LEIBNIZ'S SYNCATEGOREMATIC INFINITESIMALS, SMOOTH INFINITESIMAL ANALYSIS AND SECOND-ORDER DIFFERENTIALS

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Abstract

In contrast with some recent theories of infinitesimals as non-Archimedean entities, Leibniz's mature interpretation was fully in accord with the Archimedean Axiom: infinitesimals are fictions, whose treatment as entities incomparably smaller than finite quantities is justifiable wholly in terms of variable finite quantities that can be taken as small as desired, i.e. syncategorematically. In this paper I explain this syncategorematic interpretation, and how Leibniz used it to justify the calculus. I then compare it with the approach of Smooth Infinitesimal Analysis, as propounded by John Bell. I find some salient differences, especially with regard to higher-order infinitesimals. I illustrate these differences by a consideration of how each approach might be applied to propositions of Newton's *Principia* concerning the derivation of force laws for bodies orbiting in a circle and an ellipse.

"If the Leibnizian calculus needs a rehabilitation because of too severe treatment by historians in the past half century, as Robinson suggests (1966, 250), I feel that the legitimate grounds for such a rehabilitation are to be found in the Leibnizian theory itself."—H. J. M. Bos (1974-75, 82-3)

1. Introduction

Leibniz's doctrine of the fictional nature of infinitesimals has been much misunderstood. It has been construed as a late defensive parry, an attempt to defend the success of his infinitesimal calculus—understood as implicitly committed to the existence of infinitesimals as actually infinitely small entities—in the face of criticisms of the adequacy of its foundations. It has also been read as being at odds with other defences of the calculus Leibniz gave on explicitly Archimedean foundations. I take the position here (following Ishiguro 1990) that the idea that Leibniz was committed to infinitesimals as actually infinitely small entities is a misreading: his mature interpretation of the calculus was fully in accord with the Archimedean Axiom. Leibniz's interpretation is (to use the medieval term) *syncategorematic*: Infinitesimals are fictions in the sense that the terms designating them can be treated *as if* they refer to entities incomparably smaller than finite quantities, but really stand for variable finite quantities that can be taken as small as desired. As I (Arthur 2008) and Sam Levey (2008) have argued, this interpretation is no late stratagem, but in place already by 1676.

In section 2 I present this interpretation by tracing its development from Leibniz's early work on infinite series and quadratures to his unpublished attempt in 1701 in the tract "*Cum prodiisset*" to provide a systematic foundation for his calculus. We shall see that as early as 1676 Leibniz had succeeded in providing a rigorous foundation for Riemannian integration, based on the Archimedean Axiom. The appeal to this axiom, generalized by Leibniz into his Law of Continuity, undergirds his interpretation of infinitesimals as fictions that can nevertheless be used in calculations, and forms the basis for his foundation for differentials in "*Cum prodiisset*".

I then turn to a comparison of Leibniz's approach with the recent theory of infinitesimals championed by John Bell, Smooth Infinitesimal Analysis (SIA), of which I give a brief synopsis in section 3. As we shall see, this has many points in common with Leibniz's approach: the non-punctiform nature of infinitesimals, their acting as parts of the continuum, the dependence on variables (as opposed to the static quantities of both Standard and Non-standard Analysis), the resolution of curves into infinite-sided polygons, and the finessing of a commitment to the existence of infinitesimals. Nevertheless, there are also crucial differences. These are brought into relief in sections 4 and 5, by a consideration of how each approach might be applied to some of the central results in cosmology concerning force laws for planetary motions. In section 4 I explain how Leibniz uses differential equations to re-derive the results Newton had obtained for the v^2/r law for the centripetal force on a body orbiting around a centre of force and for the inverse square law. After briefly describing Newton's own derivations of Propositions 4, 6 and 11 in his *Principia*, I detail Leibniz's derivations of the "solicitation" *ddr* experienced by an orbiting body in a circular and then elliptical orbits. Then in section 5 I consider the question of the legitimacy of Leibniz's use of second-order differentials, by reference to his own attempts to provide a foundation. It is found that while Leibniz's syncategorematic approach is adequate to ground his derivations using second-order differentials, there is no corresponding possibility for a derivation of such expressions using nilsquare infinitesimals; that is to say, while one can define *derivatives* of all orders using SIA and thus effect proofs of all the propositions concerning orbiting bodies, and while one can also define differentials of second and higher orders in SIA, the nilsquare property precludes duplication of the calculations of Newton and Leibniz which involved dt^2. So not all the results obtained classically by the use of infinitesimals can be duplicated by SIA: even if the infinitesimals of SIA are much closer in conception to Leibnizian infinitesimals than those of Non-standard Analysis, they are not Leibnizian infinitesimals.

2. Leibniz's foundation for the infinitesimal calculus

As Leibniz explains in his mature work, the infinite should not be understood to refer to an actual entity that is greater than any finite one of the same kind, i.e. *categorematically*, but rather *syncategorematically*: in certain well-defined circumstances infinite terms may be used *as if* they refer to entities incomparably larger than finite quantities, but really stand for variable finite quantities that can be taken as large as desired. Leibniz's interpretation of infinitesimals as *fictions* is intimately linked with his doctrine of the infinite. Just as the infinite is not an actually existing whole made up of finite parts, so infinitesimals are not actually existing parts which can be composed into a finite whole. As Leibniz explained to Des Bosses at the beginning of their correspondence in 1706,

> Speaking philosophically, I maintain that there are no more infinitely small magnitudes than there are infinitely large ones, that is, no more infinitesimals than infinituples. For I hold both to be fictions of the mind through an abbreviated way of speaking, adapted to calculation, as imaginary roots in algebra are too. Meanwhile I have demonstrated that these expressions are very useful for abbreviating thought and thus for discovery, and cannot lead to error, since it suffices to substitute for the infinitely small something as small as one wishes, so that the error is smaller than any given, whence it follows that there can be no error. (GP II 305)

The roots of this syncategorematic interpretation, like the roots of Leibniz's calculus itself, can be discerned in his earliest work on quadratures and infinite series, specifically in his work on the hyperbola. Already in the Fall of 1672 in his reading notes on Galileo Galilei's *Two New Sciences* (*Discorsi*), Leibniz had formulated the basis of his later position. There he notes Galileo's demonstration that there are as many square roots of (natural) numbers as there are natural numbers, and that "therefore there are as many squares as numbers; which is impossible",[1] and then comments:

> Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent,[2] and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole. Or perhaps we should say: when distinguishing between infinities, the most infinite, i.e. all the numbers, is something that implies a contradiction, for if it were a whole it could be understood as made up of all the numbers continuing to infinity, and would be much greater than all the numbers, that is, greater than the greatest number. Or perhaps we should say that one ought not to say anything about the infinite, as a whole, except when there is a demonstration of it. (A VI iii, 168; LoC 8-9)

Two features of this commentary are seminal for Leibniz's later thought: the upholding of the part-whole axiom for the infinite, with the entailed denial that the infinite is a whole; and the more nuanced claim that one cannot assert a property of the infinite except insofar as one has an independent demonstration of it.

The first claim is graphically illustrated by a calculation Leibniz performs in 1674, in which we are confronted with an infinite whole which has a direct visual representation: the area under a hyperbola between 0 and 1. By this time, Leibniz has made great strides in summing infinite series, and applying the results to calculate "quadratures", the areas under curves. In this example, he uses his knowledge to demonstrate the inapplicability of the part-whole axiom to infinite quantities.

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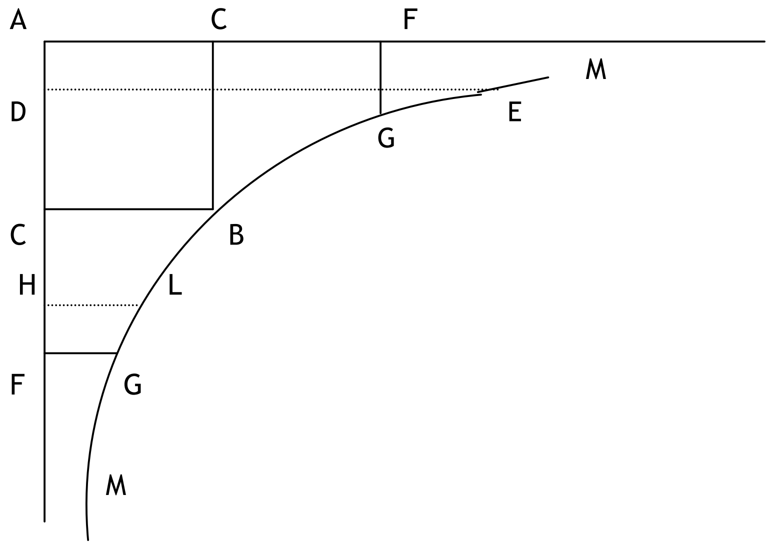
[1] Relevant extracts from Galileo's *Discorsi* may be found in (LoC 352-357).

[2] At EN 78, Salviati says: "for I believe that these attributes of greatness, smallness and equality do not befit infinities, of which one cannot be said to be greater than, smaller than, or equal to another"; and again at the end of the ensuing proof: "in final conclusion, the attributes of equal, greater, and less have no place in infinities, but only in bounded quantities" (EN 79; LoC 355-56). For Gregory of St Vincent's opinion, see his 1647, lib. 8, pr. 1, theorema, 870 ff.

Figure 1: Leibniz's hyperbola



In the above diagram, which Leibniz has deliberately drawn symmetrically, MGBEM is a hyperbola, and the variable abscissa DE = x (dashed line) is taken to vary between CB (taken as the x-axis) and (the horizontal) ACF… . Leibniz calculates the area by "applying" DE to (the vertical) AC (= 1) as a base. He expands DE as $(1 - y)^{-1} = 1 + y + y^2 + y^3 + \ldots$, and gets the result ACBEM = $1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$ In modern terms, Leibniz is evaluating the area under the curve $x = (1 - y)^{-1}$ by calculating the definite integral $\int_0^1 x\, dy$ using a series expansion, to obtain

$$\text{ACBEM} = \int_0^1 (1 - y)^{-1}\, dy = [y + y^2/_2 + y^3/_3 + y^4/_4 \ldots\,]_0^1 = 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$$

Now Leibniz applies the line HL = $(1 + y)^{-1} = 1 - y + y^2 - y^3 + \ldots$ to the line CF (= 1) to obtain the finite area CFGLB. That is,

$$\text{CFGLB} = \int_0^1 (1 + y)^{-1}\, dy = [y - y^2/_2 + y^3/_3 - y^4/_4 \ldots\,]_0^1 = 1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5 + \ldots$$

But if we now subtract the finite space CFGLB from the infinite space ACBEM, we obtain

$$\text{ACBEM} - \text{CFGLB} = (1 - 1) + (^1/_2 + {}^1/_2) + (^1/_3 - {}^1/_3) + (^1/_4 + {}^1/_4) + (^1/_5 - {}^1/_5) + \ldots$$

$$= 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$$

$$= \text{ACBEM}$$

Leibniz comments:

> This is pretty amazing (*satis mirabilis*), and shows that the sum of the series 1, $^1/_2$, $^1/_3$ etc. is infinite, and consequently that the area of the space ACBGM remains the same even when the finite space CBGF is taken away from it, i.e. that nothing noticeable is taken away.
>
> By this argument it is concluded that the infinite is not a whole, but only a fiction; for otherwise the part would be equal to the whole. (A VII 3, N. 38_{10}, p.468; October 1674)

Of interest here is the close connection between the visualizable infinite whole —the area under the hyperbola— and its expansion as an infinite series. For the infinite series is treated *as if* it were a whole, *as if* it has a definite sum, and so forth. But in this case the result leads to a contradiction, given the applicability of the part-whole axiom to the infinite. Even though this establishes the fictional nature of such infinite wholes, however, this does not mean that one cannot calculate with them; only, the viability of the resulting calculation is contingent on the provision of a demonstration.

As an example, we can look at another seminal calculation Leibniz made involving the hyperbolic series, in the process of answering a challenge set him by Huygens soon after his arrival in Paris in 1672. This was to determine the sum of the series of the reciprocal triangular numbers, $^1/_1$, $^1/_3$, $^1/_6$, $^1/_{10}$, $^1/_{15}$, ... Leibniz achieved this by noting that the successive terms are twice those of the differences between successive terms of the hyperbolic series: $1 = 2(1 - {}^1/_2)$; ${}^1/_3 = 2({}^1/_2 - {}^1/_3)$; ${}^1/_6 = 2({}^1/_3 - {}^1/_4)$; and so forth. But the sum of a series of such difference terms will, because of the cancellation of terms, equal the difference between the first term of the hyperbolic series and the last. Leibniz immediately realized the generality of this result, which he enshrined in what I have elsewhere (Arthur 2006) called his Difference Principle: "the sum of the differences is the difference between the first term and the last" (A VII, 3, 95). As is well known, after generalization to infinite series, this result will lead Leibniz to his formulation of the Fundamental Theorem of the Calculus: the sum (integral) of the differentials equals the difference of the sums (the definite integral evaluated between first and last terms), $\int Bdx = [A]_i^f$. Thus the sum of the first *n* terms of the original series,

$$\Sigma_{i=1}^{n} T_i = {}^1/_1 + {}^1/_3 + {}^1/_6 + {}^1/_{10} + {}^1/_{15} + \ldots + 2/(n^2 + n)$$

will be twice the difference between the $(n + 1)^{th}$ and the first term of the hyperbolic series:

$$\Sigma_{i=1}^{n+1} H_i = {}^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots + 1/(n + 1)$$

That is,

$$\Sigma_{i=1}^{n} T_i = 2(H_1 - H_{n+1}) = 2[1 - 1/(n + 1)]$$

Applying similar reasoning to infinite series, Leibniz calculates that if the sum of the infinite series $S(H) = \Sigma_{i=1}^{n} H_i = {}^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots$, and half the sum of the reciprocal triangular numbers ${}^1/_2 S(T) = {}^1/_2 \Sigma_{i=1}^{\infty} T_i = {}^1/_2 + {}^1/_6 + {}^1/_{12} + {}^1/_{20} + {}^1/_{30} \ldots$, then

$$S(H) - {}^1/_2 S(T) = {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + \ldots = S(H) - 1$$

giving

$$S(T) = 2.$$

Now one might of course object to this reasoning that, since the hyperbolic series is a diverging infinite series, Leibniz has effectively cancelled infinities in subtracting its sum S(H)

$= \sum_{i=1}^{n} H_i$ from both sides of the equation above. Clearly this would not be news to Leibniz, given the argument for the fictionality of the infinite treated as a whole given above. And indeed, Leibniz is sensitive to the need for rigour here. He acknowledges that this application of the Difference Principle "ought to be demonstrated to come out in the infinite" (362), and proceeds to show how this can be done. This is crucial, for in the above reasoning Leibniz has treated the infinite series *as if* they are wholes. But this can only be done, according to his commentary on Galileo, if there is a corresponding demonstration. Moreover, treating the infinite series as wholes is equivalent to treating them *as if* they had last terms, since otherwise the Difference Principle would not be applicable. This gives the all-important connection between Leibniz's doctrines of the fictionality of infinite wholes and the fictionality of infinitesimals. A justification of treating the infinite series as wholes is at the same time a justification of treating them as if they had a last, infinitely small term or *terminatio*, $1/\infty$.

Leibniz's demonstration proceeds by taking an arbitrarily small y^{th} term as the *terminatio* of the series, where "*y* signifies any number whatever". For the hyperbolic series the *terminatio* of a finite series of terms H(y) will be $1/y$, and for the series of reciprocal triangular numbers ΣT(y), it will be $2/(y^2 + y)$, since the y^{th} triangular number is $(y^2 + y)/2$. Thus when half the series ΣT(y) is subtracted from ΣH(y), the *terminatio* of the resulting series will be $1/y - 1/(y^2 + y)$, or $(y^2 + y - y)/(y^3 + y^2)$, or $1/(y + 1)$. But this is the *terminatio* for ΣH(y + 1). Leibniz does not complete this demonstration in these terms, preferring to proceed to a geometrical depiction in terms of a triangle, but it entails that for arbitrarily large y, the sum of half the series, $^1/_2$ΣT(y), is $1 - 1/(y + 1)$. So the sum of the reciprocal triangular numbers, ΣT(y), approaches 2 arbitrarily closely as y is taken arbitrarily large. Correspondingly, the *terminatio* of either this series or the hyperbolic series is not actually 0, but an arbitrarily small number.

This, then, yields the link between Leibniz's syncategorematic interpretation of the infinite and his interpretation of infinitesimals as fictions. To treat the infinite series as whole is to treat it as if it has an infinitieth term or infinitely small *terminatio*; whereas in fact the number of terms is greater than any number that can be given, and the magnitude of the *terminatio* is correspondingly smaller than any that can be given. This connection is stated explicitly by Leibniz in an annotation he made on Spinoza's Letter on the Infinite in the Spring of 1676:

> Finally those things are *infinite in the lowest degree* whose magnitude is greater than we can expound by an assignable ratio to sensible things, even though there exists something greater than these things. In just this way, there is the infinite space comprised between Apollonius' Hyperbola and its asymptote, which is one of the most moderate of infinities, to which there somehow corresponds in numbers the sum of this space: $^1/_1 + ^1/_2 + ^1/_3 + ^1/_4 + \ldots$, which is $^1/_0$. Only let us understand this 0, or nought, or rather instead a quantity infinitely or unassignably small, to be greater or smaller according as we have assumed the last denominator of this infinite series of fractions, which is itself also infinite, smaller or greater. For a maximum does not apply in the case of numbers. (A VI 3, 282; LoC 114-115)

For some time Leibniz appears to have hesitated over this interpretation, and as late as February 1676 he was still deliberating about whether the success of the hypothesis of infinities and the infinitely small in geometry spoke to their existence in physical reality too. But by April of the same year the syncategorematic interpretation is firmly in place, and we find Leibniz exploring the implications of the rejection of the infinitely small in a series of mathematico-physical reflections.[3] During the same period he was putting the finishing touches to his treatise on the infinitesimal calculus, *De quadratura arithmetica* (Leibniz,

[3] See Arthur (2009) for a discussion of the development of Leibniz's early views on infinitesimals.

1993), which he submitted for the Académie Française in the summer of 1676. Here we find his first explicit exposition of the interpretation of infinites and infinitesimals as fictions, and the provision of a theorem which, in Leibniz's words, "serves to lay the foundations of the whole method of indivisibles in the soundest way possible". Indeed, as Eberhard Knobloch has remarked, this theorem is a "model of mathematical rigour" (2002, 72).

In the treatise Leibniz promotes his new method of performing quadratures directly "without a *reductio ad absurdum*" (Prop 7, *Scholium*; Leibniz 1993, 35), by what we would now call a direct integral. This, he believes, necessarily involves the assumption of "fictitious quantities, namely the infinite and the infinitely small" (35). The traditional Archimedean method of demonstration was by a double *reductio ad absurdum*: it would be shown that a contradiction could be derived on the assumption that the quantity in question was smaller than a given value, and another contradiction on the assumption that the quantity in question was greater than that value, thus proving that it equalled it. Leibniz's method is instead to proceed by an application of the Archimedean Axiom. This axiom (actually due to Eudoxus) asserts that for any two geometric quantities x and y (with $y > x$), a natural number n can be found such that $nx > y$. This also entails that no matter how small a geometric quantity is given, a smaller can be found, and it is this property to which Leibniz appeals. Thus he prefers a justification "which simply shows that the difference between two quantities is nothing, so that they are then equal (whereas it is otherwise usually proved by a double reductio that one is neither greater nor smaller than the other)" (35). That is, he applies the Archimedean axiom to demonstrate that the error involved in calculations with infinitely small differences can be reduced to a quantity less than any given quantity by taking a difference sufficiently small, rendering it effectively null.

Moreover, this justification does not have to be effected in every case: it is enough to show that it can be done in a general case. This Leibniz does in a case that is surprisingly general, given the usual accusations about the parlous lack of justification he and Newton are alleged to have provided for their methods. For the key theorem that Leibniz successfully demonstrates in *De quadratura arithmetica* using this Archimedean method is what is now known as Riemannian Integration, as Eberhard Knobloch has shown in detail (2002).[4] This is Proposition 6; (Leibniz provides a similar justification for his Theorems 7 and 8). In his demonstration of Proposition 6 (1993, 30-33), Leibniz first identifies and then relates the sum of the "elementary rectangles [*rectangula elementales*]" (that is the Riemannian sum of unequal rectangles by which the curve is being approximated, which we may denote Q), and that of the mixtilinear figure [*spatium gradiformis*] that is the area under the curve between two ordinates ${}_1L$ and ${}_4D$, which we may denote A. Then he demonstrates that the difference between the area and the sum of the elementary rectangles, A – Q, can be no greater than the area of a certain rectangle whose height is the maximum height ψ_4D of any of the elementary rectangles, and whose width is the distance between the two ordinates ${}_1L$ and ${}_4D$. Thus $A - Q \leq {}_1L_4D \times \psi_4D$. But because the curve is assumed continuous, Archimedes' Axiom applies. Thus the height ψ_4D, even though it is greater than the heights of all the other elementary rectangles, "can be assumed smaller than any assigned quantity, for however small it is assumed to be, still smaller heights could be taken." Therefore the area of the rectangle ${}_1L_4D\psi$ "can also be made smaller than any given surface". Thus the difference A – Q too "can also be made smaller than any given quantity. QED." (30-33) There is therefore no error involved in calculating the quadrature as the sum of an infinity of infinitesimal areas, provided this is understood to mean that there are more little finite areas than can be assigned, and that their magnitude is smaller than any that can be assigned; in that case, so

[4] The exposition I give is indebted to Knobloch's (2002), but in the symbolization I have followed Leibniz's own from the *De quadratura* (Leibniz 1993).

many such areas of sufficient smallness can be taken as to make the difference between their sum and the quadrature less than any that can be assigned.

Thus it is not the case that Leibniz has two methods, one committed to the existence of infinitesimals as "genuine mathematical entities", "fixed, but infinitely small", and the other Archimedean, treating the infinitely small as arbitrarily small finite lines, ones that could be made as small as desired;[5] nor that he simply uses the infinitesimal calculus and then airily refers to the fact that one could *instead* have used an Archimedean method.[6] Rather, Leibniz's use of the Archimedean Axiom justifies proceeding *as if* there are infinitesimals, and at the same time demonstrates that what they really stand for are finite quantities which can be taken as small as desired. Once this is demonstrated in a suitably general case, it also justifies the use of these fictions in other analogous cases. As Leibniz himself writes, "Nor is it necessary always to use inscribed or circumscribed figures, and to infer by *reductio ad absurdum*, and to show that the error is smaller than any assignable; although what we have said in *Props. 6, 7 & 8* establishes that it can easily be done by those means." (Scholium to *Prop. 23*, Leibniz 1993, 69).

In effect, the application of the Archimedean Axiom enables a kind of Arithmetic of the Infinite. In his article, Knobloch identifies a number of rules which are tacitly applied by Leibniz in *De quadratura arithmetica*, "without demonstrating them, only relying on the 'law of continuity'" (2002, 67). Examples are "1. Finite + infinite = infinite", "2.1 Finite ± infinitely small = finite", "2.2 $x = (y + \text{infinitely small}) \Rightarrow x - y \approx 0$ (is unassignable)". 2.2 can

[5] Thus Henk Bos, in his classic article on Leibniz's differentials (1974-75, 55), sees Leibniz as pursuing "two different approaches to the foundations of the calculus; one connected with the classical methods of proof by *'exhaustion'*, the other in connection with a law of continuity." The quotations given in my text are from Bos 1974-75, 12, 13. See also Herbert Breger's similar criticisms of Bos on these points in his (2008, 195-197).

[6] Thus, to give a recent example of this kind of interpretation, Douglas Jesseph writes: "Leibniz often made grand programmatic statements to the effect that derivations which presuppose infinitesimals can always be re-cast as exhaustion proofs in the style of Archimedes. But he never, so far as I know, attempted anything like a general proof of the eliminability of the infinitesimal …" (Jesseph 2008, 233).

be demonstrated by Leibniz's method as follows. Suppose $x = y + dx$, where $dx > 0$, and suppose dx is a smallest assignable difference between x and y. Assuming it is a variable geometric quantity, the Archimedean Axiom will apply to it. It will therefore have an assignable ratio to another geometric quantity z such that a number $n > 0$ can be chosen to make $ndx > z$. Therefore z/n will be smaller than dx, contrary to supposition. Therefore $x - y = dx$ is unassignable; the difference between x and y is smaller than any quantity that can be assigned; it is incomparable with (has no finite ratio to) any finite quantity. What this means is that there can be no assignable error in equating $x - y$ with 0. Similar demonstrations can be given for the other rules.

Leibniz was quite explicit about this Archimedean foundation for his differentials as "incomparables". In his response to Bernard Nieuwentijt's criticisms in 1695 he wrote that when dx is added to x, the increment dx

> … cannot be exhibited by any construction. That is, I regard only those homogeneous quantities to be comparable (in agreement with Euclid, Book V, Definition 5), of which one can exceed the other when multiplied by a number, that is, a finite number. And I maintain that any entities whose difference is not such a quantity are equal. (…) This is precisely what is meant by saying that the difference is smaller than any given. (Leibniz 1695; GM V 322; also quoted in Bos 1974-75, 14)

The full "Law of Continuity" that Leibniz published for the first time in 1688 is a generalization of the method that we have just been examining:

> *When the difference between two instances in a given series, or in whatever is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought, or what results, must of*

> *necessity also be diminished or become less than any given quantity whatever*. (A VI 4, 371, 2032)[7]

But in a second formulation it also appears to be a generalization of the approach to infinite series we examined above, where the infinite limiting term or *terminatio* is included as if it were an infinitieth term in the series:

> In any proposed continuous transition that ends in a certain limiting case (*terminus*), it is permissible to formulate a general reasoning in which that final limiting case is included.[8]

It is the Law of Continuity in this form that is the basis of Leibniz's attempts to ground the rules of the calculus. His first attempt, in his first publication of the calculus, *Nova Methodus pro Maximis et Minimis* (1684)[9], was not clear to his contemporaries, as it was vitiated by a number of errors. In this article Leibniz rather confusingly begins by defining dx as "any straight line assumed arbitrarily", before giving his rules for manipulating of differentials (e.g. $d(xv) = xdv + vdx$).[10] But in a reply to the criticisms of Nieuwentijt and others, *Cum prodiisset*, drafted in or shortly after 1701 but first published by Gerhardt in 1846, he attempts "to show a little more clearly that the (so-called) algorithm of our differential calculus proposed by me in 1684 may be confirmed to be perfectly correct".[11]

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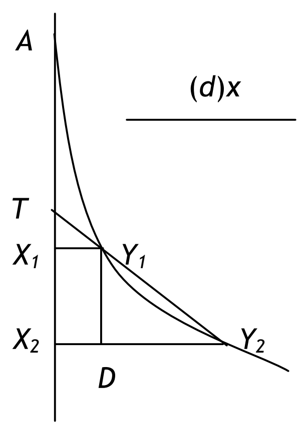
[7] Leibniz adds: "Or, to put it more commonly, *when the cases (or given quantities) continually approach one another, so that one finally passes over into the other, the consequences or events (or what is sought) must do so too*." (A VI 4, 371, 2032; translations mine.)
[8] Leibniz 1846, 40; Bos 1974-75, 56.
[9] *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus, Acta Eruditorum Lips*. 1684, GM V 220-226
[10] *Nova Methodus*, GM V 220.
[11] Henk Bos has drawn attention to the importance of *Cum prodiisset* in his article on Leibniz's differentials (Bos 1974-75). As he observes, since it contains a reference to a work by Gouye of 1701, and "deals with the problems which were discussed in 1701-1702, it is probable that it originated in or not much later than 1701" (1974-75, 56, n. 92). In a marginal note Leibniz mentions "the Parisians" and urges that "All this must be reviewed very carefully so that it can be published, leaving out the sourer things said in contradicting others." (Leibniz 1846, 39)

In this later justification Leibniz has *dx* stand for the difference between two abscissas X_1 and X_2, so that $dx = X_2 - X_1$, with *dy* the difference between the corresponding ordinates X_1Y_1 and X_2Y_2. For the sake of definiteness, he uses a particular example, the parabola $x^2 = ay$ (Figure 2 below).[12]

Figure 2: Tangent to parabola



Since $(x + dx, y + dy)$ is also a point on the parabola, "we will have

$$y + dy = (xx + 2xdx + dxdx)/a \quad (2.1)$$

and subtracting *y* from one side and x^2/a from the other, there will remain

$$dy/dx = (2x + dx)/a \quad (2.2)$$

which is the general rule expressing the ratio of the difference in ordinates to the difference in abscissas." Now Leibniz appeals to his Law of Continuity:

> Now, since by our postulate it is permissible to include in a general reasoning also the case where the ordinate X_2Y_1, having been moved closer and closer to the fixed ordinate X_1Y_1 until it finally coincides with it, it is clear that in this case *dx* will be

[12] Leibniz 1846, 44. Note that Leibniz characteristically draws his diagrams with the abscissas and *x*-axis vertical and the ordinates and *y*-axis horizontal. I have altered his ${}_1X$, ${}_2Y$, etc. to X_1, Y_2, etc., for ease of reading, and also corrected his *dx to* (*d*)*x*, as he seems to have intended, but otherwise this is his figure.

> equal to zero or should be omitted, and so it is clear that, since in this case TY_1 is the tangent, X_1Y_1 to TX_1 is as 2x to *a*. (Leibniz 1846, 44)

Leibniz's summary of how the Law of Continuity is applied here is notably clear:

> Hence it may be seen that in all our differential calculus there is no need to call equal those things that have an infinitely small difference,[13] but those things are taken as equal that have no difference at all, provided that the calculation is supposed to have been rendered general, applying equally to the case where the difference is something and to where it is zero; and only when the calculation has been purged as far as possible through legitimate omissions and ratios of non-vanishing quantities until at last application is made to the ultimate case, is the difference assumed to be zero. (44-45)

It can be appreciated, I think, how close this is to a modern justification of differentiation in terms of limits. Leibniz has here assumed that the difference *dx* is a finite difference (we would write Δx), and when an expression for $\Delta y/\Delta x$ has been found in which Δx has as far as possible been cancelled, he effectively takes the limit as $\Delta x \rightarrow 0$ to arrive at an expression for dy/dx.

The rigour of Leibniz's algorithm can be seen by applying it to the counterexample proposed by Rolle. He supposed a parabola $y^2 = ax$, and quoted Leibniz's result that

$$2y\,dy = a\,dx \quad \text{or} \quad (adx - 2ydy) = 0 \tag{2.3}$$

Now Rolle reasoned that since $(x + dx, y + dy)$ is also a point on the parabola, we will have

$$(y + dy)^2 = ax + adx = y^2 + 2ydy + dy^2 \tag{2.4}$$

$$\Rightarrow \quad dy^2 = (ax - y^2) + (adx - 2ydy) = 0 + 0 \tag{2.5}$$

[13] This is a response to Nieuwentijt's definition of an infinitesimal as something actually infinitely small: see below.

Thus, he concluded, $dy = dx = 0$, and Leibniz's infinitesimals are identically equal to zero.

But the use of (2.3) in the calculation violates Leibniz's proviso that the difference may be assumed to be zero only after "the calculation has been purged as far as possible through legitimate omissions and ratios of non-vanishing quantities" and one has reached the limiting case. Prior to that, the general rule that applies to all finite differences is $dx/dy = (2y + dy)/a$; and if one substitutes that formula into equation (2.4) in Rolle's calculation one obtains the identity $dy^2 = dy^2$. If, on the other hand, having obtained $dx/dy = (2y + dy)/a$, one applies it to the ultimate case by effectively letting $dy \to 0$, one obtains (2.3) as a result.

What this application of the Law of Continuity legitimates, then, is proceeding *as if dx* and *dy* are infinitely small quantities that can be neglected in the last step of the calculation; whereas what in fact they stand for are finite differences that are assumed to vary in accordance with the Archimedean Axiom, so that they can be made arbitrarily small. Leibniz was aware, however, that this new algorithm was a kind of shorthand for what in his time would be acceptable as a properly geometrical proof. That would have to proceed by proportions, where only quantities of the same dimensions could stand in ratios to one another, and all quantities would be finite or "assignable", even in the limit. In *Cum prodiisset* he therefore gave a proof sketch in which "*dx* and *dy* are retained in the calculation in such a way that they signify non-vanishing quantities even in the ultimate case". He lets $(d)x$ be "any assignable straight line" (i.e. a fixed finite line segment), and then defines another segment $(d)y$, by the proportion

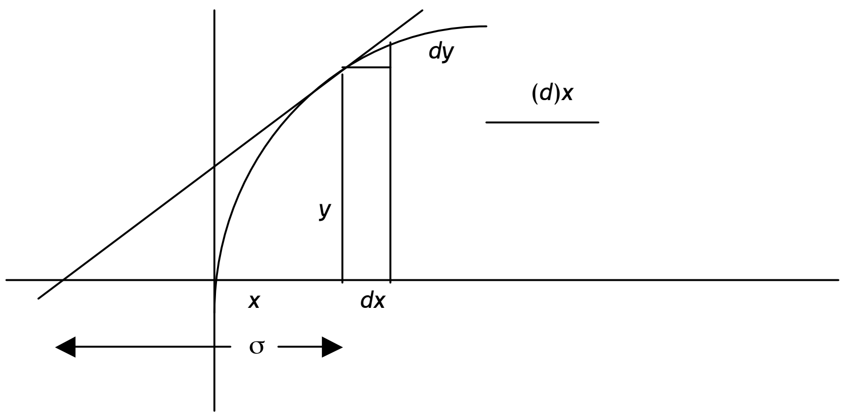
$$(d)y:(d)x = DY_2:DY_1 = dy:dx \qquad (2.1)$$

Thus $(d)y$ varies in such a way that for all finite dx and $dy > 0$ it maintains the same finite ratio to the fixed line $(d)x$ as dy has to dx. But, as Bos observes, the same $(d)y$ can also be given an interpretation in the limit when the variable $dx = 0$, namely through the proportion

$$(d)y{:}(d)x = y{:}\sigma \tag{2.2}$$

where σ is the subtangent to the curve. Bos's diagram follows:

Figure 3: Leibniz's finite surrogates



Now, since the resulting formula is still interpretable even in the case where *dx* = 0, the Law of Continuity asserts that this limiting case may also be included in the general reasoning:[14] *dy*:*dx* can be substituted for (*d*)*y*:(*d*)*x* in the resulting formulas even for the case where *dx* = 0: "in this way *dy* and *dx* will always be assignable in the ratio DY_2 to DY_1, which latter vanish in the ultimate case." As Bos explains, this explication does indeed make intelligible Leibniz's having begun his 1684 article by having *dx* and *dy* stand for finite lines. What was confusing is that there he had *dx* stand for an arbitrary finite line, corresponding to the (*d*)*x* of this later explication.

Proceeding on this foundation Leibniz offers justifications in *Cum prodiisset* for the rules of the calculus for first-order differentials that he had given in 1684: the rules for differentiating sums and differences, products, quotients, and powers. First he constructs a figure in which (as above) (*d*)*x* is a fixed straight line segment which remains constant as X_2

[14] Here I disagree with Bos, who claims that in taking the secant as the limiting position of the tangent, Leibniz "did not invoke the law of continuity; as will be seen, he used the law later, *presupposing* that the limiting position of the secant is the tangent." (Bos 1974-75, 57). But that is how the law of continuity works: it does presuppose a limiting case, it does not establish its existence.

approaches the fixed point X_1, $dx = X_2 - X_1$, $dy = X_2Y_2 - X_1Y_1$, and $(d)y = (dy/dx)\cdot(d)x$. He also introduces a third variable v and defines another variable straight line segment $(d)v$ by $(d)v = (dv/dx)\cdot(d)x$. (Leibniz 1846, 46)

Following Bos, I will give as an example his justification on this basis for the differentiation of a product: $d(xv) = xdv + vdx$. First Leibniz lets $ay = xv$ (here the purpose of the constant a is to preserve the homogeneity of the equation), and then allows x, y, and v to increase by dx, dy, and dv respectively:

"Demonstration: $ay + dy = (x + dx)(v + dv)$

$$= xv + xdv + vdx + dxdv,$$

and, subtracting from each side the equals ay and xv, we have

$$ady = xdv + vdx + dxdv$$

or $$ady/dx = xdv/dx + v + dv$$

and transposing the case, as we may, to lines that never vanish, we have

$$a(d)y/(d)x = x(d)v/(d)x + v + dv$$

so that the only remaining term which can vanish is dv, and since in the case of vanishing differences $dv = 0$, we have

$$a(d)y = x(d)v + v(d)x$$

as was asserted, *or $(d)y:(d)x = (x + v):a$.* Whence also, because $(d)y:(d)x$ always = $dy:dx$ one may feign this in the case of vanishing dy, dx and put

$$ady = xdv + vdx.$$" [15]

Thus $$d(xv) = xdv + vdx \qquad \text{QED.} \qquad (2.3)$$

[15] Leibniz 1846, 46-47; quoted from Bos (1974-75, 58). The asterisked phrase is an error for "or $(d)y:(d)x = [x(d)v/(d)x + v]:a$, but this does not figure in the remaining calculation.

This justification may be regarded as one in keeping with the standards of rigour expected by Huygens and Newton, and thus above reproach. It still depends on the same conceptions of infinitesimals as fictions: they stand for quantities that are finite but variable, and that may be made arbitrarily small.

My interpretation here contrasts with that of Bos, who reads Leibniz as having two different interpretations of differentials, one in which they are infinitely small elements of lines, and the other where they stand for finite lines. He points out that differentials are equated with finite lines by Leibniz already in his first publication, the *Nova Methodus* of 1684, apparently in preference to the approach taken in the manuscript *Elementa* "which Gerhardt identified as an alternative draft for the first publication of the rules of the calculus, in which the differentials are introduced as infinitesimals" (Bos 1974-75, 63). But in the *Elementa*, *dx* and *dy* are defined as differences of successive abscissa and ordinates which are *then* assumed to become infinitely small, so that even here they are understood as limit-elements, like Newton's nascent and evanescent quantities (Leibniz 1848, 32-33). In the *Nova Methodus*, on the other hand, Leibniz makes more explicit the finiteness of the differences prior to this limiting process. He lets *dx* stand for an arbitrary finite line, and *dy* for a variable finite line, whose ratio to *dx* in the limit as *dx* becomes evanescent equals the ratio of the ordinate to the subtangent:

$$dy{:}dx = y{:}\sigma$$

In the more sophisticated approach of *Cum prodiisset*, as we saw above, $(d)x$ is now taken as the arbitrary finite line, and $(d)y$ the variable finite line defined by (2.1) and (2.2), while *dy* and *dx* are both variable finite quantities whose ratio is always equal to the ratio $(d)y{:}(d)x$, even in the case where the finite difference *dx* goes to zero. Bos interprets this as meaning that in *Cum prodiisset* Leibniz has defined $(d)y$ as the differential. For if $y = f(x)$, and the finite differences *dy* and *dx* are written as Δy and Δx, then (2.1) becomes

$$(d)y = (\Delta y/\Delta x)(d)x = \{[f(x + \Delta x) - f(x)]/\Delta x\}\cdot(d)x \quad (2.4)$$

In the limit as $\Delta x \to 0$, the expression in braces is $f'(x)$, the derivative with respect to x of the function $f(x)$, and $(d)y$ is the differential of y as defined by Cauchy (Bos 1974-75, 59).

But this, it seems to me, is to misinterpret Leibniz's justification, the aim of which is not to define $(d)y$ as the differential, but rather to show that dy and dv may still have an assignable ratio to dx even in the limit where $dx \to 0$. In that limit, dy, dv and dx are infinitely small, fictional quantities. These are the differentials. The role of $(d)y$, $(d)v$ and $(d)x$ is to be "finite surrogates": they remain assignable quantities throughout the limiting process as dx becomes arbitrarily small, and since $(d)y$ and $(d)x$ have an assignable ratio equal to dy:dx even in that limit (likewise for $(d)v$ and $(d)x$), differential equations involving terms in dy, dv and dx to the same order are interpretable under the fiction where these quantities stand for infinitely small quantities.

Having shown this in an arbitrary case, it obviates the need to show it in every case. The strict proof operating only with assignable quantities justifies proceeding by simply appealing to the fact that dv is incomparable with respect to v: in keeping with the Archimedean axiom, it can be made so small as to render any error in neglecting it smaller than any given. Thus, in a letter to Wallis in 1699, Leibniz justifies the rule for $d(xy) = xdy + ydx$ as follows

> … there remains $xdy + ydx + dx\cdot dy$. But this $dx\cdot dy$ should be rejected, as it is incomparably smaller than $xdy + ydx$, and this becomes $d(xy) = xdy + ydx$, inasmuch as, if someone wished to translate the calculation into the style of Archimedes, it is evident that, when the thing is done using assignable qualities, the error that could accrue from this would always be smaller that any given.[16]

[16] "… *restat xdy + ydx + dxdy. Sed hic dxdy rejiciendum, ut ipsis xdy + ydx incomparabiliter minus, et fit d, xy = xdy + ydx, ita ut semper manifestum sit, re in ipsis assignabilibus peracta, errorem, qui*

As a second example of Leibniz's approach to infinitesimals, it will be instructive to examine Leibniz's criticisms of Newton's proof of his Lemma 9 in the *Principia* and the demonstration he offers in its stead. Here I will follow closely the development of Leibniz's reactions to Newton's *Principia* and its methods given by Domenico Bertoloni Meli (1993), who has analysed the evolution of Leibniz's mathematical cosmology from his initial hostile reactions to Newton's to the development of his own rival dynamics of celestial motion, in a brilliant and careful study of the surviving documents.

As Leibniz correctly recognized, Lemmas 9-11 are crucial for Newton's proofs of the inverse square law; Lemma 10 in particular, is a corollary of Lemma 9, and is appealed to by Newton (in the first edition of the *Principia*) in the proof of Proposition 6. Each of Newton's lemmas is an instance of his Method of First and Last Ratios, and establishes a ratio between quantities as they are on the very point of being generated or vanishing. The figures, accordingly, are "ultimate" ones, depicting what Whiteside has aptly termed "limit motions" (1967, 154) the motions a body would undergo during a "moment" if it continued with the velocity it had at the beginning of that moment. Leibniz had no objection to this procedure; but he did object to what he found in Proposition 6, namely the treatment of a curved trajectory in such an ultimate moment as compounded from two motions, one a rectilinear inertial motion along the tangent, and the other an acceleration towards the centre. This was an objection of his of long standing, and is closely related to his syncategorematic understanding of moments and of infinite division. In a piece written at the beginning of April 1676, he reasoned that if there were such a thing as a perfect fluid—by which he meant matter divided all the way down into points, each individuated by a differing motion—then "a new endeavour [would] be impressed at any moment whatever" on a body moving in the

inde metui queat, esse dato minorem, si quis calculum ad Archimedis stylum traducere velit." (Leibniz to Wallis, 30th March 1699; GM IV 63).

fluid. But this would be to compose a curved line from points and time from instants, and would also entail an "impossible" composition at every single instant:

> But if this is conceded, time will actually be divided into instants, which is not possible. So there will be no uniformly accelerated motion anywhere, and so the parabola will not be describable in this way. And so it is quite credible that circles and parabolas and other things of that kind are all fictitious.... For supposing a point moves in a parabolic line, it will certainly be true of it that at any instant it is moving with a uniform motion in one direction, and with a uniformly accelerated motion in another, which is impossible. (A VI 3, 492; LoC 74-77)[17]

When he first confronted Newton's *Principia* some 12 years later, Leibniz's initial reaction was therefore to believe that there was a mistake in Newton's composition of motions in deriving Proposition 6. On his understanding, in an ultimate moment only straight line motions (and equivalently the lines they would traverse in such a moment) can be compounded, since ultimately the curve is resolved into an infinite-sided polygon with fictional straight sides, each one representing a geometric "indivisible" or ultimate difference between successive values. Seeing the dependence of Proposition 6 on Lemmas 9 and 10, therefore, he concentrated his attention on them.

In Lemma 9, Newton has a curved trajectory depicted in an ultimate moment, and a claim is made that "the areas of the [curvilinear] triangles *ADB* and *AEC* will ultimately be to each other in the squared ratio of the sides", i.e. $\Delta ADB{:}\Delta AEC = AD^2{:}AE^2$ (Newton 1999, 437). Newton performs his proof by using his Method Of Finite Surrogates: here these surrogates are

[17] Here Leibniz appears to be applying a principle he had just formulated: "If a given motion can be resolved into two motions, one of them possible and the other impossible, the given motion will be impossible." (A VI 3, 492; LoC 72-73) The quoted passage should be compared with the quotation Bertoloni Meli gives from Leibniz's letter to Claude Perrault, also written in 1676: "I take it as certain that everything moving along a curved line endeavours to escape along the tangent of this curve; the true cause of this is that curves are polygons with an infinite number of sides, and these sides are portions of the tangents…" (quoted from Bertoloni Meli 1993, 75).

the lines *Ae* and *Ad*, which are finite, with *Ae* remaining fixed, and in proportion to *AD* and *AE*, which are supposed variable; likewise *Abc* is a fixed portion of curve similar to *ABC*, with the elements of the curve *AB* and *AC* supposed variable, gradually shrinking toward *A* until *B* and *C* "come together" with it. The line *AGg* is tangent to both curves at *A*. Newton's proof then proceeds by noting that in the limit *A*, *B* and *C* coincide, ∠*cAg* vanishes, and "the curvilinear areas *Abd* and *Ace* will coincide with the rectilinear areas *Afd* and *Age*, and thus (by Lemma 5) will be in the squared ratio of these sides *Ad* and *Ae*" (437). Therefore, given the assumed proportionality between these finite surrogates and the infinitesimals, "the areas ABD and ACE also are ultimately in the squared ratio of the sides *AD* and *AE*. QED" (437).

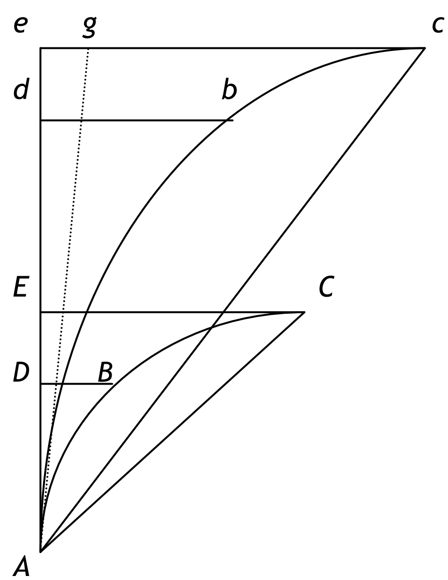


Figure 4: Newton's Lemma 9

Leibniz's reaction to Newton is fraught with irony. For he neglects Newton's use of finite surrogates, which is to all intents and purposes identical to his own way of justifying the calculus in "*Cum prodiisset*" using only assignable quantities, as explained above. (This irony is compounded by the fact that Leibniz rather arbitrarily replaces Newton's letters *E* and *C* by (*D*) and (*B*), the same bracket notation he usually uses for finite lines.) It is possible, of course, that Leibniz adopts his finite surrogate technique after further reflection on Newton's method, although, as Bos notes, Leibniz had promoted his differentials as finite variable quantities bearing a proportion to fixed lines in his first publication on the calculus (1684), (albeit in a form that was not easy for his contemporaries to understand).[18] In any case, in these first notes on Newton's *Principia* Leibniz dispenses with the finite surrogates completely, so as to concentrate on the curvilinear figures that Newton has depicted in his "ultimate moment". Here is his diagram, statement and criticism of Newton's Lemma 9, following the description given by Bertoloni Meli:

[18] See Bos (1974-75, 57, 63). Certainly some of the wording of Leibniz's later justifications seems to show Newton's influence, as Guicciardini notes in his excellent discussion (1999, 161), even if he had developed the method of finite surrogates independently.

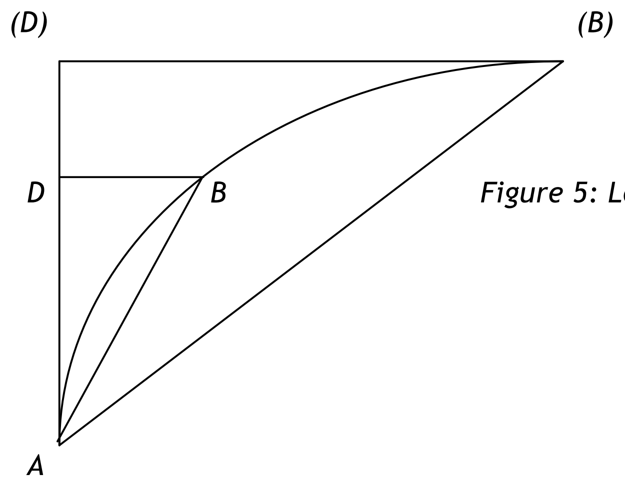


Figure 5: Leibniz: Newton's Lemma 9

> If *AB*, *A(B*) are unassignable, Δ*ADB* and Δ*A(D)(B)* will be in the squared ratio of the sides (+ namely, it must be understood that the Δs are similar, and so angle *BA(B)* has a ratio to angle *DAB* that is infinitely small, but this does not seem to be true, since in Δ*AB(B)* the side *B(B)* has an assignable ratio to *AB* or *A(B*), and so to *AD* and *DB*.+)

That is, Leibniz objects that in the limit Δs *ADB* and *A(D)(B)* will not be similar, since Newton's diagram *already* represents the ultimate moment, and in it angle *BA(B)* has a finite rather than infinitesimal ratio to the sides of the triangles. In an effort to show that Newton has made an error in calculating with infinitesimals, Leibniz proceeds with his own calculation, in which the infinitesimal order of the angle *BA(B)* with respect to *AB* and *AD* is rendered explicit by setting $AD = x$, $DB = y$, $A(D) = x + dx$, and $D(B) = y + dy$. Leibniz does not succeed in demonstrating any error on Newton's part; nor in fact is there an error, as Meli observes, since as x tends to zero so does dx, so that ultimately the triangles become similar.

Nevertheless, even though he fails to show anything wrong in Newton's procedure, Leibniz's workings provide us with a nice illustration of his approach to the differential calculus. First he expresses y as a series expansion in x:

$$y = a + bx + cx^2 + ex^3 + ... \qquad (2.4)$$

Now since $x = 0$ when $y = 0$, we have $a = 0$ —a point Leibniz first realized, and then subsequently forgot, rendering his calculation inconclusive. But as Meli shows, the lemma follows fairly easily from these premises. For

$$xy = bx^2 + cx^3 + ex^4 + \ldots \quad (2.5)$$

so that since (assuming Lemma 5) area[*ADB*] = $AD{\cdot}DB/2$,

$$\text{area}[ADB] = xy/2 = {}^1/_2(bx^2 + cx^3 + ex^4 + \ldots) \quad (2.6)$$

Meli remarks that "If *x* becomes infinitesimal, the terms in x^3, x^4, ... are negligible, and the area of *ADB* (= $xy/2$) is indeed proportional to AD^2 (= x^2)" (242). To this one might object that it seems unjustified to assume terms in x^3, x^4, ...are negligible when terms in x^2 are not. In fact, however, we can proceed more rigorously. For although Leibniz apparently thought Newton's finite surrogates could be dispensed with, they are completely in accord with his way of proceeding. Thus let *DB:AD* = *db:Ad*, etc., as Newton had assumed, with *db* and *Ad* finite, and with this ratio remaining in a finite and non-vanishing proportion to *ec* and *Ae* (with *Ae* fixed) even as $x \rightarrow 0$. Then

$$DB/AD = AD{\bullet}DB/AD^2 = b + cx + ex^2 + \ldots \quad (2.7)$$

But since this equals *db:Ad*, which is finite and non-vanishing as $x \rightarrow 0$, it follows that

$$2 \times \text{area}[ADB)]/AD^2 = AD{\bullet}DB/AD^2 = b \text{ (const.)} \quad (2.8)$$

Now the same reasoning will apply for $\Delta A(D)(B)$, giving

$$\text{area}[ADB)]:\text{area}[A(D)(B)] = AD^2:A(D)^2 \quad (2.9)$$

Newton's Lemma is therefore sound even by Leibniz's lights, as indeed he came to realize despite not having successfully completed this calculation.[19]

[19] As Meli reports (Bertoloni Meli 1993, 103), Leibniz's attitude seems to have undergone a sea change on moving on from Section 1 of the *Principia* on First and Last Ratios, to section 2, on the

Alternatively, the move from equation (2.7) to (2.8) is justifiable by the consideration that *dx* is incomparably small relative to any finite quantity of the same order as *x*, and thus ignoring it will produce an error smaller than any that can be assigned, and thus null. Thus if *x* in equation (2.7) is an infinitesimal variable, then *cx* stands for a finite line that can be made smaller than any finite quantity, and thus smaller than *b*, for any *b* and *c*, and *a fortiori* for higher order terms.

Similar considerations apply to Lemma 11, stating that the curvature of a trajectory at a given point is the same as the curvature of the osculating circle at that point. In the first stage of composition of his *Marginalia* identified by Meli, Leibniz objects to Newton's Corollary 7 to Proposition 4, in which the proportionality of centripetal forces to v^2/r, proved in Proposition 4 for bodies rotating uniformly in circular orbits, is extended to non-circular orbits for limit-motions: "Since I do not yet accept the generality of Lemma XI, I also doubt the generality of this corollary 7". (Bertoloni Meli 1993, 107) Somewhat later, according to Meli, he corrected himself: "On the contrary, this is true, because the considerations on the secant of the angle made by the radius from the centre to the curve, and that of the radius of the osculating circle to the curve, vanish on account of the similarity of the figures."[20] That is, according to Meli's analysis, "since Proposition 4 is stated in the form of a proportion between the homologous elements of two similar figures, their similarity cancels out the dependence of paracentric conatus [—the endeavour toward the centre—] on the secant of the angle and on the osculating radius" (107).

determination of central forces. Accordingly, in the first set of *Excerpts* Leibniz took from the *Principia* the following year, "lemma 9 is transcribed without commentary, and seems to be accepted without difficulty" (242).

[20] *Marginalia*, M 42 A: translated from the Latin quoted by Meli, 107.

3. Smooth infinitesimal analysis

Smooth Infinitesimal Analysis has many features in common with Leibniz's approach. It begins, like Leibniz, by eschewing the composition of the continuum from an infinity of points. In contrast to the account of the continuum established by Cantor based on Point Set Theory, according to which a continuous line is an infinite (indeed, nondenumerably infinite) set of points, SIA is rooted in Category Theory, and mappings rather than points are taken as basic. The category of smooth manifolds, ***Man***, is embedded in an enlarged category ***C*** which contains "infinitesimal" objects, and a topos $\boldsymbol{Set}^{\boldsymbol{C}}$ is then formed of sets varying over ***C***. "Each smooth topos ***E*** is then identified as a certain subcategory of $\boldsymbol{Set}^{\boldsymbol{C}}$. Any of these toposes has the property that its objects are undergoing a form of smooth variation, and each may be taken as a smooth world." (Bell 1998, 14). With this foundation secured, a great many results of differential calculus may be obtained "—with full rigour— using straightforward calculations with infinitesimals in place of the limit concept" (Bell, 1998, 4), as if infinitesimals exist.

Here I say "as if infinitesimals exist" advisedly. For another feature that SIA has in common with Leibniz's approach is that, whilst it licenses certain infinitesimal techniques, it is not committed to the existence of infinitesimals in the continuum. That is, as in Leibniz's theory, infinitesimals are *fictions* in a precisely defined sense. The sense in which they are fictions in SIA, however, is that although it is denied that an infinitesimal neighbourhood of a given point, such as 0, reduces to zero, it cannot be inferred from this that there exists any point in the infinitesimal neighbourhood distinct from 0.[21] Thus the Law of Excluded Middle, and with it the Law of Double Negation, both fail in smooth worlds. Bell explains this as follows. Define two points *a* and *b* on the real line (as represented in a smooth world Σ) as

[21] In a sympathetic but exacting analysis of SIA, Geoffrey Hellman observes that the status of infinitesimals in SIA is odd. On the one hand, "the "existence of non-zero nilsquares" is actually refutable", but on the other, "in the background one assumes their 'possibility' in making sense of the whole theory" (2006, 10). Certainly there is no constructive definition of them (11), even as fictional entities. Here, I think, Leibniz's theory is more in keeping with the constructivist spirit.

distinguishable iff they are not identical, i.e. iff $\neg a = b$, where '=' denotes identity. Now define the *infinitesimal neighbourhood* $I(0)$ of a given point 0 as the set of all those points indistinguishable from 0. That is, define $I(0)$ as follows:

$$I(0) =_{\text{def}} \{ x \mid \neg\neg x = 0\} \tag{3.1}$$

Now if the Law of Double Negation (or, equivalently, the Law of Excluded Middle) held in Σ, we could infer that $x = 0$ for each x in $I(0)$, so that the infinitesimal neighbourhood of 0 $I(0)$ would reduce to {0}. But we know that this neighbourhood does not reduce to {0} in Σ. So we cannot infer the identity of points from their indistinguishability. Again, suppose there is a point a in I that is distinguishable from 0, i.e. suppose there is a point $a \varepsilon I$ such that $\neg a = 0$. But since $a \varepsilon I$, $\neg\neg a = 0$ by definition. But this is a contradiction. Therefore it is not the case that there exists a point a in I that is distinguishable from 0. That is, the logic of smooth worlds is intuitionistic logic: the Law of Non-contradiction holds, as we have just seen; but the Laws of Excluded Middle, Double Negation, and one form of Quantifier Negation do not hold in smooth worlds. From $\neg(\forall x \in I)x = 0$ (it is not the case that all members of I are identical with 0), we can not infer that $(\exists x \in I)\neg x = 0$ (there is a member of I that is distinguishable from 0).[22] This makes precise the older conception of an infinitesimal difference as a difference smaller than any assignable, but not zero. It does so, in effect, by denying that an unassignable difference reduces to zero, but not allowing the inference from this that there exists an unassignable difference different from zero. it is in this sense that the infinitesimal intervals of SIA are fictional.

On this foundation Bell erects the theory of SIA. First he defines Δ as consisting in those points x in R such that $x^2 = 0$. The letter ε then denotes a variable ranging over Δ (21). The fundamental assumption is then that every curve is microstraight (9, 22). That is, arbitrary

[22] More precisely, the logic of smooth toposes is *free first-order intuitionistic* or *constructive logic*. See Bell 1998, 101-102.

functions f: R → R are assumed to behave locally like polynomials, so that with $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ and $\varepsilon^2 = 0$, we have

$$f(\varepsilon) = a_0 + a_1\varepsilon \text{ for any } \varepsilon \text{ in } \Delta. \quad (3.2)$$

This is the *Principle of Microstraightness*, and it implies that $(\varepsilon, f(\varepsilon)$ lies on the tangent to the curve at the point $(0, a_0)$. It also entails (if one considers only the restriction of g of f to Δ) that g is affine on Δ. This may be compared with Leibniz's statement that "… to find a *tangent* is to draw a straight line which joins two points of the curve which have an infinitely small distance, that is, the prolonged side of the infinitangular polygon which for us is the same as the *curve*." (Leibniz 1684a, GM V 223). That consideration motivates the *Principle of Microaffineness* (Bell 1998, 23):

For any map g: Δ → R, there exists a unique b in R such that, for all ε in Δ, we have

$$g(\varepsilon) = g(0) + \varepsilon b. \quad (3.3)$$

This principle allows one to define the *derivative* of an arbitrary function f: R → R as follows (1998, 26). Define the function $g_x(\varepsilon) = f(x + \varepsilon)$. By Microaffineness it follows that there is a unique b_x such that for all ε in Δ,

$$f(x + \varepsilon) = g_x(\varepsilon) = g_x(0) + \varepsilon b_x = f(x) + \varepsilon b_x \quad (3.4)$$

If we allow x to vary, the values b_x will constitute a new function, the derivative $f'(x)$:

$$f(x + \varepsilon) = f(x) + \varepsilon f'(x) \quad (3.5)$$

If the function f is a function of time, and ε is an infinitesimal time, we have as a direct consequence of this the *Principle of Microuniformity* (of natural processes): any such process may be considered as taking place at a constant rate over the "timelet" ε (1998, 9). Another important consequence of Microaffineness is the *Principle of Microcancellation*: *for any a, b in R, if* $\varepsilon a = \varepsilon b$ *for all* ε *in* Δ, *then* $a = b$ (1998, 24). I quote Bell's proof:

> Suppose that, for all ε in Δ, all $\varepsilon a = \varepsilon b$ and consider the function $g: \Delta \rightarrow R$ defined by $g(\varepsilon) = \varepsilon a$. The assumption then implies that g has both slope a and slope b: the uniqueness clause in Microaffineness yields $a = b$. (24)

I will now illustrate the neatness and simplicity of SIA by means of two examples: (i) the proof of the product rule for derivatives, and (ii) the proof of Newton's Proposition 1 of the *Principia*, the Kepler Area Law. For both of these illustrations I shall follow Bell's exposition.

(i) *The Product Rule for Derivatives*

Let $y(x) = f(x)g(x)$. We wish to prove that $y' = f'g + f g'$. By the definition of the derivative,

$$f(x + \varepsilon) = f(x) + \varepsilon f'(x), \tag{3.6}$$

and likewise for g and y. Thus since

$$y(x + \varepsilon) = f(x + \varepsilon)\cdot g(x + \varepsilon), \tag{3.7}$$

we have

$$y(x) + \varepsilon y'(x) = [f(x) + \varepsilon f'(x)]\cdot[g(x) + \varepsilon g'(x)] \tag{3.8}$$

$$= y(x) + \varepsilon[f'(x)g(x) + g'(x)f(x)] + \varepsilon^2 f'(x)g'(x) \tag{3.9}$$

Now subtracting y from both sides and assuming $\varepsilon^2 = 0$, we have

$$\varepsilon y'(x) = \varepsilon[f'(x)g(x) + g'(x)f(x)], \qquad \text{and thus} \tag{3.10}$$

$$y'(x) = f'(x)g(x) + g'(x)f(x) \tag{3.11}$$

The inference from (3.10) to (3.11) is guaranteed by the Principle of Microcancellation: if $\varepsilon a = \varepsilon b$ for all ε in Δ, then $a = b$. As we saw, this in turn depends on the nilsquare property $\varepsilon^2 = 0$.

So in the proof, this property is invoked three times: indirectly in the definition of the derivative (3.6), in the inference from (3.9) to (3.10) directly, and again indirectly in proving the Microcancellation property involved in the inference from (3.10) to (3.11).

(ii) *Newton's Proposition 1, the Kepler Area Law*

In his proof of this law, Bell assumes that the area A, the radius r and the angle θ are all functions of t, which increases by a nilsquare infinitesimal ε "with ε in Δ". "Then by Microstraightness the sector *OPQ* is a triangle of base $r(t + \varepsilon) = r + \varepsilon r'$ and height

$$r \sin[\theta(t + \varepsilon) - \theta(t)] = r \sin\varepsilon\theta' = r\varepsilon\theta'\text{" (Bell 1998, 69)} \tag{3.12}$$

Here the nilsquare property is invoked in Microstraightness, in the definitions of the derivatives r' and θ', and again in the equating of $\sin\varepsilon\theta'$ with $\varepsilon\theta'$. Since the area is half the base times the height, this gives for the area of sector *OPQ*

$$OPQ = {}^1/_2(r + \varepsilon r')r\varepsilon\theta' = {}^1/_2\varepsilon r^2\theta' \tag{3.13}$$

if the term ${}^1/_2\varepsilon^2 rr'\theta'$ is dropped, again invoking the nilsquare property. But the area *OPQ* is the increment in area produced by the motion of the radius vector,

$$OPQ = A(t + \varepsilon) - A(t) = \varepsilon A'(t) \tag{3.14}$$

so that (3.13) and (3.14) give, by Microcancellation,

$$A'(t) = {}^1/_2 r^2\theta' \tag{3.15}$$

Expressing A' as a function of x, x', y, y', Bell then proves that $A''(t) = 0$, so that, assuming $A(0) = 0$, we have the Area Law

$$A(t) = kt, \tag{3.16}$$

where k is a constant.

As a historical note, we may remark that Bell's procedure is reminiscent of that pioneered by the seventeenth century Dutch mathematician Bernard Nieuwentijt.[23] In his treatment (which was independent of Newton's and Leibniz's) Nieuwentijt began by defining infinitesimal and infinite quantities: "A quantity smaller than any given one may be called, for the sake of abbreviation, infinitesimal; greater than any given one, infinite." He then laid down a number of axioms. The first is that "That which cannot be multiplied so many times, that is, by such a great number, that it becomes equal in magnitude to any given [finite] quantity, however small, is not a quantity, but a mere nothing in geometry." (Nieuwentijt 1695, 2: Axiom 1). Axiom 2 states that any arbitrary finite quantity can be divided into arbitrarily many equal or unequal parts less than any given quantity, so that the division of a finite quantity b by an infinite number m yields an infinitesimal quantity. (This is in accord with Axiom 1, since b/m may be multiplied by the product of the infinite number m and the finite number c/b so that it does equal any other finite quantity c.) But it now follows (Nieuwentijt 1695, 4: Lemma 10) that if two infinitesimal quantities b/m and c/m are multiplied together, their product bc/mm is zero. For when multiplied by the largest possible number m, the product bc/m is still infinitesimal, and therefore cannot be made equal to any finite quantity; by Axiom 1, the product of any two infinitesimal quantities is therefore zero. Nieuwentijt's infinitesimals are nil-square infinitesimals. Nieuwentijt was proud of this result, regarding it as the "Ariadne's thread" that would provide an exit from the labyrinth of the continuum by establishing a strict criterion for which infinitesimal quantities could indeed be neglected in any calculation (Nieuwentijt 1995, 81).

Notwithstanding this agreement on nilsquare infinitesimals, there are, of course, many profound differences between SIA and Nieuwentijt's approach. For SIA, like Leibniz's approach, is based on smoothly varying geometric quantities, whereas Nieuwentijt's

[23] See the lucid account of Nieuwentijt's theory in Mancosu 1996, 158ff, and also Nagel 2008, 200-205, for a succinct account of his dispute with Leibniz.

infinitesimals are defined through division by an infinite number *m*, where *m* is the largest number. They are therefore invertible, unlike the infinitesimals of SIA; they are also actual infinitesimals by Nieuwentijt's definition, quantities smaller than any finite quantity, the inverses of categorematically infinite quantities. Moreover, SIA presupposes intuitionistic logic, whereas Nieuwentijt's logic is classical. Finally, Nieuwentijt's theory does not admit higher order differentials, whereas, as we shall see, this is possible in SIA. But the key points it has in common with Nieuwentijt's approach are these: the Principle of Microaffineness, which "entails that all curves are 'locally straight'" (110); and the Nilsquare Property, which guarantees that all squares and higher powers of an infinitesimal are intrinsically zero, and not just by comparison with other quantities.[24]

4. Propositions 4 and 6 of Newton's *Principia*

The comparison between Leibniz's syncategorematic approach and that of SIA can be set in relief by applying both to the same examples. Because of its historical importance, and also because it gave both Leibniz and Varignon some measure of grief[25] and therefore presents itself as an excellent touchstone, I shall consider Newton's Proposition 6 of Book 1 of the *Principia*, the theorem that is the basis of his derivation in Proposition 11 of the inverse square law of attraction due to gravity for a body orbiting in an ellipse. I will briefly present Newton's own proof of Proposition 6 with some commentary. Then I will proceed to a sketch of Leibniz's proof of the inverse square law for the ellipse, in which he avails himself of Newton's Proposition 1 (the area law), and its composition of motions along the chords, to give an algebraic proof using his method of differentials. But first, to motivate this, I will follow Nico Bertoloni Meli in giving an analysis of Newton's Proposition 4 according to the two

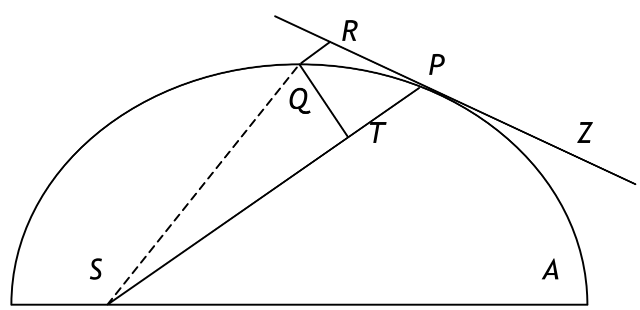
[24] Mancosu has very aptly summarized the difference between Nieuwentijt's approach and Leibniz's in his (1996, 160). One of these differences is that on Nieuwentijt's theory, the infinitesimal *b/m* cannot be eliminated from computations; but Bell's Principle of Microcancellation shows how to circumvent this problem. Bell has also provided a useful comparison of the difference between Nieuwentijt's approach and Leibniz's in his (2006, 98-100).

[25] On Varignon and Leibniz on central force see Bertoloni Meli (1993, 81-83, 201ff.).

different compositions of motion: Newton’s composition of a rectilinear uniform motion along the tangent with a uniform acceleration towards the centre, and Leibniz’s composition of a rectilinear uniform motion along the chord with a rectilinear uniform motion towards the centre (Bertoloni Meli 1993, 82-83).

It seems appropriate to consider Newton’s treatment in the first edition of the *Principia* (published in 1687), since this is the one that Leibniz’s critical comments were directed towards.[26] There he states Proposition 6 as follows:

Figure 6: Newton’s Proposition 6



> If a body *P*, revolving about a centre *S*, describes any curved line *APQ*, while the straight line *ZPR* touches that curve at any point *P*; and to the tangent from any other point *Q* of the curve, *QR* is drawn parallel to the distance *SP*, and the perpendicular *QT* is dropped onto the distance *SP*: I say that the centripetal force is as the reciprocal of the solid $SP^2 \bullet QT^2/QR$, provided that the quantity of this solid is always taken as that which is made when the points *P* and *Q* come together.

In the first edition[27] Newton gives the following proof:

[26] For the second edition (published in 1713) Newton composed a new Proposition VI, which remains the same in the third (1726), and the first edition text became Corollary 1. Although the first edition can be pieced together from the footnotes in Newton 1999, Part I is usefully presented in its entirety (in a different translation by Mary Ann Rossi) by J. Bruce Brackenridge (1995, 235-267).
[27] This proof is replaced by a new proof (of Corollary 1 to the new proposition 6) in the 2nd and 3rd editions, where *QR* is said to be “equal to the sagitta of an arc that is twice the length of an arc *QP*,

> For in the indefinitely small figure *QRPT* the nascent linelet *QR* is, given the time, as the centripetal force (by Law 2), and given the force, as the square of the time (by Lem. 10), and thus, if neither be given, as the centripetal force and the square of the time jointly, and thus the centripetal force is as the linelet *QR* directly and the square of the time inversely. But the time is as the area *SPQ*, or its double $SP \times QT$, that is, as *SP* and *QT* jointly, and thus the centripetal force is as *QR* directly and $SP^2 \times QT^2$ inversely, that is, as $SP^2 \bullet QT^2/QR$ inversely. QED. (Newton 1999, 454)

In Proposition 11 Newton went on to use this formula to derive an expression for the centripetal force holding a body in an elliptical orbit. If the minor and major axes of the ellipse are respectively b (= 2*BC*) and q (=2*AC*), then the *latus rectum* of the ellipse is $L = 2BC^2/AC = b^2/q$. By adroitly using the properties of the ellipse and the lemmas he had established for proportions in the limit motions, Newton is able to prove that $SP^2 \cdot QT^2/QR = L \cdot SP^2$. Therefore from Proposition 6, the centripetal force is the reciprocal of this, and thus as the inverse square of the altitude, *SP*.

Newton's proof of Proposition 6 involves a composition of a uniform motion with a uniformly accelerated motion, which Leibniz could not accept, as mentioned in section 2. He interprets Newton's "force" as cause of an increment in quantity of motion, so that it should be proportional to velocity, not to acceleration.[28] Newton's proof also appeals to Lemma 10, whose validity Leibniz initially doubted. Thus in his marginalia to Newton's *Principia* Leibniz altered Newton's statement in the proof of proposition 6, "the nascent linelet *QR* is, given the time, as the centripetal force (by Law 2), and given the force, as the square of the time

with *P* being in the middle; and twice the triangle *SQP* (or *SP* x *QT*) is proportional to the time in which twice that arc is described and therefore can stand for the time." (453-54). Nonetheless, Newton does write after the new proof of Prop. 6: "The proposition is easily proved by lem. 10, coroll. 4.".

[28] Bertoloni Meli notes how, in his notes on Corollary 1 to the First Law, Leibniz objected to Newton's use of "force" in stating it—writing after "… *vi*", "(*malo dicere conatu*)"— meaning that *vis* should be replaced by *conatus*, that is (directed) infinitesimal tendency to motion. See Bertoloni Meli (1993, 224, 239, 163).

(by Lem. 10)", replacing both instances of the word 'force' by 'velocity', and 'as the square of the time' simply by 'as the time'.[29]

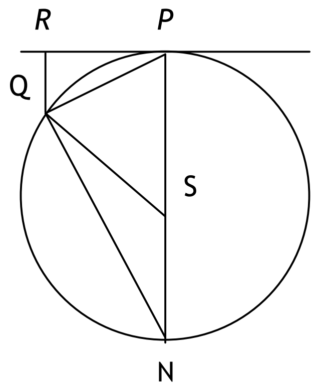
Subsequently, as we noted in Section 2 above, Leibniz came to see that Newton's way of applying his Lemmas to limit-motions involved no mathematical error. In fact, as Meli explains, the two ways of analyzing the composition of motion, Newton's composition of a rectilinear uniform motion along the tangent with a uniform acceleration towards the centre, and Leibniz's composition of a rectilinear uniform motion along the chord with a rectilinear uniform motion towards the centre, generally lead to the same mathematical result. But Leibniz still preferred a composition of two uniform motions, rejecting the physicality of Newton's instantaneous acceleration.[30] It will be worthwhile to further our understanding of Leibniz's reasons for this before proceeding to his derivation of the centripetal force law for the ellipse, by examining proofs of the centripetal force law for circular motion (Newton's Proposition 4) according to the two different compositions.

In his own proof of Proposition 4 in the first edition using a tangent composition, Newton draws two homologous and concentric circles, so that the centripetal forces are "in the first ratio of the nascent spaces *CD* and *cd*", where *CD* and *cd* are the deflections of the two revolving bodies. (The second homologous figure was necessary, since in the mathematics of his day only ratios of quantities with the same dimensions could be taken.) We will dispense with that here, and follow Meli (1993, 82-83) in giving a modernized proof of Proposition 4. For a body orbiting S in a circular arc from *P* to *Q* we have the diagram in Figure 7 below:

[29] See Meli 1993, 162.

[30] "La voye est plus simple," Leibniz wrote to Varignon in October 1706, "qui ne met pas l'acceleration dans les elemens, lorsqu'on n'en a point besoin. Je m'en suis servi depuis de 30 ans." (GM IV 150-151; Bertoloni Meli, 1993, 81).

Figure 7: Tangent Composition



Here the centripetal force is as the deflection *QR*. From the figure, with the diameter of the circle $NP = 2SP$, $\angle NQP$ is a right angle, so that $QR:PQ = PQ:NP = PQ:2SP$, giving

$$QR = PQ^2/2SP \tag{4.1}$$

Likewise, the first ratios of the velocities along the arcs of the two circles in equal times will be as the first ratio of the nascent arc *PQ* to its homologue, and therefore in the ratio of the nascent straight lines *PQ* and its homologue. Again dispensing with proportions, in a given moment dt, with v the velocity along *PQ*, $PQ = vdt$ and $SP = r$, so the centripetal force F is as

$$QR = PQ^2/2SP = {}^1/_2(v^2/r)\, dt^2 \tag{4.2}$$

In fact, however, the motion along *QR* is uniformly accelerated so that $QR = {}^1/_2 a\, dt^2$. Thus

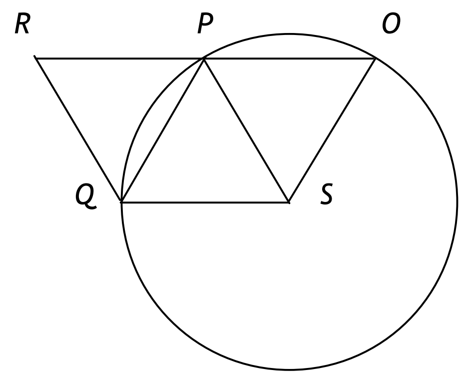
$$a = v^2/r \tag{4.3}$$

Leibniz objected to this, as he did not think there could be an acceleration in an instant. Rather, he conceived acceleration to occur by successive increments of motion, in accordance with the belief, typical of the mechanical philosophy, that all action is by contact. Thus while he had no objection to the composition of motions in Newton's Proposition 1, where the action of a force is represented by an increment of motion occurring in a moment, and the parallelogram law is applied to combine two uniform motions into a third, he could not accept a composition of a uniform inertial motion along the tangent *PR* with a uniformly accelerated motion toward the centre, along *QR*.

So he preferred the kind of composition used in Proposition 1, where the curved trajectory is conceived as the "ultimate perimeter" of a polygon as the number of sides is increased indefinitely and their width is decreased indefinitely. As is clear from the figure below, with *QR* and *SP* parallel and equal, $QR/PQ = PQ/SP$, so that

$$QR = PQ^2/SP. \tag{4.4}$$

Figure 8: Chord Composition



But *SP* is finite and *PQ* is first order infinitesimal; therefore *QR* must be second-order infinitesimal. Thus the velocity *w* of the body along *QR*, although constant because it represents a uniform motion, must be a first order differential *du*, so that

$$QR = w \bullet dt = du \bullet dt = d(dr/dt) \bullet dt = ddr \tag{4.5}$$

Meanwhile

$$PQ^2/SP = v^2 dt^2/r \tag{4.6}$$

Combining (4.4), (4.5) and (4.6) we have

$$ddr = v^2 dt^2/r \tag{4.7}$$

On a modern understanding, we would interpret this as a formula for acceleration:

$$d^2r/dt^2 = v^2/r \qquad (4.8)$$

This is not, however, how Leibniz would understand (4.7). For him it says that the solicitation in equal infinitesimal times dt is as v^2 and inversely as r. Moreover, as Leibniz explains in the *Tentamen de Motuum Coelestium Causis* published in the *Acta* of Leipzig in February 1689, "if motion is represented by a finite line which is traversed by a body in a given time, the impetus or velocity is expressed by an infinitely small line, and the element of velocity, which is solicitation of gravity or centrifugal conatus, by a line infinitely many times infinitely small".[31] That is to say, on Leibniz's way of thinking, a body only has motion in a finite time. The instantaneous velocity v is represented as a line-element dr produced in an element of time dt whose ratio dr:dt is equal to the ratio of two finite lines described in the same finite time. Solicitation, on the other hand, is an infinitesimal element of velocity in an element of time assumed constant, not an instantaneous rate of change of velocity. It is therefore not to be confused with Newton's acceleration. Thus in a time dt, velocity v is as dr, and solicitation is as dv or ddr. As Leibniz tells Burcher de Volder, "solicitations are as dx, velocities are as x [$= \int dx$], and forces as xx or $\int x dx$".[32] We shall return to this point later.

Now we are in a position to examine Leibniz's published solution in the *Tentamen* for the centripetal force toward the centre of a body orbiting that centre in an elliptical orbit. Again, this force for him will be a solicitation, a "dead force", and as such will be an element of a conatus proportional to dv and thus ddr. So as in the case of the circle above, he needs to find an expression for ddr based on the properties of the geometric figure, in this case the ellipse. Following Newton's lead, he uses the properties of the ellipse. But he will use them to set up a differential equation—indeed, the very first differential equation ever published—

[31] *Tentamen de Motuum Coelestium Causis*, Leibniz's major work on cosmology, in which he re-derives Newton's results in accordance with his own physico-mathematical principles, was published in the *Acta* of Leipzig in February 1689; I follow the English translation given by Bertoloni Meli, *An Essay on the Causes of Celestial Motions* (1993, 126-142; 131).

[32] Leibniz to De Volder, GP II 154, 156; quoted by Bertolini Meli 1993, 88.

rather than proceeding in the classical manner with ratios of quantities. Thus, for an ellipse with distance from one focus r, minor axis $BE = e$, major axis $A\Omega = q$, eccentricity e, and latus rectum $XW = a$, we have the relations

$$e^2 = q^2 - b^2 \tag{4.9}$$

$$b^2 = aq \tag{4.10}$$

To simplify, Leibniz makes the substitution

$$p = 2r - q \tag{4.11}$$

and obtains from an analysis of the geometry of the figure the proportion

$$(a\theta/r):dr :: b:\sqrt{(e^2 - p^2)} \tag{4.12}$$

where $a\theta$ is twice the area swept out by the radius in the element of time $\theta = dt$, with the constant a arbitrarily set equal to the latus rectum.[33] Leibniz rewrites (4.12) as

$$br\ dr = a\theta \sqrt{(e^2 - p^2)} \tag{4.13}$$

"which is the *differential equation*" (Meli, 137). Using the fact that with $p = 2r - q$, we have $dp = 2dr$, so that

$$d\sqrt{(e^2 - p^2)} = -2pdp/2\sqrt{(e^2 - p^2)} = -2pdr/\sqrt{(e^2 - p^2)},$$

and remembering that $a\theta$ is constant, Leibniz differentiates (4.13), to obtain:

$$b\ dr\ dr + br\ ddr = -2pa\theta\ dr\ /\sqrt{(e^2 - p^2)} \tag{4.14}$$

"Eliminating *dr* from these two equations [4.13 and 4.14]" (Meli, 137), Leibniz obtains:

$$b\ a^2\theta^2(e^2 - p^2)/b^2r^2 + br\ ddr = -2p\ a^2\theta^2/br \tag{4.15}$$

$$\Rightarrow \qquad ddr = a^2\theta^2(-e^2 + p^2 - 2pr)/b^2r^3 \tag{4.16}$$

[33] Here Leibniz makes a trivial slip, as pointed out by Bertoloni Meli (1993, 118). In order for $a\theta$ to be twice the area of the elementary triangle, a needs to be half the latus rectum. Thus $XW = 2a$, and $b^2 = 2aq$, introducing an error of a factor of 2 in the second term of equation (4.19).

Now, using the relations (4.9) $e^2 = q^2 - b^2$ and (4.11) $p = 2r - q$, we can calculate that

$$-e^2 + p^2 - 2pr = b^2 - 2qr, \tag{4.17}$$

thus giving

$$ddr = a^2\theta^2(b^2 - 2qr)/b^2r^3 \tag{4.18}$$

or, since by (4.10) $b^2 = aq$,

$$ddr = a^2\theta^2/r^3 - 2a\theta^2/r^2 \tag{4.19}$$

Here Leibniz identifies the first term as twice the centrifugal conatus, and the second as the solicitation due to gravity, in agreement with Newton's inverse square law. Taking into account Leibniz's slip in not taking a to be the *semi*-latus rectum (see footnote 37), so that b^2 should in fact be $2aq$, the result should be

$$ddr = a^2\theta^2/r^3 - a\theta^2/r^2 \tag{4.20}$$

This derivation can be compared with a modern derivation using differentials as follows. In spherical co-ordinates (r, α) the equation of the ellipse is

$$r = a/(1 + e\cdot\cos\alpha) \tag{4.21}$$

where a is the semi-latus rectum and e the eccentricity. Differentiating and rearranging, we obtain

$$dr = ea\cdot\sin\alpha\cdot d\alpha/(1 + e\cos\alpha)^2$$

$$= er^2\cdot\sin\alpha\cdot d\alpha/a \tag{4.22}$$

Now by Kepler's Law (cf. (3.15) and (3.16)), we have $r^2 d\alpha = adt$, giving

$$dr = e\cdot\sin\alpha\cdot dt$$

$$\Rightarrow \quad d^2r = e\cdot\cos\alpha\cdot d\alpha\cdot dt = ae\cdot\cos\alpha\cdot dt^2/r^2 \tag{4.23}$$

But by (4.20), $e \cdot \cos\alpha = a/r - 1$, giving

$$d^2r = (a^2/r^3 - a/r^2) \cdot dt^2 \quad (4.24)$$

agreeing with Leibniz's result (4.19), as corrected in (4.20).

Eschewing differentials and using only derivatives, an equivalent result can be derived as

$$r'' = (a^2/r^3 - a/r^2) \quad (4.25)$$

5. Second-order Differentials

This addresses the mathematical physics. But a crucial question remains: is Leibniz entitled to his second-order differentials, given his understanding of first order differentials as fictions? And can SIA achieve a similar result? In this section I shall first consider whether second-order differentials are definable on the syncategorematic interpretation of infinitesimals, and then whether they are definable in SIA. Then I will proceed to determine whether Leibniz's calculations above are justifiable on such a syncategorematic interpretation, and whether a similar calculation might be achieved using SIA. These considerations will motivate a revised treatment of differentials in SIA, as well as lead to some reflections on the relationship between Leibniz's infinitesimals and the infinitesimals of SIA.

The question of Leibniz's entitlement to second-order differentials has been subjected to a lucid examination by Henk Bos in his (1974-75). As Bos demonstrates, Leibniz's own attempt to justify rules involving second-order differentials in *Cum prodiisset* is compromised by his desire to produce differential equations that do not depend on the "progression of the variables" chosen, i.e. on the specification of an independent variable. In setting up his general scenario, Leibniz lets the variables dx and ddx vary independently, the only condition being that when $dx \rightarrow 0$, also $ddx \rightarrow 0$. But as Bos explains, he also needs ddx to be infinitely small with respect to dx, i.e. for $ddx/dx \rightarrow 0$ as $dx \rightarrow 0$. The condition that guarantees this in general is that dx be constant, so that $ddx = 0$.

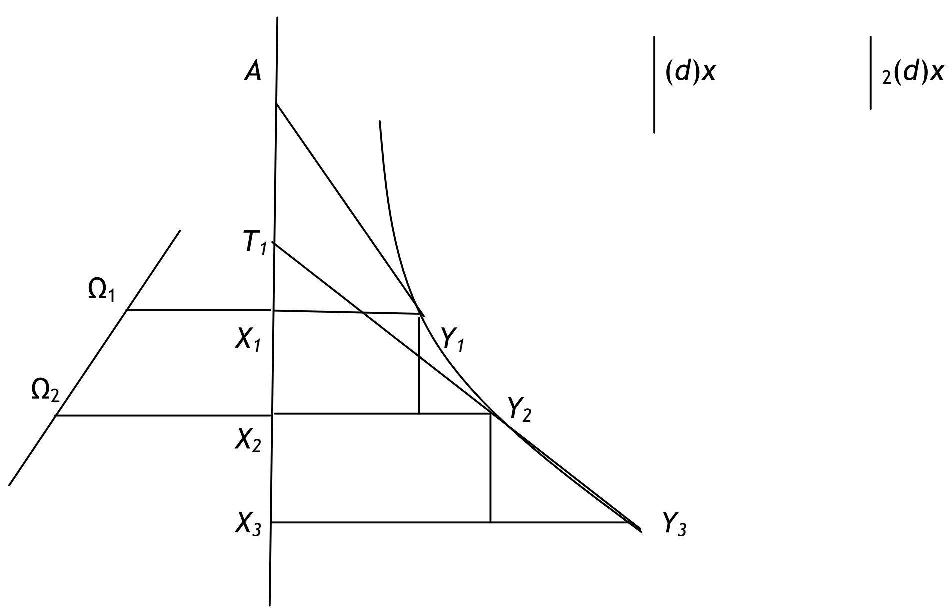
As Bos points out, Leibniz's assumption that dx and ddx vary independently is visible in his attempt to justify the rules for second-order differentials by assuming two constant lines of different lengths, $(d)x$ and $(dd)x$ —which he also writes ${}_2(d)x$— and two variable lines $(d)y$ and $(dd)y$, which are such that

$$(d)y:(d)x = dy:dx \quad \text{as before, and} \tag{5.1}$$

$$(dd)y:(dd)x = (dy + ddy):(dx + ddx) \tag{5.2}$$

He then assumed that in the limit as $dx \to 0$, $(dd)y = (d)y$. As Bos observes, however, this is the case only if in that limit $(dd)x = (d)x$. This is equivalent to assuming x as independent variable, i.e. an arithmetic progression in x, so that dx is constant and $ddx = 0$. On this assumption, however, as Bos goes on to show, Leibniz's formula may be successfully derived along the lines he proposed, consistently with the syncategorematic interpretation described above.

Figure 9: Second-order Differentials[34]



[34] This is a slightly simplified version of Leibniz's figure, with the abscissas drawn vertically as was his custom, and with his ${}_1X$, ${}_1Y$, etc., written as X_1, Y_1, etc., as before.

Here $dx = X_2 - X_1$, $ddx = X_3 - X_2$, $y = X_1Y_1$, $dy = X_2Y_2 - X_1Y_1$, and $ddy = X_3Y_3 - X_2Y_2$. In his construction Leibniz takes the varying line $(d)y$ defined by (5.1) to be equal to a second varying line $X_1\omega_1$, in such a way that, in the limit where Y_1 and Y_2 coincide, when it becomes $X_1\Omega_1$, it bears to the fixed line $(d)x$ the ratio of the ordinate X_1Y_1 to the subtangent X_1T. This accords with his definition (2.2) of $dy{:}dx$ as being equal in the limit to $(d)y{:}(d)x = y{:}\sigma$. "Since this can be done wherever the point Y_1 is assumed on the curve," he writes, "in this way a curve ΩΩ will be produced which is the differentrix of the curve *YY*" (Leibniz 1846, 49). A similar construction is then given for ${}_2(d)y$ in terms of ${}_2(d)x$ in terms of a second differentrix. Leibniz uses the same approach in his published reply to Nieuwentijt in 1695, noting that a construction of such a differentrix is always possible.[35]

The particular formula that Leibniz then proceeds to demonstrate in *Cum prodiisset* is that for the second-order differential *ddy* given that $ay = xv$, where a is a constant to preserve the homogeneity of the equation. For finite differences we have

$$a(y + dy) = (x + dx)(v + dv) \tag{5.3}$$

and

$$a(y + 2dy + ddy) = (x + 2dx + ddx)(v + 2dv + ddv) \tag{5.4}$$

giving

$$ady = xdv + vdx + dxdv \tag{5.5}$$

and, after some elementary algebraic manipulation,

$$a\cdot ddy = x\cdot ddv + v\cdot ddx + 2\,dv\cdot dx + 2dv\cdot ddx + 2dx\cdot ddv + ddx\cdot ddv \tag{5.6}$$

Since dx is constant, and $ddx = 0$, we may drop the terms in ddx. Dividing by dx^2 gives

[35] See Leibniz's reply to Nieuwentijt in his (1695), and Bos's lucid account: "Obviously, the procedure can be repeated again, by which LEIBNIZ has shown that finite line-variables can be given proportional to differentials of any order" (Bos 1974-75, 65).

$$a \cdot ddy/dx^2 = x \cdot ddv/dx^2 + 2\ dv/dx + 2dx \cdot ddv/dx^2 \qquad (5.7)$$

This is still an equation in variable quantities that may be made as small as desired. But in order to "transpose the case as far as possible to quantities that never vanish", we need to define $(d)v$ analogously to the definition of $(d)y$ in (5.1):

$$(d)v:(d)x = dv:dx \qquad (5.8)$$

In place of Leibniz's definition (5.2) and an analogous one for $(dd)v$, Bos lets

$$(dd)y:[(d)\mathrm{x}]^2 = ddy:dx^2 \qquad (5.9)$$

$$(dd)v:[(d)\mathrm{x}]^2 = ddv:dx^2 \qquad (5.10)$$

Assume further that these ratios all have an interpretation when $dx = 0$. Then each of (5.8), (5.9) and (5.10) can be substituted for its counterpart involving dx in (5.7), yielding

$$a(dd)y/(d)x^2 = x \cdot (dd)v/(d)x^2 + 2(d)v/(d)x + 2dx \cdot (dd)v/(d)x^2 \qquad (5.11)$$

By hypothesis, this formula remains interpretable when $dx = 0$, when the last term vanishes. So the Law of Continuity asserts that this limiting case may also be included in the general reasoning: dv/dx can be substituted for $(d)v/(d)x$ etc. in the resulting formulas even for the case where $dx = 0$, with dy, dx, dv, ddy, in this case interpreted as fictions. Therefore

$$a\ ddy/dx^2 = x \cdot ddv/dx^2 + 2\ dv/dx \qquad (5.12)$$

or

$$a\ ddy = x \cdot ddv + 2\ dv \cdot dx \qquad \text{QED.} \qquad (5.13)$$

This is the formula that Leibniz was aiming to derive, achieved by Bos (1974-75, 61-62) using Leibniz's method together with his own creative emendations. Although it is only a particular result, the point is that the assumptions it depends upon are quite general: an analogous construction is possible for any variable y that is differentiable in the neighbourhood of the point in question, i.e. is a smooth function of x. The particular result comes with the caveat that it is only valid for *x* as independent variable, and is not a general formula relating the second-order differentials *ddy* and *ddv*. This can be made more perspicuous by adopting the notation used by Hermann Grassmann in his *Ausdehnungslehre* of 1862 (2000, 249ff), in which (5.13) would be written

$$a\, d_x^2 y = x \cdot d_x^2 v + 2\, d_x v \cdot dx \tag{5.14}$$

Bos is of the opinion that the need to specify second derivatives with respect to an independent variable "needlessly restricts the generality of the differential calculus, as it imposes the choice of a special progression of the variables" (66). But by his own arguments, on the one hand second-order differentials are not well defined without this restriction; and, on the other, second-order differentials may in any case often be circumvented. As Bos explains, Leibniz showed, in his treatment of the evolute as envelope of the family of the normals to a curve, how higher order differentials can be eliminated by calculating the differential equation of the curve, yielding formulas for the radius of curvature that are independent of the progression of the variables (Bos 1974-75, 41). Bos suggests that the fact that this expedient, as well as Leibniz's own attempts to found the calculus (by what I have called his "finite surrogate" methods), led him to introduce differential quotients, defined as ratios of finite quantities that remain finite in the limit as $dx \to 0$, took him a long way towards the modern definitions of function and derivative of a function (66). This is perfectly true; but it is no criticism of Leibniz's founding of the calculus in terms of infinitesimals when these are understood as fictions under those very conditions: not as actually infinitely small quantities, but as finite quantities that may be made as small as desired, and which may be treated in the limit as if they are infinitely small. Thus the equation Leibniz derives at the end of *Cum prodiisset* relating second-order differentials "will hold when the points X_1 and X_2 coincide", he writes, "if, by a certain fiction, we suppose *dx, ddx, dv, ddv, dy*, and *ddy* to remain even when evanescent, as if they are infinitely small quantities (and there is no danger in this, since the matter can always be referred back to assignable quantities)" (Leibniz 1846, 50).

This establishes Leibniz's right to use second-order differentials under that fiction, provided an independent variable is identified.

Now let us turn to the question of higher order differentials in Smooth Infinitesimal Analysis. I should emphasize at the outset that SIA does not actually need them; it is perfectly able to derive a formula such as (5.12) under the assumption that *y* and *v* are functions of *x* as an independent variable, with ddy/dx^2 understood as the second derivative of *y* with respect to *x*, and similarly with ddv/dx^2 and dv/dx.

Thus if $ay = xv$, with *a* constant, by the product rule (3.11), $y'(x) = f'(x)g(x) + g'(x)f(x)$,

$$ay'(x) = xv'(x) + v(x) \tag{5.15}$$

Applying the same rule to take the derivative of this equation, we have

$$ay''(x) = xv''(x) + 2v'(x) \tag{5.16}$$

which is the analogue of (5.12) in derivative form. By the same token, SIA is easily able to derive (4.25), $r'' = (a^2/r^3 - a/r^2)$, the formula for acceleration in an elliptical orbit.

Those calculations, however, involve derivatives, not differentials. The question we are treating is whether Leibniz's infinitesimals are successfully modelled by the infinitesimals of SIA, and his derivations depend on the use of second-order differentials. Are these definable in SIA? Indeed they are, and I am indebted to John Bell for providing me with the following account of how this should be done.[36]

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[36] Bell's definition of second-order differentials here differs from that given in his (1998, 90-91), where they are defined as nilcube infinitesimals, ones such that $\varepsilon^3 = 0$. But as I pointed out in an earlier version of this paper, this makes the decision of what type of infinitesimal to adopt (nilsquare or nilcube, etc.) depend on their applicability to the problem at hand; moreover, all the principles of SIA that we have depended upon above depend critically on the nilsquare property, which fails if the

The basic idea is that, for a map h: $\mathbf{R} \to \mathbf{R}$, the n^{th} order differential $d^n h$ should be taken to be the "n^{th} order infinitesimal part" of the difference in the value of $h(x)$ when x is replaced by $x + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$, with $\varepsilon_1, \ldots, \varepsilon_n \in \Delta$. This difference is

$$h(x + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n) - h(x)$$

$$= (\varepsilon_1 + \ldots + \varepsilon_n)h'(x) + (\varepsilon_1\varepsilon_2 + \ldots + \varepsilon_{n-1}\varepsilon_n)h''(x) + \ldots + \varepsilon_1\ldots\varepsilon_n\, h^{(n)}(x). \qquad (5.17)$$

The "n^{th} order infinitesimal part" of the difference is the last term on the r.h.s., namely $\varepsilon_1\ldots\varepsilon_n\, h^{(n)}(x)$. Accordingly, for $n \geq 1$ we define

$$d^n h\colon \mathbf{R} \times \mathbf{\Delta}^n \to \Delta$$

by

$$d^n h\,(x, \varepsilon_1, \ldots, \varepsilon_n) = \varepsilon_1\ldots\varepsilon_n\, h^{(n)}(x). \qquad (5.18)$$

For example

$$dh(x, \varepsilon) = \varepsilon h'(x), \qquad d^2h(x, \varepsilon, \eta) = \varepsilon\cdot\eta\cdot h''(x) \qquad (5.19)$$

Using this definition, we may derive the formula for the second derivative of a product. Given f, g: $\mathbf{R} \to \mathbf{R}$, let $h = f.g$. Then for $\varepsilon, \eta \in \Delta$, we have

$$d^2h(x, \varepsilon, \eta) = \varepsilon\eta(fg)'' = \varepsilon\eta(f'g + fg')' = \varepsilon\eta(f''g + 2f'g' + fg'')$$

$$= \varepsilon\eta f''g + 2\varepsilon f'\eta g' + \varepsilon\eta fg''$$

$$= g\cdot d^2f + 2df\cdot dg + f\cdot d^2g \qquad (5.20)$$

From this it is easy to derive Leibniz's formula (5.13): if $ay = xv$, then $a\cdot ddy = x\cdot ddv + 2\, dv\cdot dx$. For in this case $f(x) = x$, $g(x) = v$, so that $f' = 1$, $f'' = 0$, $df = dx = \text{const.}$ and $d^2f = 0$, giving

$$a\cdot d^2y = d^2(xv) = xd^2v + 2dx\cdot dv \qquad (5.21)$$

infinitesimals are nilcube. Bell has generously offered the present account (private communication) to circumvent such difficulties.

Now let us proceed to the question of whether the formulas Leibniz derived for orbiting bodies, such as (4.7) and (4.20), are in accordance with these foundations. Taking Leibniz's infinitesimals first, we have seen that second-order differentials are well defined under his syncategorematic interpretation, provided an independent variable is identified. That is the case here. We are assuming, as both Leibniz and Newton do, that time may be taken as independent variable, so that *dt* is constant, and *ddt* = 0. Under this assumption one may derive legitimate formulas involving second-order differentials such as *ddr*, like those derived by Leibniz above. In those formulas, *ddr* implicitly depends on a choice of *t* as the independent variable. But in such a case, with $dr = r'dt$, we have

$$\begin{aligned} ddr &= dr'dt + r'ddt \\ &= dr'dt \quad (\text{since } ddt = 0) \\ &= d/dt(r')\cdot(dt)^2 = r''\cdot(dt)^2 \end{aligned} \tag{5.22}$$

For example, in Leibniz's chord composition for centrifugal solicitation in a circular motion, the solicitation *ddr* is represented by a line element which is traced out with a uniform velocity *w* in the time $\theta = dt$. Thus the velocity *w* is a linear function of the (infinitesimal) time, and we may set

$$w = f_0 + f_1\theta \tag{5.23}$$

where f_0 and f_1 are functions of *r* and the transverse velocity *v*. We may justify this analogously to the Leibnizian justification of Newton's Lemma 9 we gave above. First we expand *w* in terms of θ:

$$w = f_0 + f_1\theta + f_2\theta^2 + f_3\theta^3 + f_4\theta^4 + \ldots \tag{5.24}$$

But since $w = 0$ when $\theta = 0$, it follows that $f_0 = 0$. Now since $QR = w\cdot dt = w\theta$, we have

$$ddr = w\theta = f_1\theta^2 + f_2\theta^3 + f_3\theta^4 + f_4\theta^5 + \ldots \tag{5.25}$$

$$\Rightarrow \qquad ddr/\theta^2 = f_1 + f_2\theta + f_3\theta^2 + f_4\theta^3 + \ldots \qquad (5.26)$$

Now in the case of the circle, $QR = PQ^2/SP$, and since equal areas of the elementary triangles *PQS* are described in equal times, we have $PQ \cdot SP = a\theta$, a being a constant. Therefore

$$ddr = QR = PQ^2/SP = (a\theta)^2/SP^3 \qquad (5.27)$$

$$\Rightarrow \qquad ddr/\theta^2 = a^2/SP^3 \qquad (5.28)$$

But since a^2/SP^3 is finite and non-vanishing when $\theta \rightarrow 0$, it follows by comparison with equation (5.26), that f_2, f_3, f_4, … are all 0. Therefore

$$ddr = f_1\theta^2 \text{, with } w = f_1\theta \qquad (5.29)$$

Turning now to the problem of reproducing these results using the infinitesimals of Smooth Infinitesimal Analysis, we are immediately confronted with the difficulty that the basic principles of SIA as expounded so far, as embodied in the Principle of Microuniformity, will not countenance time variation of geometric quantities across an infinitesimal interval. As Whiteside (1966) has shown, if we want the infinitesimal elements of the curve to be rectilinear, as they are by the Principle of Microstraightness, then they must be second-order infinitesimals, in contradiction to Microuniformity. Relatedly, if we try to duplicate the Leibnizian calculation above using nilsquare infinitesimals, this means that any second-order differential, such as Leibniz's *ddr* representing solicitation, is identically zero. This can be seen as follows. If the moment *dt* is taken to be a nilsquare infinitesimal, then $PQ^2 = (vdt)^2 = v^2dt^2 = 0$. Thus

$$QR = ddr = PQ^2/SP = 0 \qquad (5.30)$$

That is, to use Leibniz's terms, there can be no solicitation along *QR*. In Newton's terms, if the force at the very beginning of the interval is "given as the square of the time inversely", i.e. as $1/(SP \bullet QT)^2$ with $SP \bullet QT$ proportional to *dt*, then the acceleration in the moment *dt* is undefined, according to SIA. For the nascent triangle *SPQ* has an area $dA = {}^1/_2 SP \bullet QT = k dt$, so $(SP \bullet QT)^2 = 4dA^2 = 4k^2 dt^2 = 0$. In sum, because $\varepsilon^2 = 0$, we are unable to apply SIA to Newton's Propositions 4, 6 and 11.

Looking back to Bell's definition of second-order differentials and his derivation of (5.21)

$$a \cdot d^2 y = d^2(xv) = x d^2 v + 2dx \cdot dv$$

we see that if x is the independent variable, then $dv = v' \cdot dx$ and $d^2 v = v'' \cdot (dx)^2$, so that

$$a \cdot d^2 y = (x \cdot v'' + 2) \cdot (dx)^2 = 0 \qquad (5.31)$$

In other words, if x is the independent variable, then the infinitesimals ε and η of (5.19) must differ at most by a constant factor k, so that $\varepsilon \cdot \eta = k\varepsilon^2 = 0$, and $d^2 h(x, \varepsilon, \eta)$ is identically zero.

How does this situation bear on Leibniz's foundation for infinitesimals? For, as Bell observes (1998, 1-5), Leibniz's non-punctiform infinitesimals are in many respects analogous to those of SIA. The Leibnizian polygonal representation of curves is closely related to Bell's Principle of Microstraightness: in each case the curve is analyzed as compounded of infinitesimal rectilinear segments that are in a certain sense fictional parts. But whereas in SIA the "area deficit" is simply stipulated to be zero (Bell 1998, 8) —that is, it is shown to be of the order of dx^2, where x is the independent variable, and thus rigorously equal to zero— in Leibniz's justification of Riemannian integration the Area Deficit is shown to be zero in the limit by an application of the Archimedean Axiom without any assumption about the nilpotency of infinitesimals. In fact, using Leibniz's method it is possible —as we saw in the above proofs of Lemma 9 and Propositions 4, 6 and 11— to have the second-order differential *ddr* proportional to the square of a first order infinitesimal. Because $QR = QP^2/SP$, with QP first order infinitesimal and *SP* finite, *QR* must be second-order infinitesimal to preserve orders of infinity. This is impossible with nilsquare infinitesimals.

As we saw, Leibniz was able to derive the result (4.7),

$$du = ddr \propto v^2/r \tag{5.32}$$

for the centrifugal solicitation of a body revolving in a circular orbit in an arbitrary infinitesimal interval of time *dt*. Here *du* is an element of velocity, and a simple integration of it gives a velocity, i.e., on Leibniz's understanding, increases its order of infinity without changing its dimension. (In his calculations he tends to leave the dependence on the time interval tacit, as here with the dependence on dt^2.)[37] Similarly, two integrations of *ddr* raise it two orders of infinity, and yield a finite line *r*.

We would read Leibniz's expression, however, as giving us a second derivative with respect to time,

$$d^2r/dt^2 = v^2/r = du/dt \tag{5.33}$$

namely an acceleration, from which the velocity *u* is obtainable by an integration with respect to time, and the radius *r* by two such integrations. So from a modern perspective a non-zero solicitation in a moment or "timelet" *dt* is equivalent to an acceleration, contrary to Leibniz's understanding. But when Leibniz's second-order differentials are recognized to be functions of an independent variable (here, time), the corresponding adjustment to the understanding of differentials strips him of his grounds for resisting the composition of a uniform motion from an inertial motion and an accelerated one: what Leibniz conceived (in (4.5)) as an inertial motion along *QR* with a velocity $w = du$ in a time *dt*, is equivalent to a uniform acceleration du/dt. Thus in motion under a central force, Microstraightness implies the failure of Microuniformity.

Thus Leibniz is able to uphold the Principle of Microuniformity because of a different understanding of how physical quantities are integrated in the calculus; but a successful grounding of second-order differentials through the syncategorematic interpretation leads to

[37] See Bos (1974-75, 5-10, 12-35) and Bertoloni Meli (1993, 66-73) for clear expositions of the difference between Leibniz's understanding of differentiation and integration and the modern conception.

an abandonment of this position, and with it, the Principle of Microuniformity. SIA, on the other hand, is able to uphold Microuniformity, à la Nieuwentijt, only by equating the squares of all first-order infinitesimals to zero, and therewith rejecting the entire apparatus underlying Newton's calculation of central forces, from Lemma 9 through to Proposition 6 and its corollaries and applications —all of which involve time variation of quantities across an infinitesimal time, and quantities depending on the squares of nascent or evanescent areas which are, as in the Nieuwentijtian calculus, necessarily equal to zero.

5. Conclusion

Despite many points in common, we have seen that Leibniz's syncategorematic approach to infinitesimals and the theory of Smooth Infinitesimal Analysis are by no means equivalent. Indeed, the conceptions of infinitesimal quantities at the heart of each approach are radically diverse. According to John Bell, "The property of being a nilsquare infinitesimal is an *intrinsic* property, in no way dependent on comparisons with other magnitudes or numbers." (Bell, 1998, 2). Leibniz's syncategorematic infinitesimals, by contrast, are essentially comparative, and are defined by an implicit limiting process. The application of the Law of Continuity presupposes the existence of a limiting value, and also the smoothness of the co-varying quantities in the neighbourhood of this value. On this approach, the vanishing of infinitesimal quantities is always a comparative affair, and is grounded on a strictly Archimedean geometry. It remains to be seen, however, whether this Leibnizian syncategorematic approach can be set on a foundation as adequate to modern mathematical standards of rigour as is SIA.

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