Lecture 13
Introduction to quasiconvex analysis

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Outline of lecture 13

I- Introduction

II- Normal approach

- a- First definitions
- b- Adjusted sublevel sets and normal operator

III- Quasiconvex optimization

- a- Optimality conditions
- b- Convex constraint case
- c- Nonconvex constraint case

Quasiconvexity

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if, for all $x, y \in K$ and all $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

Quasiconvexity

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if, for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \text{ is convex.}$$

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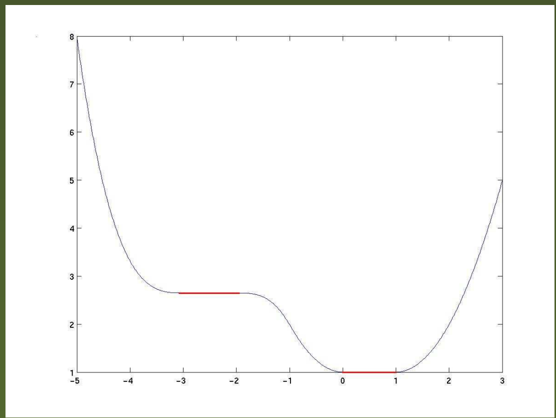
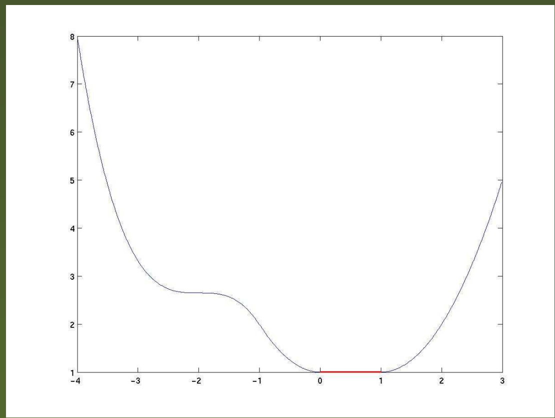
$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \text{ is convex.}$$

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* on K if, f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall\, z \in [x, y[.$$





I

Introduction

- f differentiable

  f is quasiconvex iff df is quasimonotone

  iff $df(x)(y-x) > 0 \Rightarrow df(y)(y-x) \geq 0$

- f is quasiconvex iff ∂f is quasimonotone

  iff $\exists\, x^* \in \partial f(x) \;:\; \langle x^*, y-x \rangle > 0$

  $\Rightarrow \forall\, y^* \in \partial f(y),\; \langle y^*, y-x \rangle \geq 0$

Why not a subdifferential for quasiconvex programming?

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- No (upper) semicontinuity of ∂f if f is not supposed to be Lipschitz

Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of ∂f if f is not supposed to be Lipschitz

- No sufficient optimality condition

$$\bar{x} \in S_{str}(\partial f, C) \not\Longrightarrow \bar{x} \in \arg\min_C f$$

II

Normal approach of quasiconvex analysis

II

Normal approach

a- First definitions

A first approach

Sublevel set:

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\}$$

$$S_\lambda^> = \{x \in X \ : \ f(x) < \lambda\}$$

Normal operator:

Define $N_f(x) : X \to 2^{X^*}$ by

$$\begin{aligned} N_f(x) &= N(S_{f(x)}, x) \\ &= \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \ \forall\, y \in S_{f(x)}\}. \end{aligned}$$

With the corresponding definition for $N_f^>(x)$

But ...

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^{>}(x) = N(S_{f(x)}^{>}, x)$ has no quasimonotonicity properties

But ...

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^{>}(x) = N(S^{>}_{f(x)}, x)$ has no quasimonotonicity properties

Example

Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(a,b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$

Then f is quasiconvex. Consider $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$ and $y^* = (2, 1)$.

We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ (since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0$) while $\langle x^*, y - x\rangle > 0$ and $\langle y^*, y - x\rangle < 0$. Hence $N^<$ is not quasimonotone.

But ...

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S^>_{f(x)}, x)$ has no quasimonotonicity properties

These two operators are essentially adapted to the class of semistrictly quasiconvex functions. Indeed this case, for each $x \in \mathrm{dom}\, f \setminus \arg\min f$, the sets $S_{f(x)}$ and $S^<_{f(x)}$ have the same closure and $N_f(x) = N_f^<(x)$.

II

Normal approach

b- Adjusted sublevel sets

and
normal operator

Definition

Adjusted sublevel set

For any $x \in \mathrm{dom}\, f$, we define

$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^{<}, \rho_x)$$

where $\rho_x = dist(x, S_{f(x)}^{<})$, if $S_{f(x)}^{<} \neq \emptyset$.

Definition

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where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$.

- $S_f^a(x)$ coincides with $S_{f(x)}$ if $\text{cl}(S_{f(x)}^>) = S_{f(x)}$

e.g. f is semistrictly quasiconvex

Definition

Adjusted sublevel set

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- $S_f^a(x)$ coincides with $S_{f(x)}$ if $\operatorname{cl}(S_{f(x)}^>) = S_{f(x)}$

Proposition 1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain* $\operatorname{dom} f$. *Then*

$$f \text{ is quasiconvex } \iff S_f^a(x) \text{ is convex }, \forall\, x \in \operatorname{dom} f.$$

Proof

Let us suppose that $S_f^a(u)$ is convex for every $u \in \operatorname{dom} f$. We will show that for any $x \in \operatorname{dom} f$, $S_{f(x)}$ is convex.
If $x \in \arg\min f$ then $S_{f(x)} = S_f^a(x)$ is convex by assumption.
Assume now that $x \notin \arg\min f$ and take $y, z \in S_{f(x)}$.
If both y and z belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then $y, z \in S_f^a(x)$ thus $[y, z] \subseteq S_f^a(x) \subseteq S_{f(x)}$.
If both y and z do not belong to $\overline{B}\left(S^<_{f(x)}, \rho_x\right)$, then

$$f(x) = f(y) = f(z), \quad \overline{S^<_{f(z)}} = \overline{S^<_{f(y)}} = \overline{S^<_{f(x)}}$$

and ρ_y, ρ_z are positive. If, say, $\rho_y \geq \rho_z$ then $y, z \in \overline{B}\left(\overline{S^<_{f(y)}}, \rho_y\right)$ thus

$$y, z \in S_f^a(y) \text{ and } [y, z] \subseteq S_f^a(y) \subseteq S_{f(y)} = S_{f(x)}.$$

Proof

Finally, suppose that only one of y, z, say z, belongs to $\overline{B}(S^{<}_{f(x)}, \rho_x)$ while $y \notin \overline{B}(S^{<}_{f(x)}, \rho_x)$. Then

$$f(x) = f(y), \;\; \overline{S^{<}_{f(y)}} = \overline{S^{<}_{f(x)}} \text{ and } \rho_y > \rho_x$$

so we have $z \in \overline{B}\left(S^{<}_{f(x)}, \rho_x\right) \subseteq \overline{B}\left(\overline{S^{<}_{f(y)}}, \rho_y\right)$ and we deduce as before that $[y, z] \subseteq S^{a}_{f}(y) \subseteq S_{f(y)} = S_{f(x)}$.

The other implication is straightforward. ■

Adjusted normal operator

Adjusted sublevel set:

For any $x \in \operatorname{dom} f$, we define

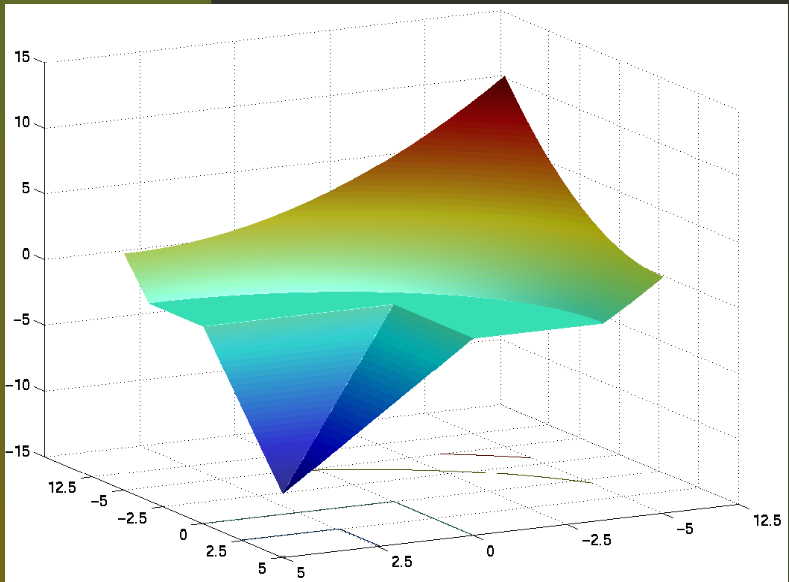
$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x)$$

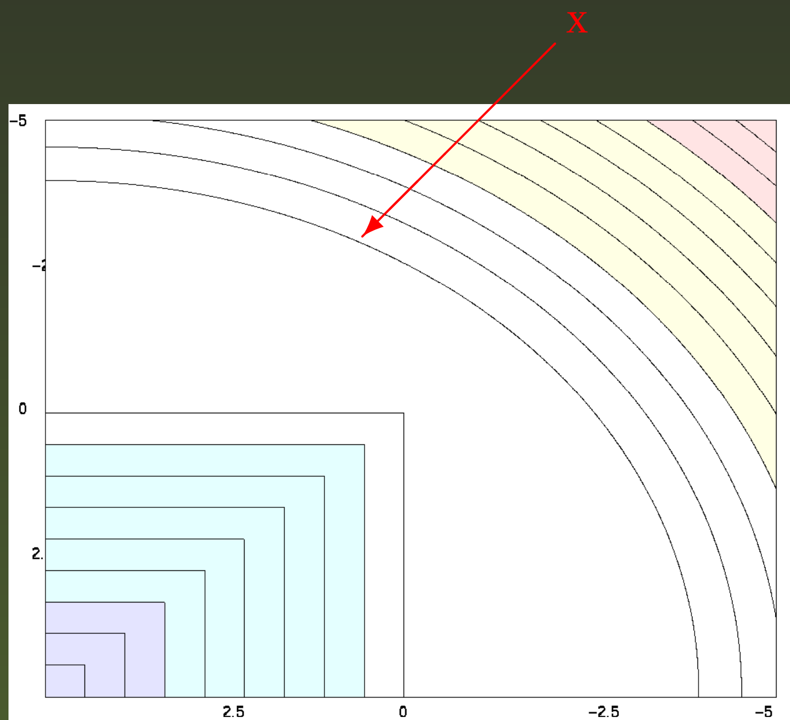
where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$.

Ajusted normal operator:

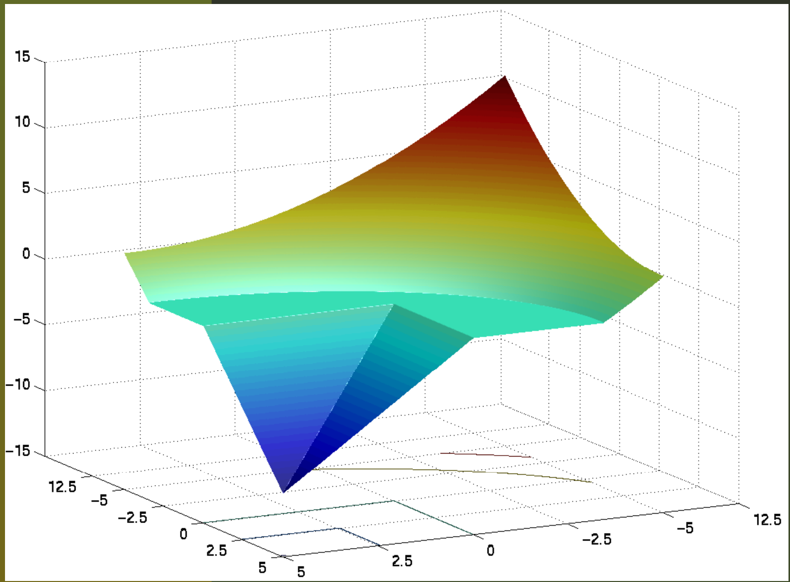
$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle \leq 0, \ \ \forall\, y \in S_f^a(x)\}$$

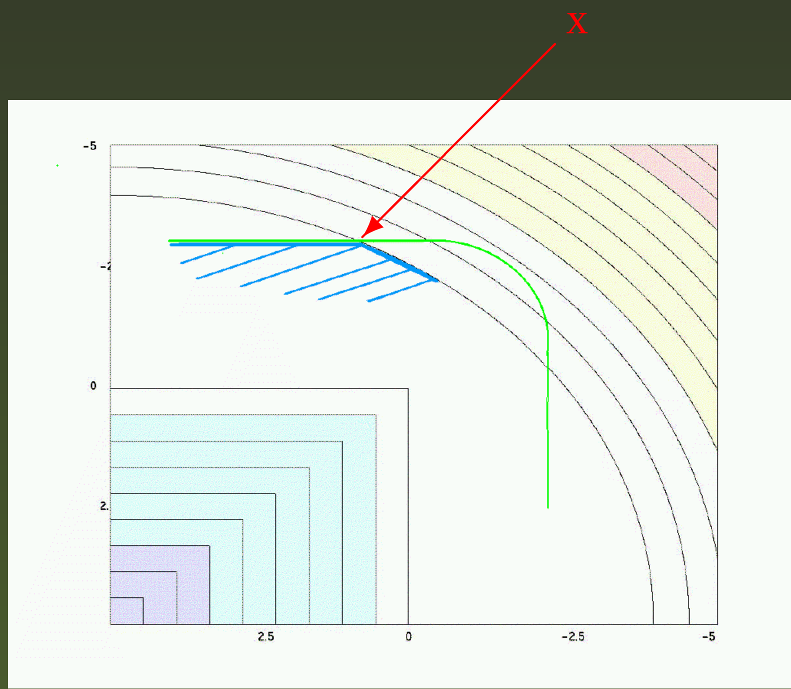
Example





Example

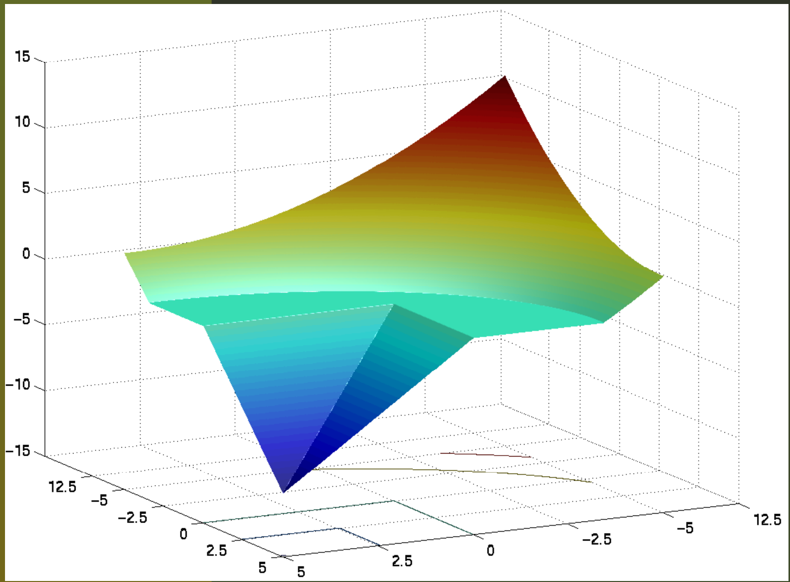


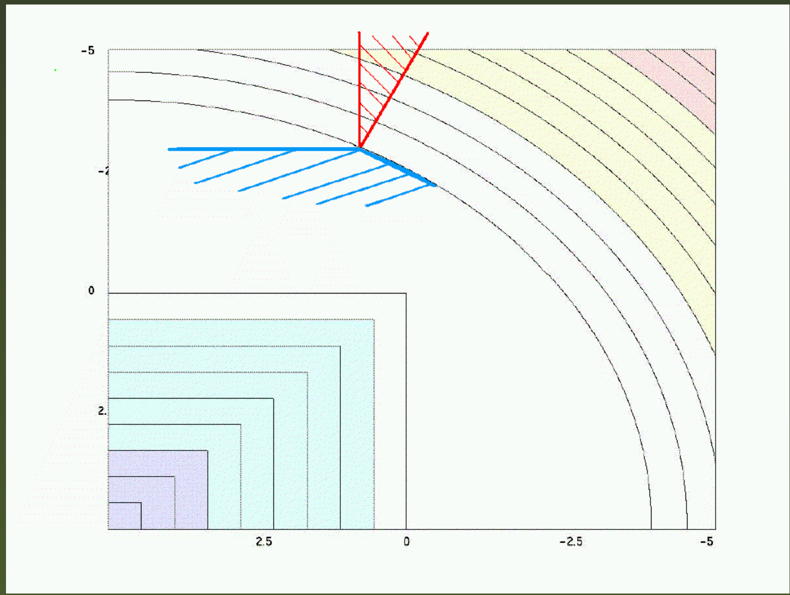


$$\overline{B}(S^{<}_{f(x)}, \rho_x)$$

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$$

Example





$$S_f^a(x) = S_f(x) \cap \overline{B}(S_{f(x)}^{<}, \rho_x)$$

$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

Subdifferential vs normal operator

One can have

$$N_f^a(x) \subsetneq \text{cone}(\partial f(x)) \quad \text{or} \quad \text{cone}(\partial f(x)) \subsetneq N_f^a(x)$$

Proposition 2

f is quasiconvex and $x \in \text{dom}\, f$

If there exists $\delta > 0$ such that $0 \notin \partial^L f(B(x,\delta))$ then

$$\left[cone(\partial^L f(x)) \cup \partial^\infty f(x)\right] \subset N_f^a(x).$$

Basic properties of N_f^a

Nonemptyness:

Proposition 3 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lsc. Assume that rad. continuous on* dom f *or* dom f *is convex and* $\mathrm{int} S_\lambda \neq \emptyset$*,* $\forall\, \lambda > \inf_X f$*. Then*

- *If f is quasiconvex* , $N_f^a(x) \setminus \{0\} \neq \emptyset, \;\; \forall\, x \in$ dom $f \setminus \arg\min f$.
- *f quasiconvex*

$$\Longleftrightarrow \text{dom}\, N_f^a \setminus \{0\} \text{ dense in dom } f \setminus \arg\min f.$$

Quasimonotonicity:

The normal operator N_f^a is always quasimonotone

Upper sign-continuity

- $T : X \to 2^{X^*}$ is said to be *upper sign-continuous* on K iff for any $x, y \in K$, one have :

$$\forall\, t \in]0,1[, \qquad \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0$$

$$\Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$$

where $x_t = (1-t)x + ty$.

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upper semi-continuous

$\Downarrow$

upper hemicontinuous

$\Downarrow$

upper sign-continuous

locally upper sign continuity

Definition 5 *Let* $T : K \to 2^{X^*}$ *be a set-valued map.*

T *est called* locally upper sign-continuous *on* K *if, for any* $x \in K$ *there exist a neigh.* V_x *of* x *and a upper sign-continuous set-valued map* $\Phi_x(\cdot) : V_x \to 2^{X^*}$ *with nonempty convex* w^**-compact values such that* $\Phi_x(y) \subseteq T(y) \setminus \{0\}, \forall\, y \in V_x$

locally upper sign continuity

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Continuity of normal operator

Proposition 7

Let f be lsc quasiconvex function such that $int(S_\lambda) \neq \emptyset, \forall\, \lambda > \inf f$.

Then N_f is locally upper sign-continuous on $\operatorname{dom} f \setminus \arg\min f$.

Proposition 8

If f is quasiconvex such that $int(S_\lambda) \neq \emptyset, \forall\, \lambda > \inf f$ and f is lsc at $x \in \operatorname{dom} f \setminus \arg\min f$,

Then N_f^a is norm-to-w* cone-usc at x.

A multivalued map with conical valued $T : X \to 2^{X^*}$ is said to be *cone-usc* at $x \in \operatorname{dom} T$ if there exists a neighbourhood U of x and a base $C\,(u)$ of $T\,(u)$, $u \in U$, such that $u \to C\,(u)$ is usc at x.

Integration of N_f^a

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex

Question: Is it possible to characterize the functions $g : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex such that $N_f^a = N_g^a$?

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Question: Is it possible to characterize the functions $g : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex such that $N_f^a = N_g^a$?

A first answer:

Let $\mathcal{C} = \{g : X \to \mathbb{R} \cup \{+\infty\}$ cont. semistrictly quasiconvex such that argmin f is included in a closed hyperplane$\}$

Then, for any $f, g \in \mathcal{C}$,

$$N_f^a = N_g^a \quad \Leftrightarrow g \text{ is } N_f^a \setminus \{0\}\text{-pseudoconvex}$$
$$\Leftrightarrow \exists\, x^* \in N_f^a(x) \setminus \{0\} \;:\; \langle x^*, y - x\rangle \geq 0 \Rightarrow g(x) \leq g(y).$$

Integration of N_f^a

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex

Question: Is it possible to characterize the functions $g : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex such that $N_f^a = N_g^a$?

General case: open question

III

Quasiconvex programming

a- Optimality conditions

Quasiconvex programming

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $K \subseteq \operatorname{dom} f$ be a convex subset.

$$(P) \qquad \text{find } \bar{x} \in K \ : \ f(\bar{x}) = \inf_{x \in K} f(x)$$

Quasiconvex programming

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Perfect case: f convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

K a nonempty convex subset of X, $\bar{x} \in K$ + C.Q.

Then

$$f(\bar{x}) = \inf_{x \in K} f(x) \quad \Longleftrightarrow \quad \bar{x} \in S_{str}(\partial f, K)$$

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What about f quasiconvex case?

$$\bar{x} \in S_{str}(\partial f(\bar{x}), K) \not\Longrightarrow \bar{x} \in \arg\min_K f$$

Sufficient optimality condition

Proposition 9

$f : X \to \mathbb{R} \cup \{+\infty\}$ *quasiconvex, radially cont. on* dom f

$C \subseteq X$ *such that* $conv(C) \subset$ dom f.

Suppose that $C \subset int($dom $f)$.

Then $\bar{x} \in S(N_f^a \setminus \{0\}, C) \implies \forall\, x \in C, f(\bar{x}) \leq f(x)$.

where $\bar{x} \in S(N_f^a \setminus \{0\}, K)$ means that there exists $\bar{x}^* \in N_f^a(\bar{x}) \setminus \{0\}$ such that

$$\langle \bar{x}^*, c - x \rangle \geq 0, \qquad \forall\, c \in C.$$

Lemma 10 *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a quasiconvex function, radially continuous on* $\operatorname{dom} f$. *Then* f *is* $N_f^a \setminus \{0\}$*-pseudoconvex on int*$(\operatorname{dom} f)$, *that is,*

$$\exists\, x^* \in N_f^a(x) \setminus \{0\} \ : \ \langle x^*, y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

Proof.

Let $x, y \in \operatorname{int}(\operatorname{dom} f)$. According to the quasiconvexity of f, $N_f^a(x) \setminus \{0\}$ is nonempty.
Let us suppose that $\langle x^*, y - x \rangle \geq 0$ with $x^* \in N_f^a(x) \setminus \{0\}$. Let $d \in X$ be such that $\langle x^*, y_n - x \rangle > 0$ for any n, where $y_n = y + \frac{1}{n} d$ ($\in \operatorname{dom} f$ for n large enough).
This implies that $y_n \notin S_f^<(x)$ since $x^* \in N_f^a(x) \subset N_f^<(x)$.

It follows by the radial continuity of f that $f(y) \geq f(x)$. ■

Necessary and Sufficient conditions

Proposition 11 *Let C be a closed convex subset of X, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be continuous semistrictly quasiconvex such that $\operatorname{int}(S_f^a(\bar{x})) \neq \emptyset$ and $f(\bar{x}) > \inf_X f$.*
Then the following assertions are equivalent:

- *i)* $f(\bar{x}) = \min_C f$
- *ii)* $\bar{x} \in S_{str}(N_f^a \setminus \{0\}, C)$
- *iii)* $0 \in N_f^a(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$.

Proposition 12 *Let C be a closed convex subset of an Asplund space X and $f : X \to \mathbb{R}$ be a continuous quasiconvex function. Suppose that either f is sequentially normally sub-compact or C is sequentially normally compact.*
If $\bar{x} \in C$ is such that

$$0 \notin \partial^L f(\bar{x}) \qquad \textit{and} \qquad 0 \notin \partial^\infty f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$$

then the following assertions are equivalent:

i) $f(\bar{x}) = \min_C f$

ii) $\bar{x} \in \partial^L f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$

iii) $0 \in N_f^a(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$

III

Quasiconvex optimization

b- Convex constraint case

III

Quasiconvex optimization

b- Convex constraint case

$$(P) \qquad \text{Find } \bar{x} \in C \text{ such that } f(\bar{x}) = \inf_C f.$$

with C convex set.

Existence results with convex constraint set

Theorem 13 *(Convex case)*

$f : X \to \mathbb{R} \cup \{+\infty\}$ *quasiconvex*

+ *continuous on* $\mathrm{dom}\,(f)$

+ *for any* $\lambda > \inf_X f$, *int*$(S_\lambda) \neq \emptyset$.

+ $C \subseteq \mathrm{int}(\mathrm{dom}\, f)$ *convex such that* $C \cap \overline{B}(0, n)$ *is weakly compact for some* $n \in \mathbb{N}$.

+ *coercivity condition*

$$\exists\, \rho > 0,\ \forall\, x \in C \setminus \overline{B}(0, \rho),\ \exists\, y \in C \text{ with } \|y\| < \|x\|$$

$$\text{such that } \forall\, x^* \in N_f^a(x) \setminus \{0\},\ \langle x^*, x - y\rangle > 0$$

Then there exists $\bar{x} \in C$ *such that* $\forall\, x \in C$, $f(x) \geq f(\bar{x})$.

Existence for Stampacchia V.I.

Theorem 14

C nonempty convex subset of X.

$T : C \to 2^{X^*}$ quasimonotone

+ locally upper sign continuous on C

+ coercivity condition:

$\exists\, \rho > 0,\ \forall\, x \in C \setminus \overline{B}(0,\rho),\ \exists\, y \in C$ *with* $\|y\| < \|x\|$

such that $\forall\, x^* \in T(x), \langle x^*, x - y\rangle \geq 0$

and there exists $\rho' > \rho$ *such that* $C \cap \overline{B}(0,\rho')$ *is weakly compact* ($\neq \emptyset$).

Then $S(T,C) \neq \emptyset$.

Proof If $\arg\min f \cap K \neq \emptyset$, we have nothing to prove.

Suppose that $\arg\min f \cap K = \emptyset$. Then N^a is quasimonotone and norm-to-w* cone-usc on K. Thus, all assumptions of Theorem 14 hold for the operator $N^a \setminus \{0\}$, so $S_{str}\left(N^a \setminus \{0\}, K\right) \neq \emptyset$. Finally, using the sufficient condition, we infer that f has a global minimum on K. ■

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Corollary 15 *Assumptions on f and K as in Theorem 13. Assume that there exists $n \in \mathbb{N}$ such that for all $x \in K$, $\|x\| > n$, there exists $y \in K$, $\|y\| < \|x\|$ such that $f(y) < f(x)$. Then there exists $x_0 \in K$ such that*

$$\forall\, x \in K, \quad f(x) \geq f(x_0).$$

Proof. If $f(y) < f(x)$ then for every $x^* \in N^a(x) \subseteq N^<(x)$, $\langle x^*, y - x\rangle \leq 0$. Hence, coercivity condition with $T = N^a$ holds. The corollary follows from Theorem 13.

III

Quasiconvex optimization

c- Nonconvex constraint case (an example)

Disjunctive programming

Let us consider the optimization problem:

$$\begin{array}{ll} \min & f(x) \\ s.t. & \left\{ \begin{array}{ll} h_i(x) \leq 0 & i = 1, \ldots, l \\ \min_{j \in J} g_j(x) \leq 0 & \end{array} \right. \end{array}$$

where $f, h_i, g_j : X \to \mathbb{R}$ are quasiconvex

J is a (possibly infinite) index set.

Disjunctive programming

Let us consider the optimization problem:

$$\min \quad f(x)$$
$$s.t. \quad \begin{cases} h_i(x) \leq 0 \quad i = 1, \ldots, l \\ \min_{j \in J} g_j(x) \leq 0 \end{cases}$$

where $f, h_i, g_j : X \longrightarrow \mathbb{R}$ are quasiconvex

J is a (possibly infinite) index set.

Example: $g : X \longrightarrow \mathbb{R}$ is continuous concave and

$$\min \quad f(x)$$
$$s.t. \quad \begin{cases} h_i(x) \leq 0 \quad i = 1, \ldots, l, \\ g(x) \leq 0 \end{cases}$$

Disjunctive programming

Let us consider the optimization problem:

$$\min \quad f(x)$$
$$s.t. \quad \begin{cases} h_i(x) \leq 0 \quad i = 1, \ldots, l \\ x \in \cup_{j \in J} C_j \end{cases}$$

where $f, h_i : X \to \mathbb{R}$ are quasiconvex
C_j are convex subsets of X
J is a (possibly infinite) index set.

A little bit of history

- Balas 1974: first paper about disjunctive prog.

  Pure and mixed 0-1 linear programming

$$
\begin{array}{ll}
Min & Z = d.x + \sum_{k=1}^{m} c_k \\
s.t. & \bigvee_{j \in J_k} \left[\begin{array}{c} Y_{jk} \\ A^{jk}x \geq b^{jk} \\ c_k = \gamma_{jk} \end{array} \right], k \in K \\
 & 0 \leq x \leq U, Y_{jk} \in \{0, 1\}
\end{array}
$$

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- puis Gugat, Grossmann, Borwein, Cornuejols-Lemaréchal,...

Duality for disjunctive prog.

- For linear Disjunctive program (Balas):

$$\begin{array}{lll} Primal & \alpha = inf & c.x \\ & s.t. & \bigvee_{j\in J} \begin{bmatrix} A^j.x \geq b^j \\ x \geq 0 \end{bmatrix} \end{array}$$

Duality for disjunctive prog.

- For linear Disjunctive program (Balas):

$$Primal \qquad \alpha = inf \quad c.x$$

$$s.t. \qquad \bigvee_{j \in J} \begin{bmatrix} A^j.x \geq b^j \\ x \geq 0 \end{bmatrix}$$

$$Dual \qquad \beta = sup \quad w$$

$$s.t. \qquad \bigwedge_{j \in J} \begin{bmatrix} w - u^j b^j \leq 0 \\ u^j.A^j \leq c \\ u^j \geq 0 \end{bmatrix}$$

Duality theorem for linear disjunctive prog.

Set $P_j = \{x : A^j.x \geq b^j,\ x \geq 0\}, U_j = \{u^j : u^j.A^j \leq c,\ u^j \geq 0\}$.

Denote $J^* = \{j \in J : P_j \neq \emptyset\}$ and $J^{**} = \{j \in J : U_j \neq \emptyset\}$

Theorem 17 *(Balas 77)*

If (P) and (D) satisfy the following regularity assumption

$$J^* \neq \emptyset,\ \ J \setminus J^{**} \neq \emptyset \Longrightarrow J^* \setminus J^{**} \neq \emptyset,$$

Then

- *either both problems are feasible, each has an optimal solution and* $\alpha = \beta$

- *or one is infeasible, the other one either is infeasible or has no finite optimum.*

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- Generalized by Borwein (JOTA 1980) to convex disjunctive program.

Computational aspects

- In the 90's: cutting plane methods (Balas-Ceria-Cornuejols, Math. Prog. 1993).

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Problem: need of representation of the convex hull of the union of polyhedral sets

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Solution: Lift-and-Project

1- representation $\tilde{P}$ of the union of polyhedra P in a higher dim. space

2- projection back in the original space such that $proj\ \tilde{P} = P$

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Problem: need of representation of the convex hull of the union of polyhedral sets

Proposition 18 *(Balas 1998)*

If $P_j = \{x \in \mathbb{R}^n : A^j x \geq b^j\} \neq \emptyset, \, j = 1, p$ *then*

$$\tilde{P} = \left\{ (x, (y^1, y_0^1), .., (y^p, y_0^p)) \in \mathbb{R}^{n+(n+1)p} : \begin{array}{r} x - \sum_{j=1}^p y^j = 0 \\ A^j y^j - y_0^j b^j \geq 0 \\ y_0^j \geq 0 \\ \sum_{j=1}^p y_0^j = 1 \end{array} \right\}$$

Computational aspects

- In the 90's: cutting plane methods (Balas-Ceria-Cornuejols, Math. Prog. 1993).

  Aim: separate x from $\cup_j P_j$ or equivalently from $P = \overline{\text{conv}}(\cup_j P_j)$

  Problem: need of representation of the convex hull of the union of polyhedral sets
- Grossmann review's on disjunctive prog. techniques (Opt. and Eng. 2002)
- Cornuejols-Lemaréchal (Math. Prog. 2006)

Let us consider the problem:

$$(P_C) \qquad \begin{array}{ll} \inf & f(x) \\ \text{s.t.} & x \in C \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex lower semicontinuous

$C \subset \text{int}(\text{dom}\, f)$ is a locally finite union of closed convex sets,

$$C = \cup_{\alpha \in A}^{lf} C_\alpha.$$

Locally finite union

A subset C of X is said to be a *locally finite union of closed sets* if

- there exists a (possibly infinite) family $\{C_\alpha \ : \ \alpha \in A\}$ of convex subsets of X such that

$$C = \cup_{\alpha \in A} C_\alpha$$

- for any $x \in C$, there exist $\rho > 0$ and a finite subset A_x of A such that

$$B(x,\rho) \cap C = B(x,\rho) \cap [\cup_{\alpha \in A_x} C_\alpha]$$

and

$$\forall\, \alpha \in A_x, \quad x \in C_\alpha.$$

Notation: $C = \cup^{lf}_{\alpha \in A} C_\alpha$

Local mapping

For any subset C of X, locally finite union of closed sets $C = \cup^{lf}_{\alpha\in A} C_\alpha$, a family $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ with $\rho_x > 0$ and A_x satisfying

$$B(x,\rho_x) \cap C = B(x,\rho_x) \cap [\cup_{\alpha\in A_x} C_\alpha]$$

and

$$\forall\, \alpha \in A_x, \quad x \in C_\alpha.$$

is called a local mapping of C.

The following subset

$$C = \{x \in X \ : \ g(x) \leq 0\}$$

is a locally finite union of convex sets, if

- g is continuous concave and locally polyhedral

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$$C = \{x \in X \ : \ g(x) \leq 0\}$$

is a locally finite union of convex sets, if

- g is continuous concave and locally polyhedral

or

- $g(x) = \min_{j \in J} g_j(x)$ with each $g_i : X \longrightarrow \mathbb{R}$ quasiconvex coercive and

(H) $\forall\, x \in X,\ \exists\, \varepsilon > 0$ and J_x finite $\subset J$ such that

$$\{j \in J \ : \ g_j(u) = g(u)\} \subset J_x, \ \ \forall_, u \in B(x, \rho).$$

Existence for lfuc

Theorem 19

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a quasiconvex function, continuous on* $\operatorname{dom} f$.
Assume that

- *for every* $\lambda > \inf_X f$, *int*$(S_\lambda) \neq \emptyset$ *subset of* C
- *for any* $\alpha \in A$, $C_\alpha \cap \bar{B}(0, n)$ *is weakly compact for any* $n \in \mathbb{N}$

and the following coercivity condition holds

$$\text{there exist } \rho > 0 \text{ such that } \forall\, x \in C_\alpha \setminus \overline{B}(0, \rho),$$
$$\exists\, y_x \in C_\alpha \cap B(0, \|x\|) \text{ such that } f(y_x) < f(x).$$

If there exists a local mapping $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ *of* C *such that the set* $\{x \in C \ : \ card(A_x) > 1\}$ *is included in a weakly compact subset of* C, *then problem* (P_C) *admits a local solution.*

Existence for disjunctive program

$$(DP) \qquad \begin{array}{ll} \min & f(x) \\ s.t. & \begin{cases} h_i(x) \le 0 \quad i = 1, \dots, l, \\ \min_{j \in J} g_j(x) \le 0 \end{cases} \end{array}$$

Existence for disjunctive program

Theorem 22

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a quasiconvex function, continuous on* $\operatorname{dom} f$.

Assume that

- (H) *holds and for every* $\lambda > \inf_X f$, *int*$(S_\lambda) \neq \emptyset$
- *for any* j, g_j *is lsc quasiconvex and coercive*
- *for any* i, h_i *is lsc quasiconvex*
- *for any* j *and any* $n \in \mathbb{N}$, *the subset*

  $S_0(g_j) \cap [\cap_i S_0(h_j)] \cap \overline{B}(0,n)$ *is weakly compact*

If there exists a local mapping $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ *of* C *such that the set* $\{x \in C \ : \ card(A_x) > 1\}$ *is included in a weakly compact subset of* C, *then problem* (DP) *admits a local solution.*

References

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Appendix 5 - Normally compactness

A subset C is said to be *sequentially normally compact* at $x \in C$ if for any sequence $(x_k)_k \subset C$ converging to x and any sequence $(x_k^*)_k$, $x_k^* \in N^F(C, x_k)$ weakly converging to 0, one has $\|x_k^*\| \to 0$.

Examples:

- X finite dimensional space
- C epi-Lipschitz at x
- C convex with nonempty interior

A function f is said to be *sequentially normally subcompact* at $x \in C$ if the sublevel set $\overline{S_{f(x)}}$ is sequentially normally compact at x.

Examples:

- X finite dimensional space
- f is locally Lipschitz around x and $0 \notin \partial^L f(x)$
- f is quasiconvex with $int(S_{f(x)}) \neq \emptyset$

Appendix 2 - Limiting Normal Cone

Let C be any nonempty subset of X and $x \in \bar{C}$. The *limiting normal cone* to C at x, denoted by $N^L(C,x)$ is defined by

$$N^L(C,x) = \limsup_{x' \to x} N^F(C,x')$$

where the Frèchet normal cone $N^F(C,x')$ is defined by

$$N^F(C,x') = \left\{ x^* \in X^* \ : \ \limsup_{u \to x',\, u \in C} \frac{\langle x^*, u - x' \rangle}{\|u - x'\|} \leq 0 \right\}.$$

The limiting normal cone is closed (but in general non convex)

Appendix - Limiting subdifferential

One can define the *Limiting subdifferential* (in the sense of Mordukhovich), and its asymptotic associated form, of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} \partial^L f(x) &= Limsup_{y \xrightarrow{f} x} \partial^F f(y) \\ &= \left\{ x^* \in X^* \ : \ (x^*, -1) \in N^L(\text{epi}\, f, (x, f(x))) \right\} \end{aligned}$$

$$\begin{aligned} \partial^\infty f(x) &= Limsup_{\substack{y \xrightarrow{f} x \\ \lambda \searrow 0}} \lambda \partial^F f(y) \\ &= \left\{ x^* \in X^* \ : \ (x^*, 0) \in N^L(\text{epi}\, f, (x, f(x))) \right\}. \end{aligned}$$