Lecture # 8

Wind driven circulation
Elementarstromsystem
Ekman transport
Ekman pumping
Sverdrup transport
Sverdrup meets Ekman

- now include also pressure gradient $\nabla_h p$
- momentum equation in vector form for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, \not{0})$$

- $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a stress vector with $\boldsymbol{\tau}(z=0) = \boldsymbol{\tau}^a$ where $\boldsymbol{\tau}^a$ is the surface wind stress in $\mathrm{N/m^2}$ acting on the ocean
- split the flow into geostrophic and frictional (Ekman) components, $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$), governed by

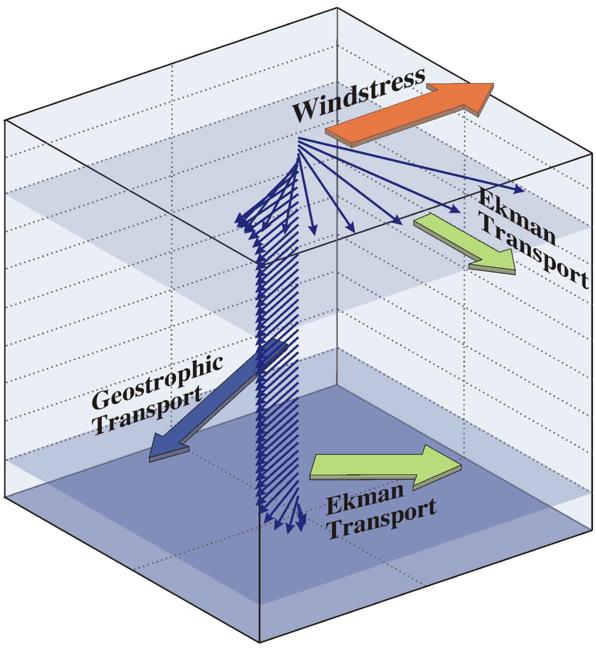
$$f\boldsymbol{k} \times \boldsymbol{u}_G = -\frac{1}{\rho_0}\nabla_h p \qquad \text{and} \qquad f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

and the same for continuity equation

$$\nabla \cdot \boldsymbol{u}_G + \frac{\partial w_G}{\partial z} = 0 \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

- sum $\boldsymbol{u}_G + \boldsymbol{u}_E$ satisfies full momentum and continuity equation

- Elementarstromsystem (for $\rho = const$)
- $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$)
  surface and bottom Ekman layers superimposed on geostrophic flow



- vertically integrated velocity

$$\boldsymbol{U} = \int_{-h}^{0} \boldsymbol{u}\, dz = \int_{-h}^{0} (\boldsymbol{u}_G + \boldsymbol{u}_E)\, dz = \boldsymbol{U}_G + \boldsymbol{U}_E$$

- with the (total) transport vector $\boldsymbol{U}$, dimension $\mathrm{m^2 s^{-1}}$
- transport by the geostrophic velocity $\rightarrow$ geostrophic transport $\boldsymbol{U}_G$
  transport by the Ekman velocity $\rightarrow$ Ekman transport $\boldsymbol{U}_E$

$$\begin{aligned} f\boldsymbol{k} \times \boldsymbol{u}_E &= \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \\ f\boldsymbol{k} \times \int_{-h}^{0} \boldsymbol{u}_E\, dz &= \frac{1}{\rho_0}(\boldsymbol{\tau}(z=0) - \boldsymbol{\tau}(z=-h)) \\ f\boldsymbol{k} \times \boldsymbol{U}_E &= \frac{1}{\rho_0}(\boldsymbol{\tau}^a - \boldsymbol{\tau}_b) \end{aligned}$$

  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$
- with $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{U}) = \boldsymbol{k} \times (-V, U, 0) = (-U, -V, 0) = -\boldsymbol{U}$

$$\boldsymbol{U}_E = -\frac{1}{f\rho_0}\boldsymbol{k} \times (\boldsymbol{\tau}^a - \boldsymbol{\tau}_b)$$

- split $\boldsymbol{U}_E$ into surface and bottom Ekman transport

- vertically integrated velocity $\boldsymbol{U} = \boldsymbol{U}_G + \boldsymbol{U}_E$ with geostrophic transport $\boldsymbol{U}_G$ and Ekman transport $\boldsymbol{U}_E$ given by

$$\boldsymbol{U}_E = -\frac{1}{f\rho_0}\boldsymbol{k}\times(\boldsymbol{\tau}^a - \boldsymbol{\tau}_b) \equiv \boldsymbol{U}_E^{top} + \boldsymbol{U}_E^{bot}$$

  with surface wind stress $\boldsymbol{\tau}^a$ and bottom stress $\boldsymbol{\tau}_b$
- split into surface Ekman transport in surface Ekman layer

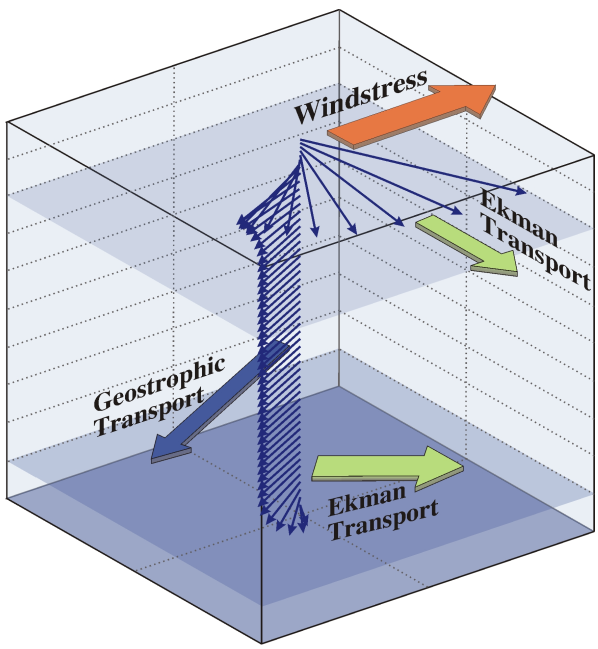
$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k}\times\boldsymbol{\tau}^a$$

  orthogonal to wind stress direction (to the right for $f > 0$)
  does not depend on parameterisation of $\boldsymbol{\tau}$ in the interior
- and bottom Ekman transport in bottom Ekman layer

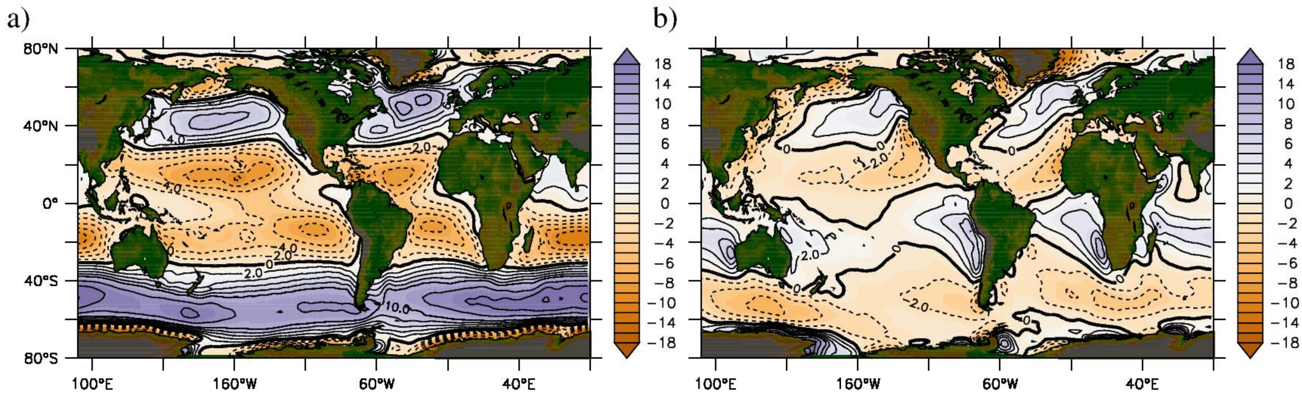
$$\boldsymbol{U}_E^{bot} = \frac{1}{f\rho_0}\boldsymbol{k}\times\boldsymbol{\tau}_b$$

  depends on parameterisation of $\boldsymbol{\tau}$ in the interior

- Elementarstromsystem
- $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$ (and $w = w_G + w_E$)
  surface and bottom Ekman layers superimposed on geostrophic flow



- zonal (left) and meridional component (right) of $\boldsymbol{\tau}^a$ in $10^{-2}\,\mathrm{N/m^2}$



- surface Ekman transport in surface Ekman layer

$$\boldsymbol{U}_E^{top} = -\frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a$$

  orthogonal to wind stress direction (to the right for $f > 0$)

- equatorward in west wind region poleward in trade wind region
- convergence between west wind and trade wind region
- divergence at high latitude and at equator

- momentum equation in vector form for $Ro \ll 1$

$$f\boldsymbol{k} \times \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k} \times \boldsymbol{u} = (-v, u, \emptyset)$$

- $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a stress vector with $\boldsymbol{\tau}(z=0) = \boldsymbol{\tau}^a$ where $\boldsymbol{\tau}^a$ is the surface wind stress in $\mathrm{N/m^2}$ acting on the ocean
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$$f\boldsymbol{k} \times \boldsymbol{u}_G = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p \qquad \text{and} \qquad f\boldsymbol{k} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}$$

  and the same for continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_G + \frac{\partial w_G}{\partial z} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

- sum $\boldsymbol{u}_G + \boldsymbol{u}_E$ satisfies full momentum and continuity equation

- integrating the continuity equation for $\boldsymbol{u}_E$ and w_E from z to $z = 0$

$$\nabla \cdot \boldsymbol{u}_E + \frac{\partial w_E}{\partial z} = 0$$

yields the vertical Ekman velocity

$$\int_z^0 \nabla \cdot \boldsymbol{u}_E \, dz + \cancel{w_E(z=0)} - w_E(z) = 0 \;\rightarrow\; w_E(z) = \nabla \cdot \int_z^0 \boldsymbol{u}_E \; dz$$

- Ekman velocity is given by

$$\boldsymbol{u}_E = D/(2A_v)e^{z/D}\left((\boldsymbol{\tau}_a - \boldsymbol{k} \times \boldsymbol{\tau}_a)\cos(z/D) + (\boldsymbol{\tau}_a + \boldsymbol{k} \times \boldsymbol{\tau}_a)\sin(z/D)\right)$$

- since $\boldsymbol{u}_E \approx 0$ below Ekman depth $D \approx 50\,\mathrm{m}$

$$w_E|_{z<-D} \approx \nabla \cdot \int_{z<-D}^0 \boldsymbol{u}_E \; dz = \nabla \cdot \boldsymbol{U}_E^{top} = -\nabla \cdot \frac{1}{f\rho_0}\boldsymbol{k} \times \boldsymbol{\tau}^a = \boldsymbol{k} \times \nabla \cdot \frac{\boldsymbol{\tau}^a}{\rho_0 f}$$

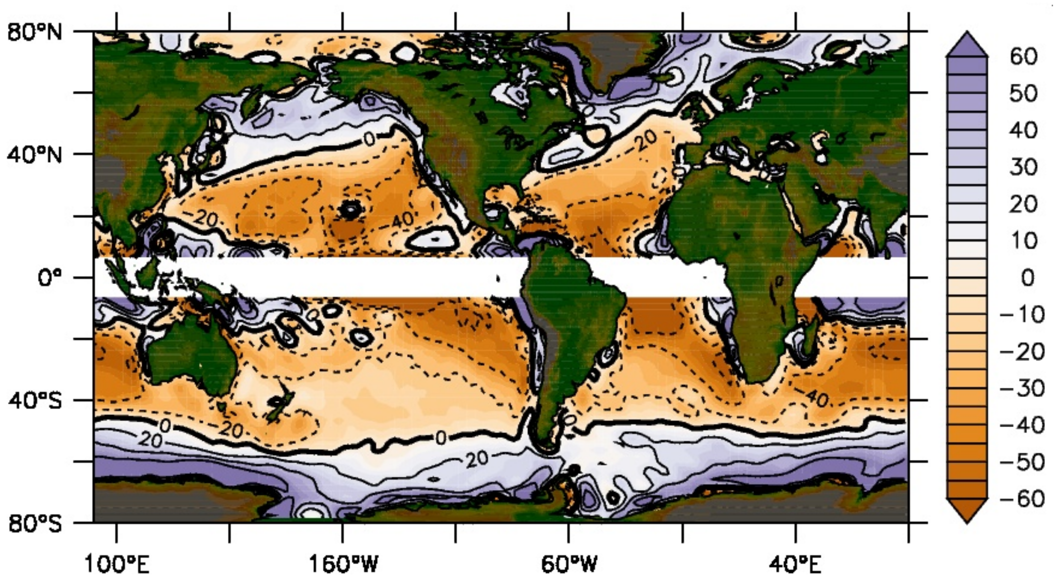
with Ekman pumping $w_E|_{z<-D}$ and

$$\begin{aligned} \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau} &= \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} \tau^{(x)} \\ \tau^{(y)} \end{pmatrix} = -\frac{\partial}{\partial y}\tau^{(x)} + \frac{\partial}{\partial x}\tau^{(y)} \\ &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \tau^{(y)} \\ -\tau^{(x)} \end{pmatrix} = \nabla \cdot (-\boldsymbol{k} \times \boldsymbol{\tau}) \end{aligned}$$

- Ekman pumping w_E in m per year

$$w_E|_{z<-D} \approx \nabla \cdot \boldsymbol{U}_E^{top} = \boldsymbol{k} \times \nabla \cdot \frac{\boldsymbol{\tau}^a}{\rho_0 f}$$

with Ekman depth $D \approx 50\,\mathrm{m}$ (depends on A_v)



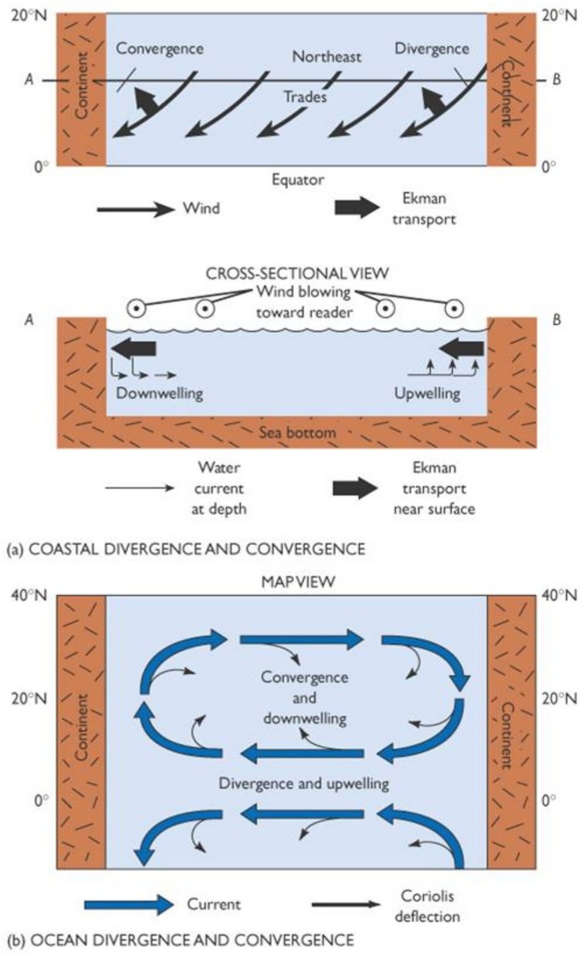
- Ekman transport $\boldsymbol{U}_E^{top}$ and pumping w_E do not depend on A_v

- Ekman pumping w_E

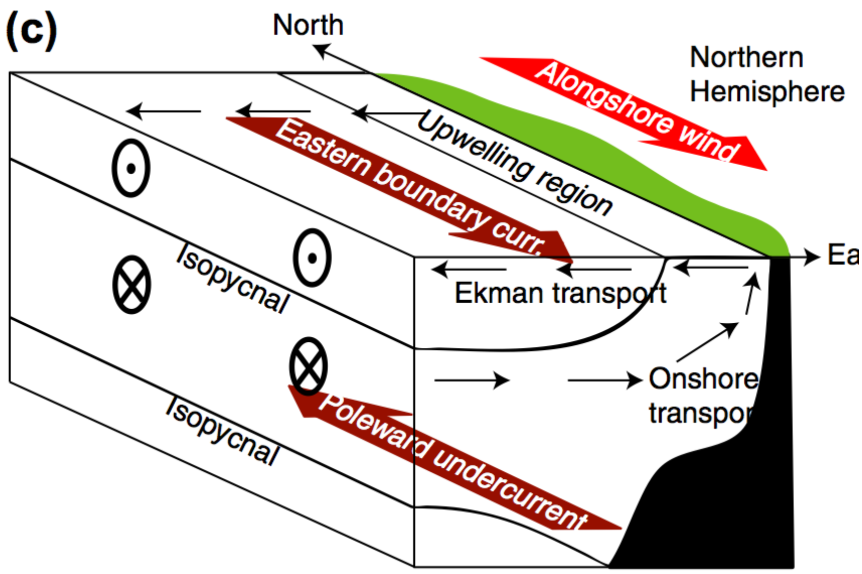
$$w_E|_{z<-D} \approx \boldsymbol{\nabla} \cdot \boldsymbol{U}_E^{top}$$

i.e. w_E from divergence of Ekman transport

- $w_E > 0$: Upwelling
  - subpolar gyre
  - at eastern boundaries
  - at equator
- $w_E < 0$: Downwelling
  - subtropical gyres

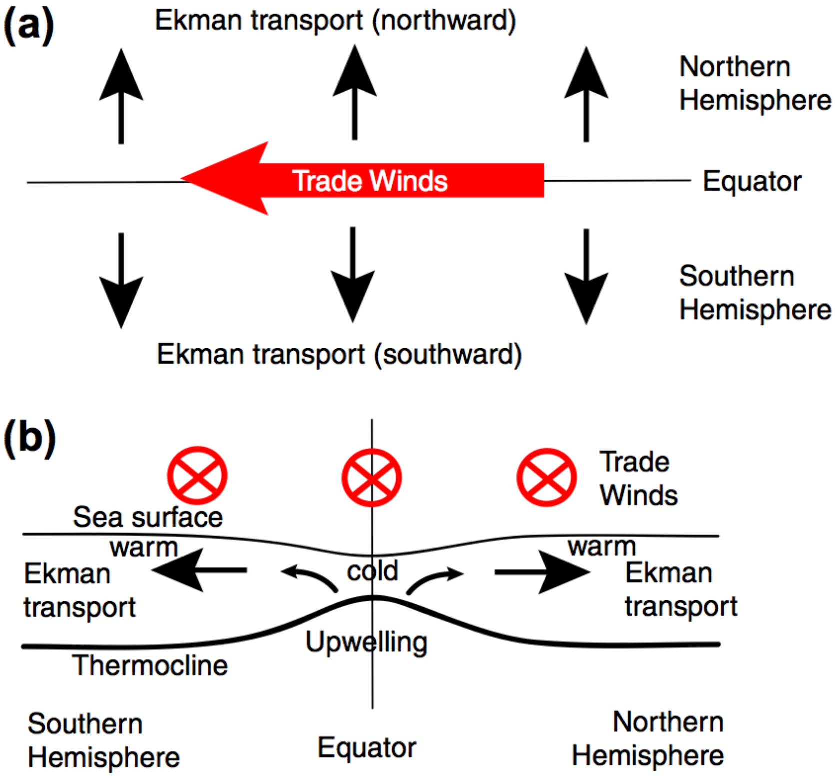


- coastal upwelling

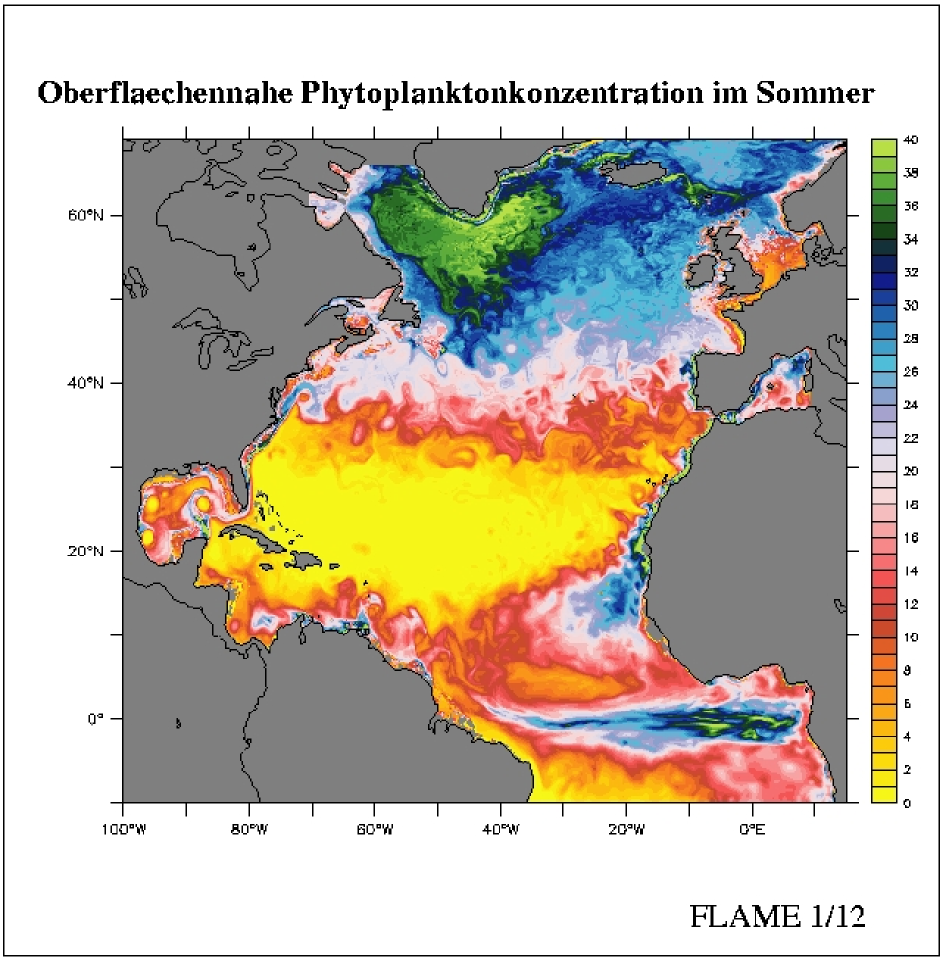


from Talley et al 2011

- equatorial upwelling



from Talley et al 2011



- momentum equation for $Ro \ll 1$

$$f\boldsymbol{k}\times\boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}_h p + \frac{1}{\rho_0}\frac{\partial\boldsymbol{\tau}}{\partial z} \quad \text{with} \quad \boldsymbol{k}\times\boldsymbol{u} = (-v, u, 0)$$

- neglect sea surface height $z = \zeta \to$ assume rigid lid at $z = 0$
- assume flat bottom at $z = -h = const$
- vertically integrate momentum equation from bottom to surface

$$\rho_0 f\boldsymbol{k}\times\boldsymbol{U} = -\boldsymbol{\nabla}_h\int_{-h}^{0} pdz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

  with transport $\boldsymbol{U} = \int_{-h}^{0}\boldsymbol{u}dz$, surface and bottom stress $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$
- take curl which yields (after a little calculation)

$$\rho_0\beta V + \rho_0 f\boldsymbol{\nabla}_h\cdot\boldsymbol{U} \quad = \quad \boldsymbol{k}\times\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)$$

  with $\beta = df/dy$
- since $\boldsymbol{\nabla}_h\cdot\boldsymbol{U} = 0$ from continuity equation $\boldsymbol{\nabla}_h\cdot\boldsymbol{u} + \partial w/\partial z = 0$ it follows the famous Sverdrup relation

$$\rho_0\beta V \quad = \quad \boldsymbol{k}\times\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)$$

- little calculation: curl of vertically integrated momentum equation

$$\rho_0 f\boldsymbol{k}\times\boldsymbol{U} = -\boldsymbol{\nabla}_h\int_{-h}^{0} pdz + \boldsymbol{\tau}_a - \boldsymbol{\tau}_b$$

- rewrite component wise

$$-\rho_0 fV = -\frac{\partial}{\partial x}\int_{-h}^{0} pdz + \tau_a^x - \tau_b^x \quad , \quad \rho_0 fU = -\frac{\partial}{\partial y}\int_{-h}^{0} pdz + \tau_a^y - \tau_b^y$$

- $\partial/\partial y$ of 1. equation minus $\partial/\partial x$ of 2. equation

$$\begin{aligned}
-\frac{\partial}{\partial y}(\rho_0 fV) &= -\frac{\partial^2}{\partial x\partial y}\int_{-h}^{0} pdz + \frac{\partial}{\partial y}(\tau_a^x - \tau_b^x) \\
\rho_0 f\frac{\partial U}{\partial x} &= -\frac{\partial^2}{\partial x\partial y}\int_{-h}^{0} pdz + \frac{\partial}{\partial x}(\tau_a^y - \tau_b^y) \\
-\rho_0\frac{df}{dy}V - \rho_0 f\frac{\partial V}{\partial y} - \rho_0 f\frac{\partial U}{\partial x} &= \frac{\partial}{\partial y}(\tau_a^x - \tau_b^x) - \frac{\partial}{\partial x}(\tau_a^y - \tau_b^y) \\
-\rho_0\beta V - \rho_0 f\boldsymbol{\nabla}_h\cdot\boldsymbol{U} &= -\boldsymbol{k}\times\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_a - \boldsymbol{\tau}_b)
\end{aligned}$$

  with $\beta = df/dy$ and $\boldsymbol{k}\times\boldsymbol{\nabla} = (-\partial/\partial y, \partial/\partial x, 0)$

- since $\nabla_h \cdot \boldsymbol{U} = 0$ introduce volume transport streamfunction, with

$$U = -\frac{\partial \psi}{\partial y} \quad , \quad V = \frac{\partial \psi}{\partial x} \quad \rightarrow \quad \boldsymbol{U} = \boldsymbol{k} \times \nabla \psi$$

transport $\boldsymbol{U}$ is parallel to contour lines of ψ

- ψ determines transport perpendicular to or "across" section $A \rightarrow B$

$$\int_A^B \boldsymbol{U} \cdot d\boldsymbol{s} = \int_A^B \boldsymbol{k} \times \nabla \psi \cdot d\boldsymbol{s} = \int_A^B \nabla_h \psi \cdot d\boldsymbol{\ell} = \psi(B) - \psi(A)$$

where $d\boldsymbol{s}$ is a line element perpendicular to section $A \rightarrow B$
and $d\boldsymbol{\ell}$ is line element along section $A \rightarrow B$

- Sverdrup relation becomes (for $\boldsymbol{\tau}_b = 0$)

$$\rho_0 \beta V = \rho_0 \beta \frac{\partial \psi}{\partial x} = \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}_a$$

- integration from x to eastern boundary ($x = x_E$) where $\psi(x_E) = 0$

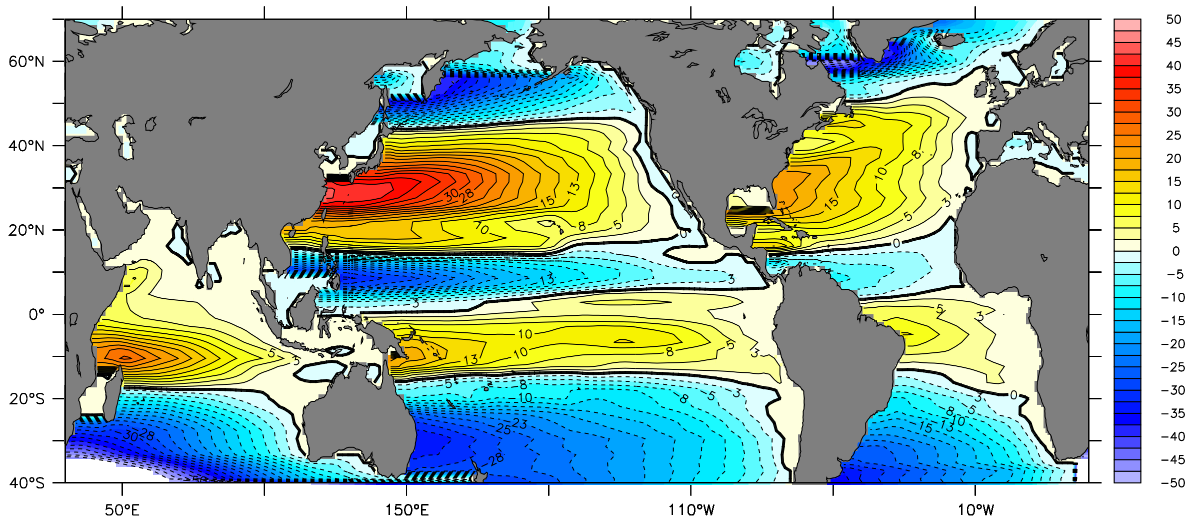
$$\psi(x, y) \quad = \quad -\frac{1}{\rho_0 \beta} \int_x^{x_e} \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}_a \, dx$$

$\rightarrow$ Sverdrup streamfunction

- Sverdrup streamfunction

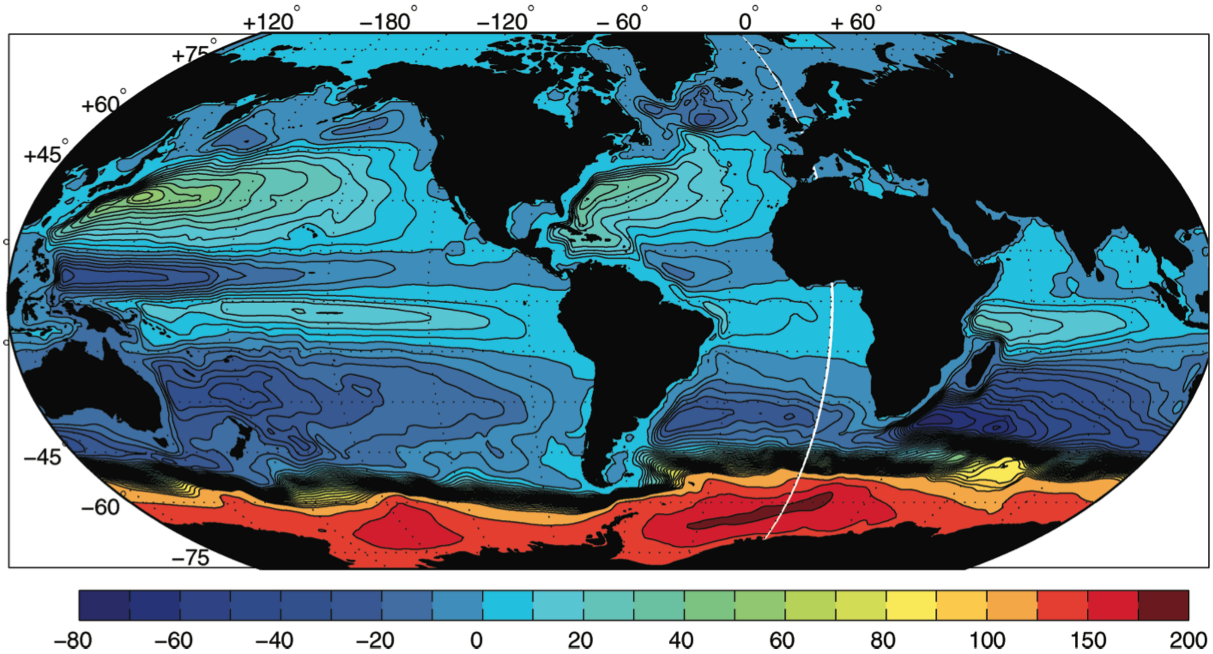
$$\psi = -\frac{1}{\rho_0 \beta} \int_x^{x_e} \boldsymbol{k} \times \nabla \cdot \boldsymbol{\tau}^a \, dx$$

from realistic wind stress in $10^6 \, \mathrm{m}^3/\mathrm{s} \equiv 1 \, \mathrm{Sv}$

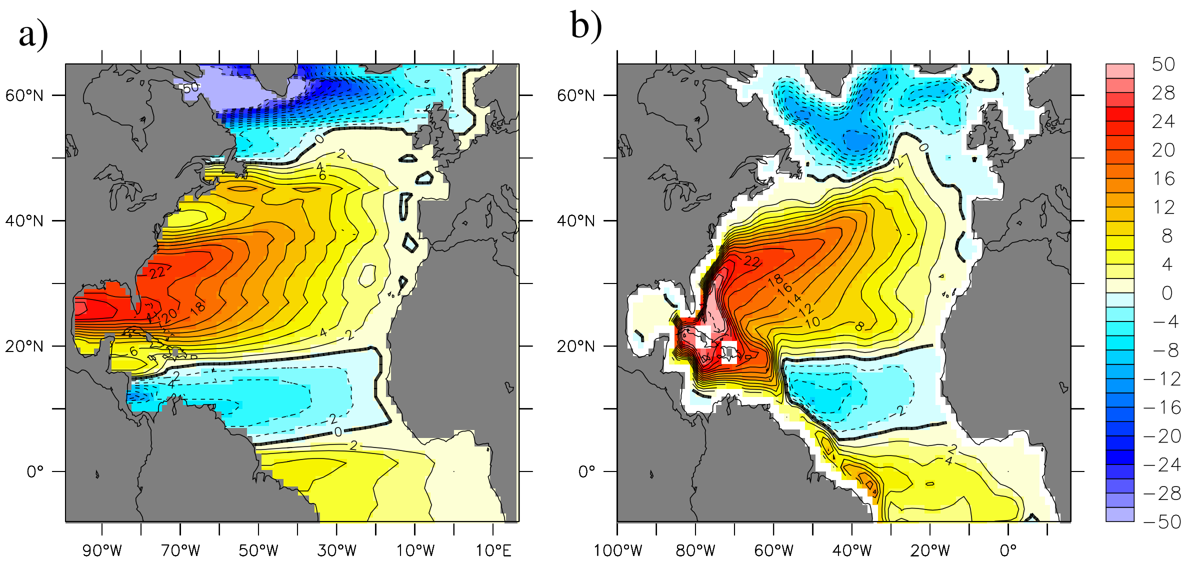


- $\psi(x_E) = 0$ along east eastern boundary but not at western boundary
$\rightarrow$ western boundary current not included
- but in the interior ψ is rather realistic

- ψ in a global state estimate in $10^6\,\mathrm{m}^3/\mathrm{s} \equiv 1\,\mathrm{Sv}$



- Streamfunction ψ in $Sv = 10^6\,\mathrm{m}^3/\mathrm{s}$ from simple Sverdrup relation
- Streamfunction ψ for a realistic model of the Atlantic Ocean



- simple Sverdrup relation works surprisingly well

- vertically integrated momentum equation

$$
\begin{aligned}
-\rho_0 f V &= -\frac{\partial}{\partial x}\int_{-h}^{0} p dz + \tau_a^x \equiv -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U &= -\frac{\partial}{\partial y}\int_{-h}^{0} p dz + \tau_a^y \equiv -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- split in Ekman transport $\boldsymbol{U}_E$ und geostrophic transport $\boldsymbol{U}_G$

$$
\begin{aligned}
-\rho_0 f V \equiv -\rho_0 f\,(V_G + V_E) &= -\frac{\partial P}{\partial x} + \tau_a^x \\
\rho_0 f U \equiv \rho_0 f\,(U_G + U_E) &= -\frac{\partial P}{\partial y} + \tau_a^y
\end{aligned}
$$

- with Ekman transport

$$
-\rho_0 f V_E = \tau_a^x \;,\; \rho_0 f U_E = \tau_a^y \;\rightarrow\; \rho_0 f \boldsymbol{k} \times \boldsymbol{U}_E = \boldsymbol{\tau}_a \;\rightarrow\; \rho_0 f \boldsymbol{U}_E = -\boldsymbol{k} \times \boldsymbol{\tau}_a
$$

- and with geostrophic transport

$$
-\rho_0 f V_G = -\frac{\partial P}{\partial x} \;,\; \rho_0 f U_G = -\frac{\partial P}{\partial y} \;\rightarrow\; \rho_0 f \boldsymbol{U}_G = \boldsymbol{k} \times \boldsymbol{\nabla}_h P
$$

- Ekman transport + geostr. transport = Sverdrup transport

- Ekman transport + geostr. transport = Sverdrup transport
- with Ekman transport

$$
-\rho_0 f V_E = \tau_a^x \quad , \quad \rho_0 f U_E = \tau_a^y \quad , \quad \rho_0 f \boldsymbol{U}_E = -\boldsymbol{k} \times \boldsymbol{\tau}_a
$$

- and geostrophic transport

$$
-\rho_0 f V_G = -\frac{\partial P}{\partial x} \quad , \quad \rho_0 f U_G = -\frac{\partial P}{\partial y} \quad , \quad \rho_0 f \boldsymbol{U}_G = \boldsymbol{k} \times \boldsymbol{\nabla}_h P
$$

- both transports are divergent

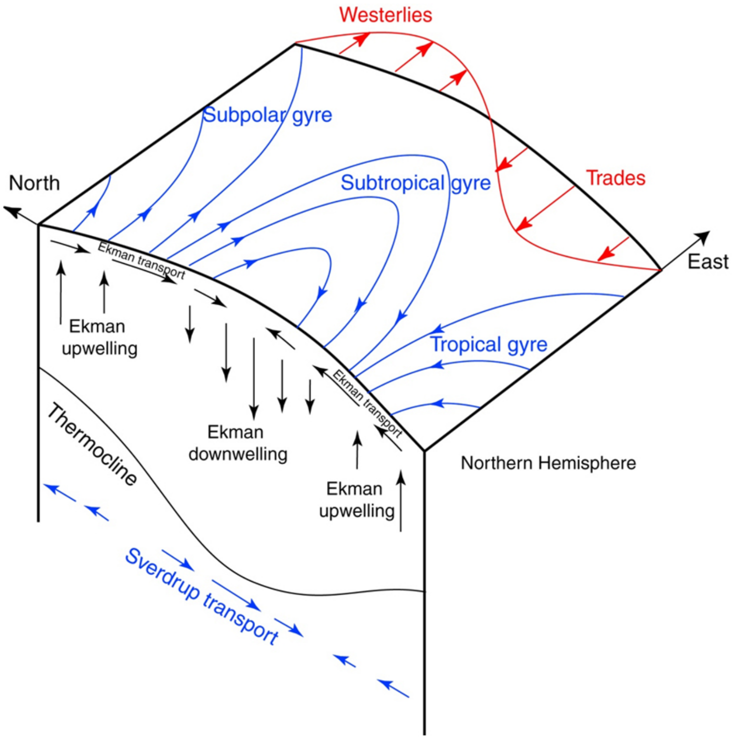
$$
\begin{aligned}
\boldsymbol{\nabla}_h \cdot \boldsymbol{U}_E &= -\boldsymbol{\nabla}_h \cdot \boldsymbol{k} \times \frac{\boldsymbol{\tau}_a}{\rho_0 f} = \boldsymbol{k} \times \boldsymbol{\nabla}_h \cdot \frac{\boldsymbol{\tau}_a}{\rho_0 f} = w_E \\
\boldsymbol{\nabla}_h \cdot \boldsymbol{U}_G &= -\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\frac{1}{\rho_0 f}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\frac{1}{\rho_0 f}\right) = \frac{\partial P}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\right) \\
&= -\frac{1}{\rho_0 f^2}\frac{df}{dy}\frac{\partial P}{\partial x} = -\frac{\beta}{\rho_0 f^2}\frac{\partial P}{\partial x} = -\frac{\beta}{f} V_G
\end{aligned}
$$

- but the total transport $\boldsymbol{U} = \boldsymbol{U}_E + \boldsymbol{U}_G$ is non-divergent

$$
\boldsymbol{\nabla}_h \cdot \boldsymbol{U} = 0 \quad \rightarrow \quad w_E^{top} = \frac{\beta}{f} V_G
$$

Ekman pumping generates southward geostr. transport (for $f > 0$)

- Ekman pumping generates southward geostr. transport (for $f > 0$)



from Talley et al 2011

- Sverdrup relation follows from potential vorticity conservation
- potential vorticity equation for a single layer

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h} \quad \text{or} \quad q \approx \zeta - \frac{f_0}{H} h + f$$

  q is conserved for fluid parcels in single layer
- w_E lead to vortex stretching and meridional motion