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The Fibonacci Quarterly

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

AN OLD THEOREM ON THE GCD AND ITS APPLICATION TO PRIMES

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1. Introduction

We show how an old theorem about the GCD can be used to define primes and to construct formulas for primes. We give new formulas for the characteristic function of the primes, the n^{th} prime p_n, the function $\pi(x)$, and the least prime greater than a given number.

These formulas are all elementary functions in the sense of Grzegorczyk [6] and Kalmar [12] (Kalmar elementary). From a theorem of Jones [11], it will follow that there exist formulas with the same range built up only from the four functions

(1.1) $\quad x + y, \qquad [x/y], \qquad x \dot{-} y, \qquad 2^x,$

by function composition (without sigma signs). There also exist polynomial formulas for the primes, but that is another subject (see [10]).

The constructions here use a theorem of Hacks [7]. (He indicates on page 207 of [7] that this result may have been known to Gauss. See also Dickson [2] vol. 1, p. 333.) Hacks [7] considered sums of the form:

Definition 1.1: $H(k, n) = 2\sum_{i=1}^{n-1}\left\lfloor\frac{k\cdot i}{n}\right\rfloor.$

Here $\lfloor x\rfloor$ denotes the floor (integer part) of x. Hacks proved that sums of this type could be used to define the GCD of k and n, i.e., (k, n).

Theorem 1.1 (Hacks [7]): $H(k, n) = nk - k - n + (k, n)$.

Proof: The proof of this theorem requires the following two lemmas.

Lemma 1.1: Suppose $k \perp n$. Then $H(k, n) = (k - 1)(n - 1)$.

Proof: $ki \equiv kj \pmod{n}$ implies $i \equiv j \pmod{n}$. Thus, the set $\{ki: i = 0, 1, 2, \ldots, n - 1\}$ is a complete residue system mod n. The sum of the remainders in this system must be equal to $n(n - 1)/2$.

Hence, let $ki \equiv r_i \pmod{n}$ where $0 \le r_i < n$, $(i = 1, 2, \ldots, n - 1)$. Then we have

$$ki = \left\lfloor\frac{k\cdot i}{n}\right\rfloor n + r_i \qquad \text{and} \qquad \sum_{i=0}^{n-1} r_i = \frac{n(n-1)}{2}.$$

Summing the first equation, we find

$$k\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} ki = \sum_{i=0}^{n-1}\left\lfloor\frac{k\cdot i}{n}\right\rfloor n + r_i = n\sum_{i=0}^{n-1}\left\lfloor\frac{k\cdot i}{n}\right\rfloor + \sum_{i=0}^{n-1} r_i.$$

Therefore,

$$k\,\frac{n(n-1)}{2} = \frac{n}{2}\cdot H(k, n) + \frac{n(n-1)}{2}.$$

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Multiplying by 2 and dividing by n gives the result:

$$k(n-1) = H(k, n) + n - 1.$$

Lemma 1.2: $H(ad, bd) = adbd - abd + dH(a, b)$.

Proof: Integers i such that $0 \le i \le bd - 1$ can be written in the form $i = bq + j$ where $0 \le q < d$ and $0 \le j < b$. Hence, we have

$$\begin{aligned} H(ad, bd) &= 2\sum_{i=1}^{bd-1}\left\lfloor\frac{ad \cdot i}{bd}\right\rfloor = 2\sum_{i=1}^{bd-1}\left\lfloor\frac{a \cdot i}{b}\right\rfloor = 2\sum_{q=0}^{d-1}\sum_{j=0}^{b-1}\left\lfloor\frac{a(bq+j)}{b}\right\rfloor \\ &= 2\sum_{q=0}^{d-1}\sum_{j=0}^{b-1} aq + 2\sum_{j=0}^{b-1}\sum_{q=0}^{d-1}\left\lfloor\frac{a \cdot j}{b}\right\rfloor = 2\sum_{q=0}^{d-1} abq + 2\sum_{j=0}^{b-1} d\left\lfloor\frac{a \cdot j}{b}\right\rfloor \\ &= 2ab\,\frac{d(d-1)}{2} + 2d \cdot \frac{1}{2}H(a, b) = abd(d-1) + dH(a, b) \\ &= adbd - abd + dH(a, b). \end{aligned}$$

Corollary 1.1 (Hacks [7]): $H(k, n) = nk - k - n + (k, n)$.

Proof: Write $k = ad$ and $n = bd$ where $a \perp b$ and $d = (k, n)$. From Lemma 1.1, we then have $H(a, b) = (a-1)(b-1)$. Using this together with Lemma 1.2, we have

$$\begin{aligned} H(k, n) &= H(ad, bd) = adbd - abd + d(a-1)(b-1) \\ &= adbd - abd + abd - da - db + d = nk - k - n + d. \end{aligned}$$

From Corollary 1.1, it follows that the function H is commutative (symmetric), $H(k, n) = H(n, k)$. The function H has other interesting properties. Using an argument similar to the proof of Lemma 1.2, it is easy to show that

(1.2) $\quad H(qk, k) = qk(k-1), \quad H(k, qk) = qk(k-1),$

(1.3) $\quad H(qk + r, k) = qk(k-1) + H(r, k).$

2. Characteristic Function of the Primes

From Lemmas 1.1 and 1.2, we see that

Lemma 2.1: $1 = (k, n) \Leftrightarrow (k-1)(n-1) = H(k, n)$.
$1 < (k, n) \Leftrightarrow (k-1)(n-1) < H(k, n)$.

Now let $m = n - 1$ (or $m = \lfloor\sqrt{n}\rfloor$, to be more economical). Then, by Lemma 2.1, n is composite if and only if

$$(\exists k)[1 \le k \le m \text{ and } (k-1)(n-1) < H(k, n)].$$

Hence, n is composite if and only if

$$(2.1) \qquad (\exists k)\left[1 \le k \le m \text{ and } 0 < 2\sum_{i=1}^{k-1}\left\lfloor\frac{i \cdot n}{k}\right\rfloor - \sum_{i=1}^{k-1}(n-1)\right].$$

It follows that n is composite if and only if

$$(2.2) \qquad (\exists k)\left[1 \le k \le m \text{ and } 0 < \sum_{i=1}^{k-1}\left(2\left\lfloor\frac{i \cdot n}{k}\right\rfloor - n + 1\right)\right], \quad m = \lfloor\sqrt{n}\rfloor.$$

When n is prime, these expressions are all 0. So, by summing over k, we can see that n is composite if and only if

$$(2.3) \qquad 0 < \sum_{0 < i < k \le m}\left(2\left\lfloor\frac{i \cdot n}{k}\right\rfloor - n + 1\right), \quad m = \lfloor\sqrt{n}\rfloor.$$

Alternatively, by summing the constant term, (2.3) can be rewritten to say that n is composite if and only if

$$(2.4)\qquad 0 < \sum_{k=1}^{m}\left(\sum_{i=1}^{k-1}\left(2\left\lfloor\frac{i\cdot n}{k}\right\rfloor\right)\right) - \frac{(n-1)(m-1)m}{2},\quad m = \lfloor\sqrt{n}\rfloor.$$

This is equivalent to the statement that n is composite if and only if

$$(2.5)\qquad 0 < m(m-1)(1-n) + \sum_{0<i<k\le m} 4\left\lfloor\frac{i\cdot n}{k}\right\rfloor,\quad m = \lfloor\sqrt{n}\rfloor.$$

Since these expressions are zero when n is a prime, they characterize primes. We summarize (2.3) in Theorem 2.1.

Theorem 2.1: Let $g(n)$ be defined by

$$(2.6)\qquad g(n) = \sum_{0<i<k\le m}\left(2\left\lfloor\frac{i\cdot n}{k}\right\rfloor - n + 1\right),$$

where $m = \lfloor\sqrt{n}\rfloor$ or $m = n - 1$. Then, for all $n > 1$, n is prime if and only if $g(n) = 0$. And n is composite if and only if $g(n) \ge 1$.

The subtraction function $x \dot{-} y$ or the $\operatorname{sgn}(x)$ function can now be used to obtain a characteristic function for the primes. A *characteristic function* for a set is a two-valued function taking value 1 on the set and value 0 on the complement of the set.

The proper subtraction function $x \dot{-} y$ is defined to be $x - y$ for $y \le x$ and 0 for $x < y$. The sign function $\operatorname{sgn}(x)$ is defined by $\operatorname{sgn}(x) = +1$ if $x > 0$, by $\operatorname{sgn}(x) = -1$ if $x < 0$ and $\operatorname{sgn}(0) = 0$.

Now define $h(n)$ to be $h(n) = 1 \dot{-} g(n)$ or define $h(n) = 1 - \operatorname{sgn} g(n)$. Then it follows from Theorem 2.1 that $h(n)$ is a characteristic function for the set of primes.

Theorem 2.2: Let $h(n)$ be defined by

$$(2.7)\qquad h(n) = 1 \dot{-} \sum_{0<i<k\le m}\left(2\left\lfloor\frac{i\cdot n}{k}\right\rfloor - n + 1\right),$$

where $m = n - 1$ or $m = \lfloor\sqrt{n}\rfloor$. Then n is prime if and only if $h(n) = 1$. And n is composite if and only if $h(n) = 0$. (These statements hold for $n > 1$.)

We can use the function h to construct a formula for the function $\pi(x)$, [$\pi(x)$ = the number of primes $\le x$]. From Theorem 2.2, we have

Theorem 2.3: The function $\pi(x)$ is given by

$$(2.8)\qquad \pi(x) = \sum_{n=2}^{x} h(n) = \sum_{n=2}^{x}\left(1 \dot{-} \sum_{0<i<k\le m}\left(2\left\lfloor\frac{i\cdot n}{k}\right\rfloor - n + 1\right)\right).$$

Proof: The idea of (2.8) is that the characteristic function h counts the primes $\le x$. [We start the sum at $n = 2$ instead of at $n = 1$ because $h(1) = 1$.]

3. Formula for the n^{th} Prime

Define $C(a, n) = 1 \dot{-} (a + 1) \dot{-} n$. Then $C(a, n)$ is the characteristic function of the relation $a < n$. That is, if $a < n$, then $C(a, n) = 1$. If $n \le a$, then $C(a, n) = 0$.

Now $\pi(i) < n$ if and only if $i < p_n$. Hence $C(\pi(i), n) = 1$ if and only if $i < p_n$. The n^{th} prime p_n is therefore given by the following formula:

$$(3.1)\qquad p_n = \sum_{\ell=0}^{k} C(\pi(\ell), n) = \sum_{\ell=0}^{k}\left(1 \dot{-} (\pi(\ell) + 1 \dot{-} n)\right)$$

when k is large enough ($k \geq p_n - 1$). It is known that

$$p_n < n(\log(n) + \log(\log(n))) \text{ for } n > 5$$

(see Rosser & Schoenfeld [13]). So we can take

$$k = n^2 \quad \text{or} \quad k = 2n\log(n+1).$$

Using Theorem 2.3 and the fact that $h(1) = 1$, we have

$$(3.2) \qquad p_n = \sum_{\ell=0}^{k}\left(1 \dot{-} \left(\left(\sum_{j=1}^{\ell} h(j)\right) \dot{-} n\right)\right).$$

Thus we have, from Theorem 2.2,

Theorem 3.1: The n^{th} prime, p_n, is given by

$$(3.3) \qquad p_n = \sum_{\ell=0}^{k}\left(1 \dot{-} \left(\left(\sum_{j=1}^{\ell}\left(1 \dot{-} \sum_{0<i<k\leq m}\left(2\left\lfloor\frac{i \cdot j}{k}\right\rfloor - j + 1\right)\right)\right) \dot{-} n\right)\right), \text{ for } n > 1.$$

4. Next Prime Greater than a Given Number

The function g of Theorem 2.1 has the property that it is nonnegative and $g(n) = 0$ if and only if n is a prime. The function h also has this property. Hence, we can use either h or g in the following construction of a formula for the next prime greater than a number q. (The number q can be any integer, it does not need to be a prime.)

Theorem 4.1: The next prime greater than q is given by the function

$$(4.1) \qquad N(q) = \sum_{j=0}^{2q}\left(1 \dot{-} \sum_{n=0}^{j}(n \dot{-} q)(1 \dot{-} g(n))\right).$$

Proof: From Bertrand's Postulate, we know that for every $q \geq 1$ there is a prime p such that $q < p \leq 2q$. Fix q and let p denote the least such prime p. Put

$$f(n) = (n \dot{-} q)(1 \dot{-} g(n)).$$

Then $f(n) \geq 0$ for $n \geq 0$. Also $f(n) > 0$ if and only if $q < n$ and $g(n) = 0$, i.e., if and only if n is a prime greater than q. Now

$$1 \dot{-} (f(0) + f(1) + \cdots + f(j)) = 1 \text{ for } j < p.$$

But

$$1 \dot{-} (f(0) + f(1) + \cdots + f(j)) = 0 \text{ for } p < j.$$

Hence, p is equal to the sum

$$(4.2) \qquad N(q) = \sum_{j=0}^{2q}\left(1 \dot{-} (f(0) + f(1) + \cdots + f(j))\right),$$

a sum of exactly p ones.

Bertrand's Postulate is a theorem that was proved by P. L. Chebychev in the nineteenth century. See Hardy [8, p. 349] for a modern proof (due to Erdös [4]).

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A COMBINATORIAL PROBLEM IN THE FIBONACCI NUMBER SYSTEM AND TWO-VARIABLE GENERALIZATIONS OF CHEBYSHEV'S POLYNOMIALS

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To my mother on the occasion of her 70th birthday.

1. Summary

We consider the following three-term recursion formula

(1.1a) $S_{-1} = 0,\ S_0 = 1$

(1.1b) $S_n = Y(n)S_{n-1} - S_{n-2},\ n \geq 1$

(1.1c) $Y(n) = Yh(n) + y(1 - h(n)),$

where $h(n)$ is the n^{th} digit of the Fibonacci-"word" 1011010110... given explicitly by (see [7], [11], [9], [20], [19])

(1.2) $h(n) = [(n + 1)\phi] - [n\phi] - 1,$

where $[a]$ denotes the integer part of a real number a, and

$$\phi := (1 + \sqrt{5})/2,$$

obeying $\phi^2 = \phi + 1,\ \phi > 1$, is the golden section [10], [9], [4].

For $Y = y$ one recovers Chebyshev's $S_n(y)$ polynomials of degree n [1]. In the general case certain two-variable polynomials $S_n(Y, y)$ emerge.

The theory of continued fractions (see [18]) shows that $(-i)^n S_n(Y, y)$ can be identified with the denominator of the n^{th} approximation of the regular continued fraction ($i^2 = -1$)

(1.3) $[0; -iY(1), -iY(2), \ldots, -iY(k), \ldots]$

$\equiv 1/(-iY(1) + 1/(-iY(2) + 1/(\ldots\ .$

The polynomials $S_n(Y, y)$ can be written as

(1.4) $$S_n(Y, y) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \sum_{k=k_{\min}}^{k_{\max}} (n;\ \ell,\ k) Y^{z(n)-\ell-k} y^{n-z(n)-\ell+k},$$

where the coefficients $(n;\ \ell,\ k)$ are defined recursively by

(1.5) $$(n;\ \ell,\ k) = (n - 1;\ \ell,\ k) + (h(n - 1) + h(n) - 1)(n - 2;\ \ell - 1,\ k - 1) + (2 - h(n - 1) - h(n))(n - 2;\ \ell - 1,\ k),$$

with certain input quantities. The range of the k index is bounded by

(1.6a) $k_{\min} \equiv k_{\min}(n,\ \ell) := \max\{0,\ \ell - (n - z(n))\},$

(1.6b) $k_{\max} \equiv k_{\max}(n,\ \ell) := \min\{z(n) - \ell,\ \min(\ell,\ p(n))\},$

with

(1.7) $$z(n) = \sum_{k=1}^{n} h(k),$$

(1.8) $$p(n) = \sum_{k=0}^{n-1} (h(k + 1) + h(k) - 1).$$

The polynomials $S_n(Y, y)$ are listed for $n = 0(1)13$ in Table 1. They are generating functions for the numbers $(n; \ell, k)$ which are shown to have a combinatorial meaning in the Fibonacci number system. This system is based on the fact that every natural number N has a unique representation (see [23], [5], [21], [11], [20]) in terms of Fibonacci numbers (see [10] and [4]):

$$(1.9) \qquad N = \sum_{i=0}^{r} s_i F_{i+2}, \quad s_i \in \{0, 1\}, \quad s_i s_{i+1} = 0.$$

(Zeckendorf's representation of the second kind in which one writes the number 1 as F_2 and not as F_1.)

Table 1. $S_n = Y(n)S_{n-1} - S_{n-2}$, $S_{-1} = 0$, $S_0 = 1$
$Y(n) = Yh(n) + y(1 - h(n))$
$h(n) = [(n + 1)\phi] - [n\phi] - 1$

n	$S_n(Y, y)$
0	1
1	Y
2	$Yy - 1$
3	$Y(Yy - 2)$
4	$Y^3y - Y(2Y + y) + 1$
5	$Y^3y^2 - Yy(3Y + y) + (2Y + y)$
6	$Y^4y^2 - Y^2y(4Y + y) + 2Y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - 2(Y + y)$
8	$Y^5y^3 - Y^3y^2(6Y + y) + Y^2y(11Y + 4y) - 2Y(3Y + 2y) + 1$
9	$Y^6y^3 - Y^4y^2(6Y + 2y) + Y^2y(11Y^2 + 9Yy + y^2) - Y(6Y^2 + 11Yy + 3y^2) + (3Y + 2y)$
10	$Y^6y^4 - Y^4y^3(7Y + 2y) + Y^2y^2(17Y^2 + 10Yy + y^2) - Yy(17Y^2 + 15Yy + 3y^2) + (6Y^2 + 7Yy + 2y^2) - 1$
11	$Y^7y^4 - Y^5y^3(8Y + 2y) + Y^3y^2(23Y^2 + 12Yy + y^2) - Y^2y(28Y^2 + 24Yy + 4y^2) + Y(12Y^2 + 18Yy + 5y^2) - (4Y + 2y)$
12	$Y^8y^4 - Y^6y^3(8Y + 3y) + Y^4y^2(23Y^2 + 19Yy + 3y^2) - Y^2y(28Y^3 + 41Y^2y + 14Yy^2 + y^3) + Y(12Y^3 + 35Y^2y + 20Yy^2 + 3y^3) - (10Y^2 + 9Yy + 2y^2) + 1$
13	$Y^8y^5 - Y^6y^4(9Y + 3y) + Y^4y^3(31Y^2 + 21Yy + 3y^2) - Y^2y^2(51Y^3 + 53Y^2y + 15Yy^2 + y^3) + Yy(40Y^3 + 59Y^2y + 24Yy^2 + 3y^3) - (12Y^3 + 28Y^2y + 14Yy^2 + 2y^3) + (4Y + 3y)$
⋮	

In this number system $N \triangleq s_r \cdots s_2 s_1 s_0 \cdot$, where the dot at the end indicates the F_1 place which is not used.

Proposition 1: $(n; \ell, k)$ gives the number of possibilities to choose, from the natural numbers 1 to n, ℓ mutually disjoint pairs of consecutive numbers such that all numbers of k of these pairs end in the canonical Fibonacci number system in an even number of zeros.

Another formulation is possible if Wythoff's complementary sequences $\{A(n)\}$ and $\{B(n)\}$ (see [22], [7], [21], [12], [8], [9], and [4]), defined by

(1.10) $A(n) := [n\phi]$, $B(n) := [n\phi^2] = n + A(n)$, $n = 1, 2, \ldots,$

are introduced.

Proposition 2: $(n;\ \ell, k)$ is the number of different possibilities to choose, from the numbers $1, 2, \ldots, n$, ℓ mutually disjoint pairs of consecutive numbers, say

$$(n_1, n_1 + 1), \ldots, (n_\ell, n_\ell + 1) \text{ with } n_j > n_{j-1} + 1 \text{ for } j = 2, \ldots, \ell,$$

such that all members of k pairs among them, say

$$(i_1, i_1 + 1), \ldots, (i_k, i_k + 1),$$

are A-numbers, i.e., $i_j = A(m_j)$ and $i_j + 1 = A(m_j + 1)$ for some m_j and all $j = 1, \ldots, k$. For $\ell = 0$, put $(n;\ 0, 0) = 1$.

From the analysis of Wythoff's sequences one learns that A-pairs $(A(m_j), A(m_j + 1) = A(m_j) + 1)$ occur precisely for $m_j = B(q_j)$ for some $q_j \in \mathbb{N}$. All remaining pairs are either of the (A, B) or (B, A) type. Thus, one may state equivalently,

Proposition 3: $(n;\ \ell, k)$ counts the number of different ways to choose, from the numbers $1, 2, \ldots, n - 1$, ℓ distinct nonneighboring numbers such that exactly k numbers among them, say $i_1, \ldots, i_k$, are AB-numbers, i.e., they satisfy for all $j = 1, \ldots, k$, $i_j = A(B(m_j))$ with some $m_j \in \mathbb{N}$.

Still another meaning can be attributed to the coefficients of the S_n polynomials based on the above findings.

Corollary: Consider the Zeckendorf representations (with 1 as F_2) of the numbers $0, 1, 2, \ldots, F_{n+1} - 1$. Then exactly $(n;\ \ell, k)$ of them need ℓ Fibonacci numbers, k of which are of the type $F_{A(B(m)+1)}$ with $m \in \{1, 2, \ldots, p(n)\}$.

The representation of 0 which does not need any Fibonacci number is included in order to cover the case $\ell = 0$, $k = 0$.

Another set of generalized Chebyshev S_n polynomials is of interest. They are defined recursively by

(1.11a) $\hat{S}_{-1} = 0$, $\hat{S}_0 = 1$,

(1.11b) $\hat{S}_n = Y(n + 1)\hat{S}_{n-1} - \hat{S}_{n-2}$, $n \geq 1$,

with $Y(n)$ defined by (1.1c). Table 2 shows $\hat{S}_n(Y, y)$ for $n = 0(1)13$. They are given as $(+i)^n$ times the denominator of the n^{th} approximation of the regular continued fraction

(1.12) $[0; -iY(2), -iY(3), \ldots, -iY(k), \ldots]$.

As far as combinatorics is concerned, one has to replace the numbers $1, 2, \ldots, n$ in the above given statements by the numbers $2, 3, \ldots, n + 1$.

The physical motivation for considering the polynomials $S_n(Y, y)$ and $\hat{S}_n(Y, y)$ is sketched in the Appendix, where a set of 2×2 matrices M_n formed from these polynomials is also introduced. In [14], [6], and [15], n-variable generalizations of Chebyshev's polynomials were introduced. For the 2-variable case, these polynomials satisfy a 4-term recursion formula and bear no relation to the ones studied in this work.

Table 2. $\hat{S}_n = Y(n+1)\hat{S}_{n-1} - \hat{S}_{n-2}$, $\hat{S}_{-1} = 0$, $\hat{S}_0 = 1$
$Y(n+1) = Yh(n+1) + y(1 - h(n+1))$
$h(n+1) = [(n+2)\phi] - [(n+1)\phi] - 1$

n	$\hat{S}_n(Y, y)$
0	1
1	y
2	$Yy - 1 = S_2(Y, y)$
3	$Y^2y - (Y + y)$
4	$Y^2y^2 - y(2Y + y) + 1$
5	$S_5(Y, y)$
6	$Y^3y^3 - Yy^2(4Y + y) + 2y(2Y + y) - 1$
7	$Y^4y^3 - Y^2y^2(5Y + y) + Yy(7Y + 3y) - (3Y + y) = S_7(Y, y) - (Y - y)$
8	$Y^5y^3 - Y^3y^2(5Y + 2y) + Yy(7Y^2 + 7Yy + y^2) - (3Y^2 + 5Yy + 2y^2) + 1$
9	$Y^5y^4 - Y^3y^3(6Y + 2y) + Yy^2(12Y^2 + 8Yy + y^2) - y(10Y^2 + 8Yy + 2y^2) + (3Y + 2y)$
10	$S_{10}(Y, y)$
11	$Y^7y^4 - Y^5y^3(7Y + 3y) + Y^3y^2(17Y^2 + 16Yy + 3y^2) - Yy(17Y^3 + 27Y^2y + 11Yy^2 + y^3) + (6Y^3 + 17Y^2y + 10Yy^2 + 2y^3) - (4Y + 2y)$
12	$Y^7y^5 - Y^5y^4(8Y + 3y) + Y^3y^3(24Y^2 + 18Yy + 3y^2) - Yy^2(34Y^3 + 37Y^2y + 12Yy^2 + y^3) + y(23Y^3 + 32Y^2y + 13Yy^2 + 2y^3) - (6Y^2 + 11Yy + 4y^2) + 1$
13	$S_{13}(Y, y) + (Y - y)$
⋮	

2. Fundamentals of Wythoff's Sequences
(see [22], [7], [21], [12], [8], [11], [9], [4], [19])

In this section we collect, without proofs, some well-known facts concerning Wythoff's pairs of natural numbers, the sequence $\{h(n)\}$, and their relation to the Fibonacci number system (1.9). We also introduce the counting sequences $\{z(n)\}$ and $\{p(n)\}$.

The special Beatty sequences $\{A(n)\}$ and $\{B(n)\}$ (see [22], [9], [4]) given by (1.10) divide the set of natural numbers into two disjoint and exhaustive sets, henceforth called A- and B-numbers. For $n = 0$ we also define the Wythoff pair $(A(0), B(0)) = (0, 0)$. The sequence h, defined in (1.2) as

(2.1) $\quad h(n) = A(n+1) - A(n) - 1,$

takes on values 0 and 1 only. Wythoff's pairs $(A(n), B(n))$ have a simple characterization in the Fibonacci number system: $A(n)$ is represented for each $n \in \mathbb{N}$ with an *even* number of zeros at the end (including the case of no zero). $B(n)$ is then obtained from the represented $A(n)$ by inserting a 0 before the dot at the end. Therefore, B-numbers end in an *odd* number of zeros in this canonical number system. It is also known how to obtain the representation of $A(n)$ from the given one for n.

The sequence $h(n)$ (2.1) distinguishes the two types of numbers:

$$(2.2)\qquad h(n) = \begin{cases} 0 & \text{iff } n \text{ is a } B\text{-number,} \\ 1 & \text{iff } n \text{ is an } A\text{-number.} \end{cases}$$

An A-number ending in a 1 in the Fibonacci system (no end zeros) has fractional part from the interval $(2 - \phi, 2(2 - \phi))$. Its fractional part is from the interval $(2(2 - \phi), 1)$ if the A-number representation ends in at least two zeros. This distinction of A-numbers corresponds to the compositions

$$A(A(n)) \equiv A^2(n) = [[n\phi]\phi] \quad\text{and}\quad AB(n) = [[n\phi^2]\phi],$$

respectively.

It is convenient to introduce the projectors

$$(2.3)\qquad k(n) := h(n) - (1 - h(n + 1)) = h(n)h(n + 1),$$

$$1 - k(n) = (1 - h(n)) + (1 - h(n + 1)),$$

k marks AB-numbers:

$$(2.4)\qquad k(n) = \begin{cases} 1 & \text{iff } n \text{ is an } AB\text{-number,} \\ 0 & \text{otherwise.} \end{cases}$$

$A(B(m) + 1) = AB(m) + 1$, i.e., $AB(m)$ is followed by an A-number. Such pairs of consecutive numbers will be called *A-pairs*. Some identities for $n \in \mathbb{N}$ which will be of use later on are:

$$(2.5a)\qquad AB(n) = A(n) + B(n) = 2A(n) + n = B(A(n) + 1) - 2,$$

$$(2.5b)\qquad BA(n) = 2A(n) + n - 1 = AB(n) - 1 = A(B(n) + 1) - 2,$$

$$(2.5c)\qquad AA(n) = A(n) + n - 1 = B(n) - 1 = A(A(n) + 1) - 2,$$

$$(2.5d)\qquad BB(n) = 3A(n) + 2n = ABA(n) + 2 = B(B(n) + 1) - 2, \\ = AAB(n) + 1.$$

No three consecutive numbers can be A-numbers, and no two consecutive numbers can be B-numbers. Among the AA-numbers $\neq 1$, we distinguish between those which are bigger members of an A-pair, viz,

$$(2.6)\qquad AB(m) + 1 = A(B(m) + 1) = AA(A(m) + 1) \quad\text{for } m \in \mathbb{N},$$

and the remaining ones which are called *A-singles*, viz,

$$(2.7)\qquad AA(B(m) + 1) = A(AB(m) + 1) = BB(m) + 1 \quad\text{for } m \in \mathbb{N}.$$

Thus, A-singles are AA-numbers having B-numbers as neighbors. $A(n)$ is an A-single if $h(n - 1) = h(n) = 1$. The AA-number 1 is considered separately because we can either count $(0, 1)$ as an A-pair or as a (B, A)-pair.

Define $z(n)$ to be the number of (positive) A-numbers not exceeding n. This is

$$(2.8)\qquad z(n) = \sum_{k=1}^{n} h(k) = [(n + 1)/\phi] = A(n + 1) - (n + 1).$$

The number of B-numbers $\neq 0$ not exceeding n is then $n - z(n) = [(n + 1)/\phi^2]$.

Define $p(n)$ to be the number of AB-numbers (0 excluded) not exceeding $n - 1$. This is

$$(2.9)\qquad p(n) = \sum_{m=1}^{n-1} k(m) = z(n) + z(n - 1) - n = 2A(n) - 3n + h(n).$$

The following identities hold:

$$(2.10)\qquad pA(n + 1) = -A(n + 1) + 2n + 1 = n - z(n) = z^2(n - 1).$$

This is just the number of B-numbers (excluding 0) not exceeding n. The last equality follows with the help of

(2.11) $A(z(n-1)+1) = A(A(n)-n+1) = n+1-h(n),$

which can be verified for A- and B-numbers n separately. Also,

(2.12) $pB(n) = A(n) - n = z(n-1),$

(2.13) $pAB(m) = pBA(m) = m - 1.$

The p-value increases by one at each argument $AB(m)+1$, due to

(2.14) $k(n) = p(n+1) - p(n).$

The p-value m appears $2h(m)+3$ times.

Another identity is

(2.15) $p(B(m)-1) = pA^2(m) = z(m-1).$

The number of A-singles ($\neq 1$) not exceeding n is

(2.16) $pA(z(n) - p(n+1)) = pAz^2(n) = pz(n).$

Finally,

(2.17) $z(n - z(n) - 1) = z(pA(n+1)-1) = z(z^2(n-1)-1) = p(n-1).$

The last equality can be established by calculating $B(n - z(n))$.

$$\begin{aligned}(2.18)\quad B(n-z(n)) &= n+1-2h(n)-h(n-1)\\ &= n - z(n) + z(n-2) + (1-h(n)) = n - h(n) - k(n-1),\end{aligned}$$

implying

(2.19) $A(n-z(n)) = z(n) + 1 - 2h(n) - h(n-1) = n - z(n) + p(n-1).$

3. Generalized Chebyshev Polynomials

Consider the recursion formula (1.1) with $h(n)$ given by (1.2). For $Y = y$, the one for Chebyshev's $S_n(y) \equiv S_n(y, y)$ polynomials [1] is found.* Their explicit form is

$$(3.1)\qquad S_n(y) = \sum_{\ell=0}^{[n/2]} (-1)^\ell \binom{n-\ell}{\ell} y^{n-2\ell}, \quad n \in \mathbb{N}_0.$$

The binomial coefficient has, for $\ell \neq 0$, the following combinatorial meaning. It gives the number of ways to choose, from the numbers $1, 2, \ldots, n$, ℓ mutually disjoint pairs of consecutive numbers. For $\ell = 0$, this number is put to 1. The sum over the moduli of the coefficients in (3.1), i.e., the sum over the "diagonals" of Pascal's triangle, is F_{n+1}. One also has

$$S_n(2) = n+1 \quad\text{and}\quad S_n(3) = F_{2(n+1)},$$

which is proved by induction.

For $Y \neq y$, a certain two-variable generalization of these S_n polynomials results. We claim that they are given by (1.4) where the new coefficients have the combinatorial meaning given in Propositions 1-3 and the Corollary of the first section.

Theorem 1: $S_n(Y, y)$ given by (1.4) with (1.5) and (1.6) is the solution of recursion formula (1.1) with (1.2) inserted.

---

*$S_n(y) = U_n(y/2)$ with $U_n(\cos\Theta) = \sin((n+1)\Theta)/\sin\Theta$, Chebyshev's polynomials of the second kind, for $|y| < 2$.

Proof: By induction over n. For $n = 0$, $k_{\min}(0, 0) = k_{\max}(0, 0) = 0$ due to $z(0) = 0$ and, therefore, $S_0 = 1$. In order to compute S_m via (1.1), assuming (1.4) to hold for $n = m - 1$ and $n = m - 2$, one writes

$$Y(m) = Y^{h(m)}y^{1-h(m)},$$

which is identical to (1.1c) due to the projector properties of the exponents. Now

$$z(m - 1) = z(m) - h(m) \quad \text{and} \quad z(m - 2) = z(m) - h(m) - h(m - 1),$$

following from (2.8) and (2.1), are employed to rewrite the Y and y exponents in the S_{m-1} term of (1.1b) such that exponents appropriate for S_m appear. In the S_{m-2} term of (1.1b) a factor $(1/Y)^{k(m-1)}(1/y)^{1-k(m-1)}$ is in excess, which, when rewritten as $k(m - 1)(1/Y) + (1 - k(m - 1)(1/y)$, produces two terms from this S_{m-2} piece. In both of them the index shift $\ell \to \ell - 1$ is performed, and in the first term $k \to k - 1$ is used. Finally, one proves that the ℓ and k range in all of the three terms which originated from S_{m-1} and S_{m-2} in (1.1b) can be extended to the one appropriate for S_m as claimed in (1.4). In order to show this, the convention to put $(n;\ \ell, k)$ to zero as soon as for given n the indices ℓ or k are out of the allowed range has to be followed. Also,

$$p(m - 2) = p(m) - k(m - 1) - k(m - 2),$$

resulting from (2.9), is used in the first term of S_{m-2} to verify that

$$k_{\max}(m - 2, \ell - 1) + 1 = k_{\max}(m, \ell).$$

In this term, $m - 1$ is always an AB-number, and

$$k_{\min}(m - 2, \ell - 1) + 1 \geq k_{\min}(m, \ell)$$

holds as well. In the second term, which originated from S_{m-2}, $m - 1$ is not an AB-number, and one can prove that

$$k_{\min}(m - 2, \ell - 1) = k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 2, \ell - 1) \leq k_{\max}(m, \ell).$$

In the S_{m-1} term one has, for even m, first to extend the upper ℓ range by one, then the k range is extended as well, using

$$k_{\min}(m - 1, \ell) \geq k_{\min}(m, \ell) \quad \text{and} \quad k_{\max}(m - 1, \ell) \leq k_{\max}(m, \ell).$$

The coefficients of the three terms can now be combined under one k-sum and are just given by $(m;\ \ell, k)$ due to recursion formula (1.5), which completes the induction proof. Our interest is now in the combinatorial meaning of the $(n;\ \ell, k)$ defined by (1.5) with appropriate inputs.

Lemma 1: S_k defined by recursions (1.1a-c) satisfies, for $k \in \mathbb{N}$,

$$S_k = Y(k) \cdots Y(1) - Y(k) \cdots Y(3)S_0 - Y(k) \cdots Y(4)S_1 - \cdots - Y(k)S_{k-3} - S_{k-2}. \tag{3.2}$$

Proof: By induction over $k = 1, 2, \ldots$.

Remark: In (3.2) each of the $k - 1$ terms with a minus sign can be obtained from the first reference term by deletion of one pair of consecutive

$$Y(i + 1)Y(i) \quad \text{for } i \in \{1, 2, \ldots, k - 1\}$$

and by replacement of all $Y(i - 1) \cdots Y(1)$ following to the right by S_{i-1}. So there is a one-to-one correspondence between these $k - 1$ terms and the $k - 1$ different pairs of consecutive numbers that can be picked out of $\{1, 2, \ldots, n\}$.

Notation: The $k - 1$ terms of $S_k - Y(k) \cdots Y(1)$ given by (3.2) are denoted by $[i, i + 1]$, with $i = 1, 2, \ldots, k - 1$. E.g., for $k = 5$, $[3, 4] \equiv -Y(5)S_2$, i.e., $Y(4)$ and $Y(3)$ do not appear.

Lemma 2: S_k of (3.2) consists in all of F_{k+1} terms if all S_i appearing on the right-hand side of (3.2) are iteratively inserted until only products of Y's occur.

Proof: By induction, using $S_0 = 1$ and $1 + \sum_{i=1}^{k-1} F_i = F_{k+1}$.

Definition 1: $Q(n)$ is the set of $F_{n+1} - 1$ elements given by the individual terms of which $S_n - Y(n) \cdots Y(1)$ consists due to Lemma 2.

Definition 2: $P_\ell(n)$, for $\ell \in \{1, 2, \ldots, [n/2]\}$, is the set of ℓ mutually disjoint pairs of consecutive numbers taken out of the set $\{1, 2, \ldots, n\}$.

Lemma 3: The elements of $Q(n)$ are given by

$$q_{\ell,i}(n) := (-1)^\ell Y(n) \cdots \overline{Y(n_{i_\ell} + 1) \cdot Y(n_{i_\ell})} \cdots \overline{Y(n_{i_1} + 1) \cdot Y(n_{i_1})} \cdots Y(1),$$

where the ℓ barred Y-pairs have to be omitted and

$$(n_{i_1}, n_{i_1} + 1), \ldots, (n_{i_\ell}, n_{i_\ell} + 1)$$

is an element of $P_\ell(n)$ for $\ell = 1, 2, \ldots, [n/2]$. The index i numerates the different ℓ pairs:

$$i = 1, 2, \ldots, \binom{n - \ell}{\ell}.$$

Proof: Let $(n_1, n_1 + 1), \ldots, (n_\ell, n_\ell + 1)$ with $n_j > n_{j-1} + 1$ for $j = 2, \ldots, \ell$ be an element of $P_\ell(n)$. Using the Notation, the corresponding element of $Q(n)$ is obtained by picking in the $[n_\ell, n_\ell + 1]$ term of S_n the $[n_{\ell-1}, n_{\ell-1} + 1]$ term of $S_{n_\ell - 1}$ which appears there, and so on, until the $[n_1, n_1 + 1]$ term of $S_{n_2 - 1}$ is reached. If $n_1 = 1$, one arrives at $S_0 = 1$. If $n_1 \geq 2$, one replaces the surviving $S_{n_1 - 1}$ by its first term, i.e., $Y(n_1 - 1) \cdots Y(1)$. In this way, each of the $\binom{n-\ell}{\ell}$ elements of $P_\ell(n)$, distinguished by the label i, is mapped to a different element of $Q(n)$. For all ℓ, there are in all $F_{n+1} - 1$ such elements, and this mapping from $\bigcup_{\ell=1}^{[n/2]} P_\ell$ to $Q(n)$ is one-to-one. It is convenient also to define $q_0 := Y(n) \cdots Y(1)$, which is the first term of S_n.

Lemma 4: (3.3) $\quad q_0 = Y^{z(n)} y^{n-z(n)}$.

Proof: Definition (2.9) of counting sequence $z(n)$.

Lemma 5: The general element $q_{\ell,i}(n) \in Q(n)$ is given by

$$(3.4) \qquad q_{\ell,i}(n) = Y^{z(n)} y^{n-z(n)} \{(-1)^\ell Y^{-(2k+\ell-k)} y^{-(\ell-k)}\},$$

if among the specific choice i of ℓ barred pairs of $q_{\ell,i}(n)$, as written in Lemma 3, k barred pairs are numerated by A-numbers.

Proof: A barred pair $Y(i + 1)Y(i)$ in $q_{\ell,i}(n)$, given in Lemma 3, corresponds to a missing factor $-Y^2$ in $Y(n) \cdots Y(1)$ iff i and $i + 1$ are both A-numbers. In all other cases a factor $-Yy$ is missing. Therefore, the reference term q_0 of (3.3) is changed as stated in (3.4).

Putting these results together, we have proved Proposition 2 given in the first section, because the elements of $Q(n) \cup q_0$ are all the terms of S_n, and the multiplicity of a term with fixed powers of Y and y given in (3.4) is just $(n; \ell, k)$ according to (1.4).

Proposition 1 is equivalent to Proposition 2 because of the characterization of A-numbers in the Fibonacci number system, as described in section 2.

If a pair of consecutive numbers is replaced by its smaller member, Proposition 3 results from either Proposition.

The Corollary follows from Proposition 3 and the Fibonacci representation explained in (1.9). The numbers 1, 2, ..., $n - 1$ indicate the places F_2, F_3, ..., F_n, respectively. In (1.9) $s_{i-1} = 1$ if the number $i \in \{1, 2, \ldots, n - 1\}$ is chosen. If $i = AB(m)$, for some $m \in \mathbb{N}$, the place of

$$F_{AB(m)+1} = F_{A(B(m)+1)}$$

is activated.

Comment: The map used in the proof of Lemma 3 never produces negative powers of Y or y. Thus,

$$\ell - (n - z(n)) \le k \le z(n) - \ell$$

is always obeyed. On the other hand, the $p(n)$ definition shows that

$$0 \le k \le \min(\ell, p(n))$$

has to hold as well. (1.6) gives the intersection of both k ranges.

The main part of this work closes with a collection of explicit formulas concerning the $(n; \ell, k)$ numbers. Here, the results listed in section 2 are used.

A necessary condition is

$$(3.5) \qquad \sum_{k=k_{\min}}^{k_{\max}} (n; \ell, k) = \binom{n - \ell}{\ell},$$

which guarantees $S_n(y, y) = S_n(y)$.

The results for $(n; \ell, k)$ for $\ell = 0, 1, 2$, are:

(3.6) $\underline{\ell = 0}$: $(n; 0, 0) = 1$,

(3.7a) $\underline{\ell = 1}$: $(n; 1, 0) = (n - 1) - p(n)$,

(3.7b) $(n; 1, 1) = p(n)$,

(3.8a) $\underline{\ell = 2}$: $(n; 2, 0) = \binom{p(n)}{2} + p(n - 1) - (n - 3)p(n) + \binom{n - 2}{2}$,

(3.8b) $(n; 2, 1) = (n - 3)p(n) - p(n - 1) - 2\binom{p(n)}{2}$,

(3.8c) $(n; 2, 2) = \binom{p(n)}{2}$.

Already the $\ell = 3$ case becomes quite involved, except for $(n; 3, 3)$, which is a special case of

$$(3.9) \qquad (n; \ell, \ell) = \binom{p(n)}{\ell}, \quad \text{for } n \ge AB(\ell) + 1.$$

This is, from the combinatorial point of view, a trivial formula, which, when derived from the recursion formula, is due to an iterative solution of

$$(n; \ell, \ell) = \sum_{k=0}^{p(n)} (BA(k); \ell - 1, \ell - 1),$$

with input $(BA(k); 0, 0) = 1$.

The last term of $S_{2\ell}$ has just the coefficient

$$(3.10) \qquad (2\ell; \ell, z(2\ell) - \ell) = 1,$$

where the input $(2; 1, 0) = 1$ was used.

Finally, we list some questions that are under investigation:

(i) What do the generating functions for S_n, $\hat{S}_n$ look like?

(ii) Which differential equations do these objects satisfy?

(iii) Are the S_n and $\hat{S}_n$ orthogonal with respect to some measure?

(iv) How does the self-similarity of the $h(n)$ sequence reflect itself in the polynomials S_n and $\hat{S}_n$?

APPENDIX

Physical Applications

The two-variable polynomials introduced in this work are basic for the solution of the discrete one-dimensional Schrödinger equation for a particle of mass m moving in a quasi-periodic potential of the Fibonacci type (see [13] and [17]). The transfer matrix for such a model is given by

$$\text{(A.1)} \qquad R_n := \begin{pmatrix} Y(n), & -1 \\ 1, & 0 \end{pmatrix},$$

with $Y(n)$ defined by (1.1c) and (1.2). $Y = E - V_1$, $y = E - V_0$, where E is the energy (in units of $\hbar^2/2ma^2$, with lattice constant a) and the potential at lattice site n is $V_n := V(n\phi)$ with

$$\text{(A.2)} \qquad V(x) = \begin{cases} V_0 & \text{for } 0 \le x < 2 - \phi \\ V_1 & \text{for } 2 - \phi \le x < 1 \end{cases} \qquad \text{and } V(x + 1) = V(x).$$

The product matrix

$$\text{(A.3)} \qquad M_n := R_n \cdots R_2 R_1,$$

which allows us to compute ψ_n, the particle's wave-function at site number n, in terms of the inputs ψ_1 and ψ_0, according to

$$\text{(A.4)} \qquad \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = M_n \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

turns out to be

$$\text{(A.5)} \qquad M_n = \begin{pmatrix} S_n, & -\hat{S}_{n-1} \\ S_{n-1}, & -\hat{S}_{n-2} \end{pmatrix}.$$

Because of $\det R_n = 1 = \det M_n$, one finds the identity

$$\text{(A.6)} \qquad \hat{S}_n S_n - \hat{S}_{n-1} S_{n+1} = 1,$$

for $n \in \mathbb{N}$, which generalizes a well-known result for ordinary Chebyshev polynomials. It allows to express $\hat{S}_n$ in terms of S_i with $i = 0, 1, \ldots, n + 1$:

$$\text{(A.7)} \qquad \hat{S}_n = \frac{1}{S_n}\left(1 + S_n S_{n+1} \sum_{i=0}^{n-1} \frac{1}{S_i S_{i+1}}\right),$$

This can be proved by induction using

$$\hat{S}_n = \frac{1}{S_n}(1 + S_{n+1}\hat{S}_{n-1}).$$

Another model that leads to the same type of transfer matrices as (A.1) is the Fibonacci chain [2] with harmonic nearest neighbor interaction built from two masses m_0 and m_1 with mass $m_{h(i)}$ at site number i. In this case

$$Y(n) = 2 - (\omega/\omega(n))^2, \text{ with } \omega^2(n) := \kappa/m_{h(n)}.$$

κ is the spring constant and ω the frequency.

One-dimensional quasi-crystal models (see [16], [3]) can be transformed to Schrödinger equations on a regular lattice with quasi-periodic potentials as considered above.

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DISTRIBUTION OF THE FIBONACCI NUMBERS MOD 2^K

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Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, denote the sequence of Fibonacci Numbers. For any modulus $m \geq 2$, and residue b (mod m), denote by $v(m, b)$ the number of occurrences of b as a residue in one (shortest) period of F_n (mod m).

If $m = 5$ with $k > 0$, then F_n (mod 5^k) has shortest period of length $4 \cdot 5^k$, and $v(5^k, b) = 4$ for all b (mod 5^k). This is so-called *uniform distribution*, and has been studied in great detail by a number of authors (e.g., [1], [4], [5], [6]). However, the study of the function $v(m, b)$ for moduli other than 5 is still relatively unexplored. Some recent work in this area can be found in [2] and [3].

In this paper we completely describe the function $v(m, b)$ when $m = 2^k$, $k \geq 1$. What makes this possible is a type of stability that occurs when $k \geq 5$. This stability does not seem to appear for primes other than $p = 2, 5$ (which somehow is not surprising). Of course, the values of $v(2^k, b)$ for $k = 1, 2, 3, 4$ are easily checked by hand. We include these values for completeness.

Main Theorem

For F_n(mod 2^k), with $k \geq 1$, the following data appertain:

For $1 \leq k \leq 4$:

$v(2, 0) = 1$,
$v(2, 1) = 2$,
$v(4, 0) = v(4, 2) = 1$,
$v(8, 0) = v(8, 2) = v(16, 0) = v(16, 8) = 2$,
$v(16, 2) = 4$,
$v(2^k, b) = 1$ if $b \equiv 3 \pmod 4$ and $2 \leq k \leq 4$,
$v(2^k, b) = 3$ if $b \equiv 1 \pmod 4$ and $2 \leq k \leq 4$, and
$v(2^k, b) = 0$ in all other cases, $1 \leq k \leq 4$.

For $k \geq 5$:

$$v(2^k, b) = \begin{cases} 1, & \text{if } b \equiv 3 \pmod 4, \\ 2, & \text{if } b \equiv 0 \pmod 8, \\ 3, & \text{if } b \equiv 1 \pmod 4, \\ 8, & \text{if } b \equiv 2 \pmod{32}, \\ 0, & \text{for all other residues.} \end{cases}$$

Most of our proofs proceed either by induction, or by invoking a standard formula for the Fibonacci sequence. Perhaps there are other proofs of our Theorem, but because of the absence in the literature of a convenient closed form for F_n (mod 2^k), our methodology is quite computational. Because of their frequent use, we record the following two standard formulas.

Addition Formula: If $m \geq 1$ and $n \geq 0$, then

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}.$$

Subtraction Forumla: If $m \geq n > 0$, then

$$F_{m-n} = (-1)^{n+1} \cdot (F_{m-1}F_n - F_mF_{n-1}).$$

The main body of this paper consists in establishing a number of congruences for $F_n \pmod{2^k}$.

Lemma 1: Let $k \geq 5$. Then

$$F_{2^{k-3}\cdot 3-1} \equiv 1 - 2^{k-2} \pmod{2^k},$$
$$F_{2^{k-3}\cdot 3} \equiv 2^{k-1} \pmod{2^{k+1}}$$

Proof: We prove these formulas simultaneously by induction on k. When $k = 5$, the results are easily checked. Now assume the result is true for $k \geq 5$, and write

$$F_{2^{k-3}\cdot 3-1} = 1 - 2^{k-2}u$$
$$F_{2^{k-3}\cdot 3} = 2^{k-1}v$$

where $u, v \equiv 1 \pmod 4$. Note that as $k \geq 5$, we have $(k - 2) + (k - 2) \geq k + 1$, and $(k - 2) + (k - 1) \geq k + 2$. Thus,

$$\begin{aligned} F_{2^{k-2}\cdot 3-1} &= F_{2^{k-3}\cdot 3-1+2^{k-3}\cdot 3} \\ &= F_{2^{k-3}\cdot 3-2}F_{2^{k-3}\cdot 3} + F_{2^{k-3}\cdot 3-1}F_{2^{k-3}\cdot 3+1} \\ &= (2^{k-1}v - 1 + 2^{k-2}u)2^{k-1}v + (1 - 2^{k-2}u)(2^{k-1}v + 1 - 2^{k-2}u) \\ &\equiv -2^{k-1}v + 2^{k-1}v + 1 - 2^{k-2}u - 2^{k-2}u \pmod{2^{k+1}} \\ &\equiv 1 - 2^{k-1} \pmod{2^{k+1}} \end{aligned}$$

and

$$\begin{aligned} F_{2^{k-2}\cdot 3} &= F_{2^{k-3}\cdot 3+2^{k-3}\cdot 3} \\ &= (1 - 2^{k-2}u)2^{k-1}v + 2^{k-1}v(2^{k-1}v + 1 - 2^{k-2}u) \\ &\equiv 2^{k-1}v + 2^{k-1}v \pmod{2^{k+2}} \\ &\equiv 2^k \pmod{2^{k+2}}. \end{aligned}$$

One consequence of this lemma is that $F_n \pmod{2^k}$ has shortest period of length $2^{k-1}\cdot 3$.

Lemma 2: Let $k \geq 5$ and $s \geq 1$. Then,

$$F_{2^{k-3}\cdot 3s-1} \equiv 1 - s\cdot 2^{k-2} \pmod{2^k}, \text{ and}$$
$$F_{2^{k-3}\cdot 3s} \equiv s\cdot 2^{k-1} \pmod{2^k}.$$

Proof: Lemma 1 is the case $s = 1$. Now proceed by induction on s, by applying the addition formula and Lemma 1 to

$$F_{2^{k-3}\cdot 3s-1} = F_{2^{k-3}\cdot 3(s-1)-1+2^{k-3}\cdot 3} \text{ and}$$
$$F_{2^{k-3}\cdot 3s} = F_{2^{k-3}\cdot 3(s-1)+2^{k-3}\cdot 3}.$$

The details are omitted.

Lemma 3: Let $k \geq 5$ and $n \geq 0$. Then,

$$F_{n+2^{k-2}\cdot 3} \equiv \begin{cases} F_n \pmod{2^k} & \text{if } n \equiv 0 \pmod 3, \\ F_n + 2^{k-1} \pmod{2^k} & \text{if } n \equiv 1,\ 2 \pmod 3. \end{cases}$$

Proof: By Lemma 1,

$$F_{2^{k-2}\cdot 3} \equiv 0 \pmod{2^k} \quad\text{and}\quad F_{2^{k-2}\cdot 3-1} \equiv 1 - 2^{k-1} \pmod{2^k}.$$

Thus,

$$\begin{aligned} F_{n+2^{k-2}\cdot 3} &= F_{n-1}F_{2^{k-2}\cdot 3} + F_n(F_{2^{k-2}\cdot 3} + F_{2^{k-2}\cdot 3-1}) \\ &\equiv F_nF_{2^{k-2}\cdot 3-1} \pmod{2^k} \equiv F_n(1 - 2^{k-1}) \pmod{2^k}. \end{aligned}$$

The result follows since F_n is even precisely when $n \equiv 0 \pmod 3$.

In our subsequent work we will frequently have need of the residues of F_n (mod 4) and F_n (mod 6). We record one period of each here, from which the reader can deduce the requisite congruences:

F_n (mod 4): 0, 1, 1, 2, 3, 1
F_n (mod 6): 0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1

Lemma 4: Let $k \geq 5$ and $n \geq 0$ and assume $n \equiv 0 \pmod 6$. Then,

$$F_{n+2^{k-3}\cdot 3} \equiv F_n + 2^{k-1} \pmod{2^k}.$$

Proof: Analogous to the previous proof. Note that $n \equiv 0 \pmod 6$ if and only if $F_n \equiv 0 \pmod 4$.

Lemma 5: If $n \equiv 3 \pmod 6$, then $F_n \equiv 2 \pmod{32}$.

Proof: Write $n = 6t + 3$ with $t \geq 0$; use induction on t together with an application of the addition formula to $F_{6(t+1)+3} = F_{6(t+3)+6}$.

Lemma 6: If $n \equiv 3 \pmod 6$ and $k \geq 5$, then for all $s \geq 1$,

$$F_{2^{k-3}\cdot 3s \pm n} \equiv F_n \pmod{2^k}.$$

Proof: We treat the two cases $\pm$ separately.

Case +:

$$\begin{aligned} F_{2^{k-3}\cdot 3s+n} &= F_{2^{k-3}\cdot 3s-1}F_n + F_{2^{k-3}\cdot 3s}F_{n+1} \\ &\equiv (1 - s\cdot 2^{k-2})F_n + s\cdot 2^{k-1} \pmod{2^k} \\ &\equiv F_n - s\cdot 2^{k-1} + s\cdot 2^{k-1} \pmod{2^k} \\ &\equiv F_n \pmod{2^k}. \end{aligned}$$

Case −: Of course, we are tacitly assuming $2^{k-3}\cdot 3s - n > 0$. We use the subtraction formula

$$\begin{aligned} F_{2^{k-3}\cdot 3s-n} &= (-1)^{n+1}\cdot (F_{2^{k-3}\cdot 3s-1}F_n - F_{2^{k-3}\cdot 3s}F_{n-1}) \\ &\equiv (1 - s\cdot 2^{k-2})F_n - s\cdot 2^{k-1}F_{n-1} \pmod{2^k} \\ &\equiv F_n - s\cdot 2^{k-1} - s\cdot 2^{k-1} \pmod{2^k} \\ &\equiv F_n \pmod{2^k}. \end{aligned}$$

Lemma 7: If $n \equiv 3 \pmod 6$ and $k \geq 6$, then,

$$F_{n+2^{k-4}\cdot 3} \equiv F_n + 2^{k-1} \pmod{2^k}.$$

Proof: By Lemma 1, write

$$F_{2^{k-4}\cdot 3} = 2^{k-2}\cdot u \quad \text{and} \quad F_{2^{k-4}\cdot 3-1} = 1 - 2^{k-3}\cdot v,$$

where $u, v \equiv 1 \pmod 4$. Then, by the addition formula and Lemma 5,

$$\begin{aligned} F_{n+2^{k-4}\cdot 3} &= F_{n-1}\cdot 2^{k-2}u + F_n(2^{k-2}u + 1 - 2^{k-3}v) \\ &\equiv 2^{k-2}u + 2^{k-1}u + F_n - 2^{k-2}v \pmod{2^k} \\ &\equiv 2^{k-2} + 2^{k-1} + F_n - 2^{k-2} \pmod{2^k} \\ &\equiv 2^{k-1} + F_n \pmod{2^k}. \end{aligned}$$

Proof of the Main Theorem

We proceed by induction on $k \geq 5$. The result is easily checked for $k = 5$, so assume $k \geq 5$ and the Theorem holds for k.

First, if $b \equiv 4, 6, 10, 12, 14, 18, 20, 22, 26, 28, 30 \pmod{32}$, then it is clear that $v(2^{k+1}, b) = 0$ since $v(2^5, b) = 0$.

<u>Case 1</u>: $b \equiv 3 \pmod 4$. Then $v(2^k, b) = 1$, so choose n such that $F_n \equiv b \pmod{2^k}$. Since b is odd, we have $n \equiv 1, 2 \pmod 3$. Now either $F_n \equiv b \pmod{2^{k+1}}$ or $F_n \equiv b + 2^k \pmod{2^{k+1}}$. In the latter case, Lemma 3 gives

$$F_{n+2^{k-1}\cdot 3} \equiv F_n + 2^k \pmod{2^{k+1}} \equiv b + 2^k + 2^k \pmod{2^{k+1}}$$
$$\equiv b \pmod{2^{k+1}}.$$

Therefore, $v(2^{k+1}, b) \geq 1$ when $b \equiv 3 \pmod 4$.

<u>Case 2</u>: $b \equiv 1 \pmod 4$. Then $(2^k, b) = 3$, so choose

$$0 < n_1 < n_2 < n_3 < 2^{k-1}\cdot 3,$$

with $F_{n_i} \equiv b \pmod{2\ }$ for all i. Then, as above, for each i, either

$$F_{n_i} \equiv b \pmod{2^{k+1}} \quad \text{or} \quad F_{n_i+2^{k-1}\cdot 3} \equiv b \pmod{2^{k+1}}.$$

So, $v(2^{k+1}, b) \geq 3$ when $b \equiv 1 \pmod 4$.

<u>Case 3</u>: $b \equiv 0 \pmod 8$. Then $v(2^k, b) = 2$, so let

$$0 < m < n < 2^{k-1}\cdot 3$$

be such that $F_m \equiv F_n \equiv b \pmod{2^k}$. Note that as $F_m \equiv F_n \equiv 0 \pmod 4$, we have $m \equiv n \equiv 0 \pmod 6$, so Lemma 4 applies. In particular,

$$F_{m+2^{k-2}\cdot 3} \equiv F_m \pmod{2^k},$$

from which it follows that $m < 2^{k-2}\cdot 3$ and $n = m + 2^{k-2}\cdot 3$.

If $F_m \equiv b \pmod{2^{k+1}}$, then by Lemma 3,

$$F_{m+2^{k-1}\cdot 3} \equiv b \pmod{2^{k+1}},$$

so $v(2^{k+1}, b) \geq 2$. Otherwise, we must have

$$F_m \equiv b + 2^k \pmod{2^{k+1}}.$$

But then by Lemma 4,

$$F_n = F_{m+2^{k-2}\cdot 3} \equiv F_m + 2^k \pmod{2^{k+1}} \equiv b \pmod{2^{k+1}},$$

and also,

$$F_{n+2^{k-1}\cdot 3} \equiv F_n \equiv b \pmod{2^{k+1}}.$$

We conclude that $v(2^{k+1}, b) \geq 2$ when $b \equiv 0 \pmod 8$.

<u>Case 4</u>: $b \equiv 2 \pmod{32}$. Assume that $v(2^k, b) = 8$. Let $F_n \equiv b \pmod{2^k}$, with $n < 2^{k-1}\cdot 3$. Then $F_n \equiv 2 \pmod 4$, so that $n \equiv 3 \pmod 6$. Now either

$$F_n \equiv b \pmod{2^{k+1}} \quad \text{or} \quad F_n \equiv b + 2^k \pmod{2^{k+1}}.$$

In the latter case, by Lemma 7 we have

$$F_{n+2^{k-3}\cdot 3} \equiv F_n + 2^k \equiv b \pmod{2^{k+1}}.$$

Thus, there is at least one index $0 < m < 2^k\cdot 3$ such that $F_m \equiv b \pmod{2^{k+1}}$. But now, by Lemma 6,

$$F_{2^{k-2}\cdot 3s\pm m} \equiv F_m \equiv b \pmod{2^{k+1}} \text{ for } s = 4, 5, 6, 7.$$

Since these eight solutions all occur in one period of $F_n \pmod{2^{k+1}}$, we conclude that $v(2^{k+1}, b) \geq 8$.

<u>Conclusion</u>: We have established inequalities in each case of the Theorem. The proof follows from a straightforward computation, using the fact that F_n has shortest period of length $2^k\cdot 3$ modulo 2^{k+1}, and the obvious identity:

$$\sum_{b \,(\mathrm{mod}\ 2^{k+1})} v(2^{k+1}, b) = 2^k \cdot 3.$$

Using the main Theorem of [2], we are now able to describe the distribution of $F_n \pmod{2^k \cdot 5^j}$. Indeed,

Theorem: For $F_n \pmod{2^k \cdot 5^j}$ with $k \geq 5$ and $j \geq 0$, we have

$$v(2^k \cdot 5^j, b) = \begin{cases} 1, & \text{if } b \equiv 3 \pmod 4, \\ 2, & \text{if } b \equiv 0 \pmod 8, \\ 3, & \text{if } b \equiv 1 \pmod 4, \\ 8, & \text{if } b \equiv 2 \pmod{32}, \\ 0, & \text{for all other residues.} \end{cases}$$

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THE GOLDEN-FIBONACCI EQUIVALENCE

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1. Introduction

The main purpose of this paper is to show the equivalence between the Golden Number and the Fibonacci Number line-sequential vector spaces. The conventions are the same as those adopted in our previous paper [6].

2. The Golden Number Line-Sequences

We shall consider the following special irrational number line-sequences:

(2.1) $F_{1,A} = F_{1,0} + AF_{0,1}$,

(2.2) $F_{1,B} = F_{1,0} + BF_{0,1}$,

where (see [6], (4.5) and (4.6):

(2.3) $A = (1 + 5^{1/2})/2$,

(2.4) $B = (1 - 5^{1/2})/2$,

(2.5) $AB = -1$, $A + B = 1$.

We shall refer to A and B as the large and the small Golden Ratios, respectively, and shall in general simply refer to these and their powers collectively as Golden Numbers.

Likewise, the ratio between the neighboring Fibonacci Numbers u_{n+1}/u_n will be called the large Fibonacci Ratio. Here, "large" means that the suffices $n + 1 > n$, without inference to the values of the u's or their ratio. Its negative reciprocal will be referred to as the small Fibonacci ratio.

The line-sequences (2.1) and (2.2) are found to be:

(2.6) $F_{1,A}$: $\ldots, A^{-3}, A^{-2}, A^{-1}, 1, A^1, A^2, A^3, \ldots;$

(2.7) $F_{1,B}$: $\ldots, B^{-3}, B^{-2}, B^{-1}, 1, B^1, B^2, B^3, \ldots .$

These are none other than a pair of divergent and convergent geometrical progressions of Golden Numbers. Henceforth, we shall refer to these two line-sequences simply as the Golden Pair. Correspondingly, the pair $F_{1,0}$ and $F_{0,1}$ will be referred to as the Fibonacci Pair.

A number of mathematical curios now begin to reveal their origin in this light.

a. On inspection of the line-sequences (2.6) and (2.7), it is obvious that Binet's formula can be obtained independently from the Golden Pair without following through the conventional algebraic derivation [7]. This is done in (4.9) below. Furthermore, as is well known, the large Fibonacci Ratio approaches the large Golden Ratio as a limit (see [7], p. 53); that is,

(2.8) $\lim u_{n+1}/u_n = A$.

b. For an arbitrary line-sequence, it has been suggested (see [2] and [3]) that the same limit as (2.8) also exists between a neighboring pair, and examples are given for $F_{1,4}$ and $F_{2,1}$. This obviously cannot be true in general, as is evidenced by the counterexample of (2.7).

3. The Golden-Fibonacci Space

By (2.8) of [6] and (2.5) above, it is obvious that both terms in the Golden Pair $F_{1,A}$ and $F_{1,B}$ are orthogonal. Hence, they form a pair of basis vectors which, like the Fibonacci Pair $F_{1,0}$ and $F_{0,1}$, spans the same 2-dimensional line-sequential vector space. Vectorally, therefore, the Golden Pair and the Fibonacci Pair are equivalent. Any line-sequence in this vector space can be expressed in terms of either of these two sets of basis vectors. In particular, the Lucas line-sequence can be expressed simply as

(3.1) $$F_{2,1} = F_{1,A} + F_{1,B}.$$

4. Some Basic Properties of the Golden Pair

Now we shall investigate some of the basic properties of the Golden Pair.

a. We note that the Golden Pair are not unit vectors; hence, they are an orthogonal but not an orthonormal pair. Their lengths are, respectively,

(4.1) $$L_{1,A} = (2 + A)^{1/2} = 1.90211\ldots,$$

(4.2) $$L_{1,B} = (2 + B)^{1/2} = 1.17557\ldots,$$

which are again in the ratio of

(4.3) $$L_{1,A}/L_{1,B} = A.$$

b. We shall investigate the linear properties of the Golden Pair. We define the following column vectors and 2×2 matrices:

(4.4) $$F = \begin{bmatrix} F_{1,0} \\ F_{0,1} \end{bmatrix}, \qquad G = \begin{bmatrix} F_{1,B} \\ F_{1,A} \end{bmatrix},$$

(4.5) $$M = \begin{bmatrix} 1 & B \\ 1 & A \end{bmatrix}, \qquad M^{-1} = (A - B)^{-1}\begin{bmatrix} A & -B \\ -1 & 1 \end{bmatrix};$$

where

(4.6) $$M^{-1}M = MM^{-1} = I,$$

(4.7) $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we have

(4.8) $$MF = G,$$

(4.9) $$M^{-1}G = F,$$

where the second element of (4.9) is just Binet's formula, as we have mentioned in Section 2a; and the transformation M is no more than a rotation followed by a dilation, or vice versa.

Also we have, for the lengths of the Golden Pair, the following linear transformation:

(4.10) $$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} L_{1,A}^2 \\ L_{1,B}^2 \end{bmatrix}.$$

c. The geometrical interpretation of the foregoing results is simple. Let $F_{1,0}$ and $F_{0,1}$ be the two unit vectors along the x- and y-axes, respectively. Then the Golden Pair $F_{1,A}$ and $F_{1,B}$ and the Lucas vector $F_{2,1}$ can be easily constructed as shown in Figure 1.

It is seen from the diagram that the angle of rotation from $F_{1,0}$ to $F_{1,B}$ is simply

(4.11) $$\tan^{-1}B = -31.72°.$$

Also see p. 164 of [4] or (1.6) of [1].

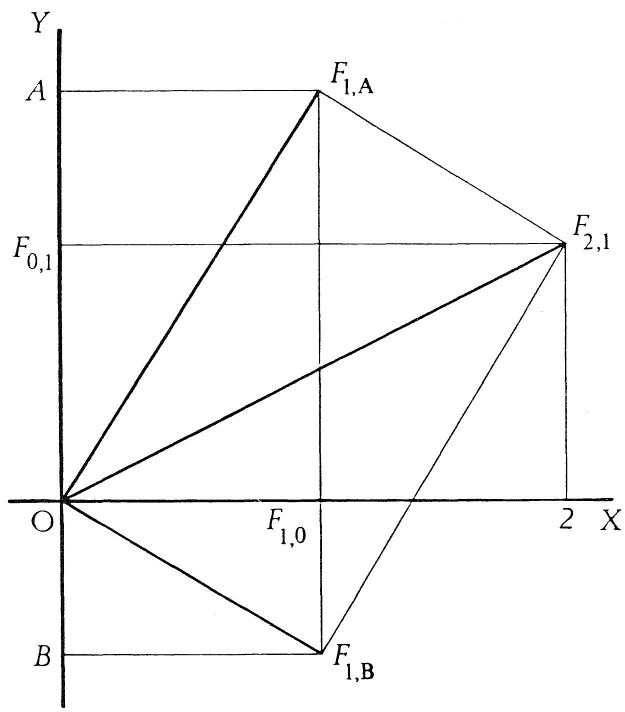

Figure 1

Geometrical Interpretation of the Fibonacci Pair and the Golden Pair

Furthermore, it is clear from (2.8) that the direction of $F_{1,A}$ is that of an asymptote toward which the Fibonacci vectors approach hyperbolically as the limit; while the vector $F_{1,B}$ lies in the direction of the other asymptote, perpendicular to the former, and alternately toward both directions of which the Fibonacci vectors recede as the limits. The Fibonacci vectors approach their limits in three different directions (see Figure 2).

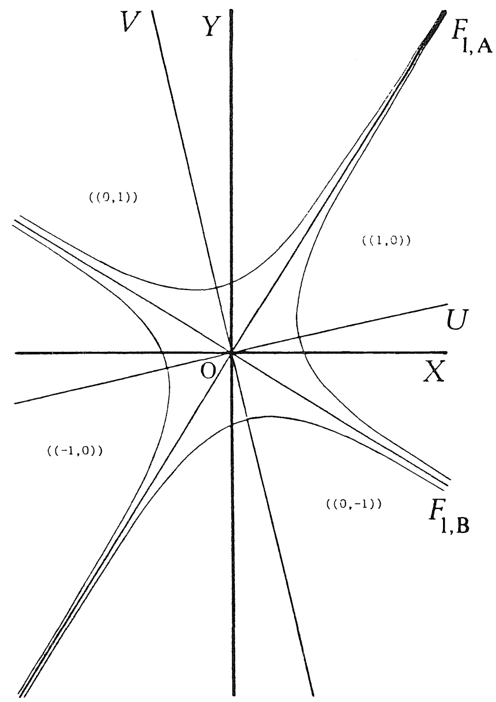

Figure 2

U, *V* are the symmetry axes; $F_{1,A}$ and $F_{1,B}$ are the asymptotes. (Drawing is not to scale.)

Up to now, the investigation on the properties of the Fibonacci hyperbolas has been based on the ray-sequence instead of the line-sequence; thus, information about the negative branch of the line-sequence has been left out [5]. When the same procedure is applied to the entire line-sequence, a complete picture emerges. For instance, on a branch of the hyperbola

(4.12) $x^2 + xy - y^2 - 1 = 0$

lie the following set of Fibonacci points:

(4.13) ((1, 0)):

$$\ldots, (5, -3), (2, -1), (1, 0), (1, 1), (2, 3), (5, 8) \ldots;$$

and on a branch of the complementary hyperbola

(4.14) $x^2 + xy - y^2 + 1 = 0$

lie the complementary set of Fibonacci points

(4.15) ((0, 1)):

$$\ldots, (-8, 5), (-3, 2), (-1, 1), (0, 1), (1, 2), (3, 5) \ldots .$$

The two sets ((1, 0)) and ((0, 1)) make up all the neighboring pairs in the line-sequence. The remaining two branches are occupied by the sets ((-1, 0)) and ((0, -1)) of the negative Fibonacci line-sequence, as shown in Figure 2. This analysis also reveals that the parity axes (see [6], Fig. 1) correspond to the symmetry axes *U* and *V*, rather than *X* and *Y*.

Acknowledgment

The author wishes to express his gratitude for the hospitality he received during the Fourth International Conference on Fibonacci Numbers and Their Applications, 1990.

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Author and Title Index for *The Fibonacci Quarterly*

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for *The Fibonacci Quarterly*. In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of *The Fibonacci Quarterly*. Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in *The Fibonacci Quarterly*. We would deeply appreciate it if all authors of articles published in *The Fibonacci Quarterly* would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

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The Editor
Gerald E. Bergum

ARMSTRONG NUMBERS: $153 = 1^3 + 5^3 + 3^3$

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A number N is an *Armstrong number of order* n (n being the number of digits) if

$$abcd\ldots = a^n + b^n + c^n + d^n + \cdots = N.$$

The number 153 is an Armstrong number of order 3 because

$$1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153.$$

Likewise, 54748 is an Armstrong number of order 5 because

$$5^5 + 4^5 + 7^5 + 4^5 + 8^5 = 3125 + 1024 + 16807 + 1024 + 32768 = 54748.$$

More generally, an n-digit number in base b is said to be a *base b Armstrong number of order n* if it equals the sum of the n^{th} powers of its base b digits. In all bases, we disregard the trivial cases where $n = 1$.

A literature search revealed very little about Armstrong numbers. This set of numbers is occasionally mentioned in the literature as a number-theoretic problem for computer solution (see Spencer [1]). Only third- and fourth-order Armstrong numbers in base ten were noted. The library search did not disclose the identity of Armstrong or any circumstances relating to the discovery of this special set of numbers. Some authors have used the term *Perfect Digital Invariant* to describe these same numbers.

We wrote a computer program to find all decimal Armstrong numbers of orders 1 through 9 simply by testing each integer for the desired property. Table 1 lists the results. It is interesting to note that there are no decimal second-order Armstrong numbers. For orders 3 through 9, there are either three or four Armstrong numbers with one exception: 548,834 is the only Armstrong number of order 6.

Table 1

Decimal Armstrong Numbers Less than One Billion

153	1741725
370	4210818
371	9800817
407	9926315
1634	24678050
8208	24678051
9474	88593477
54748	146511208
92727	472335975
93084	534494836
548834	912985153

Is the set of Armstrong numbers in any base infinite? Consider the number N of n digits in base b. Then $N \geq b^{n-1}$. The Armstrong sum, AS, $\leq n(b - 1)^n$.

Since

$$\frac{N}{AS} \geq \frac{b^{n-1}}{n(b-1)^n} = \frac{1}{bn}\left(\frac{b}{b-1}\right)^n,$$

and this tends to infinity as n increases, $N > AS$ for all sufficiently large values of n. Therefore, there are only finitely many Armstrong numbers in any base.

As an example, suppose we have a three-digit number in base two; that is, b is 2 and n is 3. Then $N \geq 2^2 = 4$ and $AS \leq 3(1)^2 = 3$. Therefore, it is not possible to have an Armstrong number in base two with three or more digits. The highest numbers that need to be tested to be sure of having all base-two Armstrong numbers would be the two-digit numbers. Excluding the trivial case of one-digit numbers, the only numbers that need to be tested are 10_{two} and 11_{two}, neither of which is an Armstrong number, since $1^2 + 0^2$ does not equal 10_{two} and $1^2 + 1^2$ does not equal 11_{two}. Therefore, there are no Armstrong numbers in base two.

Similarly, in base three: If $n = 8$, we have $N \geq 3^7 = 2187$ and $AS \leq 8(2)^8 = 2048$. Therefore, it is impossible to have a base-three Armstrong number of more than seven digits. One need only check the base-three numbers up to and including those of seven digits to be assured of having all Armstrong numbers in base three. It can be shown, in like manner, that $n = 13$ is sufficient to obtain all the Armstrong numbers in base four. Table 2 lists all the Armstrong numbers in bases three and four.

Table 2

All Armstrong Numbers in Base Three and Base Four

Base	3	4
	12	130
	22	131
	122	203
		223
		313
		332
		1103
		3303

The maximum number of digits that must be checked in any base grows rapidly as the base increases, and it becomes cumbersome to test all integers up to the theoretical maximum for the Armstrong property. Table 3 lists the bases from two to twenty and the maximum number of digits of the integers that would need to be checked to find all Armstrong numbers in that base.

Table 3

Maximum Number of Digits that Must Be Checked in Each Base To Obtain All Armstrong Numbers

Base	Maximum Digits	Base	Maximum Digits
2	2	12	78
3	7	13	87
4	13	14	97
5	20	15	106
6	28	16	116
7	35	17	126
8	43	18	136
9	52	19	146
10	60	20	156
11	69		

Our computer program for bases five through nine searched all the numbers from one to one billion. The results are reported in Table 4. Note that in base five there are no Armstrong numbers with 5, 7, 8, 10, 11, or 12 digits. There is a 13-digit Armstrong number and the computer search was terminated before 14-digit base-five numerals since such a number is greater than one billion.

Table 4

Armstrong Numbers in Bases 5 through 9, Less than One Billion

Base 5	6	7	8	9
23	243	13	24	45
33	514	34	64	55
103	14340	44	134	150
433	14341	63	205	151
2124	14432	250	463	570
2403	23520	251	660	571
3134	23521	305	661	2446
124030	44405	505	40663	12036
124031	435152	12205	42710	12336
242423	5435254	12252	42711	14462
434434444	12222215	13350	60007	2225764
1143204434402	555435035	13351	62047	6275850
		15124	636703	6275851
		36034	3352072	12742452
		205145	3352272	356614800
		1424553	3451473	356614801
		1433554	4217603	1033366170
		3126542	7755336	1033366171
		4355653	16450603	1455770342
		6515652	63717005	
		125543055	233173324	
		161340144	3115653067	
		254603255	4577203604	
		336133614		
		542662326		
		565264226		
		13210651042		
		13213642035		
		13261421245		
		23662020022		

Other interesting observations from Table 4: In the range tested, there are more base-seven Armstrong numbers than there are for any other base; there are more base-eight Armstrong numbers than there are base-ten Armstrong numbers; in bases six, seven, and eight, there are no four-digit Armstrong numbers.

Armstrong numbers provide intriguing mathematical recreation. Elementary students could be asked to find Armstrong numbers in base two, base eight, or any other nondecimal base. This activity would provide practice in the operations of addition and multiplication in these bases, and lead to a better understanding of nondecimal numbers. High school students could be challenged to write computer programs which would output Armstrong numbers in any base. This latter activity affords an excellent opportunity to discuss program efficiency, since students will likely find that their programs, though logically correct, will not go beyond numbers with only a few digits before becoming overwhelmed by time-consuming calculations. The rate at which the computing time grows as a function of the number of digits, which is an important characteristic of a computer algorithm, can be introduced here.

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WARING'S FORMULA, THE BINOMIAL FORMULA, AND GENERALIZED FIBONACCI MATRICES

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1. Introduction

Fibonacci matrices are square matrices the entries of the successive powers of which are related to Fibonacci numbers: the most celebrated among them is the 2-by-2 Q-matrix [1].

In previous papers (e.g., see [3] and [6]) properties of the *generalized Fibonacci Q-matrix*, denoted by M and defined as

$$(1.1)\quad M = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix} \quad (m \text{ a positive integer}),$$

have been used to evaluate infinite sums involving the *generalized Fibonacci* (U_n) and *Lucas* (V_n) *numbers*

$$(1.2)\quad U_n = mU_{n-1} + U_{n-2}, \quad (U_0 = 0,\ U_1 = 1),$$

$$(1.3)\quad V_n = mV_{n-1} + V_{n-2}, \quad (V_0 = 2,\ V_1 = m).$$

Note that when $m = 1$, M is the Q-matrix of [1] so that U_n and V_n are the traditional Fibonacci and Lucas numbers.

The aim of this paper is to show how, using M, M^{-1}, and some other matrices related to M, we can evaluate a variety of finite sums involving U_n and/or V_n. The underlying idea consists in using 2-by-2 *commuting* Fibonacci matrices (say, A and B) so that the matrix analogues of the *binomial formula*

$$(1.4)\quad (A + B)^n = \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j}$$

and of the *Waring formula* (e.g., see [2], formula (1.2))

$$(1.5)\quad A^n + B^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (AB)^j (A + B)^{n-2j} \quad (n > 0),$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function, remain valid. The Fibonacci-type identities are then established by equating the corresponding entries of the matrices on the right-hand side (rhs) and left-hand side (lhs) of (1.4) and (1.5). Most of the identities worked out in this paper as examples of the use of this technique are believed to be new.

Throughout this paper, boldface letters always denote matrices; for example, I denotes the 2-by-2 identity matrix.

2. Definitions

First, we recall that the numbers U_n and V_n can be expressed in closed form by means of the *Binet forms*

$$(2.1)\quad U_n = (\alpha^n - \beta^n)/\Delta$$

and

$$(2.2)\quad V_n = \alpha^n + \beta^n,$$

Work carried out in the framework of the agreement between the Italian PT Administration and the Fondazione "Ugo Bordoni."

where

$$(2.3)\qquad \begin{cases} \Delta = \sqrt{m^2 + 4} \\ \alpha = (m + \Delta)/2 \\ \beta = (m - \Delta)/2. \end{cases}$$

Observe that

$$(2.4)\qquad U_{n-1} + U_{n+1} = V_n;$$

identity (2.4) will be widely used throughout the algebraic manipulations without specific reference.

Then, we recall that (e.g., see [6])

$$(2.5)\qquad M^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix} \quad (n \geq 0)$$

and

$$(2.6)\qquad M^{-1} = M - mI = \begin{bmatrix} 0 & 1 \\ 1 & -m \end{bmatrix}.$$

From formula (2.32) of [6], it is readily seen that

$$(2.7)\qquad (M^{-1})^n = M^{-n} = (-1)^n \begin{bmatrix} U_{n-1} & -U_n \\ -U_n & U_{n+1} \end{bmatrix} \quad (n \geq 0).$$

Finally, let us define the following 2-by-2 matrices.

$$(2.8)\qquad H = R(1) = M + M^{-1} = \begin{bmatrix} m & 2 \\ 2 & -m \end{bmatrix} = 2M - mI,$$

$$(2.9)\qquad M^n + M^{-n} = R(n) = \begin{cases} V_n I & (n \text{ even}), \\ U_n H & (n \text{ odd}). \end{cases}$$

Using formulas (2.24)-(2.27) in [6] and taking into account that the eigenvalues of H are $\lambda_1 = \Delta$ and $\lambda_2 = -\Delta$, after some simple manipulations we obtain

$$(2.10)\qquad H^n = \begin{cases} \Delta^n I & (n \text{ even}) \\ \Delta^{n-1} H & (n \text{ odd}) \end{cases}$$

and

$$(2.11)\qquad H^{-n} = H^n/\Delta^{2n}.$$

Since the matrices (2.5)-(2.11) are polynomials in M, they commute so that they can replace A and B in (1.4) and (1.5) above and can be used to obtain Fibonacci-type identities, as will be shown in sections 3 and 4.

3. Use of the Binomial Formula

In this section we give some examples of the use of the matrices defined in section 2 in connection with (1.4).

Example 1: Using (2.6), we can write

$$(3.1)\qquad M^n = (M^{-1} + mI)^n = \sum_{j=0}^{n} \binom{n}{j} M^{-j} (mI)^{n-j} = m^n \sum_{j=0}^{n} \binom{n}{j} (mM)^{-j}.$$

Equating the upper right-hand entries of the matrices on the lhs and rhs of (3.1), by (2.5) and (2.7), we can write

$$\sum_{j=0}^{n} \binom{n}{j} \left(-\frac{1}{m}\right)^j U_j = -U_n/m^n$$

whence, recalling that $U_0 = 0$,

$$(3.2)\qquad \sum_{j=1}^{n-1}\binom{n}{j}\left(-\frac{1}{m}\right)^j U_j = \begin{cases} -2U_n/m^n & (n \text{ even}),\\ 0 & (n \text{ odd}).\end{cases}$$

Example 2: Using (2.8) and (2.6) and omitting the intermediate steps, we can write

$$H^n = (2M - mI)^n = (-m)^n \sum_{j=0}^{n}\binom{n}{j}\left(-\frac{2}{m}\right)^j M^j.$$

Using reasoning similar to the preceding [cf. (2.5) and (2.10)] we obtain

$$(3.3)\qquad \sum_{j=1}^{n}\binom{n}{j}\left(-\frac{2}{m}\right)^j U_j = \begin{cases} 0 & (n \text{ even}),\\ -2\Delta^{n-1}/m^n & (n \text{ odd}).\end{cases}$$

Example 3: From (2.9), we have

$$R^k(n) = \sum_{j=0}^{k}\binom{k}{j} M^{n(2j-k)}$$

whence, after some manipulations, we can write

$$R^k(n) = \begin{cases} \binom{k}{k/2} I + \sum_{j=0}^{(k-2)/2}\binom{k}{j} R(n(k-2j)) & (k \text{ even}),\\ \sum_{j=0}^{(k-1)/2}\binom{k}{j} R(n(k-2j)) & (k \text{ odd}).\end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (3.4) and taking (2.9) and (2.10) into account, we obtain

$$(3.5)\qquad \sum_{j=0}^{(k-2)/2}\binom{k}{j} V_{n(k-2j)} = V_n^k - \binom{k}{k/2} \quad (k \text{ and } n \text{ even}),$$

$$(3.6)\qquad \sum_{j=0}^{(k-2)/2}\binom{k}{j} V_{n(k-2j)} = U_n^k\Delta^k - \binom{k}{k/2} \quad (k \text{ even, } n \text{ odd}),$$

$$(3.7)\qquad \sum_{j=0}^{(k-1)/2}\binom{k}{j} V_{n(k-2j)} = V_n^k \quad (k \text{ odd, } n \text{ even}),$$

$$(3.8)\qquad \sum_{j=0}^{(k-1)/2}\binom{k}{j} U_{n(k-2j)} = U_n^k\Delta^{k-1} \quad (k \text{ and } n \text{ odd}).$$

Example 4: From (2.6), let us write

$$(3.9)\qquad m^n I = (M - M^{-1})^n = (-1)^n \sum_{j=0}^{n}\binom{n}{j}(-1)^j M^{2j-n}$$

$$= \begin{cases} \binom{n}{n/2}(-1)^{n/2} I + \sum_{j=0}^{(n-2)/2}\binom{n}{j}(-1)^j R(n-2j) & (n \text{ even}),\\ \sum_{j=0}^{(n-1)/2}\binom{n}{j}(-1)^j [M^{n-2j} - M^{-(n-2j)}] & (n \text{ odd}).\end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (3.9) and taking (2.9), (2.5), and (2.7) into account, yields

(3.10) $$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{j}(-1)^j V_{n-2j} = \begin{cases} m^n - \binom{n}{n/2}(-1)^{n/2} & (n \text{ even}), \\ m^n & (n \text{ odd}). \end{cases}$$

Example 5: Now let us consider the matrix

(3.11) $$HM^n = \begin{bmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{bmatrix} \quad \text{[from (2.5) and (2.8)]}$$

and recall [see (2.10)] that

(3.12) $$(HM)^n = H^n M^n = \begin{cases} \Delta^n M^n & (n \text{ even}), \\ \Delta^{n-1} HM^n & (n \text{ odd}). \end{cases}$$

From (2.8) we have $HM = M^2 + I$ so that we can write

(3.13) $$(HM)^n = (M^2 + I)^n = \sum_{j=0}^{n} \binom{n}{j} M^{2j}.$$

Equating the upper-right entries of the matrices on the lhs and rhs of (3.13) and taking (3.12), (2.5), and (3.11) into account gives

(3.14) $$\sum_{j=0}^{n} \binom{n}{j} U_{2j} = \begin{cases} \Delta^n U_n & (n \text{ even}), \\ \Delta^{n-1} V_n & (n \text{ odd}). \end{cases}$$

Moreover, from (2.11) and (2.7), after some manipulations it can be seen that

(3.15) $$(HM^n)^{-1} = H^{-1}M^{-n} = \frac{(-1)^n}{\Delta^2} \begin{bmatrix} -V_{n-1} & V_n \\ V_n & -V_{n+1} \end{bmatrix}.$$

From (3.11) and (3.15), equating the upper-left entries of the matrices on the lhs and rhs of the matrix equation $(HM^n)^{-1}HM^n = I$, yields

(3.16) $$V_n^2 - V_{n-1}V_{n+1} = \Delta^2(-1)^n.$$

Finally, from (3.11) and (2.7) let us write

(3.17) $$H = (-1)^n \begin{bmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{bmatrix} \begin{bmatrix} U_{n-1} & -U_n \\ -U_n & U_{n+1} \end{bmatrix}.$$

If we equate the entries of H and those of the matrix product on the rhs of (3.17), by (2.8) we can write

(3.18) $U_{n-1}V_{n+1} - U_nV_n = m(-1)^n$ (upper-left entry),

(3.19) $U_{n+1}V_n - U_nV_{n+1} = 2(-1)^n$ (upper-right entry).

From (3.18) and (3.19), the following identities involving *Pell numbers* (P_n, i.e., U_n with $m = 2$) and *Pell-Lucas numbers* (Q_n, i.e., V_n with $m = 2$) [5] can be immediately established:

(3.20) $$\begin{cases} Q_{n+1}(P_{n-1} + P_n) = Q_n(P_n + P_{n+1}) \\ P_n(Q_{n+1} - Q_n) = P_{n+1}Q_n - P_{n-1}Q_{n+1}. \end{cases}$$

4. Use of the Waring Formula

In this section we give a few examples of use of the matrices defined in section 2 in connection with (1.5). Some simple congruencial properties of the numbers U_n and V_n are then established on the basis of the identity obtained in the first of the given examples.

Example 6: By (1.5) and (2.8) we can write

$$(4.1)\qquad R(n) = M^n + M^{-n} = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} H^{n-2j} \quad (n > 0).$$

By equating the upper-left entries on the lhs and rhs of (4.1) and taking (2.9) and (2.10) into account, we obtain a rather curious formula valid [see (1.5)] for $n > 0$, namely

$$(4.2)\qquad \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} (\Delta^2)^{\lfloor n/2 \rfloor - j} = \begin{cases} V_n & (n \text{ even}), \\ U_n & (n \text{ odd}). \end{cases}$$

The curiousness of (4.2) lies in the fact that analogous formulas (e.g., see [4] formulas (1.6) and (1.7)) give *separately* all numbers U_n and V_n (independently of the parity of n).

Example 7: By (4.1) let us write

$$(4.3)\qquad M^{2n} + I = M^n(M^n + M^{-n}) = M^n R(n)$$
$$= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} M^n H^{n-2j} \quad (n > 0)$$

and observe [see (2.5), (2.8), and (2.10)] that the upper-left entry x_{11} of $M^n H^{n-2j}$ is

$$(4.4)\qquad x_{11} = \begin{cases} U_{n+1}\Delta^{n-2j} & (n \text{ even}), \\ V_{n+1}\Delta^{n-1-2j} & (n \text{ odd}). \end{cases}$$

Equating the upper-left entries of the matrices on the lhs and rhs of (4.3) and taking (2.5) and (4.4) into account gives an identity the lhs of which is the same as that of (4.2), while its rhs equals $U_{2n+1} + 1$ divided by either U_{n+1} (n even) or V_{n+1} (n odd). Comparing these identities with (4.2) yields

$$(4.5)\qquad U_{2n+1} + 1 = \begin{cases} U_{n+1}V_n & (n \text{ even}), \\ U_n V_{n+1} & (n \text{ odd}). \end{cases}$$

Observe that (4.5) also holds for $n = 0$.

Example 8: Let us replace A by mM and B by I in (1.5) and take into account that, from (1.1), (1.2), and (2.5), we have

$$(4.6)\qquad mM + I = M^2.$$

Equating the upper-right entries of the matrices on the lhs and rhs of the so-obtained matrix identity yields

$$(4.7)\qquad \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j}\binom{n-j}{j} m^j U_{2n-3j} = m^n U_n \quad (n > 0).$$

Observe that using the matrix identity $(mM)^n = (M^2 - I)^n$ [directly derived from (4.6)] in connection with (1.4) gives an alternative expression for the rhs of (4.7), namely,

$$(4.8)\qquad (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} U_{2j} = m^n U_n.$$

4.1 Some Congruencial Properties of U_n and V_n

Some congruencial properties of U_n and V_n can be derived easily from (4.2). If $p > 2$ is a prime, then from (4.2) we can write

$$(4.9) \qquad U_p = \Delta^{p-1} + \sum_{j=1}^{(p-1)/2} (-1)^j \frac{p}{p-j}\binom{p-j}{j}\Delta^{p-1-2j}$$

whence, noting that the sum on the rhs of (4.9) is divisible by p and recalling that $\Delta^2 = m^2 + 4$, we obtain the congruence

$$(4.10) \qquad U_p \equiv (m^2 + 4)^{(p-1)/2} \pmod{p}.$$

Equivalently, we can state that $U_p \equiv 0 \pmod{p}$ if $m^2 + 4 \equiv 0 \pmod{p}$ while $U_p \equiv 1$ (or -1) (mod p) if $m^2 + 4$ is (is not) a quadratic residue modulo p.

For n an arbitrary positive odd integer, let us rewrite (4.2) as

$$(4.11) \qquad U_n = (-1)^{(n-1)/2} \frac{2n}{n+1}\binom{(n+1)/2}{(n-1)/2} + \sum_{j=0}^{(n-3)/2} (-1)^j \frac{n}{n-j}\binom{n-j}{j}\Delta^{n-1-2j}$$

$$= n(-1)^{(n-1)/2} + \sum_{j=0}^{(n-3)/2} (-1)^j \frac{n}{n-j}\binom{n-j}{j}\Delta^{n-1-2j} \quad (n \text{ odd}).$$

From (4.11), the congruence

$$(4.12) \qquad U_n \equiv n(-1)^{(n-1)/2} \pmod{m^2 + 4} \quad (n \text{ odd})$$

is immediately obtained. Using the same procedure, for n even we get

$$(4.13) \qquad V_n \equiv 2(-1)^{n/2} \pmod{m^2 + 4} \quad (n \text{ even}).$$

5. Conclusions and Further Examples

In this paper it has been shown that a large number of Fibonacci-type identities can be established by using matrices related to M in connection with the binomial formula and the Waring formula. We do believe that matrices other than those defined in section 2 can be employed to obtain further identities.

On the other hand, we wish to point out that the technique discussed in section 2 can also be used profitably in connection with other formulas. For example, consider the matrix equation

$$(5.1) \qquad A^n + B^n = (A + B) \sum_{j=1}^{n} (-1)^{j-1} A^{n-j} B^{j-1},$$

which is valid if $AB = BA$ and n is odd, and the matrix equation

$$(5.2) \qquad \sum_{j=0}^{n} A^j = (A^{n+1} - I)(A - I)^{-1},$$

which is valid if all eigenvalues of A are different from 1.

If we replace A by M and B by M^{-1} in (5.1), we can write

$$R(n) = M^n + M^{-n} = (M + M^{-1}) \sum_{j=1}^{n} (-1)^{j-1} M^{n-2j+1}$$

$$= H \sum_{j=1}^{n} (-1)^{j-1} M^{n-2j+1} = H\left[(-1)^{(n-1)/2} I + \sum_{j=1}^{(n-1)/2} (-1)^{j-1} R(n - 2j + 1)\right]$$

and, by (2.9),

$$(5.3) \qquad R(n) = (-1)^{(n-1)/2} H + H \sum_{j=1}^{(n-1)/2} (-1)^{j-1} V_{n-2j+1} I \quad (n \text{ odd}).$$

Equating the upper-left entries of the matrices on the lhs and rhs of (5.3) and taking (2.9) and (2.8) into account yields

$$mU_n = (-1)^{(n-1)/2}m + m\sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{n-2j+1},$$

whence

$$(5.4)\qquad \sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{n-2j+1} = U_n - (-1)^{(n-1)/2} \quad (n \text{ odd}).$$

Of course, the lhs of (5.4) is an alternating sum of alternate V_k. Since

$$V_k = U_{k-1} + U_{k+1},$$

the sum obviously telescopes so that (5.4) has a more direct derivation.

If we replace A by HM in (5.2) and take (2.8) and (3.12) into account, for n odd we can write

$$(5.5)\qquad \sum_{j=0}^{n}(HM)^j = (H^{n+1}M^{n+1} - I)(HM - I)^{-1} = (H^{n+1}M^{n+1} - I)M^{-2} = (\Delta^{n+1}M^{n+1} - I)M^{-2}.$$

Again, by (3.12) and (2.8), the lhs of (5.5) can be rewritten as

$$(5.6)\qquad \sum_{j=0}^{n}(HM)^j = \sum_{j=0}^{(n-1)/2}\Delta^{2j}(M^{2j} + HM^{2j+1}) = \sum_{j=0}^{(n-1)/2}\Delta^{2j}(M^{2j+2} + 2M^{2j}).$$

Equating the upper-left entries of the matrices on the rhs of (5.6) and (5.5) and using (2.7) yields

$$(5.7)\qquad \sum_{j=0}^{(n-1)/2}\Delta^{2j}(U_{2j+3} + 2U_{2j+1}) = \Delta^{n+1}U_n - 1 \quad (n \text{ odd}).$$

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SOME PROPERTIES OF THE TETRANACCI SEQUENCE MODULO m

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First Wall [8] and subsequently a number of others (see, e.g., [1], [3], and [4]) have examined the properties of the Fibonacci sequence modulo m. The Tribonacci sequence modulo m was considered and a number of properties were derived in [6]. Chang [2] briefly examined higher-order sequences modulo m. Vince [5] considered the period of repetition of a general linear recurrence.

In this paper we list several basic results which follow when some of Vince's results are applied to the special case of the Tetranacci sequence. We then establish a number of additional properties. We also investigate in detail the relationship of the period of the Tetranacci sequence modulo m to the factorization of the minimum polynomial of the T-matrix defined in [7] and given below in (2).

We consider the sequence $\{M_n\}$ reduced modulo m, taking least nonnegative residues, where

(1) $\quad M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}$ $(n \geq 4)$, $M_0 = M_1 = 0$, $M_2 = M_3 = 1$.

Definitions: The *length of the period* of $\{M_n\}$ (mod m), designated $h(m)$, is the number of terms in one period of the sequence $\{M_n\}$ (mod m). A *simply periodic* sequence is periodic and repeats by returning to its initial values.

We list several results found in [7] which will be required in the development of this paper.

(2) (a) $$T^n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} M_{n+2} & N_{n+2} & S_{n+2} & M_{n+1} \\ M_{n+1} & N_{n+1} & S_{n+1} & M_n \\ M_n & N_n & S_n & M_{n-1} \\ M_{n-1} & N_{n-1} & S_{n-1} & M_{n-2} \end{bmatrix},$$

where

(3) $\quad N_n = M_{n-1} + M_{n-2} + M_{n-3}$,

(4) $\quad S_n = M_{n-1} + M_{n-2}$.

(5) (b) $|T| = -1$, where $|T|$ is the determinant of T.

(6) (c) $$|T^n| = \begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+1} & M_n \\ M_{n+2} & M_{n+1} & M_n & M_{n-1} \\ M_{n+1} & M_n & M_{n-1} & M_{n-2} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} \end{vmatrix} = (-1)^n.$$

(7) (d) $M_{n+p} = M_{n+i}M_{p-i+2} + M_{n+i-1}N_{p-i+2} + M_{n+i-2}S_{p-i+2} + M_{n+i-3}M_{p-i+1}$.

(8) (e) $$\sum_{i=0}^{n} M_i = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1).$$

(9) (f) $$\sum_{i=0}^{n} M_{2i+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} - 2).$$

Table 1 gives values of $h(m)$ for selected values of m.

Table 1

m	2	3	4	5	6	7	8	9	10	11	13	15	16	17	19	27	100
$h(m)$	5	26	10	312	130	342	20	78	1560	120	84	312	40	4912	6858	234	1560

Results in [5] may be applied to the Tetranacci sequence to yield the following theorem.

Theorem 1: The sequence $\{M_n\}$ (mod m) satisfies the following:

(a) The sequence $\{M_n\}$ (mod m) is simply periodic.

(b) If m has prime factorization $m = p_1^{t_1}p_2^{t_2} \ldots p_s^{t_s}$, then

$$h(m) = \text{LCM}[h(p_1^{t_1}),\ h(p_2^{t_2}),\ \ldots,\ h(p_s^{t_s})].$$

(c) If $h(p^2) \neq h(p)$, then $h(p^t) = p^{t-1}h(p)$.

(d) If $n > 0$ is least such that $M_{n+1} \equiv M_n \equiv M_{n-1} \equiv 0 \pmod{m}$, and if $M_{t+1} \equiv M_t \equiv M_{t-1} \equiv 0 \pmod{m}$, then $t = kn$ for some integer k.

If we examine the terms of $\{M_n\}$ (mod 5), we see that for $s = 78$ we have

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{5},$$

but $M_{s-2} \not\equiv 1 \pmod{5}$. Hence, s is not the length of the period of $\{M_n\}$ (mod 5). However, the occurrence of "triple zeros," 0, 0, 0, in $\{M_n\}$ (mod 5) and, in general, the occurrence of triple zeros in the sequence (mod m), is significant in determining, among other properties, the period structure. The following lemma states some of the results related to this phenomenon.

Lemma 1: If $s > 0$ is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{m},$$

then the following congruences are valid:

(a) $M_{s-2}^8 \equiv M_{s+2}^8 \equiv 1 \pmod{m}$,

(b) $M_{js-1} \equiv M_{js} \equiv M_{js+1} \equiv 0 \pmod{m}$ for all $j > 0$.

Proof: To prove (a), we use (6) to obtain

$$(-1)^s = |T^s| = \begin{vmatrix} M_{s+3} & M_{s+2} & M_{s+1} & M_s \\ M_{s+2} & M_{s+1} & M_s & M_{s-1} \\ M_{s+1} & M_s & M_{s-1} & M_{s-2} \\ M_s & M_{s-1} & M_{s-2} & M_{s-3} \end{vmatrix} \equiv \begin{vmatrix} M_{s+3} & M_{s+2} & 0 & 0 \\ M_{s+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{s-2} \\ 0 & 0 & M_{s-2} & M_{s-3} \end{vmatrix}$$

$$\equiv M_{s+2}^2 M_{s-2}^2 \equiv M_{s+2}^4 \pmod{m}.$$

Therefore, $M_{s+2}^4 \equiv \pm 1 \pmod{m}$ or $M_{s+1}^8 \equiv 1 \pmod{m}$ and the proof is complete.

In (b), we prove only that $M_{js} \equiv 0 \pmod{m}$. The other parts follow similarly. The proof is by induction on j. If $j = 1$, the result is clear. If we assume that $M_{js} \equiv 0 \pmod{m}$, we have, by (7) with $i = 1$,

$$M_{(j+1)s} = M_{js+s} = M_{js+1}M_{s+1} + M_{js}N_{s+1} + M_{js-1}S_{s+1} + M_{js-2}M_s \equiv 0 \pmod{m},$$

and the induction is complete.

The next theorem provides identities which involve a rather curious shift of a factor of the *subscript* of an appropriate M_n to a *power* of that M_n when the modulus is changed from m to m^2.

Theorem 2: Let $h = h(m)$ and let k be a positive integer, then the following identities hold.

(10) (a) $M_{kh-2} \equiv M_{h-2}^{k} \pmod{m^2}$,

(11) (b) $M_{kh-1} \equiv kM_{h-2}^{k-1}M_{h-1} \pmod{m^2}$,

(12) (c) $M_{kh} \equiv kM_{h-2}^{k-1}M_{h} \pmod{m^2}$,

(13) (d) $M_{kh+1} \equiv kM_{h-2}^{k-1}M_{h+1} \pmod{m^2}$.

We prove (10); the other parts follow similarly. The proof is by induction on k. If $k = 1$, the conclusion is immediate. If we assume that

$$M_{kh-2} \equiv M_{h-2}^{k} \pmod{m^2},$$

then, by (7) with $i = 2$ and the induction hypothesis,

$$M_{(k+1)h-2} = M_{(kh-2)+h} = M_{kh}M_h + M_{kh-1}N_h + M_{kh-2}S_h + M_{kh-3}M_{h-1}$$

$$(\mathrm{mod}\ m^2)\begin{cases} \equiv [M_{kh-1}(M_{h-2} + M_{h-3}) + M_{h-2}^{k}M_{h-2} + M_{h-1}(M_{kh-2} + M_{kh-3})] \\ \equiv [M_{kh-1}(M_{h+1} - M_h - M_{h-1}) + M_{h-2}^{k+1} + M_{h-1}(M_{kh+1} - M_{kh} - M_{kh-1})] \\ \equiv M_{h-2}^{k+1}, \end{cases}$$

since m divides M_{h+1}, M_h, M_{h-1} and, by Lemma 1(b), m divides M_{kh+1}, M_{kh}, M_{kh-1}. This completes the proof.

A related property is the following:

Lemma 2: If p is prime and $j = h(p^t)$ is the length of the period of $\{M_n\} \pmod{p^t}$, then

$$M_{j-2}^{p} \equiv 1 \pmod{p^{t+1}}.$$

Proof: Since $M_{j-2} \equiv 1 \pmod{p^t}$, $M_{j-2} \equiv 1 \pmod{p}$, and thus $M_{j-2}^{s} \equiv 1 \pmod{p}$ for all s. Consequently, we have

$$\begin{aligned}(M_{j-2}^{p} - 1) &= (M_{j-2} - 1)(M_{j-2}^{p-1} + M_{j-2}^{p-2} + \cdots + M_{j-2} + 1) \\ &\equiv [0 \ (\mathrm{mod}\ p^t)][(1 + 1 + \cdots + 1) \ (\mathrm{mod}\ p)] \\ &\equiv [0 \ (\mathrm{mod}\ p^t)][0 \ (\mathrm{mod}\ p)] \\ &\equiv 0 \pmod{p^{t+1}}.\end{aligned}$$

The occurrence in $\{M_n\} \pmod{m}$ of the quadruple 1, 0, 0, 0 is the signal that the end of the period has been reached and that repetition has begun. If the term immediately in front of the three zeros is M_{s-2}, where $M_{s-2} \not\equiv 1 \pmod{m}$, there are only a limited number of possibilities for the value of M_{s-2} since, by Lemma 1, we always have $M_{s-2}^{8} \equiv 1 \pmod{m}$. This implies that as an element of the finite group, $\mathbb{Z}_m$, the order of M_{s-2} is 2, 4, or 8. We now examine in detail the possibilities resulting from this implication.

Theorem 3: If s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{m} \quad \text{and} \quad M_{s-2} \not\equiv 1 \pmod{m},$$

then one of the following holds:

(a) If the order of $M_{s-2} = 2$, then $M_{2s-2} \equiv M_{s-2}^{2} \equiv 1 \pmod{m}$ and $h(m) = 2s$. An example is $\{M_n\} \pmod{31}$, where $s = 30{,}784$ and $h(31) = 61{,}568$.

(b) If the order of $M_{s-2} = 4$, then $M_{4s-2} \equiv M_{s-2}^{4} \equiv 1 \pmod{m}$ and $h(m) = 4s$. An example is $\{M_n\} \pmod{5}$, where $s = 78$ and $h(5) = 312$.

(c) If the order of $M_{s-2} = 8$, then $M_{8s-2} \equiv M_{s-2}^{8} \equiv 1 \pmod{m}$ and $h(m) = 8s$. An example is $\{M_n\} \pmod{89}$, where $s = 1165$ and $h(89) = 9320$.

Proof: The proof follows from Theorem 2 and from the fact that, if $a \equiv b \pmod{m^2}$, then $a \equiv b \pmod{m}$.

The following theorem gives further related results.

Theorem 4: If s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{m},$$

then one of the following holds:

(a) If $h(m) = 2s$, then for any r, $M_r + M_{r+s} \equiv 0 \pmod{m}$.

(b) If $h(m) = 4s$, then for any r, $M_r + M_{r+s} + M_{r+2s} + M_{r+3s} \equiv 0 \pmod{m}$.

(c) If $h(m) = 8s$, then for any r, $M_r + M_{r+s} + M_{r+2s} + \cdots + M_{r+7s} \equiv 0 \pmod{m}$.

Proof: We prove (b); the other parts follow similarly. By repeated use of (7) with $i = 1$, we have

$$\begin{aligned} & M_r + M_{r+s} + M_{r+2s} + M_{r+3s} \\ &= M_r + (M_{r+1}M_{s+1} + M_rN_{s+1} + M_{r-1}S_{s+1} + M_{r-2}M_s) \\ &\quad + (M_{r+1}M_{2s+1} + M_rN_{2s+1} + M_{r-1}S_{2s+1} + M_{r-2}M_{2s}) \\ &\quad + (M_{r+1}M_{3s+1} + M_rN_{3s+1} + M_{r-1}S_{3s+1} + M_{r-2}M_{3s}) \\ &\equiv M_r(1 + M_{s-2} + M_{2s-2} + M_{3s-2}) \pmod{m} \\ &\equiv M_r(1 + M_{s-2} + M_{s-2}^2 + M_{s-2}^3) \pmod{m} \\ &\equiv 0 \pmod{m} \end{aligned}$$

since $M_{s-2}^4 - 1 \equiv 0 \pmod{m}$ and $M_{s-2} - 1 \not\equiv 0 \pmod{m}$.

Remark: The preceding proof shows that under the hypotheses of (b),

$$\begin{aligned} M_{r+s} &\equiv M_rM_{s-2} \pmod{m}, \\ M_{r+2s} &\equiv M_rM_{2s-2} \equiv M_rM_{s-2}^2 \pmod{m}, \\ M_{r+3s} &\equiv M_rM_{3s-2} \equiv M_rM_{s-2}^3 \pmod{m}, \end{aligned}$$

whenever $M_{s+1} \equiv M_s \equiv M_{s-1} \equiv 0 \pmod{m}$.

From these congruences we conclude that whenever triple zeros, 0, 0, 0, appear in the interior of the period rather than at the end, the triple zeros divide the period into what we might call *subperiods* of equal length where the terms in each successive subperiod are a fixed multiple of the corresponding terms in the first subperiod; that is, the terms which precede the first set of triple zeros.

For example, in the sequence $\{M_n\}$ (mod 5), we have 0, 0, 0 as the terms with subscripts 77, 78, 79; 155, 156, 157; 233, 234, 235; 311, 312, 313. If we call the first 78 (length of the subperiod) terms A, then the second 78 terms are obtained as 3 times A, the third 78 as $3^2 \equiv 4 \pmod{5}$ times A, and the fourth as $3^3 \equiv 2 \pmod{5}$ times A. Further, we have $3^4 \equiv 1 \pmod{5}$ and the length of the period is $4 \times 78 = 312$.

Theorem 5: For $p > 2$, $h(p)$ is even.

Proof: Let $h = h(p)$ and use (6) to obtain

$$(-1)^h = \begin{vmatrix} M_{h+3} & M_{h+2} & M_{h+1} & M_h \\ M_{h+2} & M_{h+1} & M_h & M_{h-1} \\ M_{h+1} & M_h & M_{h-1} & M_{h-2} \\ M_h & M_{h-1} & M_{h-2} & M_{h-3} \end{vmatrix} \equiv \begin{vmatrix} M_{h+3} & M_{h+2} & 0 & 0 \\ M_{h+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{h-2} \\ 0 & 0 & M_{h-2} & M_{h-3} \end{vmatrix}$$

$$\equiv M_{h-2}^2 M_{h+2}^2 \equiv M_{h+2}^4 \equiv M_{h+2} \equiv 1 \pmod{p}.$$

Therefore, $(-1)^h \equiv 1 \pmod{p}$ and h is even.

Theorem 6: If $p > 2$ and s is least such that

$$M_{s-1} \equiv M_s \equiv M_{s+1} \equiv 0 \pmod{p},$$

but $M_{s-2} \not\equiv 1 \pmod{p}$, then one of the following holds:

(a) If $h(p) = 2s$, then s is even.
(b) If $h(p) = 4s$, then s is even.
(c) If $h(p) = 8s$, then s is odd.

Proof: We prove (c); the other parts follow similarly. If $h(p) = 8s$, then by Theorem 3(c), $M_{s+2}^8 \equiv 1 \pmod{p}$, which implies that $M_{s+2}^4 \equiv (-1) \pmod{p}$. But, from the proof of Theorem 4, $M_{s+2}^4 \equiv (-1)^s \pmod{p}$ also. Hence, $(-1) \equiv (-1)^s \pmod{p}$ and s is odd.

We now examine further the relationship of p to $h(p)$. The minimum polynomial of the matrix T,

$$f(x) = x^4 - x^3 - x^2 - x - 1,$$

and its factorization over Z_p determine what this relationship is. We begin by stating a theorem that follows from more general results in [5].

Theorem 7: If

$$f(x) = x^4 - x^3 - x^2 - x - 1 = g_1^{\alpha_1}(x) g_2^{\alpha_2}(x) g_3^{\alpha_3}(x) g_4^{\alpha_4}(x)$$

is the factorization into irreducible factors of $f(x)$ over Z_p, then

(a) $h(p) \mid p^s \text{LCM}[t_i(p^{m_i} - 1)/(p - 1)]$, where s satisfies $p^s \geq \max \alpha_i > p^{s-1}$, m_i is the degree of $g_i(x)$, and t_i is the multiplicative order of $b_i(-1)^{m_i}$ in Z_p, b_i being the constant term of $g_i(x)$.

(b) If $t_i^r \mid (p^{m_i} - 1)/(p - 1)$ for some integer r, then $t_i^{r+1} \mid h(p)$.

We now apply Theorem 7 to the cases that arise from possible factorizations of $f(x)$.

<u>Case 1</u>. $f(x)$ is irreducible. In this case we have $m_1 = 4$, $\alpha_2 = 1$, $s = 0$, $t_1 = 2$, $r = 2$.

Hence, $h(p) \mid 2(p^3 + p^2 + p + 1)$ and $8 \mid h(p)$. An example is $p = 5$, where $h(5) = 312$, which divides $2(5^3 + 5^2 + 5 + 1) = 312$ and is divisible by 8.

<u>Case 2</u>. $f(x)$ has a single linear factor.

We then have $m_1 = 1$, $m_2 = 3$, $\alpha_1 = \alpha_2 = 1$, $s = 0$; t_1, $t_2 \mid p - 1$,

$$h(p) \mid \text{LCM}[t_1,\ t_2(p^2 + p + 1)]$$

and

$$t_1 \mid h(p) \quad \text{and if } t_2^r \mid p^2 + p + 1, \quad \text{then } t_2^{r+1} \mid h(p).$$

An example is $p = 3$, where

$$f(x) = (x - 1)(x^3 - x + 1)$$

and $h(3) = 26$, $t_1 = 1$, $t_2 = 2$, $r = 0$. Then $26 \mid 2(3^2 + 3 + 1)$ and $2 \mid 26$.

<u>Case 3</u>. $f(x)$ has exactly two distinct linear factors.

We then have $m_1 = m_2 = 1$, $m_3 = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $s = 0$; t_1, t_2, $t_3 \mid p - 1$, $h(p) \mid p^2 - 1$ and if $t_3^r \mid p + 1$ for some integer r, then $t_3^{r+1} \mid h(p)$.

An example is $p = 29$, where

$$f(x) = (x - 7)(x - 15)(x^2 - 8x + 2),$$

and $h(29) = 280$, $t_1 = 7$, $t_2 = 28$, $t_3 = 4$, LCM$[7, 28, 4 \cdot 30] = 840$, which is divisible by 280. Also, 7, 28, and 4 all divide 280 and are the highest such powers.

Case 4. $f(x)$ has exactly four distinct linear factors.

We then have $m_1 = m_2 = m_3 = m_4 = 1$; $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $s = 0$,

$$h(p) \mid \text{LCM}[t_1, t_2, t_3, t_4]$$

and

$$t_i \mid p - 1 \text{ for } i = 1, 2, 3, 4.$$

An example is $p = 137$, where $h(137) = 136$ and

$$f(x) = (x - 40)(x - 52)(x - 58)(x - 125).$$

All the $t_i = 136$, so $h(137) \mid 136$ and all $t_i \mid 136$ as well.

Case 5. $f(x)$ has a repeated linear factor and two other distinct linear factors.

Then $m_1 = m_2 = m_3 = 1$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2$, $s = 1$,

$$h(p) \mid \text{LCM}[t_1, t_2, t_3]$$

and

$$t_i \mid h(p).$$

In looking for an example of this case, we consider the discriminant of $f(x) = -563$, a prime. Therefore, this case can occur only for $p = 563$. It does, in fact, occur when $p = 563$, $h(563) = 316{,}406$, and we have

$$f(x) = (x - 107)(x - 116)(x + 111)^2.$$

Then $t_1 = t_2 = t_3 = 562$ and $h(563) \mid 563 \cdot 562 = 316{,}406$. This is the only case, of course, where $f(x)$ has a repeated root.

Case 6. $f(x)$ has two distinct quadratic factors.

Then $m_1 = m_2 = 2$; $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $s = 0$,

$$h(p) \mid \text{LCM}[t_1(p + 1), t_2(p + 1)]$$

and

$$t_i^{r+1} \mid h(p) \quad \text{if } t_i^r \mid p + 1.$$

Our example in this case is $p = 13$, where $h(13) = 84$ and

$$f(x) = (x^2 + 4x - 3)(x^2 - 5x - 4).$$

Then $t_1 = t_2 = 6$, with $r = 0$, $h(13) \mid 84$, and $6 \mid 84$.

These six are the only possible cases because all other factorizations of $f(x)$ can be shown to be untenable.

Table 2 gives additional examples.

Table 2

	$h(p)$	Roots of $f(x)$ in Z_p	Factorization of $f(x)$ over Z_p
7	$342 = 7^3 - 1$	5	$(x - 5)(x^3 + 4x^2 + 5x + 3)$
11	$120 = (11^4 - 1)/122$	none	irreducible
17	$4{,}912 = 17^3 - 1$	6	$(x - 6)(x^3 + 5x^2 + 12x + 3)$
41	$240 = (41^2 - 1)/7$	3, 33	$(x - 3)(x - 33)(x^2 - 6x + 12)$
43	$162{,}800 = (43^4 - 1)/21$	none	irreducible
47	$103{,}822 = 47^3 - 1$	21	$(x - 21)(x^3 + 20x^2 - 4x + 9)$
67	$100{,}254 = (67^3 - 1)/2$	5	$(x - 5)(x^3 + 4x^2 + 19x + 27)$
73	$2{,}664 = (73^3 - 1)/2$	39, 66	$(x - 39)(x - 66)(x^2 + 31x - 23)$
109	$2{,}614{,}040 = (109^4 - 1)/54$	none	irreducible

Finally, we state a theorem which gives a number of congruences involving sums.

Theorem 8: If $h = h(m)$, then the following congruences hold:

(a) $\sum_{i=0}^{h} M_i \equiv 0 \pmod{m}$,

(b) $\sum_{i=0}^{h} M_{2i+1} \equiv 0 \pmod{m}$,

(c) $\sum_{i=0}^{h} M_{2i} \equiv 0 \pmod{m}$,

(d) $\sum_{i=0}^{h} M_{3i} \equiv 0 \pmod{m}$,

(e) $\sum_{i=0}^{h} M_{3i+1} \equiv 0 \pmod{m}$,

(f) $\sum_{i=0}^{h-1} M_{3i+2} \equiv 0 \pmod{m}$,

(g) $\sum_{i=0}^{(h-2)/2} M_{2i} \equiv 0 \pmod{m}$,

(h) $\sum_{i=0}^{(h-2)/2} M_{2i+1} \equiv 0 \pmod{m}$.

Proof: The proofs follow easily from appropriate formulas which are derived in [7], two of which have been listed earlier. By (8) we have

$$\sum_{i=0}^{h} M_i = \frac{1}{3}(M_{h+2} + 2M_h + M_{h-1} - 1) \equiv 0 \pmod{m},$$

and by (9) and Lemma 1(b), we have

$$\sum_{i=0}^{h} M_{2i+1} = \frac{1}{3}(2M_{2h+2} + M_{2h} - M_{2h-1} - 2) \equiv 0 \pmod{m}.$$

The other congruences may be proved similarly.

A number of additional congruences involving sums of terms of $\{M_n\}$ may be derived, but no attempt is made at providing an exhaustive list.

Acknowledgment

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GENERATION OF GENOCCHI POLYNOMIALS OF FIRST ORDER BY RECURRENCE RELATIONS

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1. Motivation

Genocchi polynomials of the first order, $G_n(x)$, are defined [3] by

$$(1.1)\qquad \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t + 1}e^{tx}$$

as an extension of Genocchi numbers G_n defined in [1].

Following a suggestion by the referee of [3], I show briefly how $G_{2n+1}(x)$ $(n \geq 1)$ may be generated by $x^2 - x = x(x - 1)$. Such a possibility is to be expected since by (2.2) $x = 0$ and $x = 1$ are zeros of $G_{2n+1}(x)$. For example,

$$(1.2)\qquad \begin{aligned} G_{13}(x) &= 13[x^{12} - 6x^{11} + 55x^9 - 396x^7 + 1683x^5 - 3410x^3 + 2073x] \\ &= 13[(x^2 - x)^6 - 15(x^2 - x)^5 + 135(x^2 - x)^4 - 736(x^2 - x)^3 \\ &\qquad + 2073(x^2 - x)^2 - 2073(x^2 - x)]. \end{aligned}$$

It is the main purpose of this article to establish an algorithm for deriving a result like (1.2). Equations (3.6) and (3.7) are in fact the *recurrence relations* sought for $G_{2n+1}(x)$, the Genocchi polynomials of odd order. Similarly, we obtain (3.11), a recurrence relation for $G_{2n}(x)$ of even order. Our treatment, which was excluded from [3] because of the already considerable length of that paper, follows that given in [8] for Euler polynomials $E_n(x)$.

The theory expounded here does not generalize to $G_n^{(k)}(x)$, the Genocchi polynomials of order k [3]. An examination of the $G_n^{(k)}(x)$ listed in [3] will readily reveal why this is so.

Another purpose of this article is to answer a question raised at the 1990 International Fibonacci Conference at Wake Forest University, U.S.A.

2. Some Genocchi Formulas

Properties of $G_n(x)$ required to obtain the recurrence relations include [3]

$$(2.1)\qquad \frac{dG_n(x)}{dx} = nG_{n-1}(x), \quad n \geq 1,$$

and

$$(2.2)\qquad G_{2n}\left(\frac{1}{2}\right) = G_{2n+1}(0) = G_{2n+1}(1) = 0, \quad n \geq 1.$$

It is to be noted that

$$(2.3)\qquad G_n(x) = nE_{n-1}(x),$$

from which we have *Genocchi's theorem* ([1], [3], [4])

$$(2.4)\qquad G_{2n} = 2nE_{2n-1}(0)$$

for *Genocchi numbers* $G_n \equiv G_n(0)$ given in [1], [3], and [4] (see [2] also).

However, $E_{2n-1}(0)$ are not Euler numbers, but numbers related to Euler numbers ([3], [5]). Information on Euler polynomials and Bernoulli polynomials may be found, for example, in [5]. Other material of interest relating these polynomials to angular momentum traces occurs in [6], [7], and [8].

3. The Genocchi Generation

Using induction [8] as employed in [6] for Bernoulli polynomials, we can show that

(3.1) $G_{2n+1}(x) = Y_n(u)$,

where

(3.2) $u = x^2 - x \quad \left(\dfrac{du}{dx} = 2x - 1\right).$

With the help of (2.1), (2.2), (3.1), and (3.2), from which

$$\left(\frac{du}{dx}\right)^2 = 4u + 1,$$

we can derive, after a few steps, the differential equation

(3.3) $(4u + 1)\dfrac{d^2Y_n(u)}{du^2} + 2\,\dfrac{dY_n(u)}{du} = 2n(2n + 1)Y_{n-1}(u).$

Now let

(3.4) $G_{2n+1}(x) = Y_n(u) = \sum_{i=0}^{n} A_i u^i = (2n + 1)\sum_{i=0}^{n} C_i u^i$

and

(3.5) $G_{2n-1}(x) = Y_{n-1}(u) = \sum_{i=0}^{n-1} B_i u^i = (2n - 1)\sum_{i=0}^{n-1} D_i u^i$

so that, by (2.3), the C_i and D_i are the same as for $E_{2n}(x)$ in [8].

Calculation in (3.3) - (3.5) yields (cf. [8])

(3.6) $(2n - 1)A_n = (2n + 1)B_{n-1}$

and

(3.7) $i(i + 1)A_{i+1} + 2i(2i - 1)A_i = 2n(2n + 1)B_{i-1}$,

for $1 \le i \le n - 1$, $n \ge 2$.

Solving (3.6) and (3.7) for $n = 1, 2, 3, \ldots$ gives the constants A_i and B_i in the expansions (3.4) and (3.5). Table 1 supplies an appreviated list of these.

From (2.2) and (3.1), it follows that, for $n \ge 1$,

(3.8) $Y_n(u) = G_{2n+1}(x) = 0$ when $x = 0, 1$, i.e., $u = 0$.

Thus, $Y_n(u)$, $n \ge 1$, has no constant term, i.e., $A_0 = 0$. Likewise, $B_0 = 0$.

Consequently, the recurrence relations (3.6) and (3.7) generate $G_{2n+1}(x) = Y_n(u)$, where $G_1(x) = Y_0(u) = 1 = u^0$.

Table 1

Coefficients A_i of $G_{2n+1}(x) = Y_n(u)$

$n \backslash i$	1	2	3	4	5
1	3				
2	-5	5			
3	21	-21	7		
4	-133	133	-54	9	
5	1705	-1705	605	110	11

Note that in (3.7) when $i = 1$, $n \ge 2$ ($B_0 = 0$), we obtain

(3.9) $A_2 = -A_1$.

In Table 2 of [8], we observe the apparently unnoticed fact that the elements in column 2 for the Euler polynomials $E_{2n}(x)$ are the Genocchi numbers G_4, G_6, G_8, G_{10}, ..., while those in column 1 are the negatives of these Genocchi numbers.

Why is this so?

For each $n \geq 2$,

$$(3.10)\quad \begin{aligned} G_{2n} &= 2nE_{2n-1}(0) && \text{from (2.4)},\\ &= \frac{d}{dx}E_{2n}(x)\Big|_{x=0} && \text{by (2.1), (2.3)},\\ &= (2x-1)\frac{d}{du}\left\{\sum_{i=0}^{n} C_i u^i\right\}\Big|_{\substack{x=0\\u=0}} && \text{from [8], equation (32)}\\ &= -C_1. \end{aligned}$$

Because of (3.9) and (3.10), the elements in the first and second columns of our Table 1 will be appropriate multiples of Genocchi numbers, namely,

$$(2n+1)G_{2n} = -A_1 \quad \text{for each } n \geq 2.$$

Coming now to generators of $G_{2n}(x)$ we have, from (2.1),

$$(3.11)\quad \begin{aligned} G_{2n}(x) &= \frac{1}{2n+1}\,\frac{dG_{2n+1}(x)}{dx}\\ &= \frac{2x-1}{2n+1}\,\frac{dY_n(u)}{du} \qquad \text{by (3.1), (3.2)},\\ &= (2x-1)Z_{n-1}(u), \end{aligned}$$

i.e.,

$$(3.12)\quad (2n+1)Z_{n-1}(u) = \frac{dY_n(u)}{du}$$

i.e., the $Z_{n-1}(u)$ can be derived from the known $Y_n(u)$.

For example,

$$G_6(x) = 3(2x-1)(u^2-2u+1) = (2x-1)Z_2(u)$$

with

$$\frac{dY_3(u)}{du} = 7\frac{d}{du}(3u - 3u^2 + u^3) = 7[3(1-2u+u^2)] = 7Z_2(u)$$

on using our Table 1. From this table for $Z_{n-1}(u)$, a corresponding table for $A_{n-1}(u)$ could be constructed.

4. A Question Answered

Consider $x^2 - x - 1 = u - 1$ by (3.2). This is the well-known algebraic expression for the Fibonacci recurrence, $F_{n+2} - F_{n+1} - F_n = 0$, whose zeros are $(1+\sqrt{5})/2$ and its negative reciprocal.

Next, from [1] or (1.1),

$$(4.1)\quad \begin{cases} G_5(x) = 5u(u-1)\\ G_6(x) = 3(2x-1)(u-1)^2 = 3(u-1)^2\dfrac{du}{dx}, \end{cases}$$

i.e., the term $u - 1$ in $G_5(x)$ is squared in $G_6(x)$.

At my address on Genocchi polynomials to the Fourth International Conference on Fibonacci Numbers and Their Applications held at Wake Forest University in Winston-Salem, North Carolina, U.S.A. (see [3]), I was asked: "Is there any pattern in the $G_n(x)$ for other (positive) powers of $u - 1$?"

Assume that, for some N, the Genocchi polynomial $G_N(x)$ contains a factor $(u - 1)^k$. Then, by (2.1), $G_{N-1}(x)$ contains a factor $(u - 1)^{k-1}$.

There are two cases to be investigated, namely,

I. $N = 2n$ and II. $N = 2n + 1$.

Recall that, by virtue of (2.2),

$$\begin{cases} 2x - 1 = \dfrac{du}{dx} \text{ is always a factor of } G_{2n}(x), \\ x(x - 1) = u \text{ is always a factor of } G_{2n+1}(x). \end{cases}$$

Case I. Suppose

$$(\alpha) \qquad G_{2n}(x) = n\frac{du}{dx}(u - 1)^m$$

$$(\beta) \qquad G_{2n-1}(x) = (2n - 1)u(u - 1)^{m-1},$$

the numbers $n = 2n/2$ and $2n - 1$ being necessary coefficients (see [3]). Now

$$\begin{aligned} \frac{dG_{2n}(x)}{dx} &= n\{2(u - 1)^m + (4u + 1)m(u - 1)^{m-1}\} && \text{from } (\alpha) \\ (\gamma) \qquad &= n(u - 1)^{m-1}\{(2 + 4m)u + m - 2\} \\ &= 2nG_{2n-1}(x) && \text{by } (2.1) \\ (\delta) \qquad &= 2n(2n - 1)u(u - 1)^{m-1} && \text{by } (\beta). \end{aligned}$$

For (α) and (β) to be valid, we must have (γ) = (δ). Equating these produces

$$(2 + 4m)u + m - 2 = (4n - 2)u,$$

whence

$$(4.2) \qquad \begin{cases} m = 2 \\ n = 3 \end{cases}.$$

Case II. Secondly, suppose

$$(\alpha') \qquad G_{2n+1}(x) = (2n + 1)u(u - 1)^p$$

$$(\beta') \qquad G_{2n}(x) = \frac{du}{dx}(u - 1)^{p-1}.$$

Then,

$$\begin{aligned} \frac{dG_{2n+1}(x)}{dx} &= (2n + 1)\left\{\frac{du}{dx}(u - 1)^p + up(u - 1)^{p-1}\frac{du}{dx}\right\} && \text{from } (\alpha') \\ (\gamma') \qquad &= (2n + 1)(u - 1)^{p-1}\frac{du}{dx}\{u - 1 + up\} \\ &= (2n + 1)G_{2n}(x) && \text{by } (2.1) \\ (\delta') \qquad &= (2n + 1)n\frac{du}{dx}(u - 1)^{p-1} && \text{by } (\beta'). \end{aligned}$$

Solving (γ') and (δ') leads to $p = n = -1$, which must be discarded because p and n were assumed to be positive.

Cases I and II demonstrate that, by (4.2), the only occurrence of powers of $u - 1$ is that in $G_5(x)$ and $G_6(x)$ given in (4.1).

Our answer to the question is thus: No!

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A FIBONACCI THEME ON BALANCED BINARY TREES

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In this paper we show that, when a binary tree is in a certain critical balance, there emerge the Golden Ratio and the Fibonacci numbers.

The paper consists of two sections. In the first section we find some elementary balance properties of optimal binary trees with variously weighted leaves. In the second section, a basic inequality implied by the optimality of trees is in turn used to define what we mean by "balanced" for a binary tree with leaves all weighted 1. The Fibonacci tree is then shown to be a highest balanced tree.

1. Balance Properties of Optimal Binary Trees

Consider a binary tree that has n leaves (terminal nodes) with weights or probabilities $p_i > 0$, $p_1 + \cdots + p_n = 1$, assigned to leaves. It has, then, $n - 1$ internal nodes, where an internal node is a node that has two children. We define the weight of an internal node as the sum of all leaf weights of the subtree rooted at this node. Therefore, recursively, the weight of an internal node is the sum of the weights of its children. Clearly the root has weight 1.

A node is said to be at level k if the length of the path from the root to this node is k. The root is, hence, at level 0. Let ℓ_i be the level of the leaf weighted p_i. Then the average path length is defined by

$$L = \sum p_i \ell_i .$$

In this section we shall be concerned with a binary tree that is *optimal* in the sense that it has the minimum average path length for the given leaf weights.

The well-known Huffman algorithm [3] finds an optimal tree called the *Huffman tree*. The algorithm can be stated in the following recursion form: First, find the two least weights, say x and y, in the list $p_1, p_2, \ldots, p_n$, and replace these two by z $(= x + y)$. Then construct a Huffman tree for the new list of $n - 1$ weights, and then split, in this tree, a leaf of weight z into its children of weights x and y. Note, however, that not every optimal tree is a Huffman tree.

The original motivation for minimizing average path length was to minimize expected search time to leaves. Suppose that one person thinks of $z \in \{1, \ldots, n\}$ and you attempt, knowing $\text{Prob}\{z = i\} = p_i$, to determine what it is by asking questions that can be answered "yes" or "no." Then you may use a binary tree with leaves $1, \ldots, n$ of weights $p_1, \ldots, p_n$ as follows. You ask the first question at the root: "Does z belong to the left subtree of the root?" If the answer to this question is yes [no], then you go to the left [right] child of the root, say a, where you ask the second question: "Does z belong to the left subtree of a?" If the answer to this question is yes [no], then you go to the left [right] child of a, ... The average number of questions required to find z is given by the average path length of the tree.

Lemma 1: If w_{k-1} and w_k are weights of nodes at levels $k - 1$, k in an optimal tree, then $w_{k-1} \geq w_k$.

Proof: If the node of weight w_k exists in the subtree rooted at the node of weight w_{k-1}, then the assertion is obviously true. If not, consider exchanging the subtrees rooted at these nodes. Denote by L and L' the average path lengths of the trees before and after the exchange, respectively. Then we have $L' - L = w_{k-1} - w_k$, because leaves with total weight w_{k-1} have path length one longer under L', and leaves with total weight w_k have path length one longer under L. Since the tree before the exchange is optimal, we have $L \le L'$, hence $w_{k-1} \ge w_k$. □

We say that w_{k-1}, w_k, w_{k+1} is a *weight sequence* in a binary tree if w_k is the weight of an internal node at level k, w_{k-1} is the weight of its parent, and w_{k+1} is the weight of one of its children. Also, let $\overline{w}_k$ and $\overline{w}_{k+1}$ be the weights of the "brothers" of those nodes with weights w_k, w_{k+1}, respectively.

Theorem 1: If w_{k-1}, w_k, w_{k+1} is a weight sequence in an optimal tree, then

$$w_{k-1} \ge w_k + w_{k+1}.$$

Proof: By Lemma 1, we have $\overline{w}_k \ge w_{k+1}$. Hence,

$$w_{k-1} = w_k + \overline{w}_k \ge w_k + w_{k+1}. \quad \square$$

This inequality was implicit in [4] for Huffman trees and was explicitly stated in [1]. It was shown in [1] that it also holds in a weight-balanced tree if the node with weight w_{k+1} is internal or if the sequence of leaf weights forms a valley, i.e.,

$$p_1 \ge \cdots \ge p_j \le \cdots \le p_n \text{ for some } j,\ 1 \le j \le n.$$

Theorem 2: If w_{k-1}, w_k, w_{k+1} is a weight sequence in an optimal tree, then

$$w_k/w_{k-1} \le 2/3.$$

Proof: From Theorem 1, we have

$$(1) \qquad \begin{aligned} w_{k-1} &\ge w_k + w_{k+1} \\ w_{k-1} &\ge w_k + \overline{w}_{k+1}. \end{aligned}$$

Putting $p = w_k/w_{k-1}$ and $q = w_{k+1}/w_k$, these inequalities, divided by w_{k-1}, can be written as

$$(2) \qquad \begin{aligned} 1 &\ge p + pq \\ 1 &\ge p + p(1 - q), \end{aligned}$$

which, added together, gives $p \le 2/3$. □

From this theorem, if the node with weight $\overline{w}_k$ is also internal, we have

$$1/3 \le w_k/w_{k-1} \le 2/3.$$

Otherwise, w_k/w_{k-1} can be arbitrarily small. These bounds 1/3 and 2/3 can be attained as seen from the Huffman tree for the leaf weights:

$$3^{-m-1},\ 3^{-m-1},\ 3^{-m-1},\ 3^{-m},\ 3^{-m},\ \ldots,\ 3^{-2},\ 3^{-2},\ 3^{-1},\ 3^{-1}.$$

The set of points (p, q), $0 < p < 1$, $0 < q < 1$, satisfying (2) above forms the region *ABCDO* in Figure 1. The figure may aid one to graphically understand the balance properties stated in the following. Here ψ is the Golden Section point of the unit interval:

$$\psi = (\sqrt{5} - 1)/2, \quad 1 - \psi = \psi^2.$$

Now let us say that a is α-*balanced* for $1/2 \le \alpha \le 1$ if $a \in [1 - \alpha, \alpha]$. The next theorem states that a lack of ψ-balance involving siblings at one level is immediately restored at the next lower level.

Theorem 3: If w_{k-1}, w_k, w_{k+1} is a weight sequence in an optimal tree and if $w_k/w_{k-1} > \psi$, then

(a) w_{k+1}/w_k is ψ-balanced,

(b) $(w_k/w_{k-1}) + (w_{k+1}/w_k) < 2\psi$.

Proof: From the first inequality of (2), we have

$$q \le 1/p - 1 < 1/\psi - 1 = \psi,$$

and from the second,

$$q \ge 2 - 1/p > 2 - 1/\psi = 1 - \psi.$$

Therefore, q is ψ-balanced. For (b) use $p + q = p + 1/p - 1$ obtained from (2). The function $p + 1/p - 1$ is monotonically decreasing for $p < 1$; hence, $p + q$ is less than $\psi + 1/\psi - 1 = 2\psi$ by the assumption $p > \psi$. □

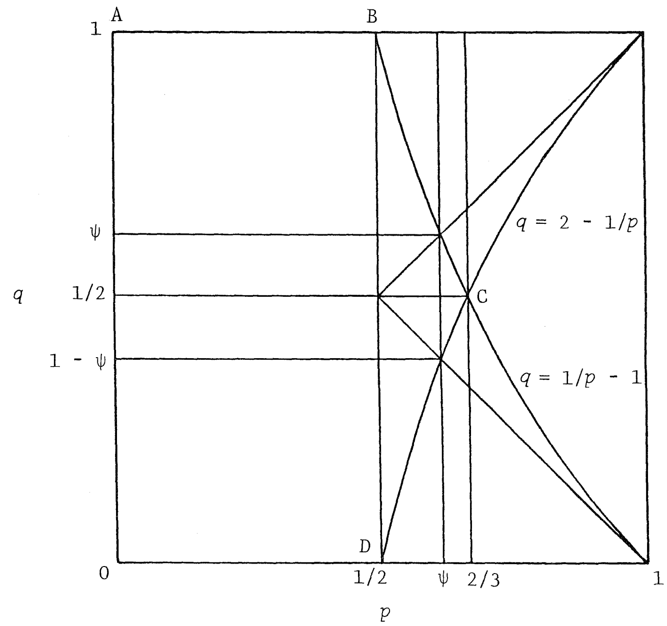

Figure 1

Notice that (1) is equivalent to the following "uncle ≥ nephew" condition:

$$(3)\quad \begin{aligned} \overline{w}_k &\ge w_{k+1}, \\ \overline{w}_k &\ge \overline{w}_{k+1}. \end{aligned}$$

Hence, the worse the balance of p (approaching 2/3), the better the balance of q (approaching 1/2). And the critical point for turning back to a better balance may be defined by the number

$$\alpha^* = \inf\{\alpha : p > \alpha \Rightarrow 1 - \alpha \le q \le \alpha\}.$$

Theorem 4: $\alpha^* = \psi$.

Proof: To determine the critical point, we set $q = p = \alpha^*$ and assume the equality $q = 1/p - 1$, i.e., $\alpha^* = 1/\alpha^* - 1$. □

What about the upper bound on w_k? Is there a bound in terms of k? Letting $F_{n+1} = F_n + F_{n-1}$ $(n \ge 1)$, $F_0 = 0$, $F_1 = 1$, be the Fibonacci numbers, we have

Theorem 5: If $w_0, \ldots, w_k, w_{k+1}$ are the weights on a path from the root in an optimal tree, then $w_k \le 2/F_{k+3}$.

Proof: By Theorem 2, we have

$$w_{k-1} \geq (3/2)w_k = (F_4/F_3)w_k.$$

Using the basic subadditivity relation of Theorem 1, we have, recursively,

$$w_{k-2} \geq w_{k-1} + w_k \geq (F_4/F_3)w_k + w_k = (F_5/F_3)w_k,$$
$$w_{k-3} \geq w_{k-2} + w_{k-1} \geq (F_5/F_3)w_k + (F_4/F_3)w_k = (F_6/F_3)w_k,$$
$$\ldots$$
$$1 = w_0 \geq w_1 + w_2 \geq (F_{k+2}/F_3)w_k + (F_{k+1}/F_3)w_k = (F_{k+3}/F_3)w_k,$$

completing the proof. □

The bound $2/F_{k+3}$ can also be attained. This is seen from one Huffman tree (there may be many) for the leaf weights that are the following divided by F_{k+3} (see Figure 2):

$$F_1, F_2, F_2, F_3, F_4, \ldots, F_k, F_{k+1}.$$

The internal node at level i has weight

$$w_i = F_{k+3-i}/F_{k+3}, \quad 0 \leq i \leq k.$$

We have $w_k/w_{k-1} = F_3/F_4 = 2/3$, and all the inequalities in the proof of Theorem 5 become equalities, and $w_k = 2/F_{k+3}$. Furthermore, we see that

$$w_i/w_{i-1} = F_{k+3-i}/F_{k+4-i}$$

approaches ψ for each i when k becomes large. This Huffman tree is a tree where the restoration of the ψ-balance is occurring "most" frequently, because, from the well-known identity $F_{n-1} - \psi F_n = (-\psi)^n$, the ratio F_{n-1}/F_n becomes larger or smaller than ψ, alternately.

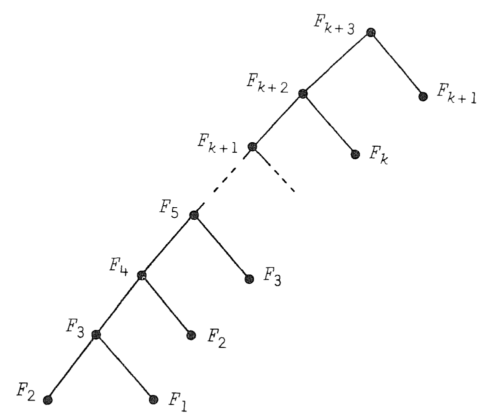

Figure 2

2. Fibonacci Tree as a Highest Balanced Tree

In the binary tree we consider here in this section, the weight of a node is defined as the number of leaves of the subtree rooted at the node. Hence, the leaf weights are all one, the weight of the root is just the total number of leaves.

When can we say that a binary tree is generally "balanced"? One natural definition may come from the inequality of Theorem 1. Since this relation is equivalent to (3) given in the previous section, let us say that a binary tree is *balanced* if it satisfies the following condition.

Balance Condition: The weight of every node is greater than or equal to the weight of each of its two "nephews" (if they exist).

A binary tree in this weight model is thus balanced if and only if the two subtrees at the children of the root are balanced and the weights of the children of the root are larger than or equal to the weights of their nephews. Also, this condition need only be checked for the child of the root with smaller weight.

There are other balance conditions that can be enforced in constructing trees, some applicable from the top down. The λ-*weight-balancing* described in [2] is such a method. Given $1/2 \le \lambda < 1$, to construct a binary tree with n leaves by λ-weight-balancing, we find the integer m such that

$$m - (1 - \lambda) < \lambda n \le m + \lambda,$$

let m and $n - m$ be the weights of the children of the root (i.e., the number of leaves assigned to each subtree), and proceed similarly to construct the two subtrees. The partition $m:(n - m)$ of n is a discrete version of the cut $\lambda:(1 - \lambda)$ of the unit interval. Notice that the λ-weight-balancing can be considered as a method to build a binary tree having a self-similar structure. We will show that the tree with n leaves built by this method is a balanced tree for every n if and only if $1/2 \le \lambda \le \psi$. First, we review a few things about the Fibonacci trees (see [2]).

The Fibonacci tree of order k, denoted by T_k, is a binary tree that has F_k leaves, and is constructed as follows: T_1 and T_2 are simply the roots only, and for $k \ge 3$ the left subtree of T_k is T_{k-1} and the right subtree is T_{k-2}. Let us denote by $T(n)$ the tree with n leaves constructed by ψ-weight-balancing. We may call $T(n)$ "the extended Fibonacci tree," for it has been shown in Theorem 5 of [2] that

$$T(F_k) = T_k.$$

We also have

Theorem 5 [2]: If $n = F_k + r$, where $0 \le r < F_{k-1}$, then the height of $T(n)$ is $k - 2$. [From $F_k \sim (1/\sqrt{5})\psi^{-k}$, we have $k - 2 \sim (\log n)/(-\log \psi)$.]

Theorem 6: If the tree with n leaves constructed by λ-weight-balancing is a balanced tree for every n, then we have $\lambda \le \psi$.

Proof: Suppose $\lambda = \psi + \varepsilon$, $\varepsilon > 0$. Let m_1 and $n - m_1$ be the weights of the children of the root, and let m_2 and $m_1 - m_2$ be the weights of the children of the node with weight m_1. The balancing rule implies $\lambda n \le m_1 + \lambda$ and $\lambda m_1 \le m_2 + \lambda$. Although the node with weight m_2 is a nephew of the node with weight $n - m_1$, we have m_2 greater than $n - m_1$, if n is taken large, as shown below:

$$\begin{aligned} m_2 - (n - m_1) &\ge \lambda m_1 - \lambda - n + m_1 \\ &\ge \lambda(\lambda n - \lambda) - \lambda - n + (\lambda n - \lambda) \\ &= \varepsilon(\sqrt{5} + \varepsilon)n - (\lambda^2 + 2\lambda), \end{aligned}$$

where we used $\lambda = \psi + \varepsilon$ and $\psi = (\sqrt{5} - 1)/2$. □

Approximately speaking, the above proof is like this: The bipartition of n by the ratio $\lambda:(1 - \lambda)$ makes children with weights λn and $(1 - \lambda)n$. And the partition of λn by the same ratio produces the node with weight $\lambda(\lambda n)$, which is a nephew of the node of weight $(1 - \lambda)n$. The balance condition requires $\lambda(\lambda n) \le (1 - \lambda)n$; hence, $\lambda^2 \le 1 - \lambda$, and we have $\lambda \le \psi$.

Next, we show

Theorem 7: The tree with n leaves constructed by ψ-weight-balancing is a balanced tree.

Proof: We prove by induction on n that $T(n)$ is balanced. Trivially, $T(2)$ is a balanced tree. Let us represent n (≥ 3) in the following form:

$$n = F_k + r,\ 0 \leq r < F_{k-1}.$$

Then $k \geq 4$. Let m_1 and $n - m_1$ be the weights of the children of the root, so that $m_1 - (1 - \psi) < \psi n \leq m_1 + \psi$. As noted in [2],

$$m_1 = F_{k-1} + s, \text{ where } s = \lceil \psi r - \psi - (-\psi)^k \rceil \text{ and } 0 \leq s < F_{k-2}.$$
$$(\lceil x \rceil = \text{the least integer} \geq x)$$

Furthermore, let m_2 and $m_1 - m_2$ be the weights of the children of the node with weight m_1, then, similarly, we have

$$m_2 = F_{k-2} + \lceil \psi s - \psi - (-\psi)^{k-1} \rceil.$$

The left and right subtrees of $T(n)$ are $T(m_1)$ and $T(n - m_1)$, which are balanced by the induction hypothesis. Since, clearly, $m_1 \geq n - m_1$ and $m_2 \geq m_1 - m_2$, we only need to show $n - m_1 \geq m_2$ or $n \geq m_1 + m_2$.

$$\begin{aligned} m_1 + m_2 &= (F_{k-1} + s) + (F_{k-2} + \lceil \psi s - \psi - (-\psi)^{k-1} \rceil) \\ &= F_k + \lceil s + \psi s - \psi - (-\psi)^{k-1} \rceil \\ &\leq F_k + \lceil (1 + \psi)(\psi r - \psi - (-\psi)^k + 1) - \psi - (-\psi)^{k-1} \rceil \\ &= F_k + \lceil r + (\psi^2 + \psi - 1)(r - 1 + (-\psi)^{k-1}) \rceil \\ &= F_k + r \\ &= n, \end{aligned}$$

completing the proof. □

Remark: If we use the rule

"$m - 1/2 < \psi n \leq m + 1/2$"

instead of the rule

"$m - (1 - \psi) < \psi n \leq m + \psi$,"

it will construct an unbalanced tree, when $n = 9$, for example. If we use the rule

"$m < \psi n \leq m + 1$,"

the tree built will not become T_8 when $n = F_8$, for example.

Now what can we say about the height of a balanced tree?

Lemma 2: Denote by n_h the minimum number of leaves that a balanced tree of height h can have. Then we have $n_h = F_{h+2}$.

Proof: Induction on h. It is immediate that $n_0 = 1 = F_2$ and $n_1 = 2 = F_3$. Consider a balanced tree of height $h \geq 2$ with n_h leaves. Let a, a' be the weights of the children of the root. We may assume that the subtree rooted at the node with weight a has height $h - 1$. We may further assume that, letting b be the weight of a child of the node with weight a, the subtree rooted at the node with weight b has height $h - 2$. Since any subtree of a balanced tree is itself a balanced tree, we have

$$a \geq n_{h-1},\quad b \geq n_{h-2}.$$

By the induction hypothesis, we have

$$n_{h-1} = F_{h+1},\quad n_{h-2} = F_h.$$

Hence,

$$a \geq F_{h+1},\quad b \geq F_h.$$

The balance condition implies

$$a' \geq b.$$

Consequently, we have

$$n_h = a + a' \geq F_{h+1} + F_h = F_{h+2}.$$

On the other hand, consider the Fibonacci tree T_{h+2}. This is a balanced tree by Theorem 7, and has height h by Theorem 5, and has F_{h+2} leaves. Hence, from the minimality of n_h, we have

$$n_h \leq F_{h+2}.$$

In conclusion, we have $n_h = F_{h+2}$. This completes the proof. □

Theorem 8: In the class of all the balanced trees with n leaves, the extended Fibonacci tree $T(n)$ is a highest one.

Proof: Let $n = F_k + r$, $0 \leq r < F_{k-1}$, and let h be the height of an arbitrary balanced tree with n leaves. From Lemma 2, we have $n \geq F_{h+2}$; hence,

$$F_{h+2} \leq F_k + r < F_k + F_{k-1} = F_{k+1}.$$

This implies $h + 2 \leq k$ or $h \leq k - 2$. However, $k - 2$ is the height of $T(n)$ by Theorem 5. □

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ANOTHER GENERALIZATION OF GOULD'S STAR OF DAVID THEOREM

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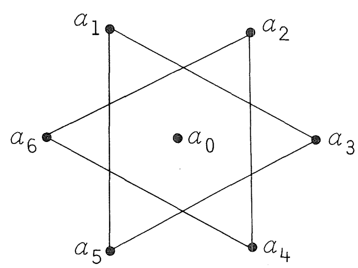
1. Introduction

Let a_1, a_2, a_3, a_4, a_5, and a_6 denote the hexagon of elements immediately surrounding any given element a_0 in Pascal's triangle.



Since the first paper by Hoggatt & Hansell [8] showing that $a_1a_3a_5 = a_2a_4a_6$ and hence that $\prod_{i=1}^{6} a_i = k^2$ for some integer k, a number of papers examining the properties of these arrays and their generalizations have appeared. Among the more surprising of these is the GCD Star of David theorem that

$$(a_1,\ a_3,\ a_5) = (a_2,\ a_4,\ a_6)$$

conjectured by Gould [4] and proved and/or generalized by Hillman & Hoggatt [5] and [6], Strauss [11], Singmaster [10], Hitotumatu & Sato [7], Ando & Sato [1], [2], and [3], and Long & Ando [9]. In the last listed paper, it was shown that

$$(a_1,\ a_3,\ \ldots,\ a_{17}) = (a_2,\ a_4,\ \ldots,\ a_{18})$$

where the a , $1 \leq i \leq 18$, are the eighteen adjacent binomial coefficients in the regular hexagon of coefficients centered on any particular coefficient $\binom{n}{r}$ and that

$$(b_1,\ b_3,\ \ldots,\ b_{11}) = t \cdot (b_2,\ b_4,\ \ldots,\ b_{12})$$

where the b , $1 \leq i \leq 12$, are the twelve adjacent binomial coefficients in the regular hexagon of coefficients centered at $\binom{n}{r}$ with $t = 1$ if r or $n - r = s$ is even, $t = 2$ if r and s are odd and $r \equiv 3 \pmod 4$ or $s \equiv 3 \pmod 4$, and $t = 4$ if $r \equiv s \equiv 1 \pmod 4$. Moreover, it was conjectured that

$$(a_1,\ a_3,\ \ldots,\ a_{2m-1}) = (a_2,\ a_4,\ \ldots,\ a_{2m})$$

if the a_i, $1 \leq i \leq 2m$, are the coefficients in a regular hexagon of binomial coefficients with edges along the rows and main diagonals of Pascal's triangle and with an even number of coefficients per edge. For such regular hexagons but with an odd number of coefficients per edge it was conjectured that

$$(a_1,\ a_3,\ \ldots,\ a_{2m-1}) = t \cdot (a_2,\ a_4,\ \ldots,\ a_{2m})$$

where t is a "simple" rational number depending on m, n, and r. In the present paper, we show that the regularity condition on the hexagons with an even number of coefficients per side is not necessary. In fact, we now conjecture that the equal gcd property holds for convex hexagons of adjacent entries along the rows and main diagonals of Pascal's triangle provided there are $2u$, $2v$, $2w$, $2u$, $2v$, and $2w$ coefficients on the consecutive sides. Being unable to prove the conjecture in general, we here prove it for the case $u = 3$, $v = 2$, and $w = 1$.

2. Some Preliminaries

Throughout the paper small Latin letters will always denote integers. Let $r + s = n$ as above, set $A = \binom{n}{r}$ and, for simplicity, set

$$(h,\ k) = \binom{n + h + k}{r + h}.$$

Let p be a prime. For any rational number α, there exists a unique integer $v = v(\alpha)$ such that $\alpha = p^v a/b$ where $(a,\ p) = (b,\ p) = 1$. If $v(n) = e$, then $p^e \| n$; i.e., $p^e | n$ and $p^{e+1} \nmid n$. Moreover, it is clear that

(1) $\quad v(1) = 0,$

(2) $\quad v(\alpha\beta) = v(\alpha) + v(\beta),$

(3) $\quad v(\alpha/\beta) = v(\alpha) - v(\beta),$

(4) $\quad v(\alpha \pm \beta) \geq \min(v(\alpha),\ v(\beta)) \quad \forall \alpha,\ \beta,$

(5) $\quad v(\alpha \pm \beta) = \min(v(\alpha),\ v(\beta)) \quad \text{if } v(\alpha) \neq v(\beta).$

Finally, if $m = m_1 m_2 \ldots m_k$, then

(6) $\quad (m_1,\ m_2,\ \ldots,\ m_k) = \prod_{p|m} p^{\min(v(m_1),\ \ldots,\ v(m_k))}.$

3. The Main Result

Now consider the eighteen binomial coefficients forming a hexagon centered at A as indicated in Figure 1. Let

$$S_1 = \{a_1,\ a_3,\ \ldots,\ a_{17}\},\quad S_2 = \{a_2,\ a_4,\ \ldots,\ a_{18}\},$$

$$\text{gcd } S_1 = (a_1,\ a_3,\ \ldots,\ a_{17}),\quad \text{gcd } S_2 = (a_2,\ a_4,\ \ldots,\ a_{18}).$$

Then, using the notation $(h,\ k)$ above,

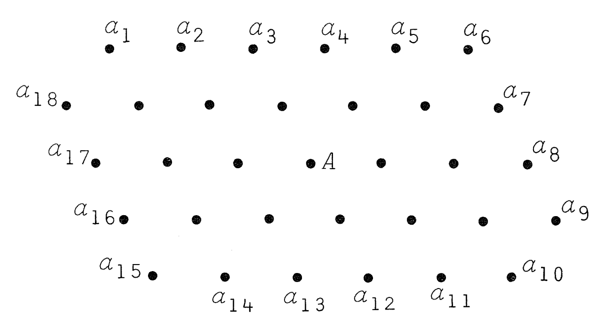


Figure 1

we can list the elements of S_1 and S_2 as in Table 1.

It is clear from the table that the product of the elements in S_1 is equal to the product of those in S_2 and it is not difficult to show by counter example that lcm S_1 = lcm S_2 is not always true. In particular, if $A = \binom{11}{5}$,

$$\text{lcm } S_1 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \quad \text{and} \quad \text{lcm } S_2 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13,$$

so lcm $S_1 \neq$ lcm S_2. However, the result shown in the Theorem below does hold.

Table 1

$S_1 = S_1(n, r)$	$S_2 = S_2(n, r)$
$(-4, 3) = \frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}A$	$(-4, 2) = \frac{r(r-1)(r-2)(r-3)}{n(n-1)(s+1)(s+2)}A$
$(-2, 3) = \frac{r(r-1)(r+1)}{(s+1)(s+2)(s+3)}A$	$(-3, 3) = \frac{r(r-1)(r-2)}{(s+1)(s+2)(s+3)}A$
$(0, 2) = \frac{(n+1)(n+2)}{(s+1)(s+2)}A$	$(-1, 3) = \frac{r(n+1)(n+2)}{(s+1)(s+2)(s+3)}A$
$(2, 0) = \frac{(n+1)(n+2)}{(r+1)(r+2)}A$	$(1, 1) = \frac{(n+1)(n+2)}{(r+1)(s+1)}A$
$(4, -2) = \frac{s(s-1)(n+1)(n+2)}{(r+1)(r+2)(r+3)(r+4)}A$	$(3, -1) = \frac{s(n+1)(n+2)}{(r+1)(r+2)(r+3)}A$
$(3, -3) = \frac{s(s-1)(s-2)}{(r+1)(r+2)(r+3)}A$	$(4, -3) = \frac{s(s-1)(s-2)(n+1)}{(r+1)(r+2)(r+3)(r+4)}A$
$(1, -3) = \frac{s(s-1)(s-2)}{n(n-1)(r+1)}A$	$(2, -3) = \frac{s(s-1)(s-2)}{n(r+1)(r+2)}A$
$(-1, -1) = \frac{rs}{n(n-1)}A$	$(0, -2) = \frac{s(s-1)}{n(n-1)}A$
$(-3, 1) = \frac{r(r-1)(r-2)}{n(n-1)(s+1)}A$	$(-2, 0) = \frac{r(r-1)}{n(n-1)}A$

Theorem: For any $n \geq 7$, $r \geq 4$, $s \geq 4$, with $r + s = n$ and S_1 and S_2 as above, gcd S_1 = gcd S_2.

Proof: Let p be any prime and, for convenience, set $v((a, b)) = v(a, b)$. Also, set

$$v_i = v_i(p) = \min_{(a, b) \in S_i} \{v(a, b)\}, \quad i = 1, 2.$$

Clearly, we must show that $v_1 = v_2$ for all p. In fact, we show that both assumptions $v_1 < v_2$ and $v_2 < v_1$ lead to contradictions, so the desired equality must hold. Actually, the proof is not elegant. Since we can use neither symmetry nor rotation arguments, it is necessary to consider individually the nine cases where we successively let $v_1 = v(a_i)$, $a_i \in S_1$, and show each time that the assumption $v_1 < v_2$ leads to a contradiction. It is also necessary to consider individually the nine cases where $v_2 = v(a_i)$, $a_i \in S_2$, and show each time that the assumption $v_2 < v_1$ leads to a contradiction. In fact, since all these arguments are very similar, we only prove case 1, where we take $v_1 = v(-4, 3) < v_2$.

For $(a, b) \in S_i$, let $u((a, b)) = u(a, b) = v(a, b) - v(A)$ and let $u_i = v_i - v(A)$ for each i. With this notation, it is clear that the assumption $v_1 < v_2$ is equivalent to $u_1 < u_2$. First, assume that p is odd. The assumption $u_1 < u_2$ implies that $u_1 < u(a_i)$ for all $a_i \in S_2$. Therefore, in particular,

$$u_1 < u(-4, 2) \quad \text{and} \quad u_1 < u(-3, 3);$$

that is,

$$(7) \qquad v\left(\frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}\right) < v\left(\frac{r(r-1)(r-2)(r-3)}{n(n-1)(s+1)(s+2)}\right)$$

and

$$(8) \qquad v\left(\frac{r(r-1)(r-2)(r-3)}{n(s+1)(s+2)(s+3)}\right) < v\left(\frac{r(r-1)(r-2)}{(s+1)(s+2)(s+3)}\right).$$

But, using (5), (7), and (8) clearly implies that

$$(9) \qquad v(s+3) > v(n-1) = v(r-4) \geq 0$$

and

(10) $\quad v(n) > v(r - 3) = v(s + 3) > 0,$

whence it follows that $p|n$, $p|(s + 3)$, and $p|(r - 3)$ since $r + s = n$. But now, since p is odd,

(11) $\quad p \nmid (n - 1)(r - 1)(r - 2)(s + 1)(s + 2)$

and it follows that

(12) $$u(-2, 0) = v\left(\frac{r(r - 1)}{n(n - 1)}\right) = v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

contrary to the assumption that $u_1 < u_2$, since $(-2, 0) \in S_2$.

Now assume that $p = 2$. Then all of the above up to, but not including (11), still holds and we may conclude that n is even and r and s are odd. Thus, $2 \nmid r(r - 2)(r + 2)s(s + 2)(n + 1)$. Also, $v(s + 3) > 0$ in (10); hence $v(n) \geq 2$. But this implies that $v(n + 2) = 1$ since every second even integer is divisible by only 2^1 and no higher power. If $v(n) \leq v(r - 1)$, then $v(r - 1) \geq 2$ and

$$u(-1, 3) = v\left(\frac{r(n + 1)(n + 2)}{(s + 1)(s + 2)(s + 3)}\right) \leq v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

contrary to the assumption that $u_1 < u_2$ since $u(-1, 3) \in S_2$. Therefore, again using (5), $v(n) > v(r - 1) = v(s + 1)$. If $v(n) \leq v(r + 1)$, then

$$u(1, 1) = v\left(\frac{(n + 1)(n + 2)}{(r + 1)(s + 1)}\right) \leq v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

since $v(n + 2) = 1 \leq v(s + 1)$ from above. Since this is again a contradiction, it follows that $v(n) > v(r + 1) = v(s - 1)$ by (5). But then

$$u(2, -3) = v\left(\frac{s(s - 1)(s - 2)}{n(r + 1)(r + 2)}\right) = v\left(\frac{r(r - 1)(r - 2)(r - 3)}{n(s + 1)(s + 2)(s + 3)}\right) = u_1$$

by (10), and this again contradicts the assumption $u_1 < u_2$ since $u(2, -3) \in S_2$.

Since similar arguments lead to contradictions in all the remaining seventeen cases, we conclude that $v_1 = v_2$ for all p and hence that gcd S_1 = gcd S_2 as claimed.

We note that this argument, as in the preceding paper [9], depends on the fact that we have only a very finite number of cases to consider. The general argument for hexagons of arbitrary size will have to be much different and much more sophisticated.

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ON THE r^{th}-ORDER NONHOMOGENEOUS RECURRENCE RELATION AND SOME GENERALIZED FIBONACCI SEQUENCES

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1. Introduction

Consider the nonhomogeneous recurrence relation

(1.1) $$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j$$

with

$$G_0 = 1;\ G_1 = 1.$$

In [1], Asveld expressed G_n in terms of Fibonacci numbers F_n and F_{n-1} and in the parameters $\alpha_0, \alpha_1, \ldots, \alpha_k$. He proved that

(1.2) $$G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)},$$

where $G_n^{(p)}$ is a particular solution of (1.1).

In this paper, we generalize this result in two ways: First, we generalize Asveld's result by taking the second-order recurrence relation as

$$T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^{k} \beta_j n^j$$

with

$$T_0 = a;\ T_1 = b.$$

Second, we prove similar results for the third-order and the r^{th}-order recurrence relations; cf. also [6].

In Section 2, we prove the results for the generalized second-order recurrence relation. In Section 3, we prove the theorem for the third-order recurrence relation. In Section 4, we mention the results for the r^{th}-order recurrence relation.

2. Generalized Second-Order Relation

Let the second-order nonhomogeneous recurrence relation be given by

(2.1) $$T_n = PT_{n-1} + QT_{n-2} + \sum_{j=0}^{k} \beta_j n^j$$

with

$$T_0 = a;\ T_1 = b.$$

Let the homogeneous relation corresponding to (2.1) be written as

(2.2) $$S_n = PS_{n-1} + QS_{n-2}$$

with the same initial conditions as for T_n, viz.,

$$S_0 = a;\ S_1 = b.$$

Whenever necessary, we denote the sequence S_n with the initial conditions $S_0 = a$, $S_1 = b$ as $S_n(a, b)$. It is well known that the solution of (2.2) is given by

$$(2.3)\qquad S_n(a,\ b) = \frac{1}{\alpha_2 - \alpha_1}[(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})]$$

where α_1 and α_2 are distinct roots of the characteristic equation of (2.2); see [5].

Note that

$$(2.4)\qquad \alpha_1 + \alpha_2 = P;\ \alpha_1\alpha_2 = -Q.$$

Also,

$$(2.5)\qquad S_n(1,\ 0) = \frac{1}{\alpha_2 - \alpha_1}[P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})],$$

$$(2.6)\qquad S_n(0,\ 1) = -\frac{1}{\alpha_2 - \alpha_1}[\alpha_1^n - \alpha_2^n],$$

and

$$(2.7)\qquad S_n(1,\ 1) = \frac{1}{\alpha_2 - \alpha_1}[(P - 1)(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})].$$

Theorem 2.1: The solution of (2.1) is given by

$$T_n = S_n(a,\ b) - S_n(1,\ 0)T_0^{(p)} - S_n(0,\ 1)T_1^{(p)} + T_n^{(p)},$$

where $S_n(a,\ b)$, $S_n(1,\ 0)$, and $S_n(0,\ 1)$ are given by (2.3), (2.5), and (2.6), respectively, and $T_n^{(p)}$ is a particular solution of (2.1).

Proof: The solution of (2.1) is given by

$$T_n = T_n^{(h)} + T_n^{(p)},$$

where $T_n^{(h)}$ is the solution of (2.2) and $T_n^{(p)}$ is a particular solution of (2.1).

Now

$$(2.8)\qquad T_n = c_1\alpha_1^n + c_2\alpha_2^n + T_n^{(p)},$$

where

$$T_0 = a;\ T_1 = b.$$

Therefore,

$$(2.9)\qquad \begin{cases} c_1 + c_2 = a - T_0^{(p)}, \\ c_1\alpha_1 + c_2\alpha_2 = b - T_1^{(p)}. \end{cases}$$

Solving (2.9) simultaneously, we get

$$c_1 = \frac{(a - T_0^{(p)})\alpha_2 - b + T_1^{(p)}}{\alpha_2 - \alpha_1} = \frac{(a - T_0^{(p)})(P - \alpha_1) - b + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Hence,

$$(2.10)\qquad c_1 = \frac{\alpha_1(T_0^{(p)} - a) + aP - b - PT_0^{(p)} + T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Similarly,

$$(2.11)\qquad c_2 = \frac{\alpha_2(-T_0^{(p)} + a) - aP + b + PT_0^{(p)} - T_1^{(p)}}{\alpha_2 - \alpha_1}.$$

Thus, by using (2.10) and (2.11) in (2.8), we have

$$\begin{aligned} T_n &= \frac{1}{\alpha_2 - \alpha_1}[(aP - b - PT_0^{(p)} + T_1^{(p)})(\alpha_1^n - \alpha_2^n) \\ &\qquad - (a - T_0^{(p)})(\alpha_1^{n+1} - \alpha_2^{n+1})] + T_n^{(p)} \\ &= \frac{1}{\alpha_2 - \alpha_1}\{[(aP - b)(\alpha_1^n - \alpha_2^n) - a(\alpha_1^{n+1} - \alpha_2^{n+1})] \\ &\qquad - [P(\alpha_1^n - \alpha_2^n) - (\alpha_1^{n+1} - \alpha_2^{n+1})]T_0^{(p)} \\ &\qquad - [-(\alpha_1^n - \alpha_2^n)]T_1^{(p)} + T_n^{(p)}. \end{aligned}$$

By using (2.3), (2.5), and (2.6) we finally obtain

$$(2.12)\quad T_n = S_n(a,\ b) - S_n(1,\ 0)T_0^{(p)} - S_n(0,\ 1)T_1^{(p)} + T_n^{(p)}.$$

Remarks:

(1) Note that, if $a = 1$, $b = 1$, $P = 1$, $Q = 1$, (2.12) reduces to Asveld's result given by (1.2). Here we use the fact that

$$S_n(1,\ 0) = -F_{n-1} + F_n = F_{n-2},\quad S_n(0,\ 1) = F_{n-1},\quad S_n(1,1) = F_n.$$

(2) To get a complete solution of (2.1), let the particular solution $T_n^{(p)}$ be given by

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i.$$

Then, from (2.1) we get

$$\sum_{i=0}^{k} A_i n^i - P\sum_{i=0}^{k} A_i(n-1)^i - Q\sum_{i=0}^{k} A_i(n-2)^i - \sum_{i=0}^{k} \beta_i n^i = 0$$

or

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k}\left(\sum_{\ell=0}^{i} A_i\binom{i}{\ell}(-1)^{i-\ell}(P + Q2^{i-\ell})n^{\ell}\right) - \sum_{i=0}^{k}\beta_i n^i = 0.$$

For each i $(0 \le i \le k)$, we have

$$(2.13)\quad A_i - \sum_{m=i}^{k}\gamma_{im}A_m - \beta_i = 0$$

where, for $m \ge i$,

$$\gamma_{im} = \binom{m}{i}(-1)^{m-i}(P + Q2^{m-i}).$$

From the recurrence relation (2.13), $A_k, \ldots, A_0$ can be computed where A_i is a linear combination of $\beta_i, \ldots, \beta_k$. To get a more explicit solution as in Asveld [1], we put

$$A_i = -\sum_{j=i}^{k} a_{ij}\beta_j,$$

where a_{ij} are as defined below. Then we get the following solution for (2.12):

$$T_n = S_n(a,\ b) + S_n(1,\ 0)\lambda_k^0 + S_n(0,\ 1)\lambda_k^1 - \sum_{j=0}^{k}\beta_j r_j(n),$$

where

$$\lambda^0 = \sum_{j=0}^{k}\beta_j a_{0j},\quad \lambda_k^1 = \sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}a_{ij},\quad\text{and}\quad r_j(n) = \sum_{i=0}^{j}a_{ij}n^i.$$

Note that

$$\gamma_{ii} = P + Q,\quad a_{ii} = \frac{1}{P + Q - 1},\quad\text{and}\quad a_{ij} = -\sum_{m=i+1}^{j}\gamma_{im}a_{mj},\quad j > i.$$

(3) If $a = 2$, $b = 1$, $P = 1$, and $Q = 1$, the sequence $S_n(a,\ b)$ reduces to the Lucas sequence L_n. Then (2.12) reduces to

$$T_n = L_n - T_0^{(p)}F_n + (T_0^{(p)} - T_1^{(p)})F_{n-1} + T_n^{(p)}.$$

(4) We are grateful to the referee for pointing out references [6], [7], and [8]. It should be noted that our results are more general than those in [6]. One can also prove results similar to those in [6] and [7] without much difficulty.

3. Third-Order Recurrence Relation

Let the third-order recurrence relation be given by

(3.1) $$T_n = P_1T_{n-1} + P_2T_{n-2} + P_3T_{n-3} + \sum_{j=0}^{k} \beta_j n^j.$$

Let the homogeneous relation corresponding to (3.1) be written as

(3.2) $$S_n = P_1S_{n-1} + P_2S_{n-2} + P_3S_{n-3}.$$

Denote the sequence S_n by S_n^1, S_n^2, S_n^3, when

(3.3) $$S_0 = 0,\ S_1 = 1,\ S_2 = P_1,$$

(3.4) $$S_0 = 1,\ S_1 = 0,\ S_2 = P_2, \quad \text{and}$$

(3.5) $$S_0 = 0,\ S_1 = 0,\ S_2 = P_3,$$

respectively.

Denote the sequence T_n with initial conditions the same as (3.3), (3.4), and (3.5) by T_n^1, T_n^2, T_n^3, respectively. If α_1, α_2, α_3 are distinct roots of the characteristic equation corresponding to (3.2), then

$$S_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n$$

with

(3.6) $$\alpha_1 + \alpha_2 + \alpha_3 = P_1; \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = -P_2; \quad \alpha_1\alpha_2\alpha_3 = P_3.$$

Using standard methods, we obtain

$$S_n^1 = \frac{1}{\Delta}[\alpha_1^{n+1}(\alpha_3 - \alpha_2) - \alpha_2^{n+1}(\alpha_3 - \alpha_1) + \alpha_3^{n+1}(\alpha_2 - \alpha_1)],$$

$$S_n^2 = \frac{1}{\Delta}[\alpha_1^{n+1}(\alpha_3^2 - \alpha_2^2) - \alpha_2^{n+1}(\alpha_3^2 - \alpha_1^2) + \alpha_3^{n+1}(\alpha_2^2 - \alpha_1^2)],$$

$$S_n^3 = \frac{P_3}{\Delta}[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)],$$

where $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} = (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1);$$ see [4].

By making use of (3.6), we easily get

$$S_n^2 = -P_1S_n^1 + S_{n+1}^1, \quad S_n^3 = P_3S_{n-1}^1.$$

For the sake of convenience, let T_n^1 be denoted by T_n in what follows.

Theorem 3.1: T_n is given in terms of S_n^1 by

$$T_n = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} - T_2^{(p)})S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)}.$$

Proof: Let $T_n^{(h)}$ be the solution of (3.2) and $T_n^{(p)}$ be a particular solution of (3.1). Then

(3.7) $$T_n = T_n^{(h)} + T_n^{(p)}$$

where

(3.8) $$T_n^{(h)} = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n$$

with initial conditions

$$T_0 = 0,\ T_1 = 1,\ T_2 = P_1.$$

Using these initial conditions, we have

$$c_1 + c_2 + c_3 = -T_0^{(p)},$$
$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 1 - T_1^{(p)},$$
$$c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 = P_1 - T_2^{(p)}.$$

Solving these equations simultaneously, we get

$$c_1 = \frac{\alpha_3 - \alpha_2}{\Delta}[-T_0^{(p)}\alpha_2\alpha_3 - (1 - T_1^{(p)})(\alpha_2 + \alpha_3) + (P_1 - T_2^{(p)})]$$
$$= \frac{\alpha_3 - \alpha_2}{\Delta}\left[-\frac{P_3}{\alpha_1}T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_1) + P_1 - T_2^{(p)}\right].$$

Similarly,

$$c_2 = -\frac{\alpha_3 - \alpha_1}{\Delta}\left[-\frac{P_3}{\alpha_2}T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_2) + P_1 - T_2^{(p)}\right]$$

and

$$c_3 = \frac{\alpha_2 - \alpha_1}{\Delta}\left[-\frac{P_3}{\alpha_3}T_0^{(p)} - (1 - T_1^{(p)})(P_1 - \alpha_3) + P_1 - T_2^{(p)}\right].$$

Hence, substituting for c_1, c_2, c_3 in (3.8) and simplifying we get

$$T_n^{(h)} = \{-P_3T_0^{(p)}[\alpha_1^{n-1}(\alpha_3 - \alpha_2) - \alpha_2^{n-1}(\alpha_3 - \alpha_1) + \alpha_3^{n-1}(\alpha_2 - \alpha_1)]$$
$$-P_1(1 - T_1^{(p)})[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)]$$
$$+ (1 - T_1^{(p)})[\alpha_1^{n+1}(\alpha_3 - \alpha_2) - \alpha_2^{n+1}(\alpha_3 - \alpha_1) + \alpha_3^{n+1}(\alpha_2 - \alpha_1)]$$
$$+ (P_1 - T_2^{(p)})[\alpha_1^n(\alpha_3 - \alpha_2) - \alpha_2^n(\alpha_3 - \alpha_1) + \alpha_3^n(\alpha_2 - \alpha_1)]\}/\Delta$$
$$= -P_3T_0^{(p)}S_{n-2}^1 - P_1(1 - T_1^{(p)})S_{n-1}^1 + (2 - T_1^{(p)})S_n^1 + (P_1 - T_2^{(p)})S_{n-1}^1.$$

On further simplification, (3.7) reduces to

(3.9) $$T_n = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} - T_2^{(p)})S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)},$$

which is the required result.

Remarks:

(1) If $P_1 = 1$, $P_2 = 1$, $P_3 = 0$, and $T_0 = 0$, $T_1 = 1$, (3.1) and (3.2) reduce to the second-order relations (2.1) and (2.2) with $P = Q = 1$ and $a = 0$, $b = 1$. With the above values of P_1, P_2, and P_3, T_n given by (3.9) reduces to

$$T_n = (T_1^{(p)} - T_2^{(p)})S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)}.$$

We verify whether this equation reduces to (2.13) with $a = 0$, $b = 1$. Now

$$T_1^{(p)} - T_2^{(p)} = T_1 - T_1^{(h)} - T_2 + T_2^{(h)},$$

since $T_n = T_n^{(h)} + T_n^{(p)}$. Also,

$$T_1 = T_2 = 1 \quad\text{and}\quad T_2^{(h)} = T_1^{(h)} + T_0^{(h)}.$$

Therefore,

$$T_1^{(p)} - T_2^{(p)} = -T_1^{(h)} + T_2^{(h)} = T_0^{(h)} = T_0 - T_0^{(p)} = -T_0^{(p)},$$

since $T_0 = 0$. Thus,

(3.10) $$T_n = -T_0^{(p)}S_{n-1}^1 + (1 - T_1^{(p)})S_n^1 + T_n^{(p)}.$$

Note that here $S_n^1 = S_n(0, 1)$. Now

$$S_n(1, 0) = S_{n-1}(0, 1).$$

Hence, (3.10) reduces to

$$T_n = -S_n(1, 0)T_0^{(p)} + (1 - T_1^{(p)})S_n(0, 1) + T_n^{(p)},$$

which is identical with (2.12).

(2) On similar lines, we can prove the following:

$$T_n^2 = P_3(1 - T_0^{(p)})S_{n-2}^1 + (P_1T_1^{(p)} + P_2 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)};$$
$$T_n^3 = -P_3T_0^{(p)}S_{n-2}^1 + (P_1T_1^{(p)} + P_3 - T_2^{(p)})S_{n-1}^1 - T_1^{(p)}S_n^1 + T_n^{(p)}.$$

(3) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i \quad \text{and} \quad A_i = -\sum_{j=i}^{k} a_{ij}\beta_j,$$

a_{ij} as defined below, the sequences T_n^1 can be expressed as

$$T_n^1 = P_3\lambda_k^0 S_{n-2}^1 + (-P_1\lambda_k^1 + \lambda_k^2)S_{n-1}^1 + (1 + \lambda_k^1)S_n^1 - \sum_{j=0}^{k}\beta_j r_j(n),$$

where

$$\lambda_k^0 = \sum_{j=0}^{k}\beta_j a_{0j}, \quad \lambda_k^1 = \sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}a_{ij}, \quad \lambda_k^2 = \sum_{j=0}^{k}\beta_j\sum_{i=0}^{j}2^i a_{ij},$$

$$r_j(n) = \sum_{i=0}^{j}a_{ij}n^i, \quad a_{ij} = -\sum_{m=i+1}^{j}\delta_{im}a_{mj}, \; j > i,$$

and

$$\delta_{im} = \binom{m}{i}(-1)^{m-i}[P_1 + P_2 2^{m-i} + P_3 3^{m-i}].$$

(4) Similar results as above can be obtained for T_n^2 and T_n^3.

4. The r^{th}-Order Recurrence Relation

Let

(4.1) $$T_n = P_1T_{n-1} + P_2T_{n-2} + \cdots + P_{r-1}T_{n-r+1} + P_rT_{n-r} + \sum_{j=0}^{k}\beta_j n^j, \; r \ge 3,$$

be the r^{th}-order recurrence relation with three sets of initial conditions as

(4.2) $T_m = 0$, for $0 \le m \le r - 3$, $T_{r-2} = 1$, $T_{r-1} = P_1$,

(4.3) $T_m = 0$, for $0 < m < r - 1$, $T_0 = 1$, $T_{r-1} = P_2$,

(4.4) $T_m = 0$, for $0 \le m \le r - 2$, $T_{r-1} = P_3$.

The homogeneous part of (4.1) is the generalized r^{th}-order Fibonacci sequence. Let it be denoted by S_n so that

$$S_n = P_1S_{n-1} + P_2S_{n-2} + \cdots + P_rS_{n-r}.$$

We take the same initial conditions as in (4.2)-(4.4). Following the same method as in Section 3, we can prove the following results:

$$\begin{aligned} T_n^1 = {} & -P_rT_0^{(p)}S_{n-2}^1 + (P_{r-2}T_1^{(p)} + \cdots + P_1T_{r-2}^{(p)} - T_{r-1}^{(p)})S_{n-1}^1 \\ & + (1 + P_{r-3}T_1^{(p)} + \cdots + P_1T_{r-3}^{(p)} - T_{r-2}^{(p)})S_n^1 + \cdots \\ & + (P_1T_1^{(p)} - T_2^{(p)})S_{n+r-4}^1 - T_1^{(p)}S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

$$\begin{aligned} T_n^2 = {} & P_r(1 - T_0^{(p)})S_{n-2}^1 + (P_{r-2}T_1^{(p)} + \cdots + P_1T_{r-2}^{(p)} + P_2 - T_{r-1}^{(p)})S_{n-1}^1 \\ & + (P_{r-3}T_1^{(p)} + \cdots + P_1T_{r-3}^{(p)} - T_{r-2}^{(p)})S_n^1 + \cdots \\ & + (P_1T_1^{(p)} - T_2^{(p)})S_{n+r-4}^1 - T_1^{(p)}S_{n+r-3}^1 + T_n^{(p)}; \end{aligned}$$

and

$$T_n^3 = -P_r T_0^{(p)} S_{n-2}^1 + (P_{r-2} T_1^{(p)} + \cdots + P_1 T_{r-2}^{(p)} + P_3 - T_{r-1}^{(p)}) S_{n-1}^1 + (P_{r-3} T_1^{(p)} + \cdots + P_1 T_{r-3}^{(p)} - T_{r-2}^{(p)}) S_n^1 + \cdots + (P_1 T_1^{(p)} - T_2^{(p)}) S_{n+r-4}^1 - T_1^{(p)} S_{n+r-3}^1 + T_n^{(p)}.$$

Here we denote S_n with initial conditions (4.2) by S_n^1 and T_n with initial conditions (4.2), (4.3), (4.4) by T_n^1, T_n^2, T_n^3, respectively.

Remarks:

(1) For $r = 3$, T_n^1 reduces to the result of Theorem 3.1.

(2) As in Remark (2) of Section 2, taking

$$T_n^{(p)} = \sum_{i=0}^{k} A_i n^i \quad \text{and} \quad A_i = -\sum_{j=i}^{k} a_{ij}\beta_j,$$

the sequence T_n^1 can be expressed as follows:

$$T_n^1 = P_r \lambda_k^0 S_{n-2}^1 - (P_{r-2}\lambda_k^1 + \cdots + P_1 \lambda_k^{r-2} - \lambda_k^{r-1}) S_{n-1}^1 + (1 - P_{r-3}\lambda_k^1 - \cdots - P_1 \lambda_k^{r-3} + \lambda_k^{r-2}) S_n^1 + \cdots + (-P_1 \lambda_k^1 + \lambda_k^2) S_{n+r-4}^1 + \lambda^1 S_{n+r-3}^1 - \sum_{j=0}^{k} \beta_j r_j(n),$$

where

$$\lambda_0^k = \sum_{j=0}^{k} \beta_j a_{0j}, \qquad \lambda_k^1 = \sum_{j=0}^{k} \beta_j \sum_{i=0}^{j} a_{ij} \ell^i, \quad \ell = 1, 2, \ldots, r-1;$$

$$r_j(n) = \sum_{i=0}^{j} a_{ij} n^i, \quad a_{ij} = -\sum_{m=i+1}^{j} \delta_{im} a_{mj}, \; j > i;$$

and

$$\delta_{im} = \binom{m}{i}(-1)^{m-i}[P_1 + P_2 2^{m-i} + \cdots + P_r r^{m-i}].$$

(3) Similarly, we can write the values of T_n^2 and T_n^3.

(4) In [3], Asveld derived expressions for the family of differential equations corresponding to (1.1).

It is natural to ask whether such results can be proved for the r^{th}-order recurrence relation. This is the subject of our next paper.

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AREA-BISECTING POLYGONAL PATHS

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Introduction

For a rather striking geometrical property of the Fibonacci sequence

$$f_0 = f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad (n = 2, 3, \ldots),$$

consider the lattice points defined by $F_0 = (0, 0)$ and

$$F_n = (f_{n-1}, f_n), \quad X_n = (f_{n-1}, 0), \quad Y_n = (0, f_n) \quad (n = 1, 2, 3, \ldots).$$

Then, as we shall prove: for each $n \geq 1$, the polygonal path

$$F_0F_1F_2 \ldots F_{2n+1}$$

splits the rectangle

$$F_0X_{2n+1}F_{2n+1}Y_{2n+1}$$

into two regions of equal area. Figure 1 illustrates this area-splitting property for $n = 0, 1, 2$.

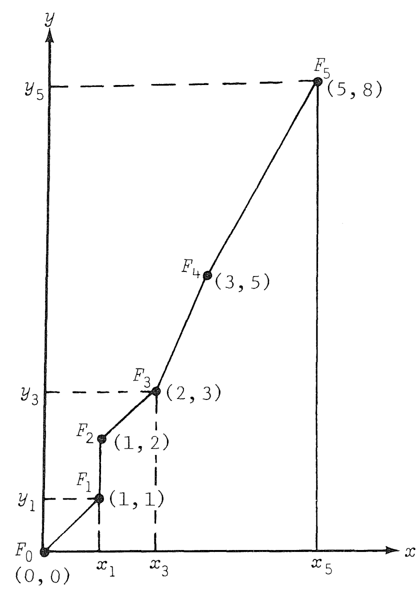

Figure 1

In view of the above, it seems only natural to ask if there exist other types of area-splitting paths, and how they may be characterized. To give some answers, it will be convenient to introduce the following notation and terminology.

We henceforth assume that every point with zero subscript is the origin. In particular, $P_0 = (0, 0)$ and each point $P_n = (x_n, y_n)$ has projections $X_n = (x_n, 0)$ and $Y_n = (0, y_n)$ on the axis. We shall also assume that a polygonal path has distinct vertices (that is, $P_n \neq P_m$ for $n \neq m$).

A polygonal path $P_0P_1P_2\ldots$ will be called *nondecreasing* if the abscissas and the ordinates of its vertices P_0, P_1, P_2, ... are each nondecreasing sequences. An *area-bisecting* k-path ($k \geq 2$) is a nondecreasing path $P_0P_1P_2\ldots$ that satisfies

$$\text{(1)} \qquad \text{Area}\{P_0P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \text{Area}\{P_0P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$$

for each integer $n \geq 0$.

An area-bisecting k-path is an area-bisecting Nk-path for each natural number N. The converse, however, is false. In Figure 2, any area-bisecting 4-path beginning with $P_0P_1P_2P_3P_4P_5$ cannot be an area-bisecting 2-path because area$\{X_1P_1P_2P_3X_3\}$ is not equal to area$\{Y_1P_1P_2P_3Y_3\}$.

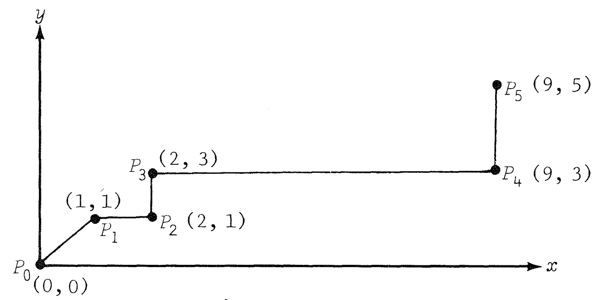

Figure 2

To characterize the situation when (1) holds, consider any segment $P_m P_{m+1}$ of the path $P_0P_1P_2\ldots$ (Fig. 3). Since

$$\text{(2a)} \qquad 2 \cdot \text{area}\{X_mP_mP_{m+1}X_{m+1}\} = x_{m+1}y_{m+1} - x_my_m - \begin{vmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{vmatrix}$$

and

$$\text{(2b)} \qquad 2 \cdot \text{area}\{Y_mP_mP_{m+1}Y_{m+1}\} = x_{m+1}y_{m+1} - x_my_m - \begin{vmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{vmatrix}$$

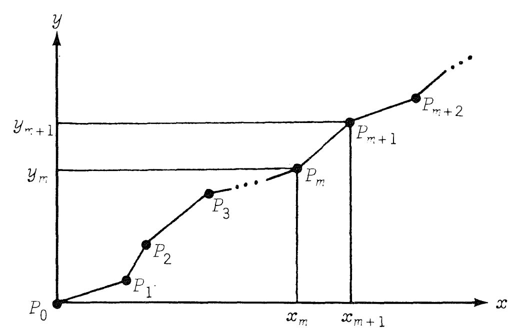

Figure 3

we see that (1) holds for each $n \geq 0$ if and only if the determinantal equation

$$\text{(3)} \qquad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{nk} & y_{nk} \\ x_{nk+1} & y_{nk+1} \end{vmatrix} = 0$$

holds for each $n \geq 1$. This can be summarized as follows.

Theorem 1: A nondecreasing path $P_0P_1P_2\ldots$ is an area-bisecting k-path if and only if

$$(4)\qquad \begin{vmatrix} x_{nk+1} & y_{nk+1} \\ x_{nk+2} & y_{nk+2} \end{vmatrix} + \begin{vmatrix} x_{nk+2} & y_{nk+2} \\ x_{nk+3} & y_{nk+3} \end{vmatrix} + \cdots + \begin{vmatrix} x_{(n+1)k} & y_{(n+1)k} \\ x_{(n+1)k+1} & y_{(n+1)k+1} \end{vmatrix} = 0$$

for each $n \ge 0$.

Remark: $P_0P_1P_2\ldots$ is an area-bisecting k-path if and only if its reflection about the line $y = x$ is an area-bisecting k-path.

To confirm that the Fibonacci path $F_0F_1F_2\ldots$ is area-bisecting, set $P_0 = F_0$ and let P_n be the point $F_n = (f_{n-1}, f_n)$ for each $n \ge 1$. For $k = 2$, condition (4) reduces to

$$\begin{vmatrix} f_{2n-2} & f_{2n-1} \\ f_{2n-1} & f_{2n} \end{vmatrix} + \begin{vmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{vmatrix} = 0$$

for each $n \ge 1$. This is clearly true since $f_i = f_{i-1} + f_{i-2}$ for each $i \ge 2$. Verification that $F_0F_1F_2\ldots$ is an area-bisecting 2-path can also be obtained by letting $\alpha = \beta = 1 = k - 1$ and setting $s_1 = f_0$ and $s_2 = f_1$ in the following.

Corollary 1.1: Let $S_n = (s_n, s_{n+1})$ for the positive sequence

$$(5)\qquad s_1,\ s_2, \text{ and } s_n = \beta s_{n-1} + \alpha s_{n-2} \quad (n \ge 3).$$

Then $S_0S_1S_2\ldots$ is an area-bisecting k-path if and only if:

(i) k is even and $\alpha = 1$ for nondecreasing $\{s_n : n \ge 1\}$

or

(ii) $s_2^2 = s_1s_3$ [which is equivalent to $S_0S_1S_2\ldots$ being embedded in the straight line $y = (s_2/s_1)x$].

Proof: First, observe that $s_2^2 = s_1s_3$ yields

$$s_3^2 = (\beta s_2 + \alpha s_1)s_3 = \beta s_2 s_3 + \alpha s_2^2 = s_2 s_4$$

and (by induction)

$$(6)\qquad s_{n+1}^2 = s_n s_{n+2} \quad (n \ge 1).$$

Since this is equavalent to

$$(7)\qquad \frac{s_{n+1}}{s_n} = \frac{s_2}{s_1} \quad (n \ge 1),$$

$s_2^2 = s_1s_3$ is equivalent to $S_0S_1S_2\ldots$ being contained in the line $y = (s_2/s_1)x$.

Conditions (i) and (ii) each ensure that $S_0S_1S_2\ldots$ is nondecreasing. Moreover, by (4), this path is an area-bisecting k-path if and only if

$$(8)\qquad \begin{vmatrix} s_{nk+1} & s_{nk+2} \\ s_{nk+2} & s_{nk+3} \end{vmatrix} + \cdots + \begin{vmatrix} s_{(n+1)k-1} & s_{(n+1)k} \\ s_{(n+1)k} & s_{(n+1)k+1} \end{vmatrix} + \begin{vmatrix} s_{(n+1)k} & s_{(n+1)k+1} \\ s_{(n+1)k+1} & s_{(n+1)k+2} \end{vmatrix} = 0$$

for each $n \ge 0$. Now observe that for each $m \ge 2$,

$$(9)\qquad s_m s_{m+2} - s_{m+1}^2 = -\alpha(s_{m-1}s_{m+1} - s_m^2)$$

follows from $s_{m+2} = \beta s_{m+1} + \alpha s_m$ and $s_{m+1} = \beta s_m + \alpha s_{m-1}$. Therefore, using (9) in successively recasting each determinant in (8) beginning with the rightmost determinant, we find that (8) is equivalent to

$$(10)\qquad (1 - \alpha + \alpha^2 - \cdots + (-\alpha)^{k-1})(s_{nk+1}s_{nk+3} - s_{nk+2}^2) = 0$$

for each $n \ge 0$. In particular, (10) holds for all $n \ge 0$ if and only if

$$\sum_{t=0}^{k-1}(-\alpha)^t = 0 \quad \text{or} \quad s_2^2 = s_1 s_3.$$

For real values of α, it is easily verified that

$$\sum_{t=0}^{k-1}(-\alpha)^t = 0 \text{ if and only if } k \text{ is even and } \alpha = 1.$$

Matrix-Generated Paths

Since the Fibonacci numbers can be generated by powers of the matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \left(\text{via } C_n = \begin{bmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{bmatrix} \text{ for each } n \geq 2\right),$$

the consecutive vertices $\{F_n = (f_{n-1}, f_n) : n = 1, 2, \ldots\}$ of the Fibonacci path are given precisely by the successive rows

$$\{F_{2n-1} = (f_{2n-2}, f_{2n-1}) \quad \text{and} \quad F_{2n} = (f_{2n-1}, f_{2n})\}$$

of $(C^2)^n$:

$$C^{2n} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{bmatrix} f_{2n-2} & f_{2n-1} \\ f_{2n-1} & f_{2n} \end{bmatrix} \quad (n \geq 1).$$

Thus, the Fibonacci path $F_0F_1F_2\ldots$ is generated by $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ in the following sense:

A path $P_0P_1P_2\ldots$ is said to be *matrix-generated* by $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n = \begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} \quad \text{for each } n \geq 1.$$

Example 1:

(i) If $S_n = (s_n, s_{n+1})$ for $(s_1, s_2) = (1, 2)$ and $s_n = s_{n-1} + 2s_{n-2}$, the area-bisecting path $S_0S_1S_2\ldots$ (contained in the line $y = 2x$) cannot be matrix-generated since the first row of

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

is not $S_3 = (4, 8)$. Note, however, that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ generates an area-bisecting path whose consecutive vertices are the successive rows of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^n$.

(ii) The area-bisecting path $P_0P_1P_2\ldots$ cannot be matrix-generated when $P_n = (f_{n-2}, f_{n-1})$ for the Fibonacci sequence beginning with $f_{-1} = 0$, or when $P_n = (\ell_n, \ell_{n+1})$ for the Lucas sequence beginning with $(\ell_1, \ell_2) = (1, 3)$ and $\ell_n = \ell_{n-1} + \ell_{n-2}$ $(n \geq 3)$.

(iii) The path in Figure 4 cannot be matrix-generated because

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & b \end{bmatrix}$$

is nonsingular, whereas points P_0, P_n, P_{n+1} are collinear if and only if $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ is singular. Indeed, P_0, P_n, P_{n+1} are collinear if and only if

$$0 = \begin{bmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n .$$

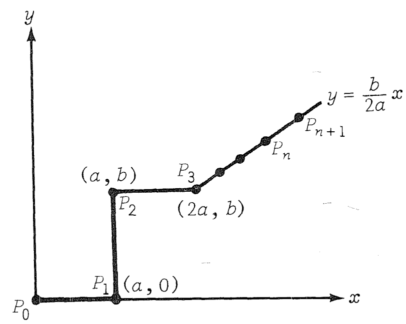

Figure 4

$ab(a - 1) \neq 0$

(iv) Suppose (Fig. 5) a nonsingular matrix U generates an area-bisecting path $P_0P_1P_2\ldots$. Then for $\theta \geq 0$, the successive rows of $\{\theta U^n : n \geq 1\}$ also produce an area-bisecting path $Q_0Q_1Q_2\ldots$, where $Q_n = \theta P_n$ for each $n \geq 1$. However, for $\theta \neq 1$, the path $Q_0Q_1Q_2\ldots$ cannot be matrix generated since $\theta U^n = (\theta U)^n$ for all $n \geq 1$ requires that U be singular.

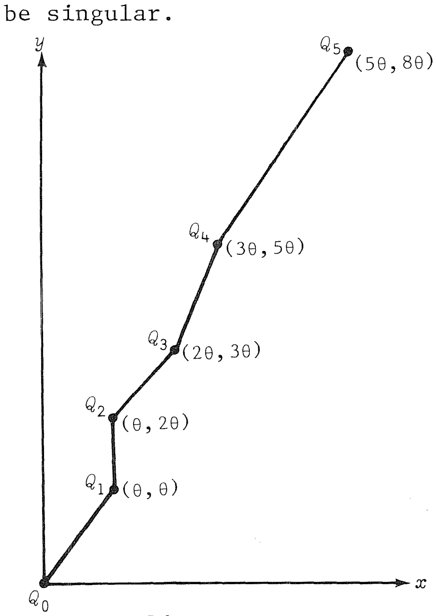

Figure 5

$Q_n = \theta F_n = (\theta f_{n-1}, \theta f_n); \ \theta(\theta - 1) \neq 0$

Under what conditions on the entries of a 2×2 real, nonnegative matrix U will the successive rows of U^n generate the consecutive vertices of an area-bisecting k-path? By definition, the path $P_0P_1P_2\ldots$ is generated by $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if and only if

$$\begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}^n \quad \text{for each } n \geq 1.$$

This is equivalent to

$$(11) \qquad \begin{bmatrix} x_{2n+1} & y_{2n+1} \\ x_{2n+2} & y_{2n+2} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{bmatrix} \quad \text{for all } n \geq 1.$$

Thus, $P_0P_1P_2\ldots$ is generated by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ if and only if for all $n \geq 1$:

(12a) $\quad x_{2n+1} = ax_{2n-1} + bx_{2n} \qquad y_{2n+1} = ay_{2n-1} + by_{2n}$

(12b) $\quad x_{2n+2} = cx_{2n-1} + dx_{2n} \qquad y_{2n+2} = cy_{2n-1} + dy_{2n}$

and (since path vertices are assumed to be distinct)

(12c) $\quad (x_{n+k}, y_{n+k}) \neq (x_n, y_n)$ for $k > 0$.

Note that (12c) requires that $(a, b) \neq (0, 1)$ and that $(c, d) \notin \{(a, b), (0, 1)\}$.

Theorem 2: The path $P_0P_1P_2\ldots$ generated by a real, nonnegative matrix

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nondecreasing if and only if

(13) $\quad a \leq c \leq a^2 + bc \quad$ and $\quad b \leq d \leq ab + bd.$

A nondecreasing U-generated path is an area-bisecting k-path if and only if:

(i) $\quad |U| = 0$

or

(14) $\quad$ (ii) $\quad (1 - a)\sum_{t=1}^{m} |U|^t = 0$ for $k = 2m$.

Proof: Since

$$U^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix},$$

the conditions in (13) are necessary for the U-generated path to be nondecreasing. To see that they are also sufficient, let $P_n = (x_n, y_n)$ for $n \geq 0$. Then (13) yields $x_2 \geq x_1 \geq 0$ and $y_2 \geq y_1 \geq 0$. From (12a) and (12b), we also obtain

$$x_{2n+2} - x_{2n+1} = (c - a)x_{2n-1} + (d - b)x_{2n} \geq 0$$

and

$$\begin{aligned} x_{2n+1} - x_{2n} &= ax_{2n-1} + (b - 1)x_{2n} \\ &= a(ax_{2n-3} + bx_{2n-2}) + (b - 1)(cx_{2n-3} + dx_{2n-2}) \\ &= (a^2 + bc - c)x_{2n-3} + (ab + bd - d)x_{2n-2} \\ &\geq 0. \end{aligned}$$

Thus, $x_{2n+2} \geq x_{2n+1} \geq x_{2n}$ for all $n \geq 0$. A similar argument establishes that $y_{2n+2} \geq y_{2n+1} \geq y_{2n}$ for all $n \geq 0$.

For the nondecreasing U-generated path, (12a) yields

$$\begin{vmatrix} x_{2n} & y_{2n} \\ x_{2n+1} & y_{2n+1} \end{vmatrix} = \begin{vmatrix} x_{2n} & y_{2n} \\ ax_{2n-1} + bx_{2n} & ay_{2n-1} + by_{2n} \end{vmatrix} = -a \begin{vmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{vmatrix}$$

for all $n \geq 1$. Thus,

(15) $$\begin{vmatrix} x_{2n-1} & y_{2n-1} \\ x_{2n} & y_{2n} \end{vmatrix} = |U|^n \quad \text{and} \quad \begin{vmatrix} x_{2n} & y_{2n} \\ x_{2n+1} & y_{2n+1} \end{vmatrix} = -a|U|^n$$

for all $n \geq 1$. We now use (15) to simplify (4). For $k = 2m$, condition (4) reduces to

$$(1 - a)|U|^{nm} \cdot \sum_{t=1}^{m} |U|^t = 0 \quad \text{for all } n \geq 0.$$

This is equivalent to (14). For $k = 2m + 1$, condition (4) reduces to

(16a) $$|U|^{nk/2} \cdot \left\{(1 - a)\sum_{t=1}^{m} |U|^t + |U|^{m+1}\right\} = 0 \quad (n \text{ even})$$

(16b) $$|U|^{n(k+1)/2} \cdot \left\{(1 - a)\sum_{t=1}^{m} |U|^t - a\right\} = 0 \quad (n \text{ odd}).$$

Conditions (16a) and (16b) can hold for all $n \geq 0$ only if $|U| = 0$. Indeed, $|U| \neq 0$ ensures that $a > 0$ and that $|U|^{m+1} = -a < 0$. But then, by equating the formulas for a in (16a) and (16b), we obtain the contradiction

$$a = |U|/a \quad \text{or} \quad |U| = a^2 > 0.$$

Corollary 2.1: Let $S_n = (s_n, s_{n+1})$ for the positive sequence

$$s_1,\ s_2, \text{ and } s_n = \beta s_{n-1} + \alpha s_{n-2} \quad (n \geq 3) \quad \text{[given as (5) above]}.$$

Then $S_0S_1S_2\ldots$ is a matrix-generated, area-bisecting k-path if and only if:

(17a) (i) k is even and $\beta = s_2 \geq s_1 = \alpha = 1$ (for $s_2^2 \neq s_1s_3$) in which case

(17b) $$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & \beta^{n+1} \end{bmatrix} \quad (n \geq 1);$$

or

(18a) (ii) $s_1^2 + s_2^2 = s_3$ and $s_2 \neq s_1$ (for $s_2^2 = s_1s_3$), in which case

(18b) $$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \left(\frac{s_2}{s_1}\right)^{2n-2} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \quad (n \geq 1).$$

Proof: An inductive argument, beginning with

$$[s_2 \quad s_3] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}$$

and

$$[s_3 \quad s_4] = [s_2 \quad s_3]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix} = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^2,$$

establishes that

$$[s_n \quad s_{n+1}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{n-1} \quad (n \geq 2).$$

In particular,

$$[s_{2n-1} \quad s_{2n}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2}$$

$$[s_{2n} \quad s_{2n+1}] = [s_1 \quad s_2]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-1} = [s_2 \quad s_3]\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2}$$

can be recast in matrix form as

$$\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}^{2n-2} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}^{n-1}$$

for all $n \geq 2$. Therefore, $S_0S_1S_2\ldots$ is $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$-generated if and only if

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^n = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}^{n-1} \quad \text{for all } n \geq 1. \tag{19}$$

(i) Assume that $s_2^2 \neq s_1s_3$. If $S_0S_1S_2\ldots$ is a matrix-generated, area-bisecting path, then (Corollary 1.1) k is even and $\alpha = 1$. Setting $n = 2$ in (19) and premultiplying by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^{-1},$$

we obtain

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha\beta \\ \beta & \beta^2 + \alpha \end{bmatrix}.$$

Since $\alpha = 1$, we see that $\beta = s_2$ and (17a) holds. Conversely, (17a) yields

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & \beta^2 + 1 \end{bmatrix}$$

and therefore (19). Since (17a) ensures that (13) and (14) hold (since $a = s_1 = 1$), the path $S_0S_1S_2\ldots$ is also area-bisecting.

(ii) Assume that $s_2^2 = s_1s_3$. Then (Corollary 1.1) the straight-line path $S_0S_1S_2\ldots$ is area-bisecting. Moreover, by (11), the path $S_0S_1S_2\ldots$ is generated by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} s_{2n+1} & s_{2n+2} \\ s_{2n+2} & s_{2n+3} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}\begin{bmatrix} s_{2n-1} & s_{2n} \\ s_{2n} & s_{2n+1} \end{bmatrix}$$

for all $n \geq 1$. Since $s_2^2 = s_1s_3$ is equivalent to (7), conditions (12a), (12b) reduce to

(20a) $s_{2n+1} = s_1 s_{2n-1} + s_2 s_{2n}$,

(20b) $s_{2n+2} = s_1 s_{2n} + s_2 s_{2n+1}$,

(20c) $s_{2n+3} = s_2 s_{2n} + s_3 s_{2n+1}$,

(20d) $s_{n+1} \neq s_n$,

for all $n \geq 1$. From (20a) and (20d), we obtain the necessary condition (18a) for $S_0S_1S_2...$ to be matrix generated. The condition $s_1^2 + s_2^2 = s_3$ in (18a) is also sufficient since, in the presence of (7), conditions (20a)-(20c) are equivalent for each fixed $n_0 \geq 1$, and condition (20c) holds for n_0 if and only if condition (20a) holds for $n_0 + 1$. Since (18a) satisfies (20a) for $n_0 = 1$, it follows that (20a)-(20c) hold for all $n \geq n_0 = 1$. This ensures that (12a)-(12c) hold for all $n \geq 1$, which means that $S_0S_1S_2...$ is generated by

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}.$$

Finally, condition (18a) also ensures that

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}^2 = \left(\frac{s_2}{s_1}\right)^2 \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

and (by induction) that (18b) holds.

Example 2: A nondecreasing path generated by matrix U that satisfies $(1 - a)|U| = 0$ is an area-bisecting $2N$-path for each natural number N. Since (14) holds for $|U| = -1$ when m is even, all nonnegative real matrices

$$\begin{bmatrix} a & b \\ c & (bc - 1)/a \end{bmatrix} \quad (a \neq 0) \quad \text{and} \quad \begin{bmatrix} 0 & b \\ 1/b & d \end{bmatrix} \quad (d \geq b > 1)$$

having determinantal value -1 that satisfy (12a)-(12c) and (13) also generate area-bisecting $4N$-paths for each natural number N. Thus,

$$\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 4 & 7/3 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}$$

generate area-bisecting 4 -paths ($n \geq 1$) that (Theorem 2) are not area-bisecting 2-paths.

Remarks: For the $\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}$-generated path $P_0P_1P_2...$, let $R_k = \text{Area}\{X_{k-1}P_{k-1}P_kX_k\}$ and $L_k = \text{Area}\{Y_{k-1}P_{k-1}P_kY_k\}$ for each $k \geq 1$. Then

$$R_k = \begin{Bmatrix} L_k, & k \text{ odd} \\ L_k - (-1)^{k/2}, & k \text{ even} \end{Bmatrix},$$

and $P_0P_1P_2...$ is an area-bisecting k-path if and only if $N = 4$ $(N \geq 1)$.

<u>δ-Splitting *k*-paths</u>

The notion of area-bisecting k-paths can be extended in several ways, the two most natural extensions being those given below as Definitions A and B. A nondecreasing path is a δ-splitting ($\delta \geq 0$) k-path if:

Definition A: $\text{Area}\{X_1P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \delta\cdot\text{Area}\{Y_1P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$ for all $n \geq 1$;

Definition B: $\text{Area}\{P_0P_1P_2\ldots P_{nk+1}X_{nk+1}\} = \delta\cdot\text{Area}\{P_0P_1P_2\ldots P_{nk+1}Y_{nk+1}\}$ for all $n \geq 1$.

Although these definitions are equivalent (to the area-bisecting property) when $\delta = 1$, they yield different results for $\delta \neq 1$. Beyond generalizing our results, motivation for investigating δ-splitting k-paths also comes from the following.

Example 3: Find an expression for $f(x)$ such that $p = \{(x, f(x)): x \geq 0\}$ is an increasing path characterized by

$$\text{Area}\{\mathscr{R}_x\} = \delta\cdot\text{Area}\{\mathscr{L}_x\}$$

for each point $(x, f(x)) \in p$. (See Fig. 6.)

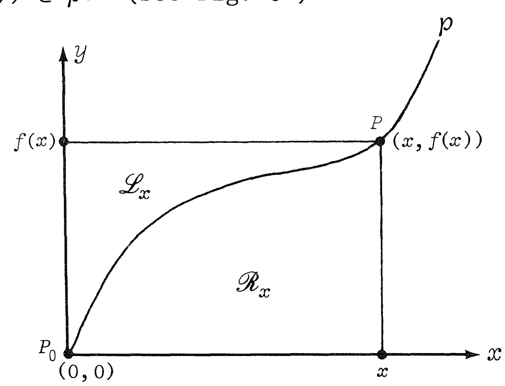

Figure 6

Our area requirement is equivalent to determining $f(x)$ such that

$$\int_0^x f(t)\,dt = \delta\cdot\int_0^{f(x)} f^{-1}(s)\,ds.$$

This can be recast as

$$(21)\qquad \left(1+\frac{1}{\delta}\right)\int_0^x f(t)\,dt = xf(x).$$

Differentiating (21) with respect to x and rearranging terms, we obtain

$$\frac{f'(x)}{f(x)} = \frac{1}{\delta x}.$$

Thus, for arbitrary positive constant C,

$$(22)\qquad f(x) = Cx^{1/\delta}.$$

For $\delta = 1$, we obtain the area-bisecting linear paths $f(x) = Cx$. Equation (22) also provides some means for constructing approximate δ-splitting k-paths. Let $\Delta(a, b)$ denote the shaded region in Figure 7. Then the trapezoidal areas $R_a = X_aP_aP_bX_b$ and $L_a = Y_aP_aP_bY_b$ satisfy

$$(23)\qquad |\text{Area}\{R_a\} - \delta\cdot\text{Area}\{L_a\}| = (1+\delta)\Delta(a, b).$$

To approximate a δ-splitting k-path, sum (23) over j consecutive points ($j = nk + 1$ for Def. A; $j = nk + 2$ for Def. B), and obtain

$$(24)\qquad \left|\text{Area}\left\{\sum_{i=1}^{j} R_{a_i}\right\} - \delta\cdot\text{Area}\left\{\sum_{i=1}^{j} L_{a_i}\right\}\right| \leq (1+\delta)\sum_{i=1}^{j}\Delta(a_i, b_i).$$

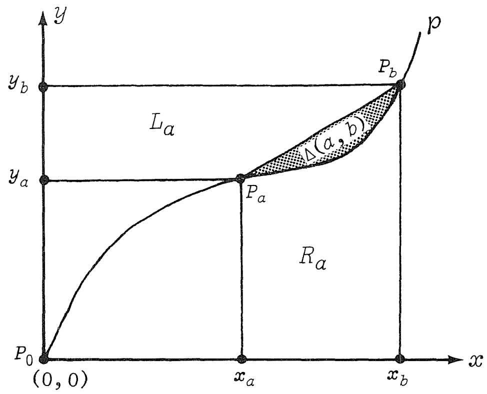

Figure 7

By way of illustration, consider $f(x) = x^2$ for the case where $\delta = 1/2$. A straightforward computation yields $\Delta(a, b) = (b - a)^3/6$ for consecutive points $P_i = (x_i, x_i^2)$ with $x_{i+1} - x_i = b - a$. Since $\text{Area}\{R_{x_i}\} = \text{Area}\{L_{x_i}\} + (b - a)^3/6$ for each such sector,

$$\text{Area}\left\{\sum_{i=1}^{j} R_{x_i}\right\} = \frac{1}{2} \cdot \text{Area}\left\{\sum_{i=1}^{j} L_{x_i}\right\} + \frac{j(b - a)^3}{12}.$$

For j such consecutive points, the error $E(j, a, b) = j(b - a)^3/12$ in constructing a 1/2-splitting k-path can be made less than ε by choosing the points on $y = x^2$ such that $x_{i+1} - x_i < (12\varepsilon/j)^{1/3}$ for each $i = 1, 2, \ldots, j - 1$.

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THE TRIANGLE OF SMALLEST PERIMETER WHICH CIRCUMSCRIBES A SEMICIRCLE

Duane W. DeTemple

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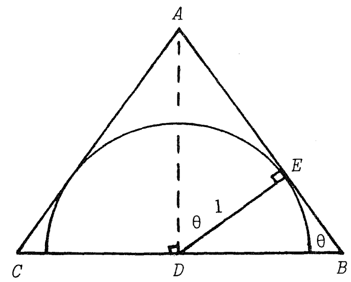
(Submitted December 1991)

Let ABC be an isosceles triangle which circumscribes a semicircle of radius 1, with the diameter of the semicircle contained in the base BC of the triangle. Such triangles may be parameterized by the base angle θ shown in the figure, where D is the circle's center and E is the point of tangency on side AB. Our objective is to determine the circumscribing isosceles triangle of smallest perimeter.



Since $DE = 1$, we see that the perimeter p of ABC is given by

$$p = 2(AE + EB + BD) = 2(\tan\theta + \cot\theta + \csc\theta).$$

The derivative is

$$p' = 2(\sec^2\theta - \csc^2\theta - \csc\theta\cot\theta),$$

which can be easily rewritten in the form

$$p' = 2(1 - \cos\theta - \cos^2\theta)(1 + \cos\theta)/\cos^2\theta\sin^2\theta.$$

It is now evident that $p' = 0$ has just one solution in $0 < \theta < \pi/2$, namely, at the point where $\cos\theta = 1/G$; here $G = (\sqrt{5} + 1)/2$ denotes the Golden Ratio. Using the relation $G^2 = G + 1$, we then have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - 1/G^2 = 1/G,$$

from which it follows that $\csc\theta = G^{1/2}$ and $\cot\theta = G^{-1/2}$. The perimeter $p_{\min}$ of the optimally circumscribed triangle is then

$$p_{\min} = 2(G^{1/2} + G^{-1/2} + G^{1/2}) = 2G^{1/2}(2 + 1/G).$$

Since $2 + 1/G = 2 + (G - 1) = G + 1 = G^2$, we see that the minimal perimeter is $2G^{5/2}$.

The triangle shown in the figure is in fact the circumscribing triangle of minimal length. The relevant dimensions are:

$$p_{\min} = 2G^{5/2} \doteq 6.66, \quad \theta = \arccos(1/G) \doteq 51.8^0,$$

$$AE = BD = G^{1/2}, \quad BE = G^{-1/2}, \quad AC = G^{3/2}, \quad AD = G.$$

The unexpected appearance of the Golden Ratio makes this minimization problem of special interest. It would also be of interest to know of other problems which give rise to the "Golden Right Triangle," whose three sides are in the proportion $1:G^{1/2}:G$.

AMS Classificaiton number: 26A06.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to 72717.3515@compuserve.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions.

Dedication. This year's column is dedicated to **Dr. A. P. Hillman** in recognition of his 27 years of devoted service as editor of the Elementary Problems Section. Devotees of this column are invited to thank Abe by dedicating their next proposed problem to Dr. Hillman.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-718 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that $[(F_n + L_n)\alpha + (F_{n-1} + L_{n-1})]/2$ is a power of the golden ratio, α.

B-719 *Proposed by Herta T. Freitag, Roanoke, VA*

Dedicated to Dr. A. P. Hillman

Let P_n be the n^{th} Pell number (defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$). Let a be an odd integer. Show how to factor $P_{n+a}^2 + P_n^2$ into a product of Pell numbers.

How should this problem be modified if a is even?

B-720 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Dedicated to Dr. A. P. Hillman

Find a closed form expression for

$$S_n = \sum_{h+k=2n} F_h F_k$$

where the sum is taken over all pairs of positive integers (h, k) such that $h + k = 2n$ and $h \leq k$.

B-721 *Proposed by Russell Jay Hendel, Dowling College, Oakdale, NY*

Dedicated to Dr. A. P. Hillman

Brittany is going to ascend an m step staircase. At any time she is just as likely to stride up one step as two steps. For a positive integer k, find the probability that she ascends the whole staircase in k strides.

B-722 *Proposed by H.-J. Seiffert, Berlin, Germany*

Dedicated to Dr. A. P. Hillman

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \geq 2.$$

Show that for all nonnegative integers n,

$$\int_0^\infty \frac{dx}{(x^2+1)F_{2n+1}(2x)} = \frac{\pi}{4n+2}.$$

B-723 *Proposed by Bruce Dearden & Jerry Metzger, University of North Dakota, Grand Forks, ND*

(a) Show that for $n \equiv 2 \pmod 4$,

$$F_{n+1}(F_n^2 + F_n - 1) \text{ divides } F_n^n(F_n^2 + F_{n+1}) - 1.$$

(b) What is the analog of (a) for $n \equiv 0 \pmod 4$?

SOLUTIONS

A Congruence for L_{2^n}

B-694 *Proposed by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA*

Prove that $L_{2^n} \equiv 7 \pmod{40}$ for $n \geq 2$.

Solution 1 by Lawrence Somer, Washington, DC

It is well known (see, for example, formula 17c of [1]) that

$$L_{2m} = L_m^2 - 2(-1)^m.$$

Letting $m = 2^{n-1}$ gives

$$L_{2^n} = L_{2^{n-1}}^2 - 2$$

for $n \geq 2$. For the case $n = 2$, we have

$$L_{2^2} = L_4 = 7.$$

Suppose that $L_{2^n} \equiv 7 \pmod{40}$ for some $n \geq 2$. Then

$$L_{2^{n+1}} = L_{2^n}^2 - 2 \equiv 7^2 - 2 \equiv 7 \pmod{40}$$

and the result is true for $n + 1$.

The result now follows for all $n \geq 2$ by mathematical induction.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Ellis Harwood Ltd., West Sussex, England, 1989.

Solution 2 by Russell Jay Hendel (paraphrased), Dowling College, Oakdale, NY

Writing down the Lucas sequence modulo 40, we obtain

$$2, 1, 3, 4, 7, 11, 18, 29, 7, 36, 3, 39, 2, 1, \ldots$$

and we thus see that the sequence repeats every 12 terms. That is,

$$L_a \equiv L_b \pmod{40} \quad \text{if } a \equiv b \pmod{12}.$$

Modulo 12, the sequence 2^n for $n \geq 2$ proceeds 4, 8, 4, 8, 4, 8, ..., so

$$2^n \equiv 4 \text{ or } 8 \pmod{12} \quad \text{for } n \geq 2.$$

Thus,

$$L_{2^n} \equiv L_4 \text{ or } L_8 \pmod{40} \quad \text{for } n \geq 2.$$

But $L_4 = 7$ and $L_8 = 47$ are both congruent to 7 modulo 40. Hence,

$$L_{2^n} \equiv 7 \pmod{40} \quad \text{for } n \geq 2.$$

None of the solvers submitted any generalizations to this problem. The problem cries out for a Fibonacci analog. Many such are possible, for example:

$$F_{2^n} \equiv 21 \pmod{42} \quad \text{for } n \geq 3.$$

The editor will normally be pleased to publish any related results or generalizations readers find for problems published in this section.

Also solved by Charles Ashbacher, A. R. Boyd, Scott H. Brown, Paul S. Bruckman, Herta T. Freitag, Ray Melham, Ioan Sadoveanu, Bob Prielipp, and the proposer.

Pell Relations

B-695 *Proposed by Russell Euler, Northwest Missouri State U., Maryville, MO*

Define the sequences $\{P_n\}$ and $\{Q_n\}$ by

$$P_0 = 0,\ P_1 = 1,\ P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \geq 0$$

and

$$Q_0 = 1,\ Q_1 = 1,\ Q_{n+2} = 2Q_{n+1} + Q_n \quad \text{for } n \geq 0.$$

Find a simple formula expressing Q_n in terms of P_n.

Solution 1 by Hans Kappus, Rodersdorf, Switzerland

Let

$$p = 1 + \sqrt{2} \quad \text{and} \quad q = 1 - \sqrt{2}$$

be the roots of the characteristic equation $t^2 - 2t - 1 = 0$ so that the Binet form for the elements of the sequences are given by

$$P_n = \frac{p^n - q^n}{2\sqrt{2}} \quad \text{and} \quad Q_n = \frac{p^n + q^n}{2}.$$

Squaring the second relation, subtracting twice the square of the first relation, and observing that $pq = -1$ yields

$$Q_n^2 - 2P_n^2 = (-1)^n.$$

From the initial conditions and the recurrence, we see that $Q_n > 0$ for all n. Hence, the desired formula is

$$Q_n = \sqrt{2P_n^2 + (-1)^n}.$$

This was the formula expected by the editor. Hendel, however, found perhaps a simpler formula which we now present.

Solution 2 by Russell Jay Hendel, Dowling College, Oakdale, NY

For $n > 0$, we have

$$Q_n = \{P_n\sqrt{2}\}$$

where $\{x\}$ denotes the integer nearest to x.

Proof: Using the Binet forms as found in Solution 1, we find

$$|Q_n - P_n\sqrt{2}| = |q^n| = |(1-\sqrt{2})^n| < \frac{1}{2}$$

for $n \geq 1$, from which the result follows. The formula may also be written as

$$Q_n = \left\lfloor P_n\sqrt{2} + \frac{1}{2}\right\rfloor, \quad \text{for } n > 0,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. This formula follows also as a particular case of Problem B-680.

Beasley found the formula $Q_n = (1+\sqrt{2})^n - P_n\sqrt{2}$.

In Lucas' seminal paper of 1878 [1], he investigated two similar recurrences $\{U_n\}$ *and* $\{V_n\}$ *defined by*

$$U_{n+2} = PU_{n+1} - QU_n,\ V_{n+2} = PV_{n+1} - QV_n,$$
$$U_0 = 0,\ U_1 = 1,\ V_0 = 2,\ V_1 = P.$$

Lucas showed (page 199) that the two sequences are related by

$$V_n^2 - \Delta U_n^2 = 4Q^n$$

where $\Delta = P^2 - 4Q$.

Many readers interpreted the problem differently. Instead of expressing Q_n *in terms of* P_n *for a given* n, *they showed how to relate the sequence* $\{Q_n\}$ *in terms of the sequence* $\{P_n\}$. *The following solution uses this interpretation.*

Solution 3 by Glenn Bookhout, N. Carolina Wesleyan Col., Rocky Mount, NC

Define the sequence $\{T_n\}$ by $T_0 = 1$, $T_1 = 0$, and

$$T_{n+2} = 2T_{n+1} + T_n$$

for $n \geq 0$. Then, clearly, $Q_n = P_n + T_n$ for all n. But $T_2 = 1$. Thus $T_n = P_{n-1}$ for $n > 0$. Hence the desired formula is $Q_n = P_n + P_{n-1}$.

Another such formula found by several of our solvers was $Q_n = P_{n+1} - P_n$. *Popov gave* $P_n = (Q_n + Q_{n-1})/2$. *The numbers* P_n *and* Q_n *are known as Pell numbers (of the first and second kind) and the relation* $Q_n = P_n + P_{n-1}$ *is well known. Sadoveanu generalized the problem to two sequences* $\{P_n\}$ *and* $\{Q_n\}$ *defined by*

$$P_0 = 0,\ P_1 = 1,\ P_{n+2} = aP_{n+1} + bP_n$$

and

$$Q_0 = 1,\ Q_1 = 1,\ Q_{n+2} = aQ_{n+1} + bQ_n$$

where a *and* b *are arbitrary constants. In this case, a simple induction argument shows that the two sequences are related by the formula*

$$Q_n = P_n + bP_{n-1}, \quad \text{for } n > 0.$$

Reference

1. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* 1 (1878):184-240, 289-321.

Also solved by Charles Ashbacher, Paul S. Bruckman, Chris Clark & H. K. Krishnapriyan, Herta T. Freitag, Pentti Haukkanen, Joe Howard, Carl Libis, Ray Melham, Blagoj S. Popov. Bob Prielipp, Ioan Sadoveanu, H.-J. Seiffert, Lawrence Somer, and the proposer (two solutions).

A Nonprimitive Pythagorean Triple

B-696 *Proposed by Herta T. Freitag, Roanoke, VA*

Let (a, b, c) be a Pythagorean triple with the hypotenuse $c = 5F_{2n+3}$ and $a = L_{2n+3} + 4(-1)^{n+1}$.

(a) Determine b.
(b) For what values of n, if any, is the triple primitive? [The elements of a primitive triple have no common factor.]

Solution to part (a) by Paul S. Bruckman, Edmonds, WA

From the well-known formulas (Identities 5 and 23 in [1]),

$$5F_a = L_{a-1} + L_{a+1} \quad \text{and} \quad L_{2a} - 2(-1)^a = 5F_a,$$

we see that $c = L_{2n+4} + L_{2n+2}$. Hence,

$$c + a = L_{2n+4} + L_{2n+3} + L_{2n+2} - 4(-1)^{n+2}$$
$$= 2(L_{2n+4} - 2(-1)^{n+2}) = 10F_{n+2}^2.$$

Also,

$$c - a = L_{2n+4} - L_{2n+3} + L_{2n+2} - 4(-1)^{n+1}$$
$$= 2(L_{2n+2} - 2(-1)^{n+1}) = 10F_{n+1}^2.$$

Then,

$$b^2 = c^2 - a^2 = (c + a)(c - a) = 10^2F_{n+2}^2F_{n+1}^2,$$

so $b = 10F_{n+1}F_{n+2}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Ellis Harwood Ltd., West Sussex, England, 1989.

Several solvers found the equivalent formula

$$b = 2(L_{2n+3} + (-1)^n) = 2a + 10(-1)^n.$$

Beasley found $b = 5(F_{2n+3} - F_n^2)$.

Solution to part (b) by Brian D. Beasley, Presbyterian College, Clinton, SC

From $c = 5F_{2n+3}$ and $b = 10F_{n+1}F_{n+2}$, it is clear that $5|c$ and $5|b$. Thus, $5^2|(c^2 - b^2)$ or $5^2|a^2$, which implies that $5|a$. Therefore, the triple is never primitive.

To show that a Pythagorean triple is primitive, it is sufficient to show that no two *elements have a common factor. Most solvers showed that* $5|a$ *using congruences or by finding explicit representations for* a. *Bruckman showed that*

$$a = 5(L_{2n+3} - 2F_{n+1}F_{n+2}).$$

Somer showed that

$$a = 5(F_{n+2}^2 - F_{n+1}^2).$$

Also solved by Charles Ashbacher, Brian D. Beasley, Paul S. Bruckman, Russell Jay Hendel, Nicola Lisi, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

A Sum of Quotients

B-697 *Proposed by Richard André-Jeannin, Sfax, Tunisia*

Find a closed form for the sum

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

where $w_n \neq 0$ for all n and $w_n = pw_{n-1} - qw_{n-2}$ for $n \geq 2$, with p and q nonzero constants.

Solution by A. P. Hillman, Albuquerque, NM

Let $\{u_n\}$ be the sequence defined by

$$u_n = pu_{n-1} - qu_{n-2}, \text{ with } u_0 = 0 \text{ and } u_1 = 1.$$

Let

$$D_n = \begin{vmatrix} u_n & u_{n+1} \\ w_{n+1} & w_{n+2} \end{vmatrix} = \begin{vmatrix} u_n & pu_n - qu_{n-1} \\ w_{n+1} & pw_{n+1} - qw_n \end{vmatrix}.$$

Subtracting p times the first column of this last determinant from the second column shows that $D_n = qD_{n-1}$. Repeated application of this formula yields

$$D_n = q^n D_0.$$

Since

$$D_0 = \begin{vmatrix} 0 & 1 \\ w_1 & w_2 \end{vmatrix} = -w_1,$$

we find that

$$u_n w_{n+2} - u_{n+1} w_{n+1} = -q^n w_1 \quad \text{or} \quad \frac{u_n}{w_{n+1}} - \frac{u_{n+1}}{w_{n+2}} = -w_1 \frac{q^n}{w_{n+1} w_{n+2}}.$$

Thus,

$$S_n = \sum_{k=1}^{n} \frac{q^{k-1}}{w_k w_{k+1}}$$

$$= -\frac{1}{w_1}\left[\left(\frac{u_0}{w_1} - \frac{u_1}{w_2}\right) + \left(\frac{u_1}{w_2} - \frac{u_2}{w_3}\right) + \cdots + \left(\frac{u_{n-1}}{w_n} - \frac{u_n}{w_{n+1}}\right)\right] = \frac{u_n}{w_1 w_{n+1}}.$$

Strictly speaking, the answer $u_n/w_1 w_{n+1}$ is not in closed form since it involves the term u_n which is defined via a recurrence. However, we can give a closed form expression for u_n by the well-known Binet formula:

$$u_n = \frac{x^n - y^n}{x - y}$$

where x and y are the roots of the characteristic equation $t^2 - pt + q = 0$.

Other equivalent formulas were found by some solvers. For example, Sadoveanu found

$$S_n = \frac{1}{q(w_1 - xw_0)}\left[\frac{x}{w_1} - \frac{x^{n+1}}{w_{n+1}}\right] \text{ and } S_n = \frac{1}{q(w_1 - yw_0)}\left[\frac{y}{w_1} - \frac{y^{n+1}}{w_{n+1}}\right].$$

Kappus found

$$S_n = \frac{1}{d}\left[\frac{w_n}{w_{n+1}} - \frac{w_0}{w_1}\right]$$

where $d = w_1^2 - w_0w_2$, *providing that* $d \neq 0$. *Also assuming* $d \neq 0$, *Popov found*

$$S_n = \frac{1}{qd}\left[\frac{w_2}{w_1} - \frac{w_{n+2}}{w_{n+1}}\right],$$

which generalizes the formula that Lucas found in 1878 for the special case in which $d = 1$ *(see page 196 of [1] or page 18 of [2]).*

References

1. Edouard Lucas. "Théorie des fonctions numériques simplement périodiques." *American Journal of Mathematics* 1 (1878):184-240, 289-321.
2. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions*. The Fibonacci Association, 1969.

Also solved by Paul S. Bruckman, Russell Jay Hendel, Hans Kappus, Blagoj S. Popov, Ioan Sadoveanu, and the proposer.

Late solution to B-684 by Nicola Lisi.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-469 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \ge 2.$$

Show that for all positive integers n and all positive reals x,

(a) $$\frac{1}{F_{2n-1}(x)} = \frac{x^2+4}{2n-1}\sum_{k=0}^{2n-2}(-1)^{k+n+1}\frac{\cos\frac{k\pi}{2n-1}}{x^2+4\cos^2\frac{k\pi}{2n-1}},$$

(b) $$\frac{1}{F_{2n}(x)} = \frac{x(x^2+4)}{4n}\sum_{k=0}^{2n-1}\frac{(-1)^{k+n}}{x^2+4\cos^2\frac{k\pi}{2n}}.$$

H-470 *Proposed by Paul S. Bruckman, Edmonds, WA*

Consider the polynomial

(1) $$G_r(z) = z^r - \sum_{k=0}^{r-1} a_k z^{r-1-k},\ r \ge 1, \text{ the } a_k\text{'s complex.}$$

Consider the r distinct sequences $(U_{n,j}^{(r)})_{n=0}^{\infty}$ satisfying the common recurrence relation:

(2) $$G_r(E)(U_{n,j}^{(r)}) = 0,\ j = 1, 2, \ldots, r;\ n = 0, 1, \ldots .$$

The sequences are specified by the initial values:

(3) $$U_{n,j}^{(r)} = \delta_{n+j,r},\ n = 0, 1, \ldots, r-1,\ j = 1, 2, \ldots, r.$$

Form the $r \times r$ matrix $U_n^{(r)}$, defined as follows:

(4) $$U_n^{(r)} = \begin{bmatrix} U_{n+r-1,1}^{(r)} & U_{n+r-1,2}^{(r)} & \cdots & U_{n+r-1,r}^{(r)} \\ U_{n+r-2,1}^{(r)} & U_{n+r-2,2}^{(r)} & \cdots & U_{n+r-2,r}^{(r)} \\ \vdots & \vdots & & \vdots \\ U_{n+1,1}^{(r)} & U_{n+1,2}^{(r)} & \cdots & U_{n+1,r}^{(r)} \\ U_{n,1}^{(r)} & U_{n,2}^{(r)} & \cdots & U_{n,r}^{(r)} \end{bmatrix} = ((U_{n+r-i,j}^{(r)})).$$

Therefore,

$$(5) \qquad U_1^{(r)} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{r-2} & a_{r-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

(A) Find the characteristic polynomial $p_r(z)$ of $U_1^{(r)}$;

(B) Prove that $(U_1^{(r)})^n = U_n^{(r)}$, $n = 1, 2, \ldots$;

(C) Let there be r sequences $(H_{n,j}^{(r)})_{n=0}^{\infty}$ satisfying the common recurrence in (2), but the arbitrary initial values. Form the $r \times r$ matrix

$$H_n^{(r)} = ((H_{n+r-i,j}^{(r)})).$$

Prove that

$$(U_1^{(r)})^{n-1} H_1^{(r)} = H_n^{(r)}, \quad n = 1, 2, \ldots .$$

SOLUTIONS

Woops

H-451 *Proposed by T. V. Padmakumar, Trivandrum, South India (Vol. 29, no. 1, February 1991)*

If p is a prime and x and a are positive integers, show

$$\binom{x + ap}{p} - \binom{x}{p} \equiv a \pmod{p}.$$

Editorial Note: Many readers pointed out that this problem was published in an earlier issue of this Quarterly as B-643. Also, this result readily follows from B-666. In spite of this, we offer one more solution.

Solution by Guo-Gang Gao, University of Montreal, Montreal, Canada

Lemma 1: Let z be a positive integer. If $z + 1 \not\equiv 0 \pmod{p}$, then

$$\binom{z}{p-1} \equiv 0 \pmod{p}.$$

Proof: If $z + 1 \not\equiv 0 \pmod{p}$, then *only* one of $z, z - 1, \ldots, z - p + 2$ must be divisible by p, by the pigeonhole principle. Hence, $\binom{z}{p-1}$ always contains a factor of p because p is a prime, and the lemma follows. □

Lemma 2: Let z be a positive integer. Then, for $1 \le k \le p - 1$,

$$\binom{zp - k - 1}{p - k} \equiv 0 \pmod{p}.$$

Proof: Since $zp - k - 1 - (p - k) = (z - 1)p - 1$, $zp - k - 1 \ge (z - 1)$, and $0 < p - k < p$, thus

$$\binom{zp - k - 1}{p - k} = \frac{(zp - k - 1)!}{(zp - k - 1 - p + k)!(p - k)!}$$

always contains a factor of p, i.e., the lemma follows. □

Lemma 3: Let z be a positive integer. Then

$$\binom{zp-1}{p-1} \equiv 1 \pmod{p}.$$

Proof: (a) If $z = 1$, it is trivial; (b) let $z > 1$, then by repetitively applying Lemma 2, and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

we have

$$\begin{aligned}\binom{zp-1}{p-1} &\equiv \binom{zp-2}{p-1} \pmod{p} + \binom{zp-2}{p-2} \pmod{p}\\ &\equiv \binom{zp-3}{p-2} \pmod{p} + \binom{zp-3}{p-3} \pmod{p}\\ &\vdots\\ &\equiv \binom{zp-p}{0} \pmod{p}\\ &\equiv 1 \pmod{p}.\end{aligned}$$

We now come to the proof of the statement. By repetitively applying

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

we have

$$\begin{aligned}\binom{x+ap}{p} - \binom{x}{p} &= \binom{x+(a-1)p}{p} - \binom{x}{p} + \sum_{i=(a-1)p}^{ap-1}\binom{x+i}{p-1}\\ &\vdots\\ &= \sum_{j=1}^{a}\sum_{i=(j-1)p}^{jp-1}\binom{x+i}{p-1}.\end{aligned}$$

For any fixed j $(1 \le j \le a)$, $x + i$ can be one of p consecutive integers, $x + (j - 1)p, \ldots, x + jp - 1$. Of these p consecutive integers, there always exists *only* one $x + i$ such that $x + i + 1 \equiv 0 \pmod{p}$, by the pigeonhole principle. Therefore, by Lemmas 1 and 3, for any fixed j,

$$\sum_{i=(j-1)p}^{jp-1}\binom{x+i}{p-1} \equiv 1 \pmod{p},$$

that is,

$$\binom{x+ap}{p} - \binom{x}{p} \equiv a \pmod{p},$$

completing the proof. □

Also solved by K. Atanassov, P. Bruckman, P. Filipponi, R. Hendel, J. Kostal, Y. H. H. Kwong, B. Prielipp, H.-J. Seiffert, and the proposer.

Divide and Conquer

H-452 *Proposed by Don Redmond, Southern Illinois U., Carbondale, IL (Vol. 29, no. 2, May 1991)*

Let $p_r(m)$ denote the m^{th} r-gonal number $(m/2)\{2 + (r - 2)(m - 1)\}$. Characterize the values of r and m such that

$$p_r(m) \Bigg| \sum_{k=1}^{m} p_r(k).$$

Solution by C. Georghiou, University of Patras, Patras, Greece

Let $S_r(m) := \sum_{k=1}^{m} p_r(m)$. Then it is easy to see that

$$S_r(m) = \frac{m(m+1)}{12}[(r-2)(2m+1) - 3(r-4)].$$

Now, since $p_r(1) = 1$ and $S_r(1) = 1$, the given property is trivially true for all r and $m = 1$. So, we are interested in the case $m > 1$ (and, of course, $r > 1$). Then the given property is true only if

$$r = 2 \text{ and } m \equiv 1 \bmod 2 \quad \text{or} \quad r = 3 \text{ and } m \equiv 1 \bmod 3.$$

Indeed, we have

$$S_2(m)/p_2(m) = (m+1)/2 \quad \text{and} \quad S_3(m)/p_3(m) = (m+2)/3.$$

It remains to show that $p_r(m) \nmid S_r(m)$ for $r > 3$ (and $m > 1$). We have

$$S_r(m)/p_r(m) = \frac{(m+1)[(r-2)m - (r-5)]}{3[(r-2)m - (r-4)]}.$$

Since 3 must divide either factor of the numerator, we have the following three possibilities: (i) $m = 3n - 1$; (ii) $m = 3n + 1$; (iii) $r = 3s - 1$ and $m = 3n$.

In Case (i), we get

$$S_r(m)/p_r(m) = n + n/[(3r-6)n - (2r-6)],$$

and since $0 < n/[(3r-6)n - (2r-6)] < 1$ for $n > 0$ and $r > 3$ we conclude that $p_r(3n-1) \nmid S_r(3n-1)$ for any $r > 3$ and any $n > 0$.

In Case (ii), we get

$$S_r(m)/p_r(m) = n + [(2r-3)n + 2]/[(3r-6)n + 2],$$

and it is easy to see that the second term lies (strictly) between 0 and 1 for $r > 3$ and $n > 0$.

Finally, in Case (iii), we get

$$S_r(m)/p_r(m) = n + [3s-2)n - (s-2)]/[(9s-9)n - (3s-5)],$$

and again the second term is positive and less than unity for any $n > 0$ and $s > 1$.

Also solved or partially solved by P. Bruckman, N. Jensen, S. Rabinowitz, and the proposer.

Sum Formulae!

H-453 *Proposed by James E. Desmond, Pensacola Jr. College, Pensacola, FL (Vol. 29, no. 2, May 1991)*

Show that for positive integers m and n,

$$\frac{L_{(2m+1)n}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} + (-1)^{m(n+1)}$$

and

$$\frac{F_{2mn}}{L_n} = \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)}.$$

Solution by Stanley Rabinowitz, Westford, MA

Lemma:

$$S(n, a, b, r) \equiv \sum_{j=1}^{n} r^j F_{aj+b} = \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)b} - (-1)^a r^2 F_b + rF_{a+b}}{(-1)^a r^2 - rL_a + 1}.$$

Proof: Let

$$G(x, n) \equiv \sum_{j=1}^{n} x^j = x\left(\frac{x^n - 1}{x - 1}\right).$$

Now

$$r^j F_{aj+b} = r^j\left(\frac{\alpha^{aj+b} - \beta^{aj+b}}{\sqrt{5}}\right) = \frac{\alpha^b}{\sqrt{5}}(r\alpha^a)^j - \frac{\beta^b}{\sqrt{5}}(r\beta^a)^j.$$

Thus,

$$\begin{aligned}
S(n, a, b, r) &= \frac{\alpha^b}{\sqrt{5}} G(r\alpha^a, n) - \frac{\beta^b}{\sqrt{5}} G(r\beta^a, n) \\
&= \frac{\alpha^b}{\sqrt{5}} r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \frac{\beta^b}{\sqrt{5}} r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right) \\
&= \frac{r}{\sqrt{5}}\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) - \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{\alpha^{a+b}(r\beta^a - 1)(r^n\alpha^{an} - 1) - \beta^{a+b}(r\alpha^a - 1)(r^n\beta^{an} - 1)}{(r\alpha^a - 1)(r\beta^a - 1)}\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{r^{n+1}(\beta^a\alpha^{a(n+1)+b} - \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) - r(\alpha^{a+b}\beta^a - \alpha^a\beta^{a+b}) + \alpha^{a+b} - \beta^{a+b}}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right] \\
&= \frac{r}{\sqrt{5}}\left[\frac{r^{n+1}(\alpha\beta)^a(\alpha^{an+b} - \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} - \beta^{a(n+1)+b}) - r(\alpha\beta)^a(\alpha^b - \beta^b) + (\alpha^{a+b} - \beta^{a+b})}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right] \\
&= r\left[\frac{r^{n+1}(-1)^a F_{an+b} - r^n F_{a(n+1)+b} - r(-1)^a F_b + F_{a+b}}{(-1)^a r^2 - rL_a + 1}\right] \\
&= \frac{(-1)^a r^{n+2} F_{an+b} - r^{n+1} F_{a(n+1)+b} - (-1)^a r^2 F_b + rF_{a+b}}{(-1)^a r^2 - rL_a + 1}
\end{aligned}$$

which was to be proved.

Using this lemma, we have

$$\begin{aligned}
&\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)} \\
&= (-1)^{(n+1)m} S(m, 2n, -n, (-1)^{n+1}) \\
&= (-1)^{(n+1)m} \frac{(-1)^{(n+1)(m+2)} F_{2mn-n} - (-1)^{(n+1)(m+1)} F_{2n(m+1)-n} - F_{-n} + (-1)^{n+1} F_n}{2 - (-1)^{n+1} L_{2n}} \\
&= \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}
\end{aligned}$$

where we have used the fact that $F_{-n} = (-1)^{n+1} F_n$.

Thus, it remains to prove that our answer,

$$(1) \qquad \sum_{j=1}^{m} (-1)^{(n+1)(m-j)} F_{n(2j-1)} = \frac{F_{n(2m-1)} + (-1)^n F_{n(2m+1)}}{2 + (-1)^n L_{2n}}$$

is equivalent to the proposer's answer of F_{2mn}/L_n. Cross multiplying, we see that this would be equivalent to showing that

(2) $$F_{n(2m-1)}L_n + (-1)^n F_{n(2m+1)} = 2F_{2mn} + (-1)^n F_{2mn} L_{2n}.$$

Applying the well-known identity,

$$F_x L_y = F_{x+y} + (-1)^y F_{x-y}$$

to equation (2), we find that all the terms drop out; hence, equation (2) is true. Thus, our answer (1) is equivalent to the proposer's answer.

In the same manner, we can prove a similar lemma for the Lucas numbers:

$$\begin{aligned}
T(n, a, b, r) &\equiv \sum_{j=1}^{n} r^j L_{aj+b} \\
&= \alpha^b G(r\alpha^a, n) + \beta^b G(r\beta^a, n) \\
&= \alpha^b r\alpha^a\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^b r\beta^a\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right) \\
&= r\left[\alpha^{a+b}\left(\frac{r^n\alpha^{an} - 1}{r\alpha^a - 1}\right) + \beta^{a+b}\left(\frac{r^n\beta^{an} - 1}{r\beta^a - 1}\right)\right] \\
&= r\left[\frac{\alpha^{a+b}(r\beta^a - 1)(r^n\alpha^{an} - 1) + \beta^{a+b}(r\alpha^a - 1)(r^n\beta^{an} - 1)}{(r\alpha^a - 1)(r\beta^a - 1)}\right] \\
&= r\left[\frac{r^{n+1}(\beta^a\alpha^{a(n+1)+b} + \alpha^a\beta^{a(n+1)+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) - r(\alpha^{a+b}\beta^a + \alpha^a\beta^{a+b}) + (\alpha^{a+b} + \beta^{a+b})}{r^2(\alpha\beta)^a - r(\alpha^a + \beta^a) + 1}\right] \\
&= r\left[\frac{r^{n+1}(\alpha\beta)^a(\alpha^{an+b} + \beta^{an+b}) - r^n(\alpha^{a(n+1)+b} + \beta^{a(n+1)+b}) - r(\alpha\beta)^a(\alpha^b + \beta^b) + (\alpha^{a+b} + \beta^{a+b})}{(\alpha\beta)^a r^2 - r(\alpha^a + \beta^a) + 1}\right] \\
&= \frac{(-1)^a r^{n+2}L_{an+b} - r^{n+1}L_{a(n+1)+b} - (-1)^a r^2 L_b + rL_{a+b}}{(-1)^a r^2 - rL_a + 1}.
\end{aligned}$$

Using this result, we have

$$\begin{aligned}
&\sum_{j=1}^{m} (-1)^{(n+1)(m-j)} L_{2nj} \\
&= (-1)^{(n+1)m} T(m, 2n, 0, (-1)^{n+1}) \\
&= (-1)^{(n+1)m}\left[\frac{(-1)^{(n+1)(m+2)}L_{2mn} - (-1)^{(n+1)(m+1)}L_{2n(m+1)} - L_0 + (-1)^{n+1}L_{2n}}{2 - (-1)^{n+1}L_{2n}}\right] \\
&= \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)}L_{2n}}{2 + (-1)^n L_{2n}}.
\end{aligned}$$

To show that our answer is equivalent to the proposer's, we must show that

$$\frac{L_{(2m+1)n}}{L_n} - (-1)^{m(n+1)} = \frac{L_{2mn} + (-1)^n L_{2n(m+1)} - 2(-1)^{(n+1)m} + (-1)^{(n+1)(m+1)}L_{2n}}{2 + (-1)^n L_{2n}}$$

or, equivalently,

$$2L_{n(2m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^n L_{2n}L_{n(2m+1)} - (-1)^{m(n+1)+n}L_nL_{2n}$$
$$= L_nL_{2mn} + (-1)^n L_nL_{2n(m+1)} - 2(-1)^{m(n+1)}L_n + (-1)^{(n+1)(m+1)} L_nL_{2n}.$$

Again, this falls out by applying the well-known identity,

$$L_xL_y = L_{x+y} + (-1)^y L_{x-y}.$$

Also solved by P. Bruckman, N. Jensen, B. Prielipp, H.-J. Seiffert, and the proposer.

Editorial Note: Several readers have pointed out that H-462 was published earlier as H-449.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

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