Journal of Convex Analysis
Volume 15 (2008), No. 3, 459–472

Quasimonotone Variational Inequalities and Quasiconvex Programming: Qualitative Stability

M. Ait Mansour
Université Cadi Ayyad, Faculté Poly-Disciplinaire,
Route Sidi Bouzid, 4600 Safi, Morocco
maitmansour@hotmail.com

D. Aussel
Dep. Math., University of Perpignan,
52 av. P. Alduy, 66860 Perpignan, France
aussel@univ-perp.fr

Received: April 3, 2007
Revised manuscript received: January 17, 2008

In this paper we consider quasimonotone Stampacchia variational inequalities subject to perturbations, both on the set-valued operator and on the constraint set.

We establish continuity properties, upper semicontinuity and lower semicontinuity, of the (possibly set-valued) solution map. Application to quasiconvex optimization is given throughout the concept of normal operator.

Keywords: Quasimonotone variational inequality, stability, upper semicontinuity, lower semicontinuity, quasiconvex programming, Mosco convergence

1991 Mathematics Subject Classification: Primary 49J40; Secondary 49K40, 49J52

1. Introduction

Let X be a Banach space, U and Λ be metrizable topological spaces and consider, for $\mu \in U$ and $\lambda \in \Lambda$, the variational inequality problem:

$$(P_{\lambda,\mu}) \quad \text{Find } \bar{x} \in K(\mu) \text{ such that there exists } \bar{x}^* \in T(\bar{x}, \lambda) \\ \text{with } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0,\ \forall y \in K(\mu).$$

where $T : X \times \Lambda \to 2^{X^*}$ and $K : U \to 2^X$ are two set-valued maps. In such a setting, λ and μ play the role of a perturbation respectively on the operator and on the constraint set defining the variational inequality.

The analysis of the stability (or sensitivity) of the parametric variational inequality consists in evaluating the influence of those perturbations on the solution set $S(\lambda, \mu)$ of problem $(P_{\lambda,\mu})$. This analysis can be *qualitative* (continuity properties of the solution map $S(\cdot, \cdot)$) or *quantitative* (Hölder-type evaluations of the distance between two solution sets $S(\lambda, \mu)$ and $S(\lambda', \mu')$ in term of the norm of the perturbations $\|\lambda - \lambda'\|$ and $\|\mu - \mu'\|$, whenever Λ and U are normed spaces).

Both the qualitative and the quantitative points of view have been extensively studied for different types of variational inequalities. Some monotonicity properties are always re-

ISSN 0944-6532 / $ 2.50

quired, namely the operator is assumed to be monotone or pseudomonotone for the qualitative analysis [11, 13, 15, 17, 20, 23] and strongly monotone or strongly pseudomonotone for the quantitative approach, see [2, 1, 4, 11, 26, 27] and references therein. But for many applications (mathematical economics, quasiconvex programming,...) monotonicity or pseudomonotonicity constitutes a too strong hypothesis.

This paper is devoted to the qualitative analysis of the variational inequality problem $(P_{\lambda,\mu})$ and the obtained results extend to the quasimonotone setting previous results for monotone or pseudomonotone variational inequalities. The quantitative analysis of quasimonotone variational inequalities has been developped by the authors in [1]. For both parts of the analysis, an appropriate notion of solution to be able to obtain stability results in the quasimonotone case is the so-called star-solution.

The paper is organized as follows. In Section 2 we fix our notation and we define the concept of star-solution while in Section 3 we bring to the fore some properties of Mosco convergence of sets which will be useful for the stability analysis. In Section 4 our main results (upper semicontinuity and lower semicontinuity of the solution map) are stated and proved. Finally, thanks to the concept of normal operator, we derive, in Section 5, a stability result for quasiconvex optimization problems.

2. Preliminaries

Let X be a real Banach space, X^* its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. The topological closure, the interior and the boundary of a set A will be denoted respectively by $\text{cl}(A)$, $\text{int}(A)$ and $\text{bd}\,(A)$. Given any nonempty subset A of X and a point $x \in X$, the distance from x to A will be denoted by $\text{dist}(x, A) = \inf\{\|x - y\| : y \in A\}$.

For any $x, y \in X$, we will use the notation $[x, y]$, $]x, y[$ and $]x, y]$ for the segments $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$, $]x, y[= \{tx + (1-t)y : t \in]0, 1[\}$ and $]x, y] = \{tx + (1-t)y : t \in]0, 1]\}$. The domain and the graph of a set-valued operator $T : X \to 2^{X^*}$ will be denoted, respectively, by $\text{dom}\,(T)$ and $\text{gr}(T)$.

A set-valued operator $T : K \to 2^{X^*}$ is said to be *quasimonotone* on a subset K if, for all $x, y \in K$,

$$\exists\, x^* \in T(x) : \langle x^*, y - x\rangle > 0 \Rightarrow \forall\; y^* \in T(y) : \langle y^*, y - x\rangle \geq 0.$$

In the sequel we will also use some generalized convexity assumptions. So let us recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be:

- *quasiconvex* on a subset $K \subset \text{dom}\, f$ if, for any $x, y \in K$ and any $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

- *semistrictly quasiconvex* on a subset $K \subset \text{dom}\, f$ if, f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y),\ \forall\, z \in [x, y[.$$

Let us denote, for any $\alpha \in \mathbb{R}$, by $S_\alpha(f)$ and $S^<_\alpha(f)$ the sublevel set and the strict sublevel set associated to f and α:

$$S_\alpha(f) = \{x \in X : f(x) \leq \alpha\} \quad \text{and} \quad S^<_\alpha(f) = \{x \in X : f(x) < \alpha\}.$$

It is well known that the quasiconvexity of a function f is characterized by the convexity of the sublevel sets (or the convexity of the strict sublevel sets). Analougously, it is easy to check that any lower semicontinuous function f, semistrictly quasiconvex on its domain $\operatorname{dom} f$ satisfies the following property:

$$\forall \alpha > \inf_X f, \ \operatorname{cl}(S^<_\alpha(f)) = S_\alpha(f). \tag{1}$$

Roughly speaking it means that a lower semicontinuous semistrictly quasiconvex function f does not have any "flat part" with nonempty interior on $\operatorname{dom} f \setminus \operatorname{argmin}_X f$.

As announced in the introduction, along this paper we will consider the following perturbed (or parametric) Stampacchia variational inequality problem

$$(P_{\lambda,\mu}) \quad \begin{array}{l} \text{Find } \bar{x} \in K(\mu) \text{ such that there exists } \bar{x}^* \in T(\bar{x}, \lambda) \\ \text{with } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \ \forall y \in K(\mu). \end{array}$$

where $T : X \times \Lambda \to 2^{X^*}$ and $K : U \to 2^X$ are two set-valued maps.

As already observed in [6, 7, 1] the classical solution set

$$S(\lambda, \mu) = \{x \in K(\mu) : \exists\, x^* \in T(x, \lambda) \text{ with } \langle x^*, y - x \rangle \geq 0, \ \forall\, y \in K(\mu)\}$$

is not adapted to the case of quasimonotone variational inequalities and their applications (in particular to obtain stability results and optimality conditions for quasiconvex programming) whereas the concept of star-solution carries enough information to handle this quasimonotone setting. The set of star-solutions of the perturbed problem $(P_{\lambda,\mu})$ is

$$S^*(\lambda, \mu) = \left\{ x \in K(\mu) \ : \ \begin{array}{l} \exists\, x^* \in T(x, \lambda) \setminus \{0\} \text{ with} \\ \langle x^*, y - x \rangle \geq 0, \ \forall\, y \in K(\mu). \end{array} \right\}$$

It is interesting to observe that, in the case of a set-valued operator T with conical values (as in Section 5), considering star-solutions corresponds to work with classical solutions of the Stampacchia variational inequality defined by a truncated form $\tilde{T}$ of T, for example $\tilde{T}(x) = T(x) \cap S^*(0, 1)$ where $S^*(0, 1)$ stands for the unit sphere of the dual space X^*.

It is also important to notice that all star-solutions of $(P_{\lambda,\mu})$ are on the boundary of the constraint set $K(\mu)$.

3. Int-Mosco convergence of sets

The concept of Mosco convergence of a sequence of subsets has been introduced in [23] to study the convergence properties of the solutions of various variational inequalities. This concept has proved to be extremely useful in different fields of analysis.

In this section we investigate some properties of the Mosco convergence which will play a crucial role for establishing convergence properties for quasimonotone variational inequalities (see Section 4).

Let us first recall, for any sequence $(S_n)_n$ of subsets of the Banach space X, the definitions of *Lower limit* and *Upper limit* in the sense of Kuratowski:

$$\operatorname{Lim\,inf}_n S_n = \{x \in X : \lim_{n\to\infty} d(x, S_n) = 0\}$$
$$\operatorname{Lim\,sup}_n S_n = \{x \in X : \liminf_{n\to\infty} d(x, S_n) = 0\}.$$

or, in other words, $\operatorname{Lim\,inf}_n S_n$ is the set of limits of sequences $(x_n)_n$ with $x_n \in K_n$, for any n, while the upper limit is the set of cluster points of such sequences $(x_n)_n$. From the definitions it is clear that the sets $\operatorname{Lim\,inf}_n S_n$ and $\operatorname{Lim\,sup}_n S_n$ are closed sets. Moreover, for any sequence $(S_n)_n$ one has $\operatorname{Lim\,inf}_n \operatorname{cl}(S_n) = \operatorname{Lim\,inf}_n S_n$.

Finally we will also consider the weak version of the upper limit, denoted by $w - \operatorname{Lim\,sup}_n S_n$, and defined as the set of weak limits of sequences $(x_{n_k})_k$, $x_{n_k} \in S_{n_k}$ where $(S_{n_k})_k$ is a subsequence of $(S_n)_n$.

According to Mosco [23], we say that a sequence $(S_n)_n$ of subsets of X converges to a subset S in the sense of Mosco, if both of the following equalities hold:

$$w - \operatorname*{Lim\,sup}_n S_n = S \quad \text{and} \quad \operatorname*{Lim\,inf}_n S_n = S.$$

Since one always has $\operatorname{Lim\,sup}_n S_n \subset w - \operatorname{Lim\,sup}_n S_n$, the Mosco convergence of $(S_n)_n$ to S turns out to be simply equivalent to the inclusions

$$\text{a)} \quad w - \operatorname*{Lim\,sup}_n S_n \subset S \quad \text{and} \quad \text{b)} \quad S \subset \operatorname*{Lim\,inf}_n S_n.$$

Let us introduce the following slight variant of Mosco convergence.

Definition 3.1. Let S be a subset of X and $(S_n)_n$ be a sequence of subsets of X. We say that the sequence $(S_n)_n$ *Int-Mosco converges* to S if and only if

$$\text{a)} \quad w - \operatorname*{Lim\,sup}_n S_n \subset S \quad \text{and} \quad \text{b}') \quad S \subset \operatorname*{Lim\,inf}_n \operatorname{int}(S_n).$$

One should note that if the Int-Mosco limit exists, it is unique.

Clearly, the Int-Mosco convergence implies the Mosco convergence, but the converse is not true in general, even for nonempty interior subsets (consider, e.g., the sequence of closed subsets of $\mathbb{R}^2$ defined by $S_n = \overline{B}(0,1) \cup ([1, 2 + 1/n] \times \{0\})$ where $\overline{B}(0,1)$ denotes the closed unit ball). Nevertheless, as shown by the following proposition, both concepts coincide for sequences of convex subsets with nonempty interior.

Proposition 3.2. *Let $(S_n)_n$ be a sequence of convex subsets of X, with $\operatorname{int}(S_n) \neq \emptyset$ and S be a subset of X. Assume that $\operatorname{int}(S) \neq \emptyset$. Then the following assertions are equivalent:*

i) $(S_n)_n$ *Mosco converges to* S

ii) $w - \operatorname{Lim\,sup}_n S_n \subset S$, $\operatorname{int}(S) \subset \operatorname{Lim\,inf}_n \operatorname{int}(S_n)$ *and* S *is convex*

iii) $(S_n)_n$ int-*Mosco converges to* S.

Proof. Clearly *iii)* $\Rightarrow$ *ii)*. On the other hand, if *ii)* holds, due to the closedness of the set $\operatorname{Lim\,inf}_n \operatorname{int}(S_n)$, one has $\operatorname{cl}(\operatorname{int}(S)) \subset \operatorname{Lim\,inf}_n \operatorname{int}(S_n)$. But S is convex with nonempty interior and therefore, $S \subset \operatorname{cl}(S) = \operatorname{cl}(\operatorname{int}(S))$ which ensures *iii)*.

Finally *i)* $\Leftrightarrow$ *iii)* since, due to the convexity of S_n, for any n, one has

$$\operatorname*{Lim\,inf}_n S_n = \operatorname*{Lim\,inf}_n \operatorname{cl}(S_n) = \operatorname*{Lim\,inf}_n \operatorname{cl}(\operatorname{int}(S_n)) = \operatorname*{Lim\,inf}_n \operatorname{int}(S_n).$$

$\square$

Let us now consider the important case of constraint sets $K(\mu)$ described by inequalities: given a family of q functions $g_i : X \to \mathbb{R}$, $i = 1, \dots, q$, define the map $K : \mathbb{R}^q \to 2^X$ as follows

$$K(\mu) = \{x \in X : g_i(x) \leq \mu^i, \ \forall\, i = 1 \dots, q\}.$$

Our aim, in the forthcoming Proposition 3.3, is to give sufficient conditions ensuring that, for any sequence $(\mu_n)_n$ of perturbations converging to some μ_0, the sequence $(K(\mu_n))_n$ Mosco converges (to $K(\mu_0)$). This Mosco convergence of the constraint sets will be an essential assumption for the stability results of Sections 4 and 5.

Proposition 3.3. *Assume that the functions g_i are continuous and semistrictly quasiconvex on X. Then for any sequence $(\mu_n)_n$ converging to some μ_0, $K(\mu_n)$ Mosco converges to $K(\mu_0)$, provided that $K(\mu_0)$ and the subsets $K(\mu_n)$ have a nonempty interior.*

This proposition extends Theorem 2 of [16] where the strict quasiconvexity of the functions g_i is required. See also [10, Th. 3.1.6] where strict quasiconvexity of the constraint functions g_i is assumed in order to obtain convergence properties of the map K.

Let us observe that quasiconvexity (even coupled with continuity) of the functions g_i is not enough to obtain this Mosco convergence of the constraint sets as shown by the following simple example.

Example 3.4. On $\mathbb{R}^2$ let us consider that $q = 1$ and g_1 is defined by $g_1(x, y) = x^2 + y^2$ if $x^2 + y^2 \leq 1$, $g_1(x, y) = 1$ if $1 < x^2 + y^2 \leq 2$ and $g_1(x, y) = x^2 + y^2 - 1$ otherwise. Then clearly, for $\mu_0 = 1$ and $\mu_n = 1 - 1/n$, $n \in \mathbb{N}$, the sequence of closed convex subsets $(K(\mu_n))_n$ doesn't Mosco converge to $K(\mu_0)$.

Proof. The Mosco convergence of the sequence $(K(\mu_n))_n$ to $K(\mu_0)$ will be proved using its equivalent formulation *ii*) of Proposition 3.2.

a) The upper part of the Mosco convergence is a simple consequence of the lower semicontinuity of the functions g_i. Indeed, let $(K(\mu_{n_k}))_k$ be any subsequence of $(K(\mu_n))_n$ and $(x_k)_k \subset X$ be weakly converging to x with $x_k \in K(\mu_{n_k})$, for any $k \in \mathbb{N}$. Since, for any $i \in \{1, \dots, q\}$, g_i is quasiconvex and lower semicontinuous, it is weakly lower semicontinuous and

$$g_i(x) \leq \liminf_{k \to \infty} g_i(x_k) \leq \liminf_{k \to \infty} \mu^i_{n_k} = \mu^i_0$$

which clearly implies that $x \in K(\mu_0)$.

b) To prove the lower part of the Mosco convergence of $(K(\mu_n))_n$ to $K(\mu_0)$, let y be any element of $\mathrm{int}(K(\mu_0))$. We claim that, for n large enough, y is an element of $\mathrm{int}(K(\mu_n))$.

Let us define $I = \{i \in \{1, \dots, q\} : \inf_X g_i = \mu^i_0\}$ and $I^c = \{1, \dots, q\} \setminus I$. Since each function g_i is continuous semistrictly quasiconvex and using (1), we obtain, for any $i \in I^c$ and any $\alpha > \inf_X g_i$,

$$\mathrm{int}(S_\alpha(g_i)) = \mathrm{int}(\mathrm{cl}(S^<_\alpha(g_i))) = S^<_\alpha(g_i)$$

and thus

$$\begin{aligned} y \in \mathrm{int}(K(\mu_0)) &= \mathrm{int}\left(\cap_{i=1}^q S_{\mu^i_0}(g_i)\right) = \cap_{i=1}^q \mathrm{int}\left(S_{\mu^i_0}(g_i)\right) \\ &= \left[\cap_{i\in I} \mathrm{int}(\operatorname*{argmin}_X g_i)\right] \cap \left[\cap_{i\in I^c} S^<_{\mu^i_0}(g_i)\right]. \end{aligned}$$

From this formula, it is clear that, for any $i \in I^c$, $g_i(y) < \mu_0^i$. Therefore, for n large enough (say $n > N$) and for any $i \in I^c$, y is also an element of $S^<_{\mu_n^i}(g_i)$.

On the other hand, if $i \in I$, then, for any n, $y \in \text{int}(\text{argmin}_X g_i) \subset \text{int}(S_{\mu_n^i}(g_i))$. Thus finally

$$y \in \Big[\cap_{i \in I_n} \text{int}(\operatorname*{argmin}_X g_i)\Big] \cap \Big[\cap_{i \in I^c \cup [I \setminus I_n]} S^<_{\mu_n^i}(g_i)\Big] = \text{int}(K(\mu_n))$$

where $I_n = \{i \in I : \mu_n^i = \inf_X g_i\}$. The claim is proved and this completes the proof of the Mosco convergence of $(K(\mu_n))_n$ to $K(\mu_0)$. □

Even if the Mosco convergence result established in Proposition 3.3 is deeply linked to the problem of the convergence of sublevel sets of a function, its conclusion differs from the approach developped in Beer-Rockafellar-Wets [12] and Rockafellar-Wets [24] where, for a given function, it was proved that it is possible to recover a fixed sublevel set of this function as a limit of a sequence of sublevel sets of an appropriate sequence of functions.

4. Semicontinuity of the solution map

In this section we investigate continuity properties of the solution map S^* of quasimonotone variational inequalities. More precisely, in Subsection 4.1, (sequential) closedness and upper semicontinuity of S^* are considered while Subsection 4.2 is devoted to the study of lower semicontinuity.

4.1. Upper semicontinuity

Before establishing the (graph) closedness and upper semicontinuity of S^*, let us recall from [6] a very weak kind of continuity for set-valued maps: given a convex subset $K \subseteq X$ and a map $T : K \to 2^{X^*}$ with nonempty values, T is called *upper sign-continuous* on K if for any $x, y \in K$, the following implication holds:

$$(\forall\, t \in]0,1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x\rangle \geq 0) \Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0$$

where $x_t = (1-t)x + ty$. If for example T is upper hemicontinuous (i.e., the restriction of T to every line segment of K is usc with respect to the w*-topology in X^*), then T is upper sign-continuous. Any strictly positive real function is upper sign-continuous.

Unfortunately, for the applications that we have in scope in Section 5, the upper sign-continuity is even too strong and we will use a "local" form of it.

Definition 4.1. Let K be a convex subset of X and $T : K \times \Lambda \to 2^{X^*}$ be a map with nonempty values. For any $\lambda \in \Lambda$, $T(\cdot, \lambda)$ is called *locally upper sign-continuous* on K if for any $x \in K$ there exist a neighbourhood V_x of x and an upper sign-continuous map $\Phi_x(\cdot, \lambda) : V_x \cap K \to 2^{X^*}$ with nonempty convex w^*-compact values satisfying $\Phi_x(y, \lambda) \subseteq T(y, \lambda) \setminus \{0\}$, $\forall\, y \in V_x \cap K$.

In the monotone or pseudomonotone case, some closedness (or upper semicontinuity) results for the solution set of variational inequalities can be found in [11, 13, 15, 17, 20]. See also [25], [3], [10] for alternative approaches.

To our knowledge, the recent paper of He [18] is the only attempt to obtain stability results under a monotonicity assumption weaker than pseudomonotonicity. In this paper, closedness of the solution map is proved in a reflexive Banach space assuming the properly quasimonotonicity of the operator. In the forthcoming Theorem 4.2 and Corollary 4.4, we will only assume quasimonotonicity of the operator T.

Theorem 4.2. *Let us suppose that for any $(\lambda,\mu)\in\Lambda\times U$, $S^*(\lambda,\mu)$ is nonempty and*

i) *for all $\mu\in U$, $K(\mu)$ is convex with nonempty interior;*

ii) *for all $\lambda\in\Lambda$ and all $\mu\in U$, $T(\cdot,\lambda)$ is quasimonotone and locally upper sign-continuous on $K(\mu)$;*

iii) *for all $y_n\to y$, all $(\lambda_n,\mu_n)\to(\lambda,\mu)$, and all $z_n\to z$ with $y\in\operatorname{int}(K(\mu))$, $y_n\in\operatorname{int}(K(\mu_n))$, $z\in K(\mu)$, $z_n\in K(\mu_n)$,*

$$\sup_{y^*\in T(y,\lambda)\setminus\{0\}}\langle y^*,z-y\rangle\le\liminf_n\sup_{y_n^*\in T(y_n,\lambda_n)\setminus\{0\}}\langle y_n^*,z_n-y_n\rangle;$$

iv) *for any $\mu_n\to\mu$, $K(\mu_n)$ Mosco converges to $K(\mu)$.*

Then the set-valued map S^* is closed.

Remark 4.3. a) In fact one can express Theorem 4.2 in a more detailed way. Indeed let us consider the concept of *weak-int solution* for $(P_{\lambda,\mu})$ linked to the weak Stampacchia variational inequality defined by $T(\cdot,\lambda)$ and $\operatorname{int}(K(\mu))$ which is defined by: $x\in S^*_{w,int}(\lambda,\mu)$ if and only if

$$x\in K(\mu)\text{ and }\forall\, y\in\operatorname{int}(K(\mu)),\ \exists\, x^*\in T(x,\lambda)\setminus\{0\}\text{ with }\langle x^*,y-x\rangle\ge 0.$$

Then, the formulation of the theorem will be:

Assume that conditions i), iii), iv) hold, together with

ii′) *for all $\lambda\in\Lambda$ and all $\mu\in U$, $T(\cdot,\lambda)$ is quasimonotone on $K(\mu)$ and for any $x\in\operatorname{dom}T(\cdot,\lambda)$ there exist a neighbourhood V_x of x and an upper sign-continuous map $\Phi_x(\cdot,\lambda):V_x\cap K(\mu)\to 2^{X^*}$ with nonempty w^*-compact values satisfying $\Phi_x(y,\lambda)\subseteq T(y,\lambda)\setminus\{0\}$, $\forall\, y\in V_x\cap K(\mu)$;*

Then the set-valued map $S^*_{w,int}$ is closed. If, additionally, the operators Φ_x are convex valued, the set-valued map S^* is closed.

The proof of Theorem 4.2 is built on this formulation.

b) In the particular case where $T(\cdot,\lambda)\setminus\{0\}$ is single-valued for any $\lambda\in\Lambda$, the rather technical hypothesis *iii)* simply expresses the lower semicontinuity of the function $(y,z,\lambda)\mapsto\langle T(y,\lambda)\setminus\{0\},z-y\rangle$.

Proof. According to Remark 4.3 a), let $(\lambda_n,\mu_n)_n$ and (x_n) be sequences of $\Lambda\times U$ and X respectively such that,

$$x_n\to x,\ (\lambda_n,\mu_n)\to(\lambda_0,\mu_0)\quad\text{and}\quad x_n\in S^*_{w,int}(\lambda_n,\mu_n),\ \forall\, n.$$

Let V_x be a convex neighbourhood of x and $\Phi_x(\cdot,\lambda_0):V_x\cap K(\mu_0)\to 2^{X^*}$ be a set-valued map, upper sign-continuous at x and such that, for any $v\in V_x\cap K(\mu_0)$, $\Phi_x(v,\lambda_0)$ is a nonempty w^*-compact subset of $T(v,\lambda_0)\setminus\{0\}$.

Due to the Mosco convergence of $(K(\mu_n))_n$ and since $x_n \in K(\mu_n)$, for any n, one immediately obtains $x \in K(\mu_0)$. Let y be an arbitrary point of $[V_x \cap \text{int}(K(\mu_0))] \setminus \{x\}$. Since $K(\mu_0)$ and V_x are convex, the segment $[y, x[$ is included in $[V_x \cap \text{int}(K(\mu_0))]$. Let $z_t = ty + (1-t)x$ ($t \in]0,1]$) be an element of $[y, x[$.

We claim that, for any $z_t^* \in T(z_t, \lambda_0) \setminus \{0\}$,

$$\langle z_t^*, x - z_t \rangle \leq 0. \tag{2}$$

Indeed, according to Proposition 3.2 *i*) $\Rightarrow$ *ii*), one can find a sequence $(z_n)_n$ converging to z_t such that $z_n \in \text{int}\,(K(\mu_n))$, $z_n \neq x_n$, $\forall\, n \in \mathbb{N}$. Since $x_n \in S^*_{w,int}(\lambda_n, \mu_n)$, there exists $x_n^* \in T(x_n, \lambda_n) \setminus \{0\}$ verifying

$$\langle x_n^*, z_n - x_n \rangle \geq 0.$$

But since $x_n^* \neq 0$ and z_n is an element of $\text{int}(K(\mu_n))$, one can assume, without loss of generality, that the previous inequality is strict, for any $n \in \mathbb{N}$. Thus, by quasimonotonicity of $T(\cdot, \lambda_n)$, we deduce that

$$\langle z_n^*, z_n - x_n \rangle \geq 0,\ \forall\, z_n^* \in T(z_n, \lambda_n) \setminus \{0\}.$$

Hypothesis *iii*) now leads to

$$\sup_{z_t^* \in T(z_t, \lambda_0) \setminus \{0\}} \langle z_t^*, x - z_t \rangle \leq \liminf_n \sup_{z_n^* \in T(z_n, \lambda_n) \setminus \{0\}} \langle z_n^*, x_n - z_n \rangle \leq 0$$

and the claim is proved.

From (2) we derive that, for any $t \in]0,1]$ and any $z_t^* \in \Phi_x(z_t, \lambda_0)$,

$$0 \leq t\langle z_t^*, y - z_t \rangle + (1-t)\langle z_t^*, x - z_t \rangle \leq t\langle z_t^*, y - z_t \rangle$$

which yields

$$\inf_{z_t^* \in \Phi_x(z_t, \lambda_0)} \langle z_t^*, y - x \rangle \geq 0,\ \forall\, t \in]0,1[$$

Therefore, according to the upper sign-continuity of $\Phi_x(\cdot, \lambda_0)$ on $V_x \cap K(\mu_0)$ and to the w^*-compactness of $\Phi_x(x, \lambda_0)$, one has

$$\max_{x^* \in \Phi_x(x, \lambda_0)} \langle x^*, y - x \rangle \geq 0$$

which means that for any $y \in [V_x \cap \text{int}(K(\mu_0))]$ there exists $x^* \in T(x, \lambda_0) \setminus \{0\}$ such that $\langle x^*, y - x \rangle \geq 0$. The latter still holds for any $y \in \text{int}(K(\mu_0))$ since in this case, $K(\mu_0)$ being convex, the point $x + \rho/\|y - x\|[y - x]$ is an element of $[V_x \cap \text{int}(K(\mu_0))]$ for ρ sufficiently small. Consequently x is an element of $S^*_{w,int}(\lambda_0, \mu_0)$, completing the first part of the proof.

If the operator $\Phi_x(\cdot, \lambda_0)$ is convex valued, using the above proof, we obtain that,

$$\inf_{y \in [V_x \cap \text{int}(K(\mu_0))]} \sup_{x^* \in \Phi_x(x, \lambda_0)} \langle x^*, y - x \rangle \geq 0.$$

and therefore, using a Sion minimax theorem, there exists an element $x^* \in \Phi_x(x, \lambda_0) \subset T(x, \lambda_0) \setminus \{0\}$ such that,

$$\langle x^*, y - x \rangle \geq 0,\ \forall\, y \in [V_x \cap \text{int}(K(\mu_0))].$$

But $K(\mu_0)$ being convex and x^* being continuous, the previous inequality still holds for any $y \in K(\mu_0)$ and consequently x is an element of $S^*(\lambda_0, \mu_0)$. $\square$

Now invoking classical arguments (e.g. [5, Prop. 1.4.8]) we obtain in finite dimensions the upper semicontinuity of the set-valued map S^*.

Corollary 4.4. *If, additionally to the assumptions of Theorem 4.2, the set $K(U) = \cup_{\mu\in U} K(\mu)$ is compact and $\dim(X) < \infty$, then the set-valued map S^* is upper semicontinuous on $\Lambda \times U$.*

4.2. Lower semicontinuity

It is well known that general lower semicontinuity results need quite restrictive assumptions. In our case, as it will be enlightened below by Example 4.6, even if the parametrized constraint sets $K(\mu_n)$ are convex polyhedra and the operator map T is constant, the solution map S^* may not be lower semicontinuous.

Nevertheless in the forthcoming Theorem 4.5 we will show, in finite dimensions, that the solution map S^* is lower semicontinuous at some particular points (λ_0, μ_0) where the map S^* coincides with the map $S^>(\lambda_0, \mu_0)$ of strict solutions.

Let us recall from [1] the concept of strict solution: $x \in K(\mu)$ is a strict solution of $(P_{\lambda,\mu})$ if there exists $x^* \in T(x, \lambda)$ such that $\langle x^*, y - x\rangle > 0$, for any $y \in K(\mu) \setminus \{x\}$. The set of strict solutions of $(P_{\lambda,\mu})$ is denoted by $S^>(\lambda, \mu)$ and is included in $\mathrm{bd}\,(K(\mu))$, as a subset of $S^*(\lambda, \mu)$.

Theorem 4.5. *Let $K : U \to 2^{\mathbb{R}^k}$ and $T : X \times \Lambda \to 2^{\mathbb{R}^k}$ be two set-valued maps, $\mathcal{V}_0$ a neighbourhood of $\lambda_0 \in \Lambda$ and $\mathcal{U}_0$ a neighbourhood of $\mu_0 \in U$. Assume that*

i) *for any $\mu \in \mathcal{U}_0$, $K(\mu)$ has a nonempty interior;*
ii) *the map $T(\cdot, \lambda)$ is quasimonotone, for any $\lambda \in \mathcal{V}_0$;*
iii) *$K(\mathcal{U}_0)$ is compact;*
iv) *for any $\mu_n \to \mu_0$, $K(\mu_n)$ int-Mosco converges to $K(\mu_0)$;*
v) *for all $y_n \to y$, all $(\lambda_n, \mu_n) \to (\lambda_0, \mu_0)$ and all $z_n \to z$ with $y_n \in \mathrm{int}(K(\mu_n))$, $y \in K(\mu)$, $z \in K(\mu)$, $z_n \in K(\mu_n)$,*

$$\sup_{y^*\in T(y,\lambda_0)\setminus\{0\}} \langle y^*, z - y\rangle \le \liminf_n \sup_{y_n^*\in T(y_n,\lambda_n)\setminus\{0\}} \langle y_n^*, z_n - y_n\rangle;$$

vi) *$S^*(\lambda, \mu)$ is nonempty, for all $(\lambda, \mu) \in \mathcal{V}_0 \times \mathcal{U}_0$;*
vii) *$S^*(\lambda_0, \mu_0) = S^>(\lambda_0, \mu_0)$.*

Then, S^* is lower semicontinuous at (λ_0, μ_0).

Proof. Let $x_0 \in S^*(\lambda_0, \mu_0)$ and a sequence $(\lambda_n, \mu_n)_n$ converging to (λ_0, μ_0). According to the lower part of the int-Mosco convergence of the sequence $(K(\mu_n))_n$ to $K(\mu_0)$, one can find a sequence $(x_n)_n$ converging to x_0 such that $x_n \in \mathrm{int}(K(\mu_n))$ for any n.

On the other hand, from hypothesis *vi)*, one can also construct a sequence $(\bar{x}_n)_n$ such that, for any n, $\bar{x}_n \in S^*(\lambda_n, \mu_n)$ and thus, together with [1, Prop. 2.1], there exists, for any n, an element $\bar{x}_n^* \in T(\bar{x}_n, \lambda_n) \setminus \{0\}$ such that

$$\langle \bar{x}_n^*, y - \bar{x}_n\rangle > 0, \ \forall\, y \in \mathrm{int}(K(\mu_n)) \setminus \{\bar{x}_n\}. \tag{3}$$

As $K(\mathcal{U}_0)$ is compact, we immediately obtain, possibly considering a subsequence, the convergence of $(\bar{x}_n)_n$ to a point $\bar{x}_0$ of $K(\mathcal{U}_0)$. In fact, $\bar{x}_0$ is an element of $K(\mu_0)$ by the upper part of the int-Mosco convergence of $(K(\mu_n))_n$ to $K(\mu_0)$.

Without loss of generality one can assume that there exists $N \in \mathbb{N}$ such that, for any $n > N$, $\bar{x}_n \neq x_n$. Indeed, otherwise one can select a subsequence $(\bar{x}_{n_k})_k$ converging to x_0 and the set-valued map S^* is proved to be lower semicontinuous at (λ_0, μ_0).

Obviously the map S^* is also lower semicontinuous at (λ_0, μ_0) if $\bar{x}_0 = x_0$. So let us assume, for the rest of the proof, that $\bar{x}_0 \neq x_0$. By (3), for any $n > N$, one has $\langle \bar{x}_n^*, x_n - \bar{x}_n \rangle > 0$, and therefore, by quasimonotonicity of $T(\cdot, \lambda_n)$,

$$\langle x_n^*, \bar{x}_n - x_n \rangle \leq 0, \ \forall x_n^* \in T(x_n, \lambda_n).$$

According to the continuity hypothesis $v)$, this clearly forces

$$\langle x^*, \bar{x}_0 - x_0 \rangle \leq 0, \ \forall x^* \in T(x_0, \lambda_0). \tag{4}$$

But, according to assumption $vii)$, x_0 is an element of $S^>(\lambda_0, \mu_0)$. A contradiction with (4). □

Example 4.6. Let us observe that assumption $vii)$ could not be easily avoided. Indeed, let us consider the very simple following case of a constant operator T defined on $\mathbb{R}^2$ by $T(x) = \{(1, 0)\}$ and of a perturbation map of the constraint set defined on $\mathbb{R}$ by

$$K(\mu) = \begin{cases} \{(x, y) \in]-\infty, 1] \times [0, 1] : y \geq x/\mu\} & \text{if } \mu < 0 \\ [0, 1] \times [0, 1] & \text{if } \mu = 0 \\ \{(x, y) \in]-\infty, 1] \times [0, 1] : y \leq x/\mu\} & \text{otherwise.} \end{cases}$$

Clearly the multivalued map S^* is not lower semicontinuous at 0 since

$$S^*(\mu) = \begin{cases} \{(\mu, 1)\} & \text{if } \mu < 0 \\ \{0\} \times [0, 1] & \text{if } \mu = 0 \\ \{(0, 0)\} & \text{otherwise} \end{cases}$$

but all hypothesis of Theorem 4.5 are satisfied, except $vii)$ since $S^>(0)$ is empty.

Let us now give a geometrical sufficient condition for the problem (P_{λ_0,μ_0}) to satisfy hypothesis $vii)$ of the above theorem, that is $S^*(\lambda_0, \mu_0) = S^>(\lambda_0, \mu_0)$.

Let $\mu \in U$ be such that $K(\mu)$ is a convex polyhedron of $\mathbb{R}^k$, i.e. $K(\mu) = \cap_{i=1}^n \{x \in \mathbb{R}^k : \langle a_i^*, x \rangle \leq \mu^i\}$ where $a_i^* \in \mathbb{R}^k$, for any $i = 1, \ldots, n$. For any i, we denote by $H_i(\mu)$ the closed hyperplane defined by $H_i(\mu) = \{x \in \mathbb{R}^k : \langle a_i^*, x \rangle = \mu^i\}$. For any $x \in \text{bd}\,(K(\mu))$, we will denote by $F\,(K(\mu), x)$ the set $F\,(K(\mu), x) = \cup\{H_i(\mu) \cap K(\mu) : x \in H_i(\mu)\}$.

The problem $(P_{\lambda,\mu})$ is said to be *facely non normal* if for all $x \in S^*(\lambda, \mu)$, all $x^* \in T(x, \lambda) \setminus \{0\}$ and all $y \in F\,(K(\mu), x) \setminus \{x\}$, one has $\langle x^*, y - x \rangle \neq 0$.

Proposition 4.7. *Let (λ_0, μ_0) be an element of $\Lambda \times U$ such that $K(\mu_0)$ is convex, polyhedral with nonempty interior. If the problem (P_{λ_0,μ_0}) is facely non normal then $S^*(\lambda_0, \mu_0) = S^>(\lambda_0, \mu_0)$.*

If, moreover, the operator $T(\cdot, \lambda_0)$ is quasimonotone then the solution set $S^*(\lambda_0, \mu_0)$ contains at most one element.

Proof. It is sufficient to prove that $S^*(\lambda_0, \mu_0) \subset S^>(\lambda_0, \mu_0)$. So let x be an element of $S^*(\lambda_0, \mu_0)$ and $x^* \in T(x, \lambda_0) \setminus \{0\}$ be such that

$$\langle x^*, y - x\rangle \geq 0, \ \forall y \in K(\mu_0).$$

Since (P_{λ_0,μ_0}) is facely non normal it follows that $\langle x^*, y-x\rangle > 0$, for all $y \in F(K(\mu_0), x) \setminus \{x\}$. On the other hand, according to [1, Proposition 2.1],

$$\langle x^*, y - x\rangle > 0, \ \forall y \in \mathrm{int}(K(\mu_0)). \tag{5}$$

So let y be any element of $K(\mu_0) \setminus [F(K(\mu_0), x) \cup \mathrm{int}(K(\mu_0))]$. Then the open segment $]x, y[$ is included in $\mathrm{int}(K(\mu_0))$. Indeed, if there exist $t \in]0, 1[$ and $i_0 \in \{1, \ldots, n\}$ such that $tx + (1 - t)y \in H_{i_0}(\mu_0) \cap K(\mu_0)$, then one immediately obtains that x and y are elements of $H_{i_0}(\mu_0) \cap K(\mu_0)$ and thus $y \in F(K(\mu_0), x)$. A contradiction.

It follows from (5) that

$$\langle x^*, y - x\rangle = 2\langle x^*, \frac{x + y}{2} - x\rangle > 0$$

and the proof of the first part is completed. As observed in [1], the uniqueness of the (strict) solution of (P_{λ_0,μ_0}) is straightforward provided that $T(\cdot, \lambda_0)$ is quasimonotone. □

5. Application to quasiconvex programming

Let X be a Banach space, U be a metrizable topological space and consider, for $\mu \in U$, the following parametrized optimization problem

$$\begin{array}{lll} (Q_\mu) & \inf & f(x) \\ & st. & x \in K(\mu) = \{x \in X : g_i(x) \leq \mu^i, \ i = 1, \ldots, q\}. \end{array}$$

where the functions $f : X \to \mathrm{I\!R}$ and $g_i : X \to \mathrm{I\!R}$, $i = 1, \ldots, q$ are quasiconvex.

Denote by *Opt* the (possibly set-valued) solution map of the perturbed optimization problem (Q_μ):

$$\begin{array}{rccl} Opt: & U & \to & 2^X \\ & \mu & \mapsto & Opt(\mu) = \mathrm{argmin}_{K(\mu)} f \end{array}$$

It is very natural to try to give conditions ensuring some regularity of the solution map *Opt*. This question has been well treated in [14] where closedness and upper semicontinuity was established in a general framework. The aim of this short section is to show how the previous stability results for quasimonotone variational inequalities can be applied to the perturbed quasiconvex programming problem (Q_μ), thus providing an alternative approach to the one developed in [14].

As shown in [7, 8], an efficient tool to study the properties of quasiconvex functions is the so-called *normal operator* N_f^a defined as the normal cone to the adjusted sublevel sets S_f^a, that is:

$$N_f^a(x) = \left\{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in S_f^a(x)\right\}$$

where $S_f^a(x) = S_{f(x)} \cap \overline{B}\left(S_{f(x)}^<, \rho_x\right)$ (with $\rho_x = \mathrm{dist}(x, S_{f(x)}^<)$) if $x \notin \mathrm{argmin}\, f$, and $S_f^a(x) = S_{f(x)}$ otherwise. Many precious properties of the operator have been proved for quasiconvex functions (see [7], [8]). In the case of a semistrictly quasiconvex function f, the associated normal operator admits a simplified definition $N_f^a(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall\, y \in S_{f(x)}(f)\}$.

Theorem 5.1. *Let $f : X \to \mathbb{R}$ be a continuous semistrictly quasiconvex function. Let $g_i : X \to \mathbb{R}$, $i = 1, \dots, q$ be continuous semistrictly quasiconvex functions. Assume that for any $\mu \in U$, the constraint set $K(\mu)$ has a nonempty interior and $K(\mu) \cap \operatorname{argmin}_X f = \emptyset$. Suppose moreover that, for any $\mu \in U$, (Q_μ) admits at least one solution.*

If the normal operator N_f^a satisfies the following regularity assumption: for all $y_n \to y$, all $\mu_n \to \mu$ and all $z_n \to z$ with $y \in \operatorname{int}(K(\mu))$, $y_n \in \operatorname{int}(K(\mu_n))$, $z \in K(\mu)$, $z_n \in K(\mu_n)$,

$$\sup_{y^* \in N_f^a(y)\setminus\{0\}} \langle y^*, z - y\rangle \leq \liminf_n \sup_{y_n^* \in N_f^a(y_n)\setminus\{0\}} \langle y_n^*, z_n - y_n\rangle$$

then the solution map Opt is closed.

If, moreover, $K(U) = \cup_{\mu\in U} K(\mu)$ is compact, the map Opt is upper semicontinuous.

Proof. According to [8, Theorem 5.1], for any $\mu \in U$, $Opt(\mu) = \tilde{S}^*(\mu)$ where $\tilde{S}^*(\mu)$ is the set of star-solutions of the perturbed variational inequality

$$(\tilde{P}_\mu) \quad \text{Find } \bar{x} \in K(\mu) \text{ such that there exists } \bar{x}^* \in N_f^a(\bar{x}) \setminus \{0\}$$
$$\text{with } \langle \bar{x}^*, y - \bar{x}\rangle \geq 0,\ \forall y \in K(\mu).$$

By [7, Proposition 3.3], $N_f^a(x) \setminus \{0\}$ is nonempty, for any $x \in K(U) \subset (X \setminus \operatorname{argmin}_X f)$ while, from [7, Proposition 3.4], the map $N_f^a(x)$ is quasimonotone. Moreover, according to [7, Proposition 3.5], the map N_f^a, which is conical valued, is norm-to-w^* cone upper semicontinuous on $K(U)$, that is, for every $x \in U$ there exist a neighbourhood V_x of x and a convex base $B(u)$ of $N_f^a(u)$ for each $u \in V_x$ such that $S_x : u \mapsto B(u)$ is upper semicontinuous at x. As shown in [7] (Lemma 3.6 and beginning of the proof of Proposition 3.5) one can construct such bases $B(u)$ to be w^*-compact. Therefore the map N_f^a is locally upper sign-continuous on $K(U)$.

Finally, by Proposition 3.3 and since each constraint set $K(\mu)$ has a nonempty interior, the sequence of subsets $(K(\mu_n))_n$ Mosco converges to $K(\mu_0)$, for any sequence $(\mu_n)_n$ converging to μ_0. Thus, all assumptions of Theorem 4.2 hold for the perturbed variational inequality $(\tilde{P}_\mu)$ and the map $\mu \mapsto Opt(\mu)$, is (graph) closed. And if $K(U)$ is compact, the upper semicontinuity of the map *Opt* follows from Corollary 4.4. □

For semicontinuity results of the solution map of convex optimization problems, the interested reader can refer to [22] and references therein. For quasiconvex programming, similar results have been obtained in [10] in finite dimensions. See also [28] and [19] for different approaches.

One can also deduce some convergence properties of the solution map by proving epiconvergence of the sequence $\left(f + \Psi_{K(\mu_n)}\right)_n$. This approach has been used in [24, Th. 7.33 and Cor. 7.55] to prove the closedness of the mapping *Opt*, in finite dimensions and assuming some compactness hypothesis (level coercivity or eventually boundedness). See also [9] where the same approach is proposed for sequences of quasiconvex functions and convex subsets. In [21] the convergence results of [9] have been extended to the infinite dimensional setting. Closedness of operator *Opt* was not considered in [21] but it was shown ([21, Th. 3.4]) that, if $(f_n)_n$ is a sequence of quasiconvex functions epi-converging to f, quasiconvex upper semicontinuous, and C_n is a sequence of subsets converging (Kuratowski) to C, then $M \subset \limsup_n M_n$ where $M = \{x : f(x) \leq \inf_C f\}$ and $M_n = \{x : f_n(x) \leq \inf_{C_n} f_n\}$.

Acknowledgements. The authors are grateful to the referee for his constructive suggestions and comments.

References

[1] M. Ait Mansour, D. Aussel: Quasimonotone variational inequalities and quasiconvex programming: quantitative stability, Pacific J. Optim. 2 (2006) 611–626.

[2] M. Ait Mansour, H. Riahi: Sensitivity analysis for abstract equilibrium problems, J. Math. Anal. Appl. 306 (2005) 684–691.

[3] Y. Alber, D. Butnariu, I. Ryazantseva: Regularization of monotone variational inequalities with Mosco approximations of the constraint sets, Set-Valued Anal. 13 (2005) 265–290.

[4] H. Attouch, R. J.-B. Wets: Quantitative stability of variational systems. II. A framework for nonlinear conditioning, SIAM J. Optim. 3 (1993) 359–381.

[5] J.-P. Aubin, H. Frankowska: Set-Valued Analysis, Birkhäuser, Boston (1990).

[6] D. Aussel, N. Hadjisavvas: On quasimonotone variational inequalities, J. Optimization Theory Appl. 121 (2004) 445–450.

[7] D. Aussel, N. Hadjisavvas: Adjusted sublevel sets, normal operator, and quasiconvex programming, SIAM J. Optim. 16 (2005) 358–367.

[8] D. Aussel, J. J. Ye: Quasiconvex programming with starshaped constraint region and application to quasiconvex MPEC, Optimization 55 (2006) 433–457.

[9] D. Azé, M. Volle: A stability result in quasi-convex programming, J. Optimization Theory Appl. 67 (1990) 175–183.

[10] B. Bank, J. Guddat, D. Klatte, B. Kummer, K. Tammer: Non-Linear Parametric Optimization, Birkhäuser, Basel (1983).

[11] M. Bianchi, R. Pini: A note on stability for parametric equilibrium problems, Oper. Res. Lett. 31 (2003) 445–450.

[12] G. Beer, R. T. Rockafellar, R. J.-B. Wets: A characterization of epi-convergence in terms of convergence of level sets, Proc. Amer. Math. Soc. 116 (1992) 753–761.

[13] L.-J. Chu, T.-S. Lin: Existence and asymptotical convergence of solutions of variational inequalities, Far East J. Math. Sci. 5 (1997) 1–18.

[14] G. B. Dantzig, J. Folkman, N. Shapiro: On the continuity of the minimum set of a continuous function, J. Math. Anal. Appl. 17 (1967) 519–548.

[15] A. Domokos: Solution sensitivity of variational inequalities, J. Math. Anal. Appl. 230 (1999) 382–389.

[16] D. Dugosija: On stability in quasiconvex semi-infinite optimization, Yugosl. J. Oper. Res. 1 (1991) 135–140.

[17] J. Gwinner: Stability of monotone variational inequalities with various applications, in: Variational Inequalities and Network Equilibrium Problems, F. Giannessi, A. Maugeri (eds.), Plenum Press, New York (1995) 123–142.

[18] Y. He: Stable pseudomonotone variational inequality in reflexive Banach spaces, J. Math. Anal. Appl. 330 (2007) 352–363.

[19] D. Klatte, B. Kummer: Nonsmooth Equations in Optimization, Kluwer, Dordrecht (2002).

[20] M. B. Lignola, J. Morgan: Generalized variational inequalities with pseudomonotone operators under perturbations, J. Optim. Theory Appl. 101 (1999) 213–220.

[21] M. B. Lignola, J. Morgan: Epiconvergence and continuous convergence, Atti Semin. Mat. Fis. Univ. Modena 42 (1994) 467–475.

[22] R. Lucchetti: On the continuity of the minima for a family of constrained optimization problems, Numer. Funct. Anal. Optimization 7 (1985) 349–362.

[23] U. Mosco: Convergence of convex sets and of solutions of variational inequalities, Adv. Math. 3 (1969) 510–585.

[24] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (2004).

[25] A. Shapiro: Sensitivity analysis of parameterized variational inequalities, Math. Oper. Res. 30 (2005) 109–126.

[26] N. D. Yen: Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint, Math. Oper. Res. 20 (1995) 695–707.

[27] N. D. Yen: Hölder continuity of solutions to a parametric variational inequality, Appl. Math. Optimization 31 (1995) 245–255.

[28] J. Zhao: The lower semicontinuity of optimal solution sets, J. Math. Anal. Appl. 207 (1997) 240–254.