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On Generalized Wielandt Subgroup

[1]Fazli Amin, [2]Asif Ali and [3]Muhammad Arif

[1]Department of Mathematics, Hazara University, Mansehra, Pakistan
[2]Department of Mathematics, Quaid-e-Azam University, Islamabad, Pakistan
[3]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan

Abstract: In this paper, we define a generalized Wielandt subgroup, local generalized Wielandt subgroup and its series for finite group and discuss its different basic properties which explain the notion of generalized Wielandt subgroup in a better way. We bound generalized Wielandt length as a function of nilpotency classes of its Sylow subgroups.

Key words: Wielandt subgroup . subnormal subgroup . sylow subgroup . soluble subgroup

INTRODUCTION

The Wielandt subgroup of a group G is denoted by $\omega(G)$, consists of those elements of G which normalize each subnormal subgroup of G. The Wielandt subgroup is the generalization of the idea of centre of a group. In [4] Helmut Wielandt, for whom the subgroup named, showed that for any minimal normal subgroup N of G satisfying minimal condition on subnormal subgroups, $\omega(G)$ contains N. Thus for a finite non-trivial group the Wielandt subgroup is always a non-trivial characteristic subgroup. In the same paper he define a series in G by setting $\omega_0(G) = 1$ and for $i \geq 1$

$$\omega_i(G)/\omega_{i-1}(G) = \omega(G/\omega_{i-1}(G))$$

The smallest n such that $\omega_n(G) = G$ is called the Wielandt length of G and denoted by $\omega l(G)$, consequently Wielandt length is well-defined. In [1] A. Ali bound Wielandt length as a function of numerical invariants of Sylow subgroups and the invariants for the Sylow subgroups he considered are Wielandt length and nilpotency class. The groups having Wielandt length one are called T-groups, those in which every subnormal subgroup is normal. The structure of T-groups is investigated by Gashutz [7], Zacher [3], Robinson [2] and Peng [6]. In particular finite soluble T-groups are completely characterized by Robinson [2]. In [5] Bryce and Cossey give the idea of local Wielandt subgroup, it is the normalizer of all p'-perfect subnormal subgroups of G and denotes it by $\omega^p(G)$ for some prime p. Also the references [1, 5] have influenced much of the current work, which provides a substantial generalization of these two papers.

In this paper we define a generalized Wielandt subgroup and its series for finite group, "it is the set of elements of a group G which normalize all subnormal subgroups of G which are contained in N is the generalized Wielandt of G with respect to N and we denote it by $\omega_N(G)$". It is clearly a normal subgroup of G but in general it may be different than N. For example $\omega_{A_3}(S_3) = S_3$. It is also obvious that $\omega(G) \subseteq \omega_N(G)$ and in particular $\omega(G) = \omega_N(G)$ if $N = G$ or $N = \omega(G)$ or N is the unique maximal normal subgroup. More (i) If M, N are normal subgroups of G such that $M \subset N$, then $\omega_N(G) \subset \omega_M(G)$. (ii) If M is a subgroup and N is a normal subgroup of a group G such that $N \subset M$, then $\omega_N(G) \cap M = \omega_N(M)$. (iii) If G is a T-group and N is a normal subgroup of G, then $\omega_N(G) = G$. However if M is normal in (ii) then $\omega_N(G)$ is normal in G. Also we see that the converse of (iii) is not true since if

$$G = D_8 = \langle x, y : x^8 = y^2 = (xy)^2 = 1 \rangle$$
$$N_1 = \langle x^2, x^4 \rangle \text{ and } N_2 = \langle x, x^2, x^4 \rangle$$

then $\omega_{N_1}(G)$ and $\omega_{N_2}(G) = G$ but G is not a T-group. Since $\omega(G) \subset \omega_N(G)$ so generalized Wielandt subgroup is non-trivial for any finite group G. We define the ascending generalized Wielandt series for G as follows. Write $\omega_{0,N}(G) = 1$, $\omega_{1,N}(G) = \omega_N(G)$ and for $i > 1$, define the subgroup $\omega_{i,N}(G)$ of G inductively by:

$$\omega_{N\omega_{i,N}(G)/\omega_{i,N}(G)}(G/\omega_{i,N}(G)) = \omega_{i+1,N}(G)/\omega_{i,N}(G)$$

Corresponding Author: Fazli Amin, Department of Mathematics, Hazara University Mansehra, Pakistan

The smallest n such that $\omega_{n,N}(G) = G$ is called the generalized Wielandt length of G with respect to N and we denote it by $\omega_N l(G)$. We observe that a number of results are more transparent if framed in terms of local generalized Wielandt subgroup, defined for each prime p as "let G be a finite group and N be a normal subgroup of G. The set of those elements of G which normalize all p'-perfect subnormal subgroups of G that are contained in N, is called the local generalized Wielandt subgroup of G with respect to N and we denote it by $\omega_N^p(G)$". It is important to note that the groups we deal with throughout in this paper are finite.

In §2 we discussed different basic properties of generalized Wielandt subgroup which explain the notion of generalized Wielandt subgroup in a better way, among other things we prove that, "if G is a group satisfying minimal condition on those subnormal subgroups which are contained in N, where N is normal subgroup of G, then $\omega_N(G)$ has finite index in G".

In §3 we use the technique which was developed in [1], for calculating the generalized Wielandt subgroup of the semi-direct product of two groups of co-prime order. In §4 we have shown the variation of local generalized Wielandt subgroup and establish a relation between the generalized Wielandt subgroup and local generalized Wielandt subgroup for a finite soluble group. In §5 we use the technique of §3 and the relation of §4 to prove among other things, that "if a normal Sylow p-subgroup of G has nilpotency class n > 1, a normal Sylow p-subgroup of $G/\omega_N(G)$ has nilpotency class at most n-1". Also we establish a relation between the generalized Wielandt length of a supersoluble and other invariants of its normal Sylow subgroups, we succeed to prove a general result that, "if G is a supersoluble group and n is the maximum of the nilpotency classes of the normal Sylow subgroups of G, then G has generalized Wielandt length at most n+1 for all n".

GENERALIZED WIELANDT SUBGROUP

In the following proposition we have establish a relation between N and generalized Wielandt subgroup with respect to N and investigate the condition under which N is contained in $\omega_N(G)$.

Proposition 2.1: Let G be a group and N be any normal subgroup of G, then $\omega_N(G) \cap N = \omega(N)$ and N is a T-group if and only if N is contained in $\omega_N(G)$.

Proof: Let n be any arbitrary element of $\omega(G)$ and S_1 be a subnormal subgroup of G contained in N, clearly S_1 is subnormal in N, so $(S_1)^n = S_1$ and so n belongs to $\omega_N(G)$, so $\omega(N) \subseteq \omega_N(G)$ and hence

$$\omega(N) \subseteq \omega_N(G) \cap N$$

Conversely let g be any arbitrary element of $\omega_N(G) \cap N$. Let S be a subnormal subgroup of N, so S is subnormal in G contained in N, therefore $S^x = S$ and so g belongs to $\omega(G)$, this means that $\omega_N(G) \cap N \subseteq \omega(N)$. Thus $\omega_N(G) \cap N = \omega(N)$. Now we will prove the second part of the proposition, for this let us suppose that N is a T-group and let n be any arbitrary element of N, let S be a subnormal subgroup of G contained in N, clearly S is subnormal in N and as N is a T-group therefore $S \triangleleft N$ so $S^n = S$ which implies that n belongs to $\omega_N(G)$ therefore $N \subseteq \omega_N(G)$. Conversely let us suppose that $N \subseteq \omega_N(G)$, we will show that N is a T-group, for this let S be a subnormal subgroup of N, since N is normal in G so S is subnormal in G contained in N therefore $S^g = S$ for all g belongs to $\omega_N(G)$ and hence $S^n = S$ for all n belongs to N therefore $S \triangleleft N$, thus the result follows

Theorem 2.2: If M and N are normal subgroups of G such that $(|M|,|N|) = 1$, then

$$\omega_{MN}(G) = \omega_M(G) \cap \omega_N(G)$$

Proof: Let h be any arbitrary element of $\omega_M(G) \cap \omega_N(G)$ and let S be any subnormal subgroup of G contained in MN as M, N are of coprime order so by lemma 2.1 of [1], we have $S = (S \cap M)(S \cap N)$. Now consider

$$S^g = \left((S \cap M)(S \cap N)\right)^g = (S \cap M)^g (S \cap N)^g$$
$$= (S \cap M)(S \cap N) = S$$

Therefore g belongs to $\omega_{MN}(G)$. Hence

$$\omega_M(G) \cap \omega_N(G) \le \omega_{MN}(G)$$

Conversely, since $M \subseteq MN$ and $N \subseteq MN$ so by property (i) we have

$$\omega_{MN}(G) \le \omega_M(G) \text{ and } \omega_{MN}(G) \le \omega_N(G) \text{ thus}$$

$$\omega_{MN}(G) \le \omega_M(G) \cap \omega_N(G)$$

Hence the result follows

The following is an immediate corollary of theorem 2.2.

Corollary 2.3: If G is nilpotent and each P_i's are the Sylow p-subgroups of G, then

$$\cap_{i \in I} \omega_{P_i}(G) = \omega(G)$$

Proposition 2.4: Let G be a group. Suppose $N_i;i\in I$ are all normal subgroups of G, then $\cap_{i\in I}\omega_{N_i}(G)=\omega(G)$.

Proof: First we show that $\cap_{i\in I}\omega_{N_i}(G)\subseteq\omega(G)$. For this, let g be any arbitrary element of $\cap_{i\in I}\omega_{N_i}(G)$ this implies that g belong to $\omega_{N_i}(G)$ for all $i\in I$, let S be an arbitrary subnormal subgroup of G. Then $S\subseteq N_{i_0}$ for some $i_0\in I$, so $S^g = S$ therefore g belongs to $\omega(G)$, hence $\cap_{i\in I}\omega_{N_i}(G)\subseteq\omega(G)$. Conversely, it is an immediate consequence of the definition that $\omega(G)\subseteq\omega_{N_i}(G)$ for all $i\in I$. Hence $\cap_{i\in I}\omega_{N_i}(G)=\omega(G)$. The next result shows that $\omega_N(G)$ is non- trivial in general.

Theorem 2.5: If N is a normal subgroup of G and A is a minimal normal subgroup of N then A normalizes every subnormal subgroup of G which is contained in N. That is A is contained in $\omega_N(G)$.

Proof: We proceed by induction on |N|. Let H is subnormal in G, such that $H\subseteq N$, $H\neq N$ and put $H_1 = H^N$, so $H_1\leq N$. If $A\nleq H_1$, then $A\cap H_1 = I$ and thus $[A,H_1] = I$. Hence $[A,H] = I$. On the other hand, if $A\leq H_1$, then $A = N_1^N$ for some minimal normal subgroup N_1 of H_1. Indeed each conjugate N_1^n, for $n\in N$, will be a minimal normal subgroup of H_1 and so will normalize H, by induction. Therefore A normalizes H and hence is contained in $\omega_N(G)$.

Theorem 2.6: Suppose that G is any arbitrary group. Then $\omega_N(G)$ contain every simple non-abelian subnormal subgroup of N, where N is a normal subgroup of G.

Proof: Suppose that H is simple non-abelian subnormal subgroup of N, as N is normal in G, this implies that H is subnormal in G, also let K be any subnormal subgroup of G, such that $K\leq N$, so K is subnormal in N. Now if $H\cap K\neq 1$ then since $H\cap K$ is subnormal in H and H is simple we must have $H\leq K$ and so H normalizes K trivially. If $H\cap K = 1$, then $[H,K] = 1$ and hence the result follows.

In groups which satisfy minimal condition on those subnormal subgroups which are contained in N, where N is normal in that group, then generalized Wielandt subgroup is larger than one might expect in the following sense.

Theorem 2.7: Suppose that G is a group satisfying minimal condition on those subnormal subgroups which are contained in N, where N is normal subgroup of G, then $\omega_N(G)$ has finite index in G.

Proof: Let R denote the finite residual of G such that $R\subseteq N$ and let H is subnormal in G, such that $H\subseteq N$. To prove that $H^R = H$ will be conclusive. Accordingly assume that this is false and let the subnormal subgroup such that $H\subseteq N$, be chosen minimal subject to $H^R\neq H$. Denote by P the joint of all proper subnormal subgroups of H: then $P^R = P$ by minimality of H. Moreover $P\lhd H$ and clearly H/P must be simple. Since $P\lhd HR$ and HR is subnormal in G, such that $HR\subseteq N$, the group HR/P inherits the property minimal subnormal from G. We may therefore by using theorem 13.3.6 of [2], to conclude that HR/P possesses only finitely many minimal subgroups. If $x\in R$, then $P\leq H^x$ and H^x/P is a simple and therefore a minimal, subnormal subgroup of HR/P. Consequently the number of conjugate of H in R is finite or equivalently $|R: N_R(H)|$ is finite. However, R has no proper subgroups of finite index, so $R: N_R(H)$ and $H^R = H$, a contradiction. Hence the result follows.

The following is an immediate corollary of the above theorem.

Corollary 2.8: If the group G satisfies the minimal condition on those subnormal subgroups which are contained in N, where N is normal in G, there is an upper bound for the defects of those subnormal subgroups of G.

Proof: We write $W = \omega_N(G)$ and let H is subnormal in G, such that $H\subseteq N$. Then $H\lhd HW$, while G/W is finite by theorem 2.7, now HW/W is subnormal in the finite group G/W, so certainly

$$s(G:HW)\leq|G:W| = m$$

Hence

$$s(G:HW)\leq m+1$$

GENERALIZED WIELANDT SUBGROUP OF SEMIDIRECT PRODUCT OF TWO GROUPS OF COPRIME ORDER

We begin with a theorem which proves very useful in calculating the generalized Wielandt subgroup of the semidirect product of two groups of coprime order. To prove this theorem we need the following two lemmas.

Lemma 3.1: Let G = BA be the semidirect product of subgroups A,B of coprime order with A normal in G. If N is a normal subgroup of G which contains A, then

$$\omega_N(G) \cap B \subseteq \omega_{N \cap B}(B)$$

Proof: Let S be a subnormal subgroup of B contained in N. Then $SA \subseteq N$ is subnormal in G. Let x be an element of $\omega_N(G) \cap B$. Then $(SA)^x = SA$, which implies that $S^x A = SA$. Now for all $s \in S$ there is some $s_1 \in S$ so that $s^x \in s_1 A$ and so $s_1^{-1} s^x \in A \cap B = 1$. This means that $s^x = s_1 \in S$ for all $x \in \omega_N(G) \cap B$. Hence we get $\omega_N(G) \cap B \subseteq \omega_{N \cap B}(B)$

Lemma 3.2: Let G = BA be a semidirect product of subgroups A,B of coprime order, with A normal in G. If N is a normal subgroup of G such that A is the unique maximal normal subgroup of N, then $\omega_N(G) \cap A = \omega(A)$.

Proof: Let x be any arbitrary element of $\omega_N(G) \cap A$ and let S be a subnormal subgroup of A. Clearly S is a subnormal subgroup of G contained in N. Then $S^x = S$ and so x belongs to $\omega(A)$. Therefore, $\omega_N(G) \cap A \subseteq \omega(A)$.

Conversely, let x be any arbitrary element of $\omega(A)$ and let S be a subnormal subgroup of G contained in N. Since A is the unique maximal normal subgroup of N, so we have $S \subseteq A$. Therefore $S^x = S$ and so x belongs to $\omega_N(G) \cap A$. This means $\omega(A) \subseteq \omega_N(G) \cap A$ and hence $\omega_N(G) \cap A = \omega(A)$

The following is an easy corollary of the above lemma.

Corollary 3.3: Let G = BA be a semidirect product of subgroups A,B of coprime order, with A normal in G. If N is a normal subgroup of G which is contained in A, then $\omega_N(G) \cap A = \omega_N(A)$.

Now we use these results to prove the following theorem.

Theorem 3.4: Let G = BA be a semidirect product of subgroups A, B of coprime order with A nilpotent and normal. If N is a normal subgroup of G which contains A and P is the set of those element of $\omega_{N \cap B}(B)$ which act by conjugation as power automorphism on A, then $\omega_N(G) = P\omega(A)$.

Proof: Suppose that G satisfies the hypothesis of the theorem so that G = BA, (|A|,|B|) = 1 and A is normal and nilpotent in G. By corollary 4.1.2 of [1] we can write

$$\omega_N(G) = (\omega_N(G) \cap B)(\omega_N(G) \cap A)$$

Using lemma 3.1, we get

$$\omega_N(G) \subseteq \omega_{N \cap B}(B)(\omega_N(G) \cap A)$$

By lemma 3.2, the above result becomes

$\omega_N(G) \subseteq \omega_{N \cap B}(B)\omega(A)$. Since A is normal and nilpotent, every subgroup of A is subnormal in G. Next we know that $A \subseteq N$, therefore $\omega_N(G)$ normalizes all subgroups of A. Hence $\omega_N(G) \cap B \subseteq P$. We claim that $P \subseteq \omega_N(G) \cap B$. By definition $P \subseteq B$ and we only have to show that $P \subseteq \omega_N(G)$. To prove this let S is a subnormal subgroup of G such that $S \subseteq N$. Corollary 4.1.2 of [1], we have $S = (S \cap B)(S \cap A)$. Since $S \cap B$ is subnormal in B and $S \cap B \subseteq N \cap B$, it is normalized by P. It follows from the definition of P, that P normalizes $S \cap A$ and hence S. This proves our claim and we get $\omega_N(G) \cap B = P$ and hence $\omega_N(G) \cap B = P\omega(A)$

LOCAL GENERALIZED WIELANDT SUBGROUP

In the following results we have shown the variation of local generalized Wielandt subgroup for a finite soluble group.

Lemma 4.1: If M, N are normal subgroups of a soluble group G, then

(i) $\omega_{MN}(G)N/N \subseteq \omega_{MN/N}(G/N)$

(ii) $\omega^p_{MN}(G)N/N \subseteq \omega^p_{MN/N}(G/N)$

Proof

(i) Let g be an arbitrary element of $\omega_{MN}(G)N/N$, so g = xN, where $x \in \omega_{MN}(G)$. Let S/N be a subnormal subgroup of G/N contained in MN/N. This implies that S is a subnormal subgroup of G that is contained in MN. Now consider

$$(S/N)^g = g^{-1}(S/N)g = (xN)^{-1}(S/N)(xN)$$
$$= x^{-1}N(S/N)xN = x^{-1}Sx/N = S^x/N = S/N$$

This means that $g \in \omega_{MN/N}(G/N)$ and hence $\omega_{MN}(G)N/N \subseteq \omega_{MN/N}(G/N)$

(ii) Let g be an arbitrary element of $\omega^p_{MN}(G)N/N$, so g = xN, where $x \in \omega^p_{MN}(G)$. Let S/N be a p'-perfect subnormal subgroup of G/N contained in MN/N. This implies that S is a p'-perfect subnormal subgroup of G and is contained in MN. Now consider

$$(S/N)^g = g^{-1}(S/N)g = (xN)^{-1}(S/N)(xN) = x^{-1}N(S/N)xN = x^{-1}Sx/N = S^x/N = S/N$$

This means $g \in \omega^p_{MN/N}(G/N)$ and therefore

$$\omega^p_{MN}(G)N/N \subseteq \omega^p_{MN/N}(G/N)$$

Theorem 4.2: Let M, N be normal subgroups of a soluble group G.

If N is a p'-group, then $N \subseteq \omega^p_{MN}(G)$ and $\omega^p_{MN/N}(G/N) = \omega^p_{MN}(G)/N$.

If G/N is p'-group, then $\omega^p(N) = \omega^p_N(G) \cap N$.

(iii) If G/N is p'-group, then

$$O_p(\omega^p(N)) = O_p(\omega^p_N(G) \cap N)$$

Proof: (i) It follows directly from lemma 4.1(ii) that

$$\omega^p_{MN}(G)N/N \subseteq \omega^p_{MN/N}(G/N)$$

Conversely we prove this relation by induction on |N|. First let us suppose that N is a minimal normal subgroup of G. Then $N \subseteq \omega^p_{MN}(G)$. Let xN be an arbitrary element of $\omega^p_{MN/N}(G/N)$. If S is a p'-perfect subnormal subgroup of G contained in MN. Then x belongs to $N_G(NS)$ and so by lemma 2.2.(i) of [5], x belongs to $N_G(S)$. Hence $x \in \omega^p_{MN}(G)$ therefore $xN \in \omega^p_{MN}(G)/N$ thus $\omega^p_{MN/N}(G/N) \subseteq \omega^p_{MN}(G)/N$. Now let us consider the case if N is not the minimal normal let $N_1 \subseteq N$ be a minimal normal subgroup of G, then $N/N_1 \subseteq \omega^p_{MN/N_1}(G/N_1)$ by induction, so

$$N/N_1 \subseteq \omega^p_{MN/N_1}(G/N_1) = \omega^p_{MN}(G)/N_1$$

and so $N \subseteq \omega^p_{MN}(G)$.

Let $\theta : G/N \rightarrow \frac{G/N_1}{N/N_1}$ be the natural isomorphism. Then

$$\omega^p_{MN/N}(G/N)\theta = \omega^p_{\frac{MN/N_1}{N/N_1}}\left(\frac{G/N_1}{N/N_1}\right)$$
$$= \frac{\omega^p_{MN/N_1}(G/N_1)}{(N/N_1)}, \quad \text{by induction}$$
$$= \frac{\omega^p_{MN}(G)/N_1}{N/N_1}, \quad \text{by induction}$$
$$= (\omega^p_{MN}(G)/N)\theta$$

Therefore

$$\omega^p_{MN/N}(G/N) = \omega^p_{MN}(G)/N$$

and hence the induction is completed

(ii) Let x be an arbitrary element of $\omega^p(N)$ and S is a p'-perfect subnormal subgroup of G contained in N, this implies that S is a p'-perfect subnormal subgroup of N, so $S^x = S$. Thus $\omega^p(N) \subseteq \omega^p_N(G) \cap N$.

Conversely, let x be an arbitrary element of $\omega^p_N(G) \cap N$ and S is a p'-perfect subnormal subgroup of N so S is a p'-perfect subnormal subgroup of G contained in N. This means that $S^x = S$ for all $x \in \omega^p_N(G) \cap N$, so $\omega^p_N(G) \cap N \subseteq \omega^p(N)$. Thus the result follows.

(iii) As $O_p(\omega^p_N(G) \cap N) \le N$, therefore

$$O_p(\omega^p_N(G) \cap N) \le N \cap O_p(\omega^p_N(G) \cap N)$$
$$= N \cap \omega^p_N(G) \cap N \cap O_p(G)$$
$$= N \cap \omega^p_N(G) \cap O_p(G)$$
$$= \omega^p(N) \cap O_p(G) \quad \text{by(ii)}$$
$$= O_p(\omega^p(N))$$

Now by using the above results we have established the following important relation between the generalized Wielandt subgroup and local generalized Wielandt subgroup.

Theorem 4.3: If N be a normal subgroup of a soluble group G, then

$$\omega_N(G) = \cap_{p \in \pi} \omega^p_N(G)$$

Proof: It is obvious that $\omega_N(G) \subseteq \omega^p_N(G)$.

Conversely let x be an arbitrary element of $\cap_{p \in \pi} \omega^p_N(G)$ and S be a subnormal subgroup of G contained in N. Now S can be written as a product of subnormal subgroups each of which has a unique maximal normal subgroup and hence a p'-perfect for some prime p and so each of which is normalized by x therefore

$$\cap_{p \in \pi} \omega^p_N(G) \subseteq \omega_N(G)$$

Hence

$$\omega_N(G) = \cap_{p \in \pi} \omega^p_N(G)$$

Theorem 4.4: Let G be a soluble group and G/N is a p′-perfect, then for a prime p

$$\omega_{MO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) = \omega^{p}_{MO_{p'}(G)}(G)/O_{p'}(G)$$

Where $M \lhd G$ contain N.

Proof: By theorem 4.2(i) we have

$$\omega^{p}_{MO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) = \omega^{p}_{MO_{p'}(G)}(G)/O_{p'}(G)$$

So we need only to show that

$$\omega_{MO_{p'}(G)/O_{p}(G)}(G/O_{p'}(G)) = \omega^{p}_{MO_{p'}(G)/O_{p}(G)}(G/O_{p'}(G))$$

Let $A = O_{p'}(G)$ and assume for convenience that A = 1, let $B = \omega^{p}_{M}(G)$, let q≠p be a prime and S is a q′-perfect subnormal subgroup of G contained in M. Put $N = O^{p'}(S)$ so N is a p′-perfect subnormal subgroup and normalized by B and so it is a normal subgroup of $C = \langle S^{x} : x \in B \rangle$. Further C/N is generated by subnormal p′-subgroups so it is a p′-group. Therefore [C,B]≤C∩B≤N. It follows that [S,B]≤N≤B it means that B normalizes S. Therefore $B \subseteq \omega^{q}_{M}(G)$ for all primes q. Now by theorem 4.3, we have

$$B \subseteq \cap_{q \in \pi} \omega^{q}_{M}(G) = \omega_{M}(G)$$

so

$$B = \omega^{p}_{M}(G) \subseteq \omega_{M}(G)$$

Converse is obvious one, hence $\omega^{p}_{M}(G) = \omega_{M}(G)$ therefore

$$\omega_{MO_{p'}(G)/O_{p}(G)}(G/O_{p'}(G)) = \omega^{p}_{MO_{p'}(G)/O_{p}(G)}(G/O_{p'}(G))$$
$$= \omega^{p}_{MO_{p'}(G)}(G)/O_{p'}(G)$$

MAIN RESULTS

In this section our aim is to find a relation between the generalized Wielandt length of a supersoluble group with the invariants of the normal Sylow sub-groups of it. We bound the generalized Wielandt length in term of the nilpotency classes of the normal Sylow subgroups. We begin with a result which gives information about the Sylow p subgroups of generalized Wielandt subgroup of a supersoluble group.

Theorem 5.1: If A is a Sylow p-subgroup of a supersoluble group G, then

$$\omega_{MN}(AM) \cap F(G) = O_{p}(\omega_{MN}(G))$$

where $M, N \lhd G$, such that N⊆A and M is a p′-subgroup.

Proof: Consider

$$(\omega_{MN}(AM) \cap F(G))M/M \subseteq \omega_{MN}(AM)M/M$$

Since G/M is supersoluble, it has a normal Sylow subgroup which is a p-group, because F(G/M) is a p-group by theorem 5.4.8 of [2]. Thus AM/M, being a Sylow p-subgroup of G/M, is normal. Also

$$\omega_{MN}(AM)M/M \subseteq \omega_{MN/M}(AM/M)$$

by lemma 4.1(i). Again since AM/M is a normal Hall subgroup of G/M, by corollary 3.3,

$$\omega_{MN/M}(AM/M) \subseteq \omega_{MN/M}(G/M)$$

and so Now by theorem 4.4, the local generalized Wielandt subgroup of G/M is given by

$$\omega_{MN/M}(G/M) = \omega^{p}_{MN}(G)/M$$

and therefore

$$(\omega_{MN}(AM) \cap F(G))M/M \subseteq \omega^{p}_{MN}(G)/M$$

and so

$$\omega_{MN}(AM) \cap F(G) \subseteq \omega^{p}_{MN}(G)$$

Since $\omega_{MN}(AM) \cap F(G)$ is subnormal p-subgroup of G, so

$$\omega_{MN}(AM) \cap F(G) \subseteq O_{q'}(G),$$

for all primes q≠p. But by theorem 4.2, $O_{q'}(G) \subseteq \omega^{q}_{MN}(G)$. Therefore, by using theorem 4.3, we have

$$\omega_{MN}(AM) \cap F(G) \subseteq \cap_{r \in \pi} \omega^{r}_{MN}(G) = \omega_{MN}(G)$$

where the intersection is over all primes r. It follows that, since $\omega_{MN}(AM) \cap F(G)$ is a p-group and subnormal,

$$\omega_{MN}(AM) \cap F(G) \subseteq O_{p}(\omega_{MN}(G))$$

Conversely, we claim that

$$O_p(\omega_{MN}(G)) \subseteq \omega_{MN}(AM)$$

To prove this claim suppose that S is subnormal subgroup of AM, such that S is contained in MN. Then S is subnormal subgroup of G and contained in MN. Thus for all $x \in O_p(\omega_{MN}(G)) \subseteq AM$, we have $S^x = S$ and therefore

$$O_p(\omega_{MN}(G)) \subseteq \omega_{MN}(AM)$$

Now $O_p(\omega_{MN}(G))$ is nilpotent, characteristic subgroup of $\omega_{MN}(G)$ therefore normal in G. So it is contained in F(G). Thus we have

$$O_p(\omega_{MN}(G)) \subseteq \omega_{MN}(AM) \cap F(G)$$

Therefore

$$\omega_{MN}(AM) \cap F(G) = O_p(\omega_{MN}(G))$$

The following is an easy corollary of the above theorem. Corollary 5.2. Let G be a supersoluble group and suppose a normal Sylow p-subgroup of G has nilpotency class n>1. Then a normal Sylow p-subgroup of $G/\omega_N(G)$ has nilpotency class at most n-1. (Where $N \lhd G$, contained in every Sylow p-subgroup of G).

Proof: Let A be a normal Sylow p-subgroup of G. Since A has nilpotency class n, $\gamma_n(A) \subseteq Z(A) \subseteq \omega(A) \subseteq \omega_N(A)$ and by corollary 3.3, $\omega_N(A) \subseteq \omega_N(G)$. Since $A\omega_N(G)/\omega_N(G)$ is normal Sylow p-subgroup of $G/\omega_N(G)$ and

$$\gamma_n(A\omega_N(G)/\omega_N(G)) = \gamma_n(A)\omega_N(G)/\omega_N(G) = 1$$

Therefore $A/\omega_N(G)/\omega_N(G)$ has nilpotency class at most n-1.

In the following results we have tried to relate the generalized Wielandt length of a supersoluble group with its Sylow subgroups, we begin with the following theorem.

Theorem 5.3: Let G be a supersoluble group. If all Sylow p-subgroups of G are abelian except for p = 2 and if the Sylow 2-subgroups have class at most two, then G has generalized Wielandt length at most two.

Proof: We begin by supposing that, for some prime p dividing |G|, $O_{p'}(G) = 1$ In this case $O_p(G)$ is a normal Sylow p-subgroup of G, by theorem 5.4.8 [2]. Consider the case $p \neq 2$. Then by hypothesis, $O_p(G)$ is abelian and so by lemma 3.2, $O_p(G) \subseteq \omega_N(G)$. Moreover $O_p(G) = F(G)$ and using theorem 5.4.10 of [2], $G/O_p(G)$ is abelian. It follows that $G/\omega_N(G)$ is abelian and $\omega_{2,N}(G) = G$. That is G has generalized Wielandt length two. Now consider the case when p = 2. In this case $O_p(G) = G$ using theorem 5.4.8 of [2] and so G has nilpotency class at most two. But $Z(G) \subseteq \omega_N(G)$ and hence $G/\omega_N(G)$ is again abelian. Thus, under the assumption $O_{p'}(G) = 1$, G has generalized Wielandt length two. Now consider the general case and, for some prime p dividing |G|, put $H = G/O_{p'}(G)$ and let N be a normal subgroup of G and let $K = NO_{p'}(G)/O_{p'}(G)$ is a normal subgroup of H. Note that $O_{p'}(H) = 1$. Using the theorem 4.2(i) and theorem 4.4, the local generalized Wielandt subgroup of H is given as follows:

$$\omega_k^p(H) = \omega^p_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) = \omega^p_{NO_{p'}(G)}(G)/O_{p'}(G)$$
$$= \omega_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G)) = \omega_k(H)$$

From above case $H/\omega_k(H)$ is abelian and so is $H/\omega_k^p(H)$. Now

$$H/\omega_k^p(H) = (G/O_{p'}(G)) \Big/ (\omega^p_{NO_{p'}(G)/O_{p'}(G)}(G/O_{p'}(G))$$
$$= (G/O_{p'}(G)) \Big/ \omega^p_{NO_{p'}(G)}(G)/O_{p'}(G)$$

again by theorem 4.2

which is isomorphic to $G/\omega^p_{NO_{p'}(G)}(G)$. Thus $G/\omega^p_{NO_{p'}(G)}(G)$ is abelian. This implies that $G' \subseteq \omega^p_{NO_{p'}(G)}(G)$ for all p dividing |G|. Hence $G' \subseteq \cap_{p\in\pi}\omega^p_{NO_{p'}(G)}(G)$ and so by theorem 4.3, $G' \subseteq \omega_{NO_{p'}(G)}(G)$. Thus $G/\omega_{NO_{p'}(G)}(G)$ is abelian. This shows that G has generalized Wielandt length for a normal subgroup $NO_{p'}(G)$ at most two.

In the next result we have shown that the generalized Wielandt length of a supersoluble group is bounded by the nilpotency class of its normal Sylow subgroups.

Theorem 5.4: Let N be a normal subgroup of a supersoluble group G if n is the maximum of the nilpotency classes of normal Sylow subgroups of G which contains N, then $G \in W_{n+1,N}$. ($W_{n,N}$: Class of groups having generalized Wielandt length at most n)

Proof: We prove this relation by induction on n. Its follows immediately from theorem 5.3 that the relation holds for n = 1. Let us suppose that it is true for n = k≥1 so that, if all the normal Sylow subgroups contains N have nilpotency class at most k, then $G \in W_{n+1,N}$. Now for n = k+1, that is, when all normal Sylow subgroups of G contains N have nilpotency class at most k+1. By corollary 5.2, this implies that all the normal Sylow subgroups of $G/\omega_N(G)$ have nilpotency class at most k. By our supposition $G/\omega_N(G) \in W_{k+1,N}$, in other words

$$\omega_{N\omega_{k+1,N}(G)/\omega_{k+1,N}(G)}(G/\omega_N(G)) = G/\omega_N(G)$$

But by definition

$$\omega_{N\omega_{k+1,N}(G)/\omega_{k+1,N}(G)}(G/\omega_N(G)) = \omega_{k+2,N}(G)/\omega_N(G)$$

Therefore $\omega_{k+2,N}(G) = G$. It means that $G \in W_{k+2,N}$. So the theorem holds for n = k+1 and hence for all n.

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