Dispersion relations for three-pion decays and transition form factors

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with B. Kubis and F. Niecknig, EPJC **72**, 2014 (2012); arXiv:1206.3098 [hep-ph]

Outline

Motivation

- why dispersive analyses?
- physics case

Dispersive framework for three-pion decays

- some essentials
- integral equations for $\omega/\phi \to 3\pi$
- effects of crossed-channel rescattering
- comparison to $\phi \to 3\pi$ experiment

Dispersive framework for the $\omega/\phi \to \pi^0\gamma^*$ transition form factor

- integral equation
- numerical results

Conclusions and outlook

Why dispersive analyses?

- advent of high-statistics experiments allows for accurate measurements of decay amplitudes
  BES-III, WASA-at-COSY, MAMI-B/-C, CLAS@JLAB, CMD, KLOE, ELSA
  $\Rightarrow$ need to match this accuracy on the theoretical side
- final-state interactions in hadronic three-body decays play essential role in precision amplitude analyses
- perturbative approaches (ChPT, NREFT,...): implement final-state-interactions up to a certain order in a small power-counting parameter
- goal of dispersion relations: resum effects of hadronic rescattering to all orders $\Rightarrow$ precise implementation of final-state interactions, allows extension to higher energies
- high-accuracy parametrizations of phase-shifts required $\Rightarrow$ now available in some cases ($\pi\pi$, πK,...)

Physics case

$\omega/\phi \to 3\pi$:

- most simple imaginable system with physical relevance
  $\Rightarrow$ P-wave interactions only (Neglecting F- and higher waves)
  $\Rightarrow$ ideal testing ground for the approach
- large existent (ϕ: KLOE/CMD-2) and upcoming (ω: WASA) database
- $\phi \to 3\pi$: study crossed-channel effects on resonances in the decay region

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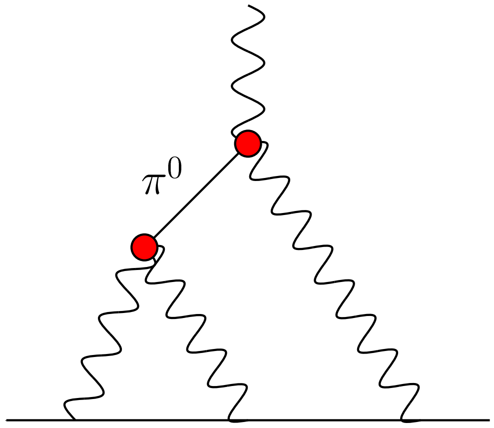
$\omega/\phi \to \pi^0\gamma^*$ transition form factor:

- can help to constrain pseudoscalar pole terms (π^0, η, η') in hadronic light-by-light
- strength determined by decay $\pi^0 \to \gamma^*\gamma^*$:
  doubly-virtual form factor $F_{\pi^0\gamma^*\gamma^*}(M_{\pi^0}^2, q_1^2, q_2^2)$
- for fixed isocalar photon virtuality: can extract $F_{\pi^0\gamma^*\gamma^*}(M_{\pi^0}^2, q_1^2, M_\omega^2)$ from $\omega \to \pi^0\ell^+\ell^-$



The framework: Some fundamentals

- was applied in $\eta \to 3\pi$ decays before, but also $\eta' \to \eta\pi\pi, K_{\ell 4}, \ldots$ possible Anisovich, Leutwyler '98; Lanz '12; Stoffer '12

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- integral equations based on two fundamental principles: unitarity, analyticity and crossing symmetry
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- integral equations based on two fundamental principles: unitarity, analyticity and crossing symmetry
- assume elastic $\pi\pi$ rescattering
- decay amplitude can be decomposed according to:

$$\omega/\phi \to 3\pi : \mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$$

$$s + t + u = M_{\omega/\phi}^2 + 3M_\pi^2 \doteq 3s_0$$

▷ $\mathcal{F}(s)$ functions of one variable with only a right-hand cut

▷ decomposition exact only if $l \geq 3$ partial waves are real

From unitarity to integral equations

- the unitarity condition:

$$\text{Im}\,\mathcal{F}(s,t,u) = \frac{1}{2}\sum_{n'}(2\pi)^4\delta(p_n - p_{n'})T^*_{n'n}\mathcal{F}_{n'}(s',t',u')$$

From unitarity to integral equations

- from the unitarity condition:

$$\text{disc}\,\mathcal{F}(s) = 2i\,\theta(s-4\,M_\pi^2)\{\mathcal{F}(s)+\hat{\mathcal{F}}(s)\}\,\sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

How can we solve this?

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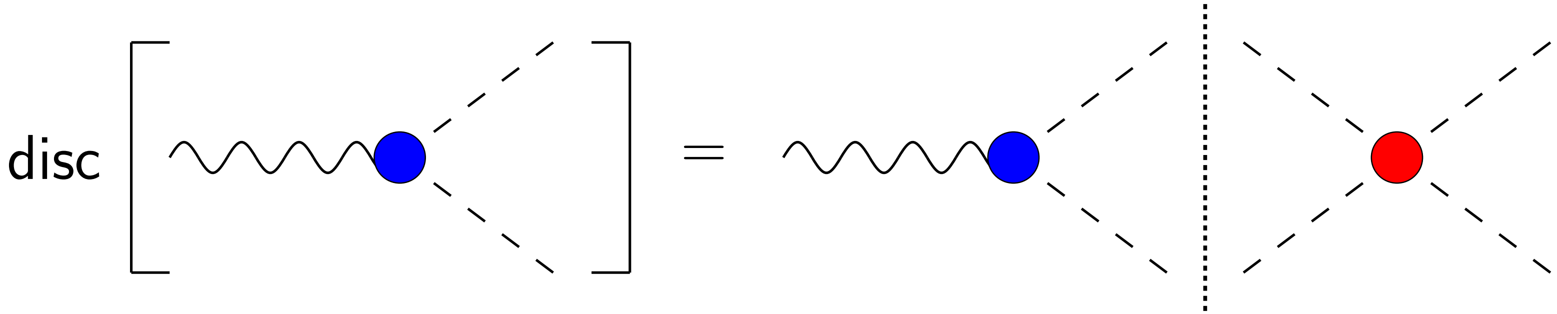
- consider two particles in the final-state for the moment



$\Rightarrow$ Watson's final-state theorem

Watson '54

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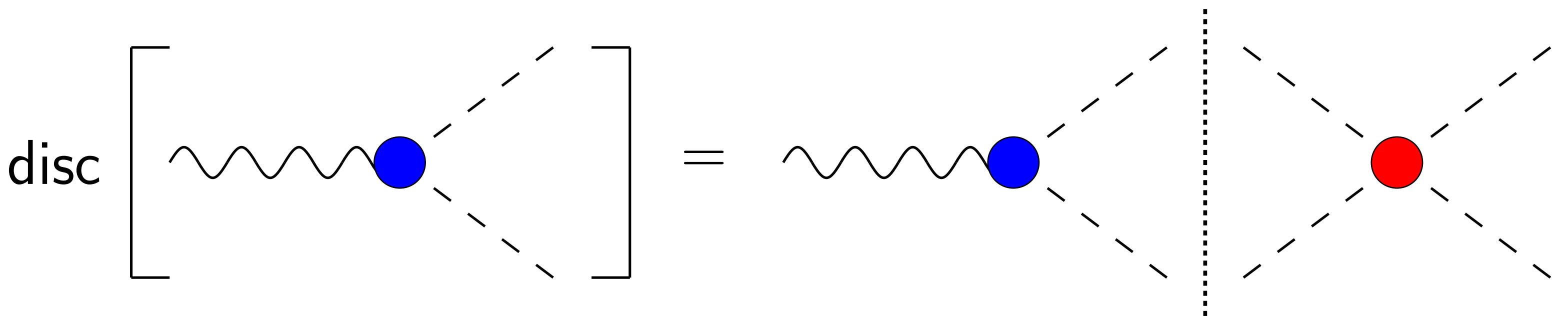

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Watson '54

- solution to this homogeneous integral equation known:

$$\mathcal{F}(s) = P(s)\Omega(s)\,, \quad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

$P(s)$ polynomial, $\Omega(s)$ Omnès function

Omnès '58

completely determined by phase-shift $\delta_1^1(s)$

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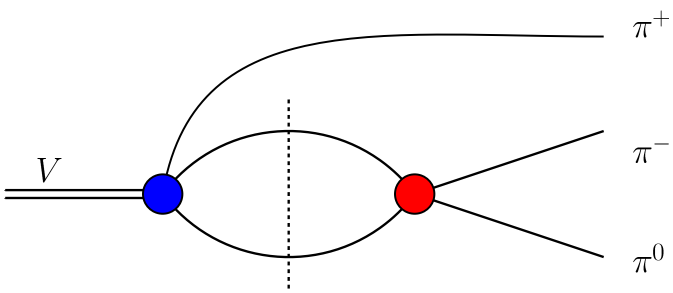

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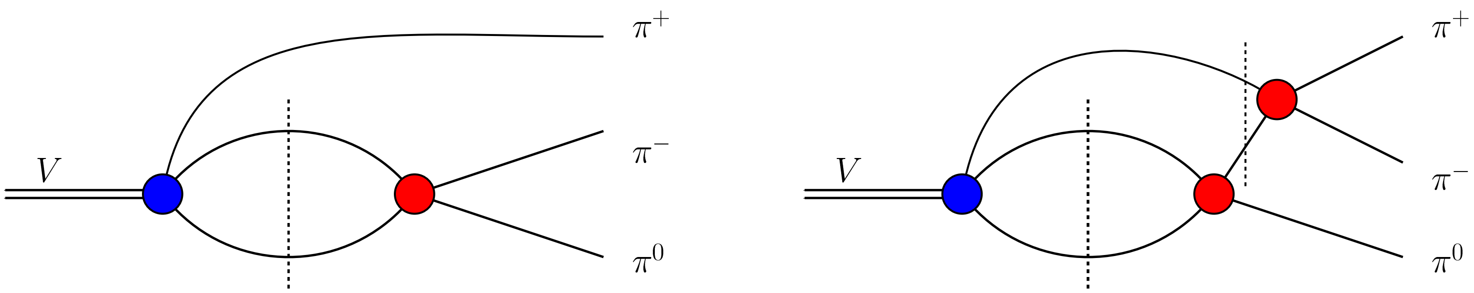

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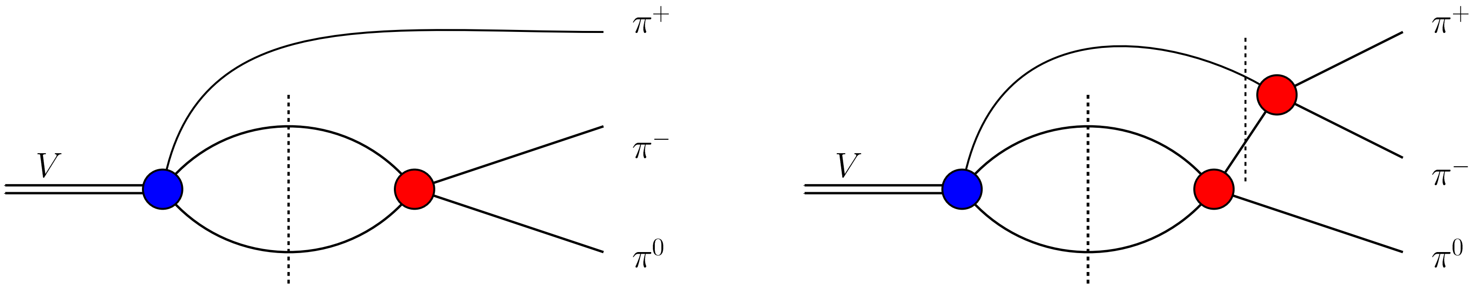

- crossed-channel scattering between s-, t-, and u-channel
  $\Rightarrow$ inhomogeneities $\hat{\mathcal{F}}(s)$: angular integration over $\mathcal{F}(s)$

The inhomogeneities $\hat{\mathcal{F}}(s)$

$$\hat{\mathcal{F}}(s) = \frac{3}{\kappa(s)} \int_{s_-(s)}^{s_+(s)} ds' \left[1 - \left(\frac{2s' - 3s_0 + s}{\kappa(s)}\right)^2\right] \mathcal{F}(s')$$

$$s_\pm(s) = \frac{1}{2}(3s_0 - s \pm \kappa(s))$$

$$\kappa(s) = \sqrt{\frac{s - 4\,M_\pi^2}{s}} \times \sqrt{(s - (M_V + M_\pi)^2)(s - (M_V - M_\pi)^2)}$$

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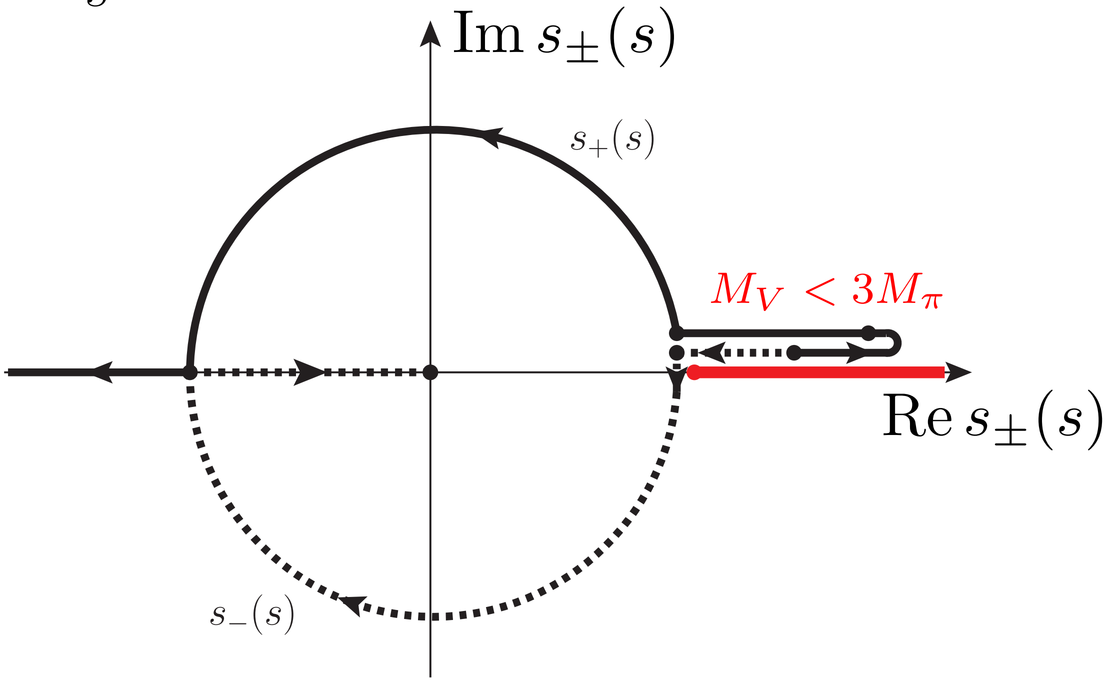

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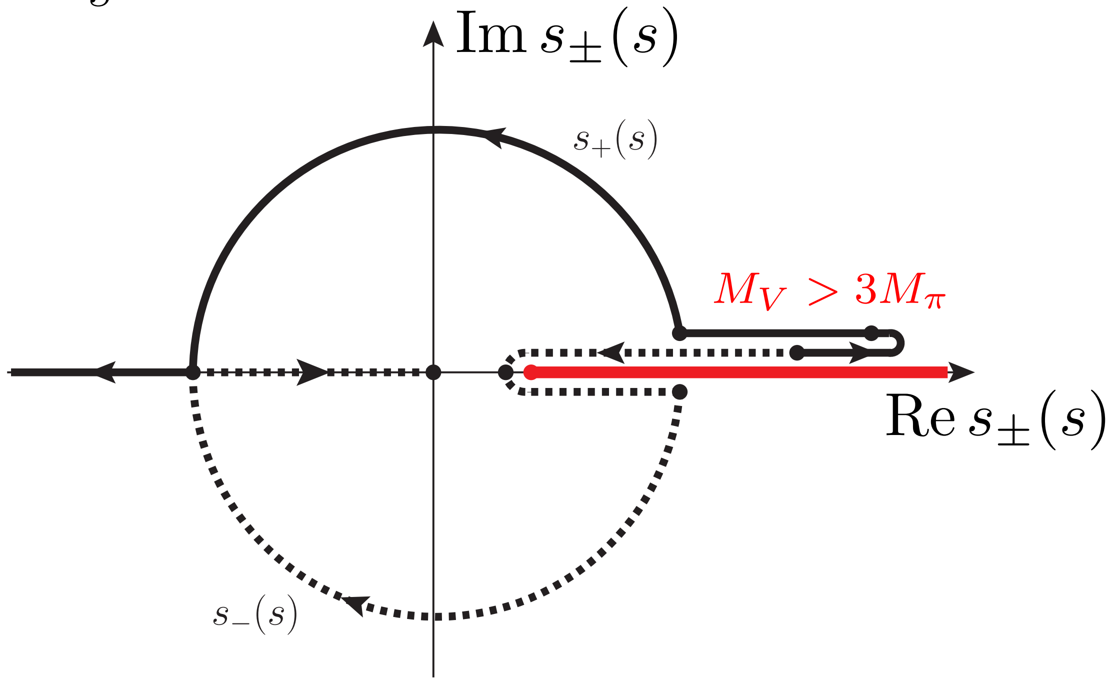

- the vector particle V is unstable $\Rightarrow$ 3-particle cuts become manifest in $\kappa(s)$ $\Rightarrow$ generates complex analytic structure

Integral equations

- Solution to:

$$\text{disc}\,\mathcal{F}(s) = 2i\,\theta(s-4\,M_\pi^2)\{\mathcal{F}(s)+\hat{\mathcal{F}}(s)\}\,\sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

$$\mathcal{F}(s) = \Omega(s)\Big\{a + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\,\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\Big\}$$

Niecknig, Kubis, SPS ’12
Anisovich, Leutwyler ’98

Khuri,Treiman ’60; Aitchison, Pasquier ’66

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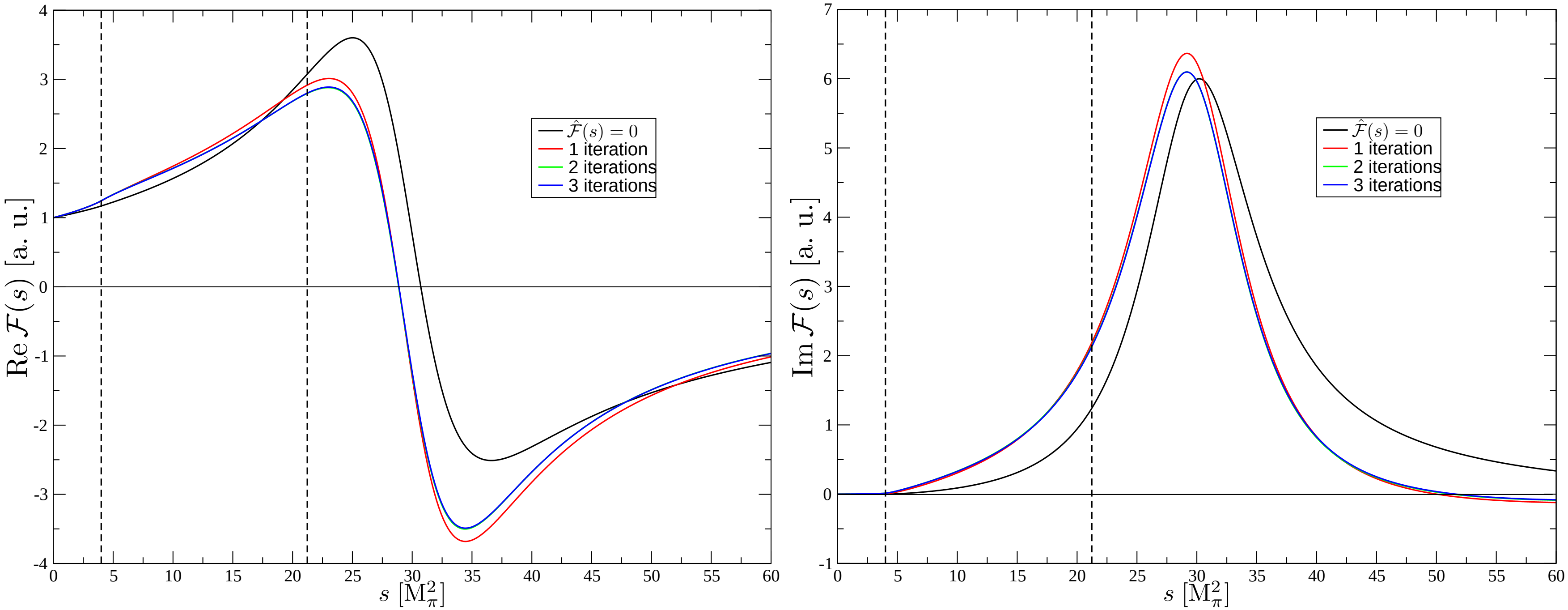
- only one subtraction constant in this system
  - ▷ dynamics (Dalitz Plot) do not depend on the specific choice of this subtraction constant!
  - ▷ a matched to reproduce the $\omega/\phi \to 3\pi$ partial width
- $\delta_1^1(s)$ from phenomenological analyses (Roy equations)

Caprini (in preparation), García-Martín et al. ’11

- solve these equations by an iterative numerical procedure

$\omega/\phi \to 3\pi$ amplitude

- iteration steps of $\mathcal{F}(s)$ in $\omega \to 3\pi$:



- convergence reached after two iterations
- crossed-channel effects sizeable
- “no crossed-channel effects” always refers to $\hat{\mathcal{F}}(s) = 0$

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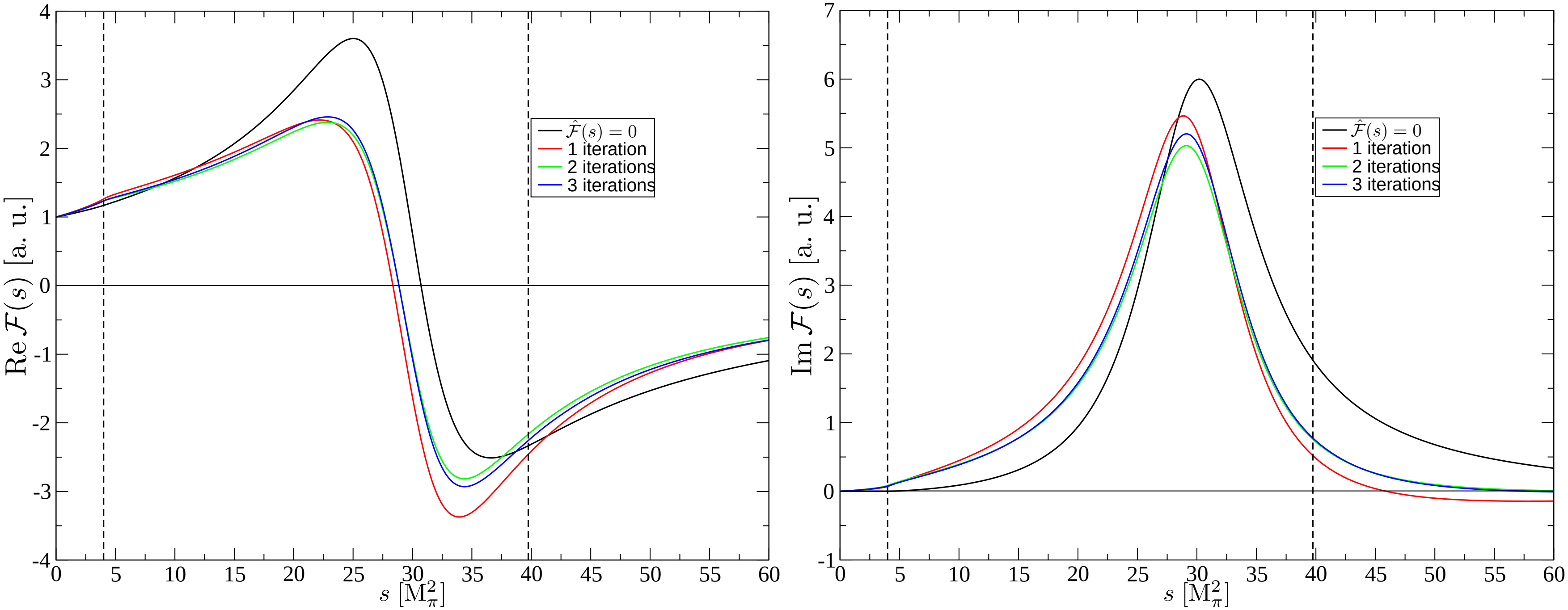

- convergence reached after three iterations
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$\omega/\phi \to 3\pi$ Dalitz plot

- normalized Dalitz plot ($y = \frac{3(s_0 - s)}{2\,M_V\,(M_V - 3M_\pi)}$, $x = \frac{\sqrt{3}(t-u)}{2\,M_V\,(M_V - 3M_\pi)}$):

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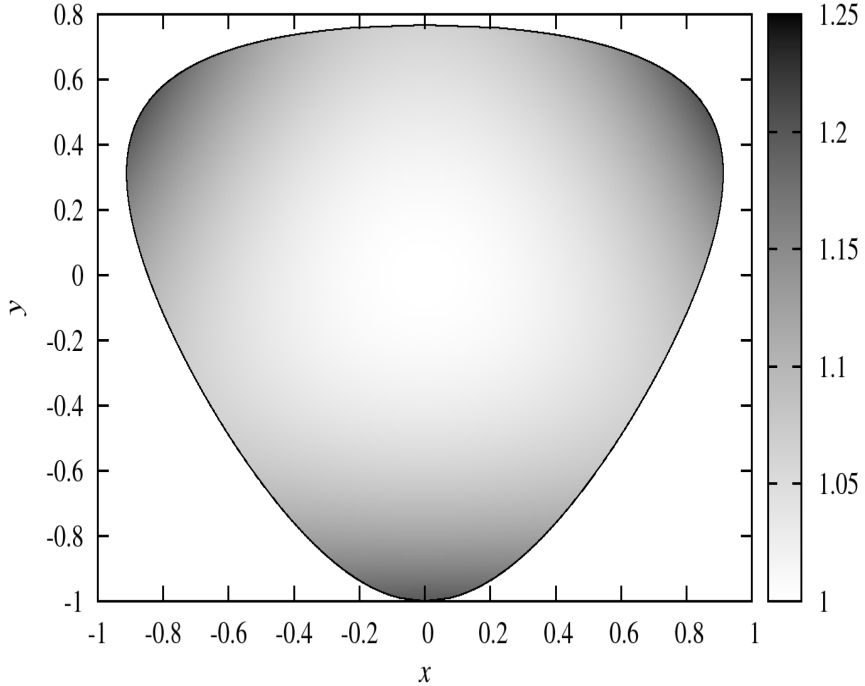

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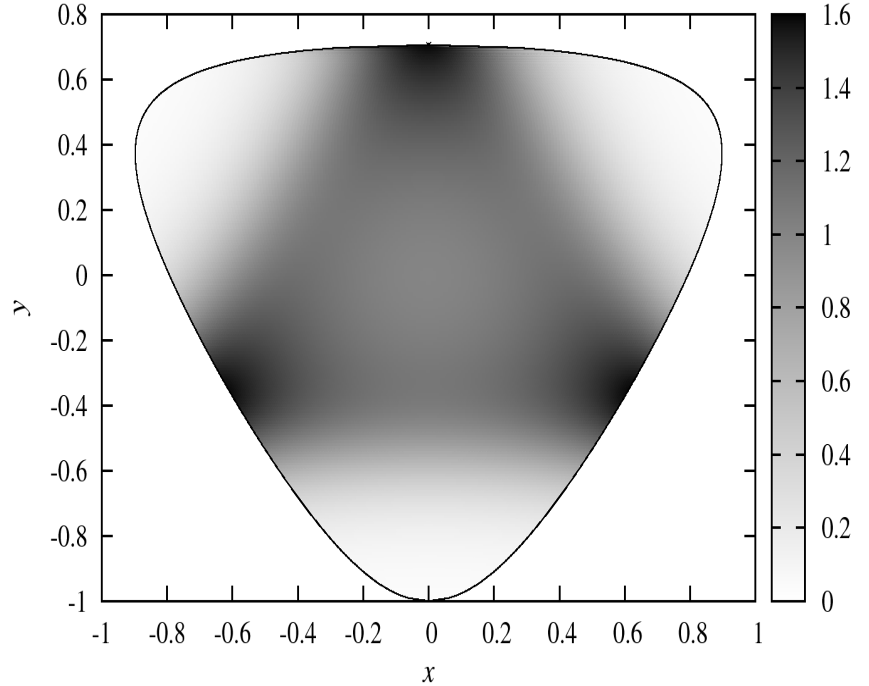

- normalized Dalitz plot is independent of the subtraction constant!
- ω Dalitz plot is relatively smooth
- ϕ Dalitz plot clearly shows ρ resonance bands
- so what are the effects of crossed-channel rescattering?

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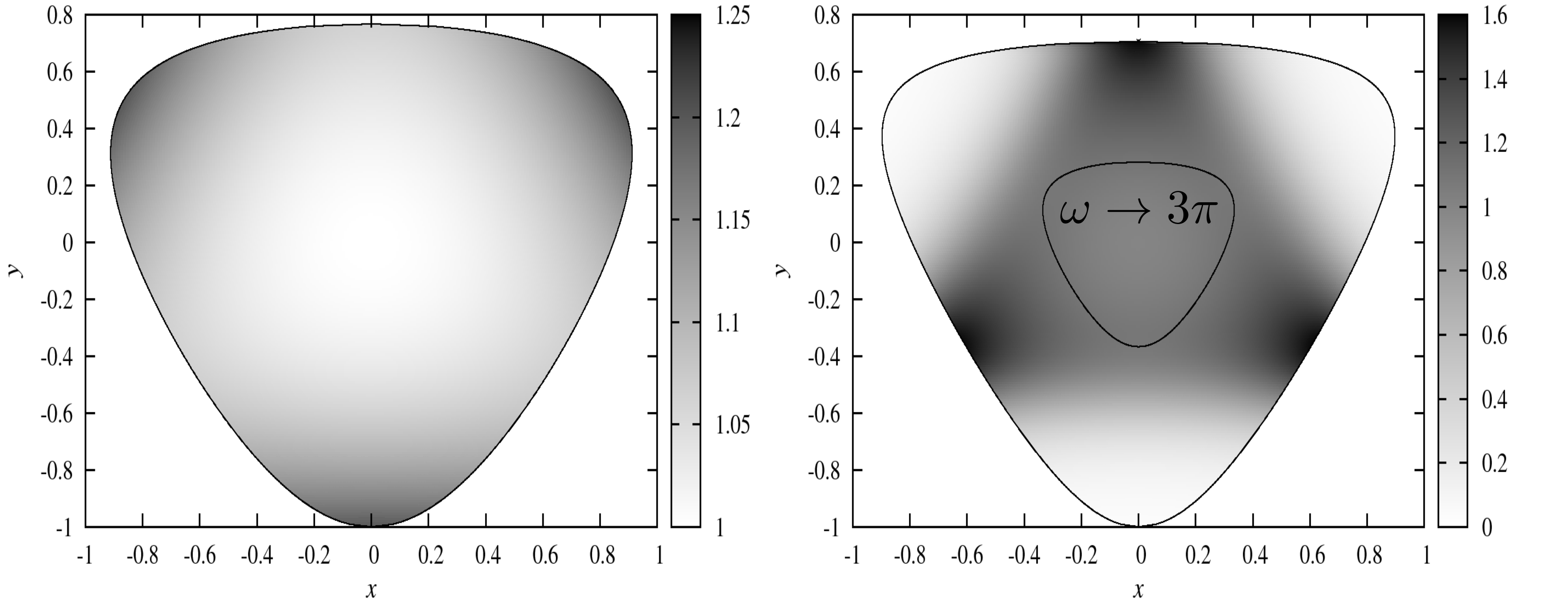
$\omega \to 3\pi$: $\phi \to 3\pi$:



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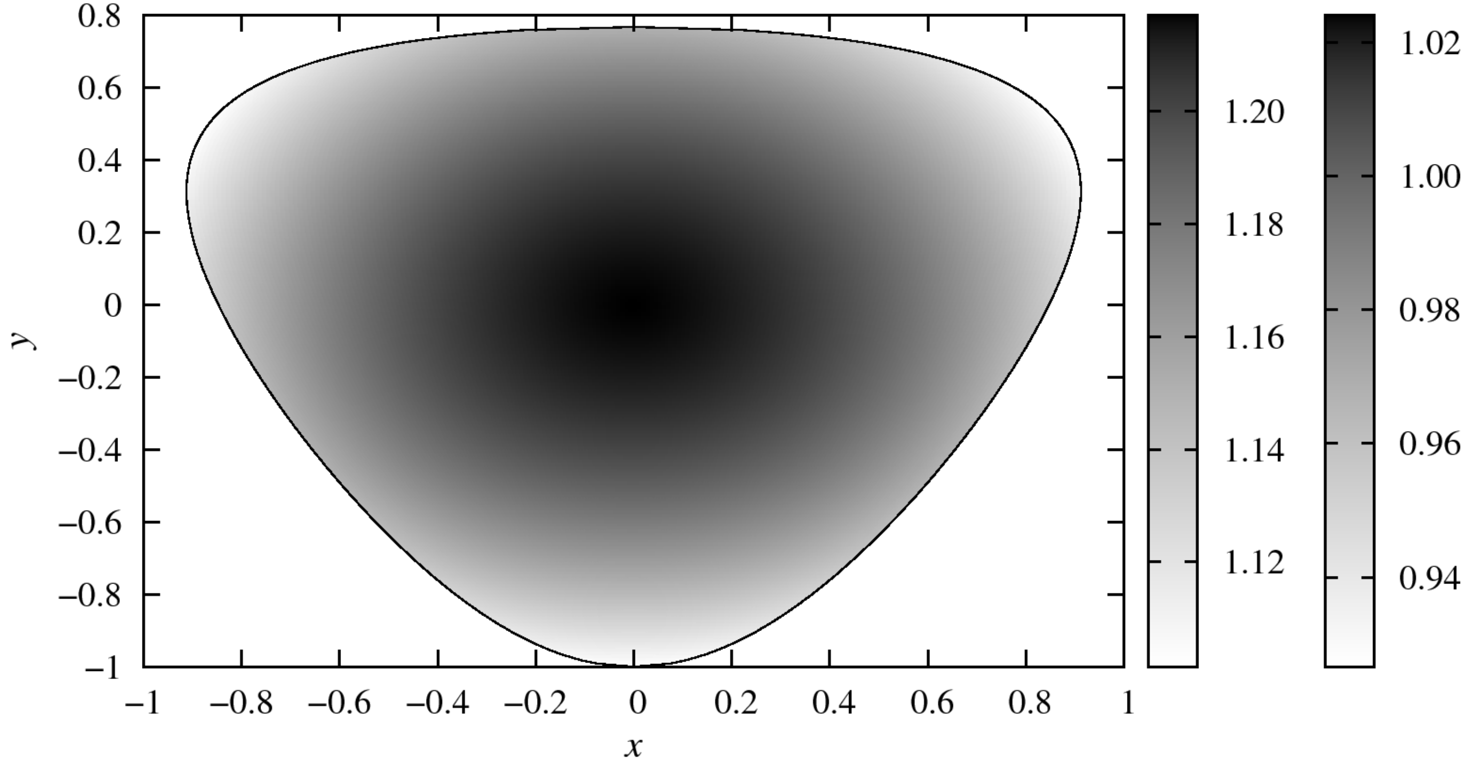
Crossed-channel effects in $\omega \to 3\pi$

- shown is $|\mathcal{F}_{\rm full}(s,t,u)|^2/|\mathcal{F}_{\hat{\mathcal{F}}=0}|^2$



- left scale: a fixed to decay rate before iteration
  - ▷ partial width increased by about 16%
- right scale: a fixed to decay rate before and after iteration
  - ▷ significant part of changes absorbed in overall normalisation

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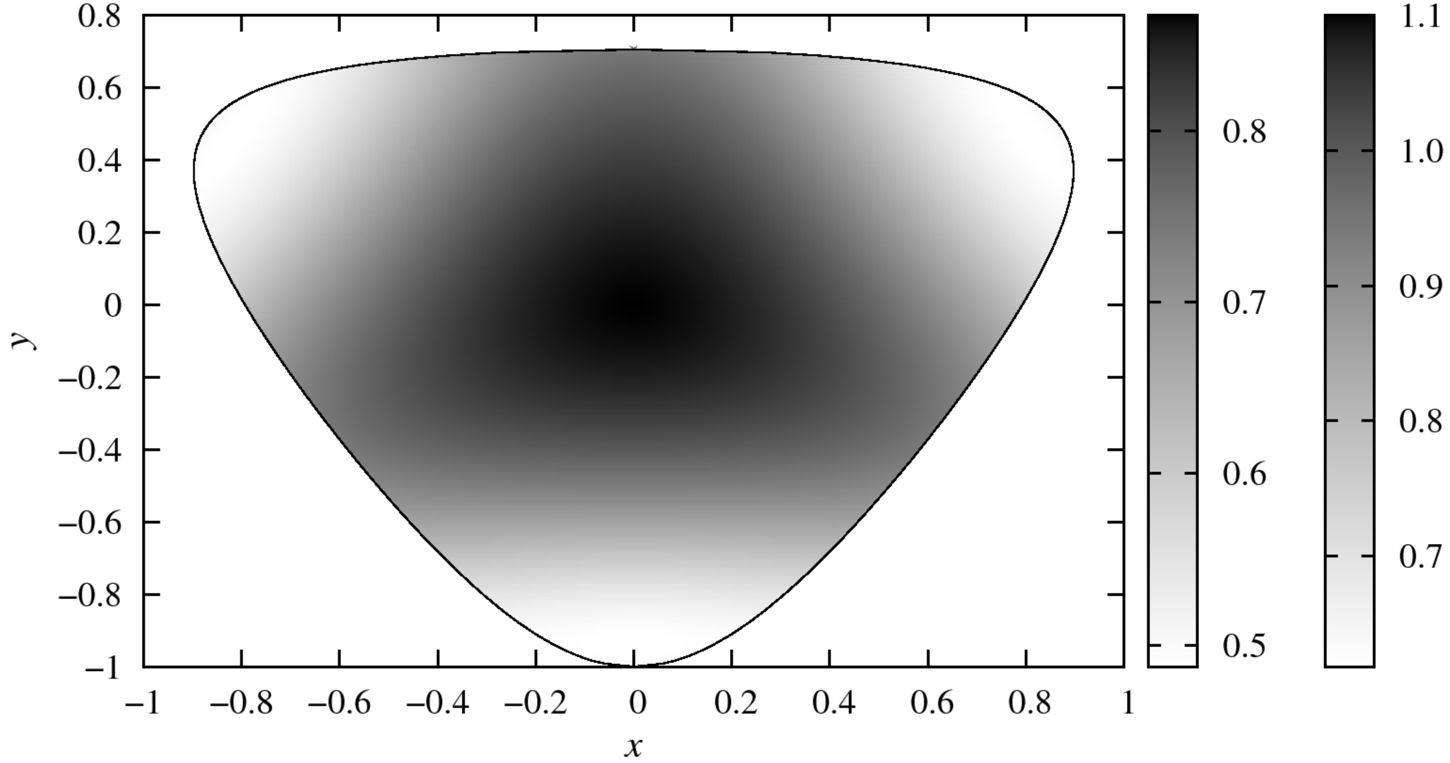
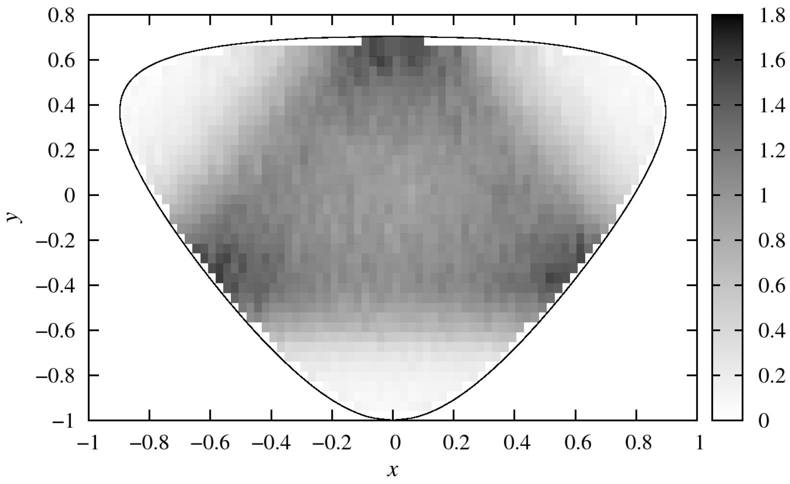
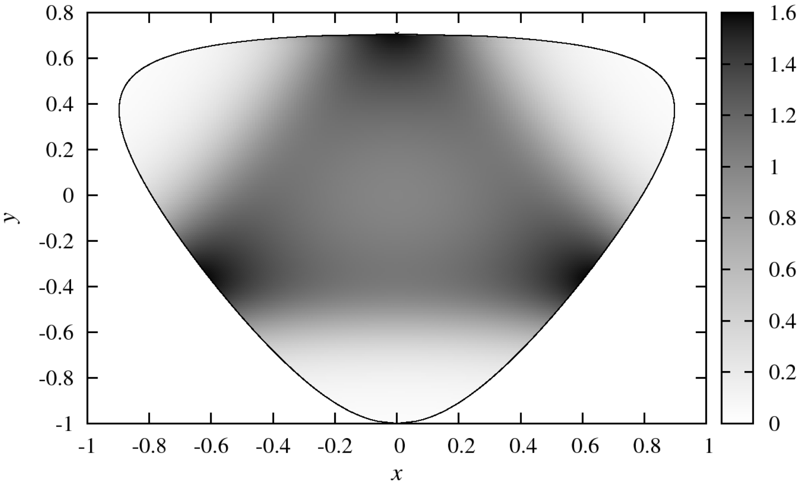


- left scale: a fixed to decay rate before iteration
  - ▷ partial width decreased by about 20%
- right scale: a fixed to decay rate before and after iteration
  - ▷ significant part of changes absorbed in overall normalisation
  - ▷ ρ bands relatively unaffected

Comparison with experiment

Compare to experimental $\phi \to 3\pi$ data:





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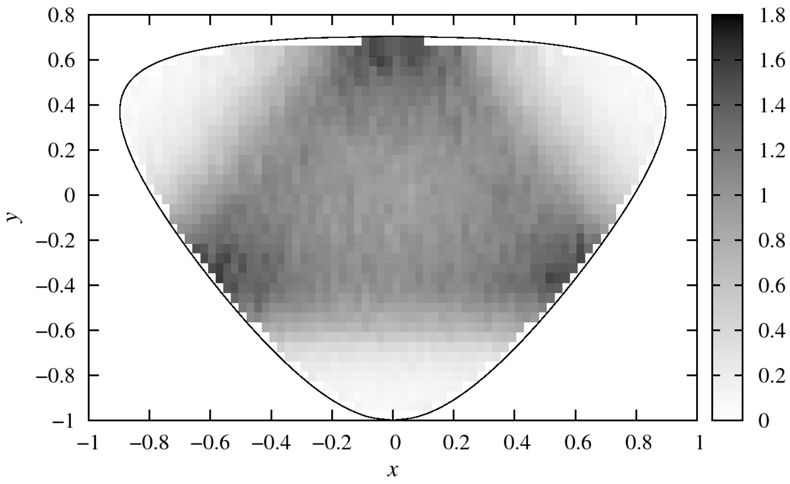

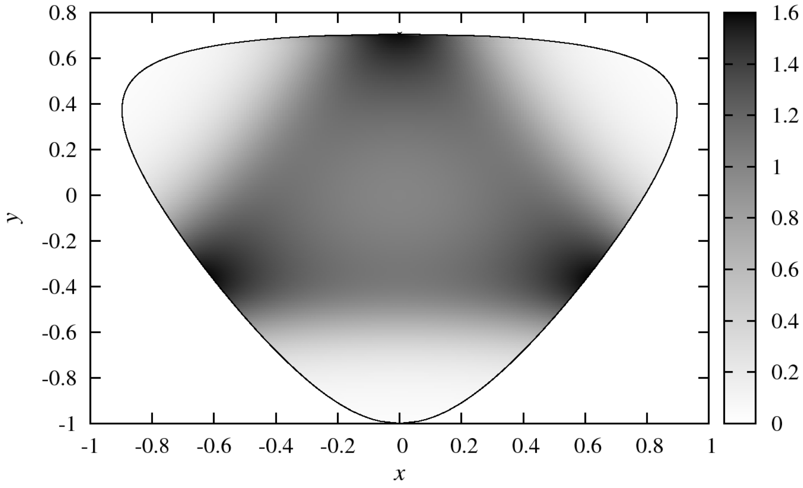

"Fit" results KLOE:

	$\hat{\mathcal{F}} = 0$	full, once-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$

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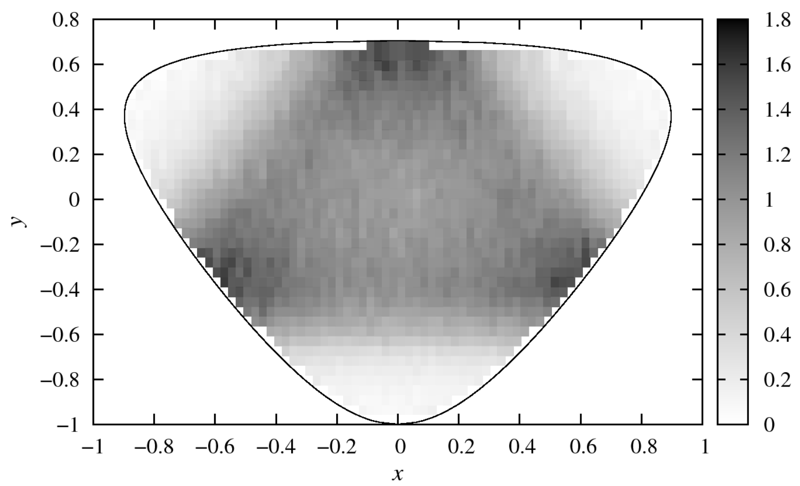

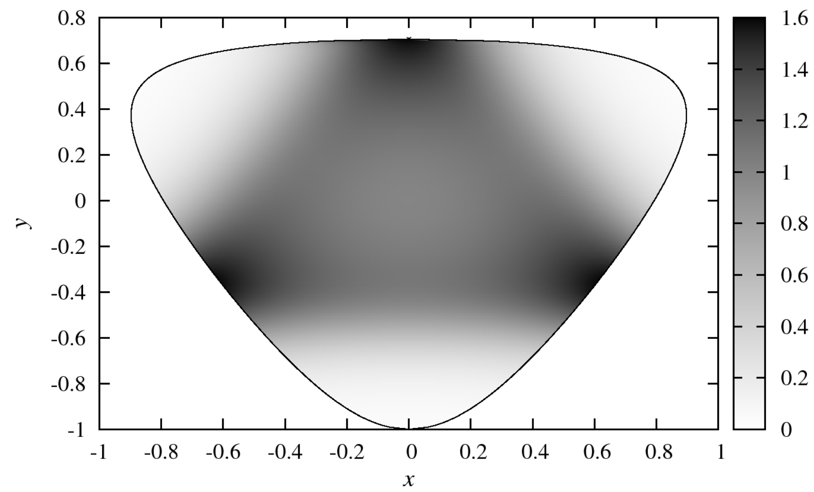

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"Fit" results CMD-2:

	$\hat{\mathcal{F}} = 0$	full, once-subtracted
χ^2/ndof	$1.00 \ldots 1.01$	$1.50 \ldots 1.80$

Two subtractions

- twice-subtracted dispersion relation

$$\mathcal{F}(s) = \Omega(s)\Big\{a + b\,s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\Big\}$$

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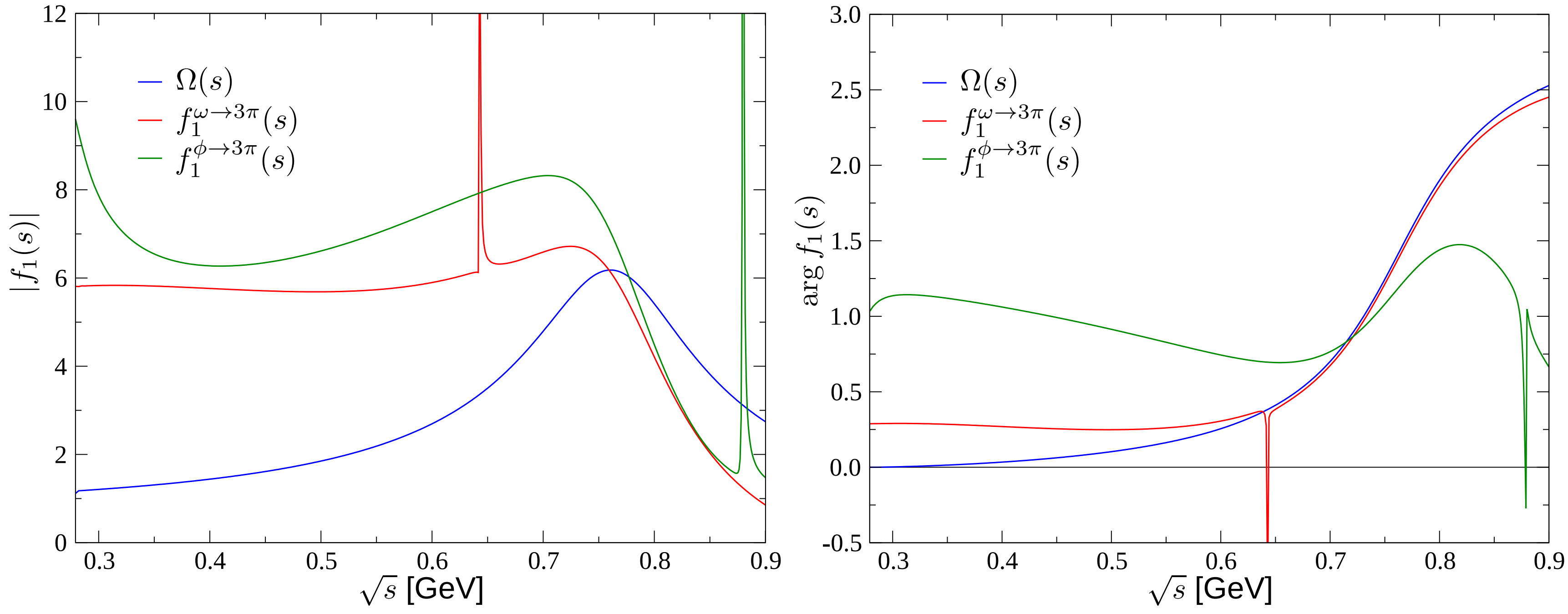
	KLOE		CMD-2	
	Bern	Madrid-Krakòv	Bern	Madrid-Krakòv
χ^2/ndof	1.02	1.03	0.96	0.94
$\lvert b\rvert \times \text{GeV}^2$	0.97 ± 0.03	0.94 ± 0.03	$0.97^{+0.16}_{-0.13}$	$0.95^{+0.15}_{-0.12}$
$\arg\, b$	0.52 ± 0.03	0.42 ± 0.03	0.00 ± 0.016	-0.18 ± 0.18

- perfect fits in both cases $\Rightarrow$ representation respects unitarity, analyticity, and crossing symmetry
- apparent disagreement between KLOE and CMD-2 $\Rightarrow$ systematics?

$V \to 3\pi$ partial-wave amplitude

- partial-wave projection $f_1(s)$:

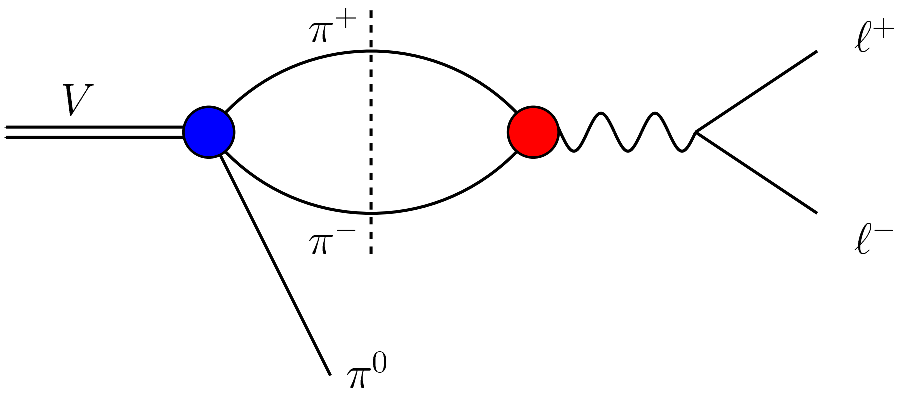
$$f_1(s) = \frac{3}{4}\int_{-1}^{1} dz\left(1 - z^2\right)\mathcal{F}(s,t,u)$$



- phase of the partial-wave amplitude does not vanish at threshold
- divergence at pseudo-threshold expected
  $\Rightarrow$ does not generate non-analytic structure in the TFF

Transition form factor: unitarity implications

$V \to \pi^0\gamma^*$ transition form factor is dominated by $\pi\pi$ intermediate states (γ^* is an isovector $\Rightarrow$ no 3π intermediate states):



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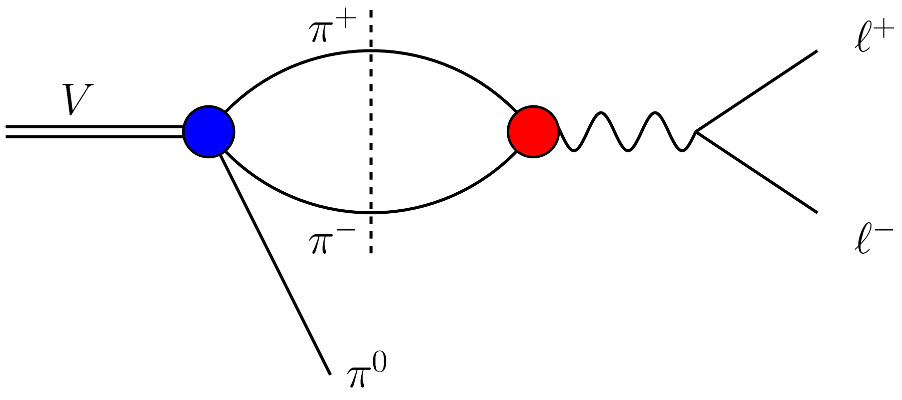

$\Rightarrow$ discontinuity of the TFF from unitarity

$$\text{disc}\, f_{V\pi^0}(s) = \frac{i q_{\pi\pi}^3(s)}{6\pi\sqrt{s}} f_1(s) F_\pi^{V*}(s)\,, \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

Köpp '74

- $F_\pi^{V*}(s)$ pion vector form factor
- $f_1(s)$ $l=1$ partial-wave amplitude for $V \to 3\pi$

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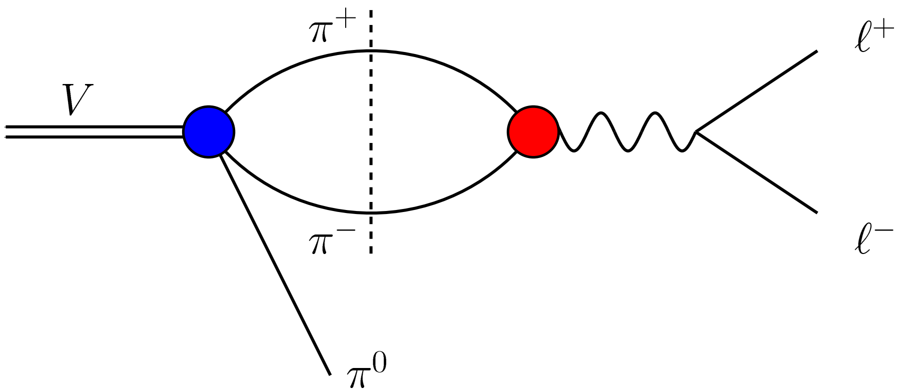


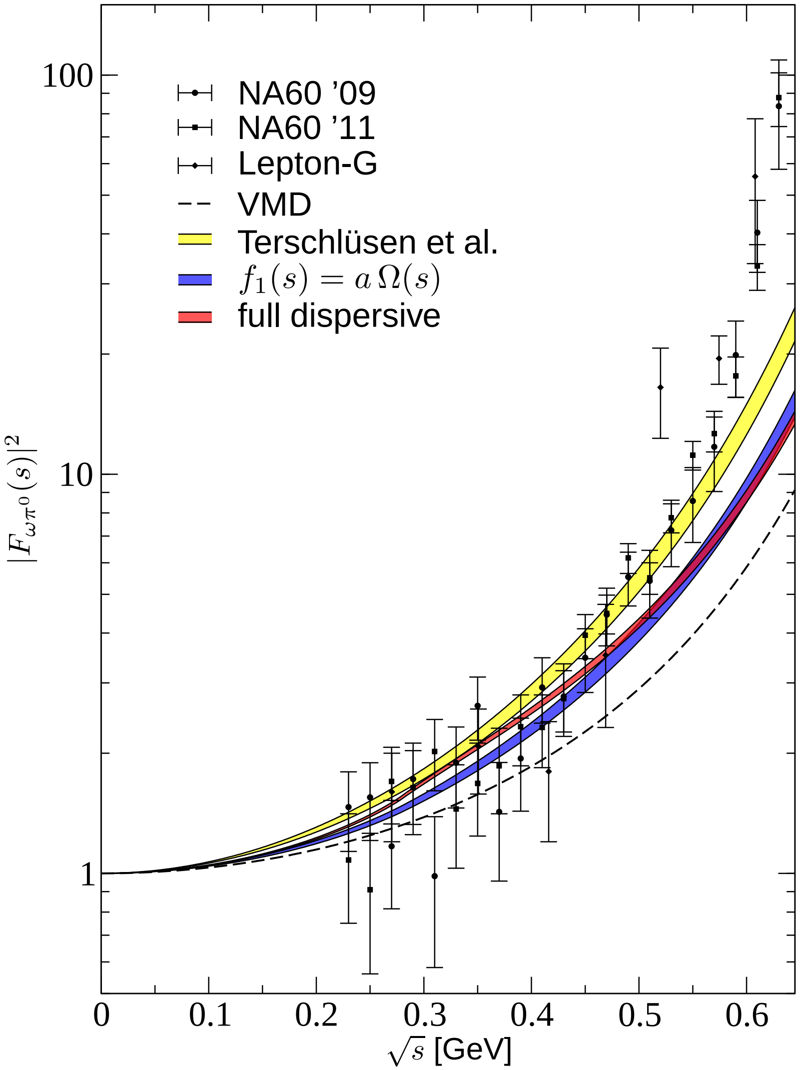
$\Rightarrow$ discontinuity of the TFF from unitarity

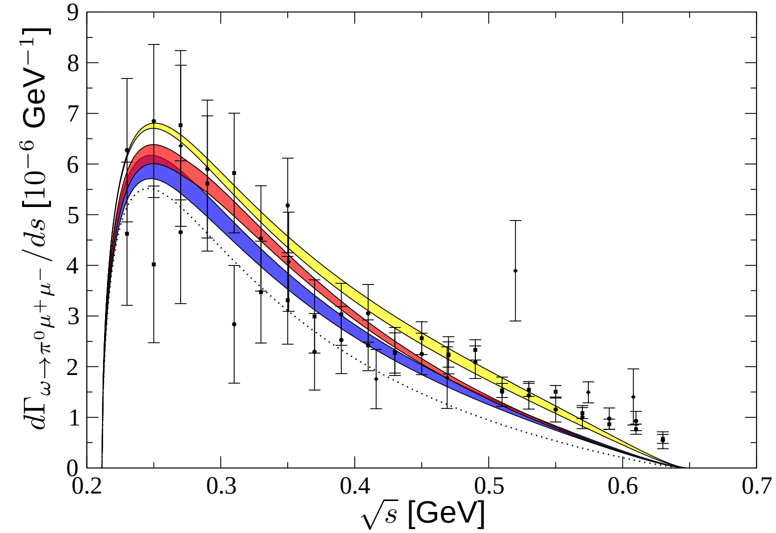
$$\Rightarrow f_{V\pi^0}(s) = f_{V\pi^0}(0) + \frac{s}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds' \frac{q_{\pi\pi}^3(s') F_\pi^{V*}(s') f_1(s')}{s'^{3/2}(s'-s)}$$

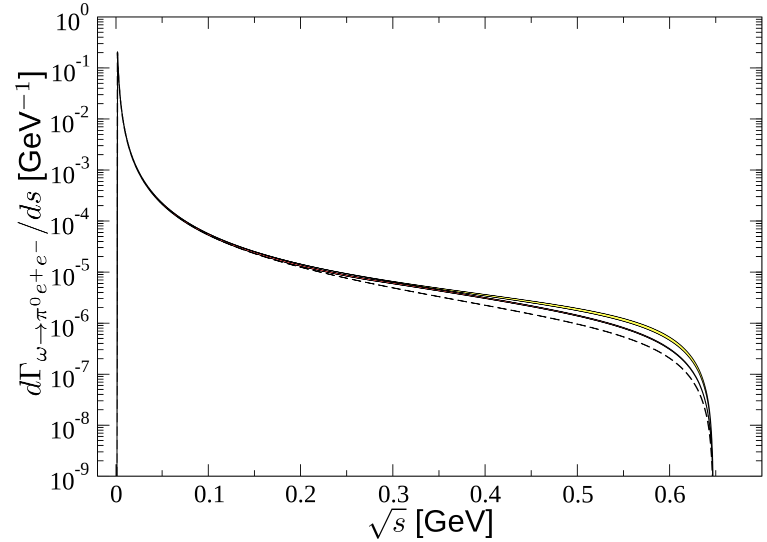
Köpp '74

- $F_\pi^{V*}(s)$ pion vector form factor
- $f_1(s)$ $l=1$ partial-wave amplitude for $V \to 3\pi$
- determine $f_{V\pi^0}(0)$ from $\Gamma_{V\to\pi^0\gamma}$

Numerical results: $\omega \to \pi^0 \gamma^*$

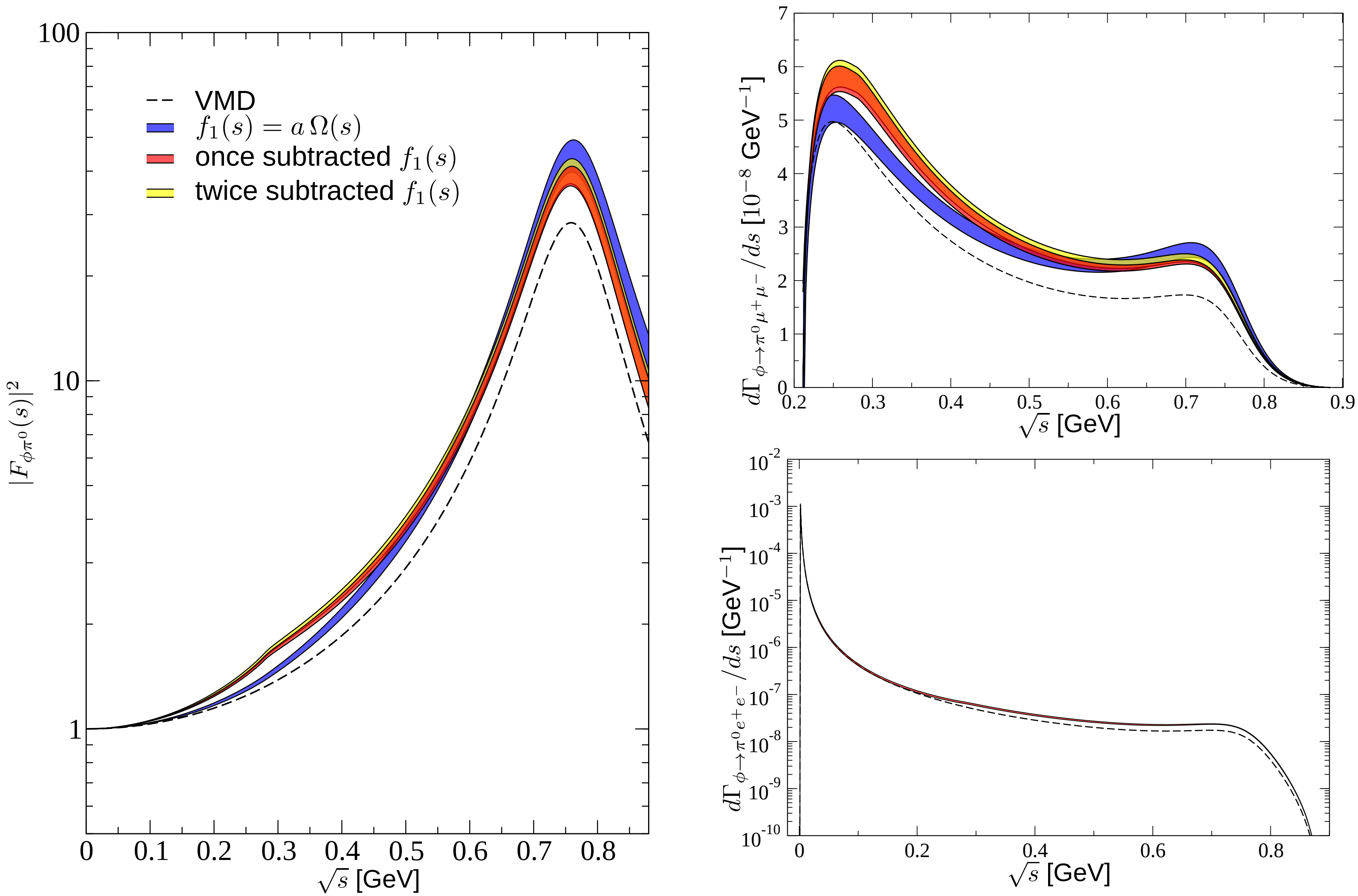






- unable to account for steep rise $\Rightarrow$ pole structure???
- partial-wave amplitude not backed up by $\omega \to 3\pi$ experiment

Numerical results: $\phi \to \pi^0\gamma^*$



- measurement extremely helpful $\Rightarrow$ investigate “pole structure”
- partial-wave amplitude backed up by experiment

Conclusions and outlook

Conclusions...

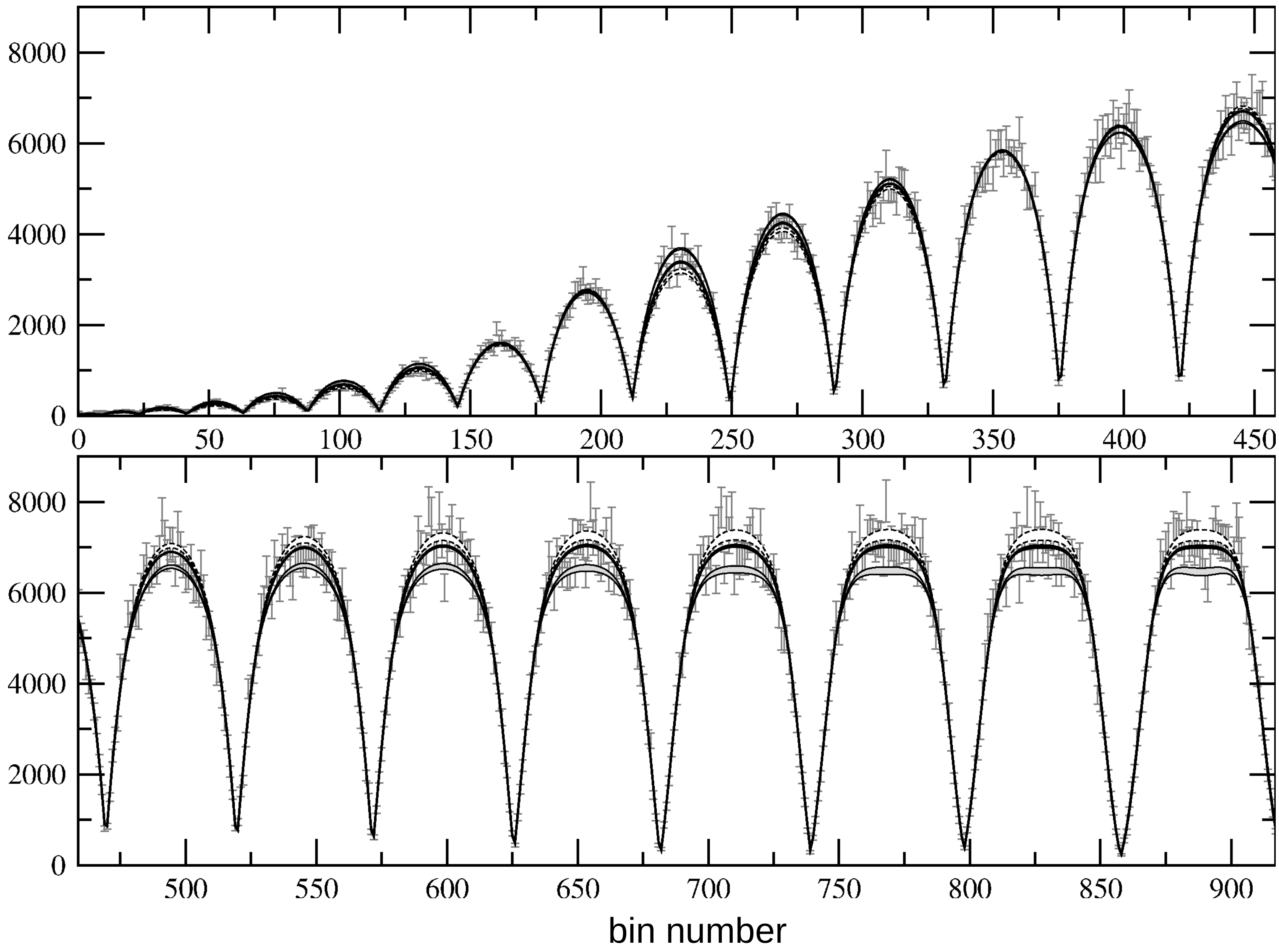
- dispersion relations are a strong tool to study hadronic interactions
- based on fundamental principles of unitarity, analyticity and crossing symmetry
- application in $\phi \to 3\pi$ produces promising results
  $\Rightarrow$ perfect fit to KLOE data
- steep rise in $\omega \to \pi^0\gamma^*$ transition form factor cannot be explained in our framework
- measurement of $\phi \to \pi^0\gamma^*$ should give insights!

...and Outlook

- short term: $\eta' \to \eta\pi\pi$ and $D^+ \to K^-\pi^+\pi^+$ in progress
- long term: CP violation in D decays

Spares

Fit results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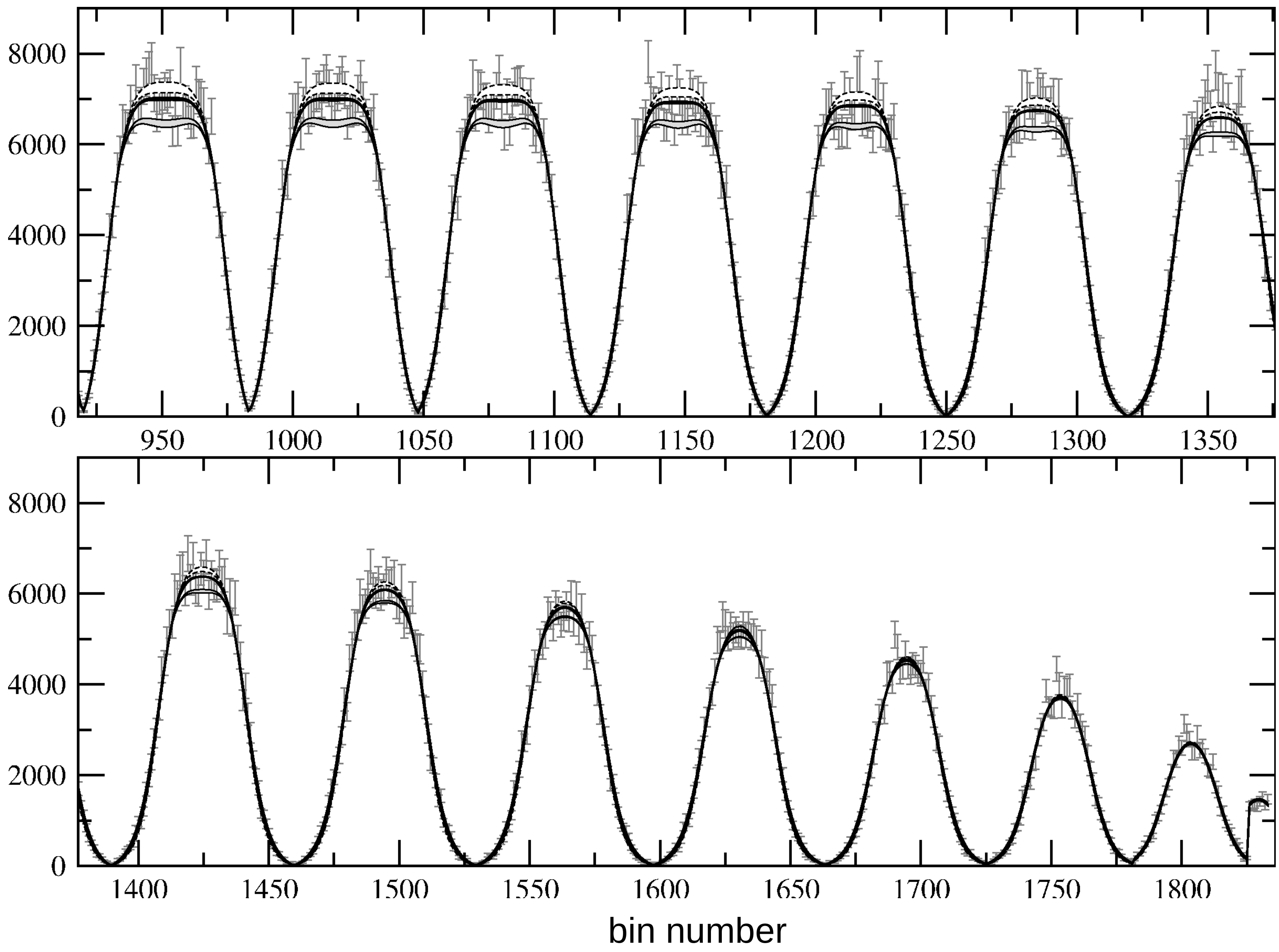
Fit results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Integral equation for the TFF

- Reminder: discontinuity of the TFF:

$$\text{disc}\, f_{V\pi^0}(s) = \frac{i q_{\pi\pi}^3(s)}{6\pi\sqrt{s}} f_1(s) F_\pi^{V*}(s)$$

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$$\Rightarrow \boxed{f_{V\pi^0}(s) = f_{V\pi^0}(0) + \frac{s}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds' \frac{q_{\pi\pi}^3(s') F_\pi^{V*}(s') f_1(s')}{s'^{3/2}(s'-s)}}$$

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- sum-rule for the subtraction constant:

$$f_{V\pi^0}(0) = \frac{1}{12\pi^2} \int_{4M_\pi^2}^{\infty} ds' \frac{q_{\pi\pi}^3(s')}{s'^{3/2}} F_\pi^{V*}(s') f_1(s')$$

  - ▷ related to the $\pi^0\gamma$ decay width:

$$\Gamma_{V\to\pi^0\gamma} = \frac{\alpha(M_V^2 - M_{\pi^0}^2)^3}{24 M_V^3} |f_{V\pi^0}(0)|^2 \,.$$

$V \to \pi^0\gamma$ branching ratios

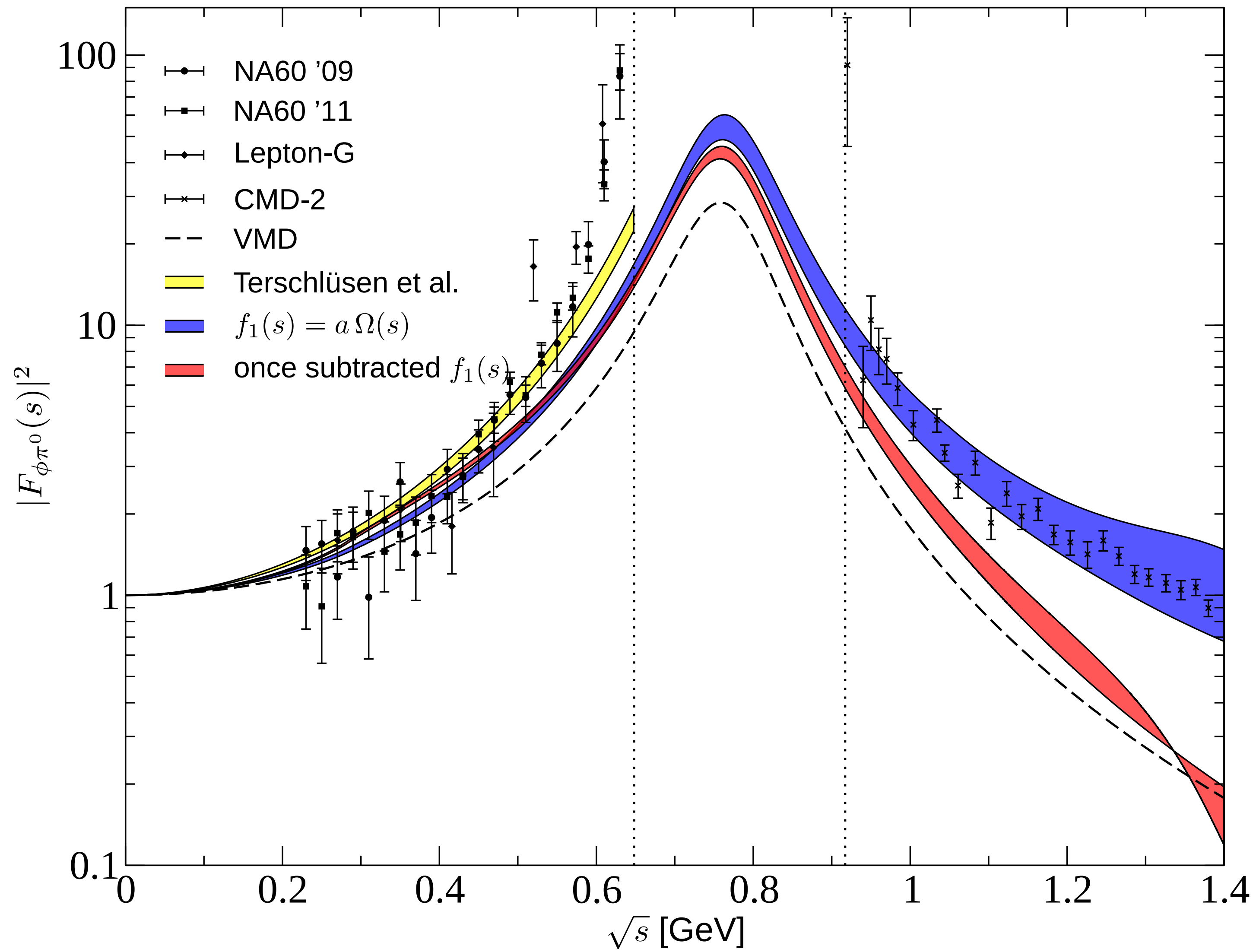
- Estimates for the branching ratios:

$$\mathcal{B}(\omega \to \pi^0\gamma) = (7.48 \ldots 7.75) \times 10^{-2}$$
$$\mathcal{B}^{\text{exp}}(\omega \to \pi^0\gamma) = (8.28 \pm 0.28) \times 10^{-2}$$

$$\mathcal{B}(\phi \to \pi^0\gamma) = (1.28 \ldots 1.37) \times 10^{-3}$$
$$\mathcal{B}^{\text{exp}}(\phi \to \pi^0\gamma) = (1.27 \pm 0.06) \times 10^{-3}$$

- **But:** integrand of $f_{V\pi^0}(0)$ not very well converging
  - ▷ benchmark for approximation of two-pion intermediate states
  - ▷ expected to work better for once-subtracted DR
    $\Rightarrow s$-dependence
  - ▷ fix $f_{V\pi^0}(0)$ by using $\Gamma_{V\to\pi^0\gamma}$ as input

Form factor beyond the $\pi\omega$ threshold



- full solution too low
- coupled-channel treatment of $\omega\pi \to \pi\pi$ might be necessary

Numerical results: $\omega \to \pi^0\gamma^*$

- slope $b_{V\pi^0} = \frac{dF_{V\pi^0}}{ds}|_{s=0}$:
  - ▷ VMD: $b_{\omega\pi^0} = 1\, M_\rho^{-2}$
  - ▷ Terschlüsen et al.: $b_{\omega\pi^0} \approx 2\, M_\rho^{-2}$
  - ▷ our result: $b_{\omega\pi^0} = (1.41 \ldots 1.45)\, M_\rho^{-2}$
  - ▷ NA60 '09: $b_{\omega\pi^0} = (3.72 \pm 0.10 \pm 0.03)\, M_\rho^{-2}$
  - ▷ NA60 '11: $b_{\omega\pi^0} = (3.73 \pm 0.04 \pm 0.05)\, M_\rho^{-2}$

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- branching ratios from integrated spectra:

$$\mathcal{B}(\omega \to \pi^0\mu^+\mu^-) = (0.94\ldots1.00) \times 10^{-4}$$
$$\mathcal{B}^{\text{exp}}(\omega \to \pi^0\mu^+\mu^-) = (1.3 \pm 0.4) \times 10^{-4}$$

$$\mathcal{B}(\omega \to \pi^0 e^+e^-) = (7.6\ldots8.1) \times 10^{-4}$$
$$\mathcal{B}^{\text{exp}}(\omega \to \pi^0 e^+e^-) = (7.7 \pm 0.6) \times 10^{-4}$$

Numerical results: $\phi \to \pi^0\gamma^*$

- slope $b_{V\pi^0} = \frac{dF_{V\pi^0}}{ds}|_{s=0}$:
  - ▷ VMD: $b_{\phi\pi^0} = 1\, M_\rho^{-2}$
  - ▷ our result: $b_{\phi\pi^0} = (1.52 \ldots 1.61)\, M_\rho^{-2}$

Numerical results: $\phi \to \pi^0\gamma^*$

- slope $b_{V\pi^0} = \frac{dF_{V\pi^0}}{ds}|_{s=0}$:
  - ▷ VMD: $b_{\phi\pi^0} = 1\, M_\rho^{-2}$
  - ▷ our result: $b_{\phi\pi^0} = (1.52 \ldots 1.61)\, M_\rho^{-2}$
- branching ratios from integrated spectra:

$$\mathcal{B}^{\text{once}}(\phi \to \pi^0\mu^+\mu^-) = (3.7 \ldots 4.0) \times 10^{-6}$$

$$\mathcal{B}^{\text{twice}}(\phi \to \pi^0\mu^+\mu^-) = (3.8 \ldots 4.1) \times 10^{-6}$$

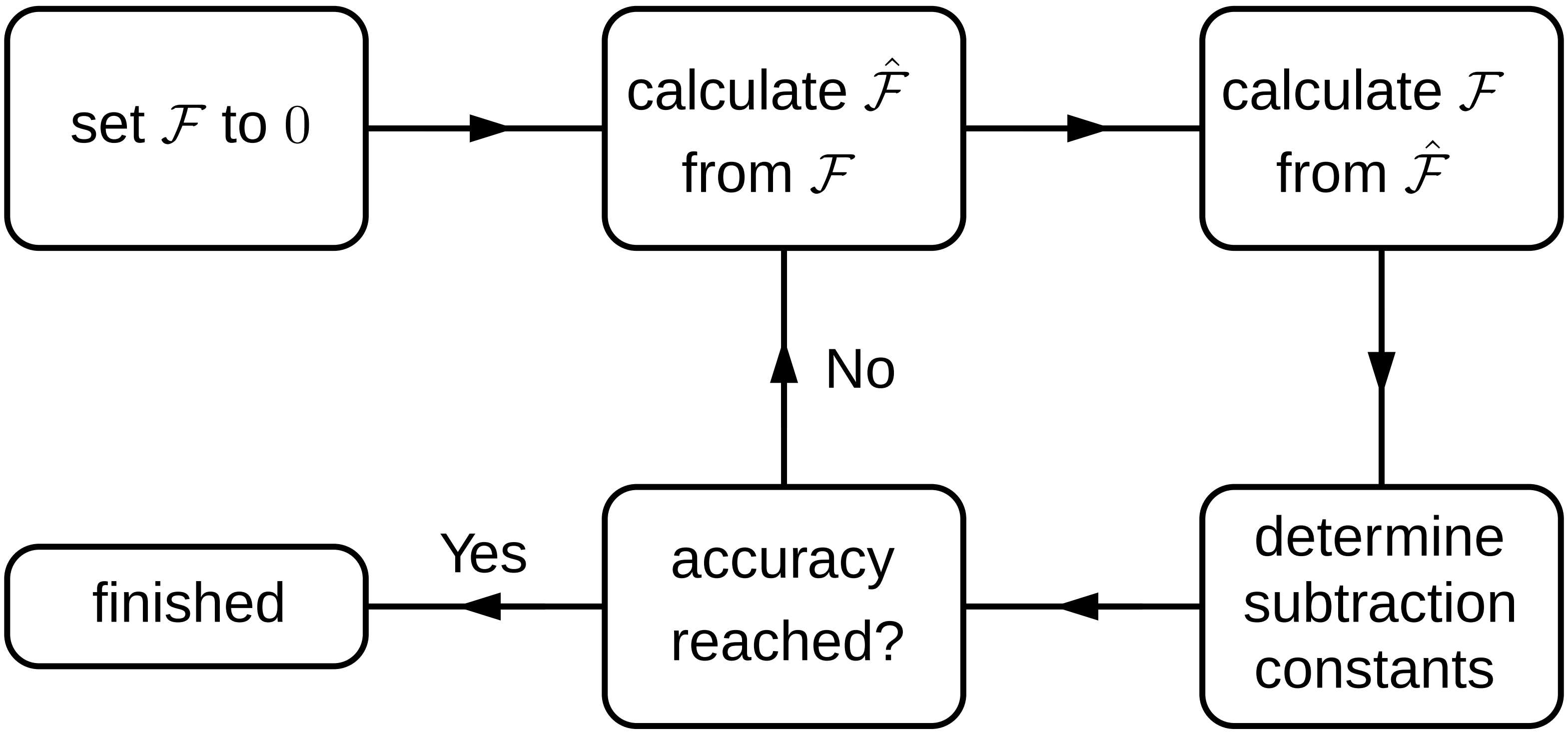
$$\mathcal{B}^{\text{exp}}(\phi \to \pi^0\mu^+\mu^-) = \;???$$

$$\mathcal{B}^{\text{once}}(\phi \to \pi^0 e^+e^-) = (1.39 \ldots 1.51) \times 10^{-5}$$

$$\mathcal{B}^{\text{twice}}(\phi \to \pi^0 e^+e^-) = (1.40 \ldots 1.53) \times 10^{-5}$$

$$\mathcal{B}^{\text{exp}}(\phi \to \pi^0 e^+e^-) = (1.12 \pm 0.28) \times 10^{-5}$$

Iterative procedure for solving integral equations



Input for the pion vector form factor

Input for the phase shifts:

- Roy equation solutions — Caprini (in preparation), García-Martín et al. '11
- Fit to $\tau \to \pi^- \pi^0 \nu_\tau$ form factor data (purely elastic ρ', ρ'') — Roig '12