p) < U^\psi(q)$, then we can pick some $r \succ p \succ q$ and $\beta \in (0,1)$ such that $U^\psi(\beta p + (1-\beta)r) = U^\psi(q)$. This would imply by Lemma 4 that $\beta p + (1-\beta)r \asymp q$, but betweenness implies $r \succ \beta p + (1-\beta)r \succ p \succ q$. Thus we must have $U^\psi(p) > U^\psi(q)$ for every $\psi \in \Psi$.

Now suppose $U^\psi(p) > U^\psi(q)$ for every $\psi \in \Psi$. Then by Lemma 4 $\neg(p \asymp q)$ and by the argument above $\neg(p \prec q)$, so we conclude that $p \succ q$. We conclude that $p \succ q$ if and only if $U^\psi(p) > U^\psi(q)$ for every $\psi \in \Psi$. Letting $\mathcal{V} = \{(u^\psi, w^\psi) : \psi \in \Psi\}$ establishes the representation.

Thus $\succ$ satisfies (A.1)-(A.3) only if it has a multiple weighted expected utility representation.

This completes the proof. □

6.4 Proof of Theorem 2

(Necessity) Suppose that $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle \equiv \mathcal{V}^*$, then for every $p \in \Delta(X)$ we have that $\langle \hat{\mathcal{U}}^1_p \rangle = \langle \hat{\mathcal{U}}^2_p \rangle = \{u_p = wu + (1-w)\bar{u}_p : (u, w) \in \mathcal{V}^*\} \equiv \mathcal{U}^*_p$. By Proposition 7, this implies that for any $q \in \Delta(X)$,

$$p \succ^1 q \iff \sum_{x \in X} p(x) u_p(x) > \sum_{x \in X} q(x) u_p(x),\ \forall\, u_p \in \mathcal{U}^*_p \iff p \succ^2 q.$$

This implies that $\succ^1 = \succ^2$.

(Sufficiency) Suppose, without loss of generality, that there is $(u^*, w^*) \in \langle \hat{\mathcal{V}}^1 \rangle \setminus \langle \hat{\mathcal{V}}^2 \rangle$. Then for some $p \in \Delta(X)$ we have that $u^*_p = w^* u^* + (1-w^*)\bar{u}^*_p \in \langle \hat{\mathcal{U}}^1_p \rangle \setminus \langle \hat{\mathcal{U}}^2_p \rangle$. For $j = 1, 2$, define the local domination cone $\mathcal{D}^j_p = \{\lambda(p-q) : q \prec^j p, \lambda \geq 0\}$. By definition $d \in \mathcal{D}^j_p$ if and only if there are $\lambda \geq 0$ and $q \in \Delta(X)$ such that $d = \lambda(p-q)$, and since $p \succ^j q$, by Proposition 7 we have that

$$\sum_{x \in X} d(x) u_p(x) = \lambda \sum_{x \in X} [p(x) - q(x)] u_p(x) > 0,\ \forall\, u_p \in \langle \hat{\mathcal{U}}^j_p \rangle$$

Since $u^*_p \notin \langle \hat{\mathcal{U}}^2_p \rangle$, by the separating hyperplane theorem there is $d \in \mathcal{D}^2_p$ such that

$$\sum_{x \in X} d(x) u_p(x) > 0 \geq \sum_{x \in X} d(x) u^*_p(x),\ \forall\, u_p \in \langle \hat{\mathcal{U}}^2_p \rangle$$

Hence there are $\lambda \geq 0$ and $q \in \Delta(X)$ such that $d = \lambda(p-q)$. This implies on one hand that $\sum_{x \in X} p(x) u_p(x) > \sum_{x \in X} q(x) u_p(x)$ for every $u_p \in \langle \hat{\mathcal{U}}^2_p \rangle$, so that $p \succ^2 q$, but on the other hand that $\sum_{x \in X} p(x) u^*_p(x) \leq \sum_{x \in X} q(x) u^*_p(x)$, so that as $u^*_p \in \langle \hat{\mathcal{U}}^1_p \rangle$, we have $\neg(p \succ^1 q)$. Hence, $\succ^1 \neq \succ^2$.

Therefore, we conclude that $\succ^1 = \succ^2$ if and only if $\langle \hat{\mathcal{V}}^1 \rangle = \langle \hat{\mathcal{V}}^2 \rangle$. □

6.5 Proof of Theorem 3

(Necessity) Suppose we have $\mathcal{U}$ and w that represent $\succ$. Then for every $u^k \in \mathcal{U}$, let

$$U^k(p) = \frac{\sum_{x \in X} p(x) w(x) u^k(x)}{\sum_{x \in X} p(x) w(x)}, \qquad W(p) = \sum_{x \in X} p(x) w(x).$$

Letting $\mathcal{V} = \mathcal{U} \times \{w\}$, by Theorem 1 we have that (A.1) and (A.2) are satisfied. To show that (A.4) is satisfied, pick $p, q \in \Delta(X)$ such that $p \asymp q$, then there are $u^1, u^2 \in \mathcal{U}$ such that $U^1(p) \geq U^1(q)$ and $U^2(p) \leq U^2(q)$. For every $\beta \in (0,1)$, fix $\tau = \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{W(p)}{W(q)}$, then for every $r \in \Delta(X)$,

$$\begin{aligned} U^1(\beta p + (1-\beta) r) &= \frac{\beta W(p) U^1(p) + (1-\beta) W(r) U^1(r)}{\beta W(p) + (1-\beta) W(r)} \\ &\geq \frac{\beta \frac{W(p)}{W(q)} W(q) U^1(q) + (1-\beta) W(r) U^1(r)}{\beta \frac{W(p)}{W(q)} W(q) + (1-\beta) W(r)} \\ &= \frac{\gamma W(q) U^1(q) + (1-\gamma) W(r) U^1(r)}{\gamma W(q) + (1-\gamma) W(r)} = U^1(\gamma q + (1-\gamma) r). \end{aligned}$$

Likewise, $U^2(\beta p+(1-\beta)r) \leq U^2(\gamma q+(1-\gamma)r)$, which implies that $\beta p+(1-\beta)r \asymp \gamma q+(1-\gamma)r$. To show that the substitution ratio $\tau = \frac{W(p)}{W(q)}$ is unique let $r = \delta_{x_n}$, then for $\tau' < \frac{W(p)}{W(q)}$ there is $\beta \in (0,1)$ such that $U^k(\beta p+(1-\beta)r) < U^k\left(\frac{\beta\tau' q+(1-\beta)r}{\beta\tau'+(1-\beta)}\right)$ for all $u^k \in \mathcal{U}$, and likewise for $\tau' > \frac{W(p)}{W(q)}$ there is $\beta \in (0,1)$ such that $U^k(\beta p+(1-\beta)r) > U^k\left(\frac{\beta\tau' q+(1-\beta)r}{\beta\tau'+(1-\beta)}\right)$ for all $u^k \in \mathcal{U}$. This implies that τ, and therefore γ, is unique, so that (A.4) is satisfied.

(Sufficiency) Suppose that $\succ$ satisfies (A.1),(A.2),(A.4). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^\psi, w^\psi) : \psi \in \Psi\}$. By Lemma 5, for every $x_i \in X$ there is $\tau_i > 0$ such that $\Phi(\delta_{x_i}) = A(\delta_{x_i}) \times \{\tau_i\}$, and thus $w^\psi(x_i) = \tau_i \equiv w(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{U} = \{u^\psi : \psi \in \Psi\}$, we have that $\mathcal{V} = \mathcal{U} \times \{w\}$, so that $\succ$ has the desired representation.

This completes the proof. □

6.6 Proof of Theorem 4

We begin by proving the uniquness of the utility assigned to the outcomes.

Claim Given (A.1), (A.2) and (A.3), axiom (A.5) implies that for all $x \in X$, there is a unique $\alpha \in [0,1]$ such that $\delta_x \asymp \zeta_\alpha$.

Proof Suppose that there exist $\alpha, \alpha' \in (0,1)$, $\alpha > \alpha'$ such that $\delta_x \asymp \zeta_\alpha$ and $\delta_x \asymp \zeta_{\alpha'}$. Define $\bar{\alpha} = \sup\{\alpha \in (0,1) \mid \delta_x \asymp \zeta_\alpha\}$. By Proposition 2, $\delta_x \asymp \zeta_{\bar{\alpha}}$. By definition $\bar{\alpha} \geq \alpha > \alpha'$. Hence, by (A.5), for all $\alpha \in (\bar{\alpha}, 1)$, $\zeta_\alpha \succ \delta_x$. Since $\alpha' \in (0, \bar{\alpha}]$, by (A.5) $\delta_x \succ \zeta_{\alpha'}$. A contradiction of $\delta_x \asymp \zeta_{\alpha'}$. Thus, there is a unique $\alpha \in [0,1]$ such that $\delta_x \asymp \zeta_\alpha$.

We turn now to the proof of the theorem.

(Necessity) Suppose we have u and $\mathcal{W}$ that represent $\succ$. Then for every $w^k \in \mathcal{W}$, let

$$U^k(p) = \frac{\sum_{x\in X} p(x)w^k(x)u(x)}{\sum_{x\in X} p(x)w^k(x)}, \qquad W^k(p) = \sum_{x\in X} p(x)w^k(x).$$

Letting $\mathcal{V} = \{u\} \times \mathcal{W}$, by Theorem 1 we have that (A.1)-(A.3) are satisfied. To show that (A.5) is satisfied, for every $x_i \in X$ set $\alpha_i = \frac{u(x_i)-u(x_1)}{u(x_n)-u(x_1)}$, so that for every $w^k \in \mathcal{W}$ we have $U^k(\delta_{x_i}) = u(x_i) = \alpha u(x_n) + (1-\alpha)u(x_1) = U(\zeta_\alpha)$ if and only if $\alpha = \alpha_i$.

(Sufficiency) Suppose that $\succ$ satisfies (A.1)-(A.3),(A.5). Then by Theorem 1 we have a representation by $\mathcal{V} = \{(u^\psi, w^\psi) : \psi \in \Phi\}$. By (A.5), for every $x_i \in X$ there is a α_i such that $\Phi(\delta_{x_i}) = \{\alpha_i\} \times T(\delta_{x_i}, \zeta_{\alpha_i})$, and thus $u^\psi(x_i) = \alpha_i \equiv u(x_i)$ for every $\psi \in \Psi$. Thus letting $\mathcal{W} = \{w^\psi : \psi \in \Psi\}$, we have that $\mathcal{V} = \{u\} \times \mathcal{W}$, so that $\succ$ has the desired representation.

This completes the proof. □

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