

Separately subharmonic functions and quasi-nearly subharmonic functions

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ABSTRACT

First, we give the definition for quasi-nearly subharmonic functions. Second, after recalling the existing subharmonicity results of separately subharmonic functions, we give corresponding counterparts for separately quasi-nearly subharmonic functions, thus generalizing previous results of Armitage and Gardiner, of ours, of Arsove, of Avaniessian, and of Lelong.

Key words: Subharmonic, Quasi-nearly subharmonic, Separately subharmonic, Integrability condition.

1. INTRODUCTION

1.1 Previous results

It is a well-known problem whether a separately subharmonic function is subharmonic or not. As far as we know, it was Lelong [9, Théorème 1 bis, p. 315] who gave the first result related to this problem. Much later Wiegerinck [24], see also [25, Theorem 1, p. 246], showed that a separately subharmonic function need not be subharmonic. On the other hand, Armitage and Gardiner [1, Theorem 1, p. 256], gave an “almost sharp” condition, which ensures that a separately subharmonic function u of a domain Ω in \mathbb{R}^{m+n} , $m \geq n \geq 2$, is subharmonic. Armitage’s and Gardiner’s condition was the following:

$\phi(\log^+ u^+)$ is locally integrable in Ω , where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing function such that

$$\int_1^{+\infty} s^{(n-1)/(m-1)} (\phi(s))^{-1/(m-1)} ds < +\infty.$$

The purpose of this paper is the following. First, we list the existing results on this problem. Second, we extend these results to the more general setup of so called quasi-nearly subharmonic functions. We begin with the notation and necessary definitions.

1.2 Notation.

Our notation is rather standard, see e.g. [21] and [7]. m_N is the Lebesgue measure in the Euclidean space \mathbb{R}^N , $N \geq 2$. We write v_N for the Lebesgue measure of the unit ball $B^N(0, 1)$ in \mathbb{R}^N , thus $v_N = m_N(B^N(0, 1))$. D is a domain of \mathbb{R}^N . The complex space \mathbb{C}^n is identified with the real space

\mathbb{R}^{2n} , $n \geq 1$. Constants will be denoted by C and K . They will be nonnegative and may vary from line to line.

1.3 Nearly subharmonic functions

We recall that an upper semicontinuous function $u : D \rightarrow [-\infty, +\infty)$ is *subharmonic* if for all $\overline{B^N(x, r)} \subset D$,

$$u(x) \leq \frac{1}{v_N r^N} \int_{B^N(x, r)} u(y) dm_N(y).$$

The function $u \equiv -\infty$ is considered subharmonic.

We say that a function $u : D \rightarrow [-\infty, +\infty)$ is *nearly subharmonic*, if u is Lebesgue measurable, if $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$, and for all $\overline{B^N(x, r)} \subset D$,

$$u(x) \leq \frac{1}{v_N r^N} \int_{B^N(x, r)} u(y) dm_N(y).$$

Observe that in the standard definition of nearly subharmonic functions one uses the slightly stronger assumption that $u \in \mathcal{L}_{\text{loc}}^1(D)$, see e.g. [7, p. 14]. However, our above, slightly more general definition seems to be more practical, see [21, Proposition 2.1 (iii) and Proposition 2.2 (vi) and (vii)].

1.4 Quasi-nearly subharmonic functions

Let $K \geq 1$. A Lebesgue measurable function $u : D \rightarrow [-\infty, +\infty)$ is *K-quasi-nearly subharmonic*, if $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$ and if there is a constant $K = K(N, u, D) \geq 1$ such that for all $\overline{B^N(x, r)} \subset D$,

$$u_M(x) \leq \frac{K}{v_N r^N} \int_{B^N(x, r)} u_M(y) dm_N(y)$$

for all $M \geq 0$. Here $u_M := \max\{u, -M\} + M$. A function $u : D \rightarrow [-\infty, +\infty)$ is *quasi-nearly subharmonic*, if u is *K-quasi-nearly subharmonic* for some $K \geq 1$.

A Lebesgue measurable function $u : D \rightarrow [-\infty, +\infty)$ is *K-quasi-nearly subharmonic n.s. (in the narrow sense)*, if $u^+ \in \mathcal{L}_{\text{loc}}^1(D)$ and if there is a constant $K = K(N, u, D) \geq 1$ such that for all $\overline{B^N(x, r)} \subset D$,

$$u(x) \leq \frac{K}{v_N r^N} \int_{B^N(x, r)} u(y) dm_N(y).$$

A function $u : D \rightarrow [-\infty, +\infty)$ is *quasi-nearly subharmonic n.s.*, if u is *K-quasi-nearly subharmonic n.s.* for some $K \geq 1$.

Quasi-nearly subharmonic functions (perhaps with a different terminology) have previously been considered at least in [13], [12], [16], [18], [19], [14], [21], [8] and [4]. We

recall here only that this function class includes, among others, subharmonic functions, and, more generally, quasiharmonic (see e.g. [9, p. 309], [3, p. 136], [7, p. 26]) and also nearly subharmonic functions (see e.g. [7, p. 14]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, and polyharmonic functions. Also, the class of Harnack functions is included, thus, among others, nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations; in particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see [23]. Observe that already Domar in [5, p. 430] has pointed out the relevance of the class of (nonnegative) quasi-nearly subharmonic functions. For, at least partly, an even more general function class, see [6].

For basic properties of quasi-nearly subharmonic functions, see the above references, especially [14] and [21]. Here we recall only the following:

- (i) A K -quasi-nearly subharmonic function n.s. is K -quasi-nearly subharmonic, but not necessarily conversely.
- (ii) A nonnegative Lebesgue measurable function is K -quasi-nearly subharmonic if and only if it is K -quasi-nearly subharmonic n.s.
- (iii) A Lebesgue measurable function is 1-quasi-nearly subharmonic if and only if it is 1-quasi-nearly subharmonic n.s. and if and only if it is nearly subharmonic (in the sense defined above).
- (iv) If $u: D \rightarrow [0, +\infty)$ is quasi-nearly subharmonic and $\psi: [0, +\infty) \rightarrow [0, +\infty)$ is permissible, then $\psi \circ u$ is quasi-nearly subharmonic in D .
- (v) Harnack functions are quasi-nearly subharmonic.

Recall that a function $\psi: [0, +\infty) \rightarrow [0, +\infty)$ is permissible, if there exist an increasing (strictly or not), convex function $\psi_1: [0, +\infty) \rightarrow [0, +\infty)$ and a strictly increasing surjection $\psi_2: [0, +\infty) \rightarrow [0, +\infty)$ such that $\psi = \psi_2 \circ \psi_1$ and such that the following conditions are satisfied:

- (a) ψ_1 satisfies the Δ_2 -condition,
- (b) ψ_2^{-1} satisfies the Δ_2 -condition,
- (c) the function $t \mapsto \frac{\psi_2(t)}{t}$ is quasi-decreasing, i.e. there is a constant $C = C(\psi_2) > 0$ such that

$$\frac{\psi_2(s)}{s} \geq C \frac{\psi_2(t)}{t}$$

for all $0 \leq s \leq t$.

Recall also that a function $\phi: [0, +\infty) \rightarrow [0, +\infty)$ satisfies the Δ_2 -condition, if there is a constant $C = C(\phi) \geq 1$ such that $\phi(2t) \leq C\phi(t)$ for all $t \in [0, +\infty)$.

Examples of permissible functions are: $\psi_1(t) = t^p$, $p > 0$, and $\psi_2(t) = ct^{p\alpha}[\log(\delta + t^{p\gamma})]^\beta$, $c > 0$, $0 < \alpha < 1$, $\delta \geq 1$, $\beta, \gamma \in \mathbb{R}$ such that $0 < \alpha + \beta\gamma < 1$, and $p \geq 1$. And also functions of the form $\psi_3 = \phi \circ \phi$, where $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is a concave surjection whose inverse ϕ^{-1} satisfies the Δ_2 -condition and $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is an increasing, convex function satisfying the Δ_2 -condition. See e.g. [16], [18], [19] and [14, Lemma 1 and Remark 1].

2. ON THE SUBHARMONICITY OF OF SEPARATELY SUBHARMONIC FUNCTIONS

2.1 Armitage's and Gardiner's result

Armitage and Gardiner 1993: Let Ω be a domain in \mathbb{R}^{m+n} , $m \geq n \geq 2$. Let $u: \Omega \rightarrow [-\infty, +\infty)$ be such that

- (a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is subharmonic,

- (b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is subharmonic,

- (c) $\phi(\log^+ u^+)$ is locally integrable in Ω , where $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is an increasing function such that

$$\int_1^{+\infty} s^{(n-1)/(m-1)} (\phi(s))^{-1/(m-1)} ds < +\infty.$$

Then u is subharmonic in Ω .

2.2 Previous results

Below we list the previous results of Lelong [9, Théorème 1 bis, p. 315], of Avanissian [3, Théorème 9, p. 140], see also [7, Theorem, p. 31], of Arsove [2, Theorem 1, p. 622] and of ours [15, Theorem 1, p. 69]. Observe that though Armitage's and Gardiner's result includes all of them, its proof is, however, based either on the previous result of Avanissian, or of Arsove, or of Riihentausta. Observe here that all these three results have different and independent proofs.

Lelong 1945: Let Ω be a domain in \mathbb{C}^n , $n \geq 2$, and let $u: \Omega \rightarrow [-\infty, +\infty)$ be separately subharmonic (i.e. subharmonic with respect to each complex variable z_j , when the other variables $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$ are fixed, $j = 1, 2, \dots, n$). If u is locally bounded above in Ω , then u is subharmonic.

See also [10, Théorème 1 b, p. 290] and [11, Proposition 3, p. 24].

Avanissian 1961: As the result of Armitage and Gardiner 1993, except that (c) is now replaced with the stronger condition: (c') u is locally bounded above in Ω .

Arsove 1966: As the result of Armitage and Gardiner 1993, except that (c) is now replaced with the stronger condition: (c'') $u \in \mathcal{L}_{\text{loc}}^1(\Omega)$.

Riihentausta 1989: As the result of Armitage and Gardiner 1993, except that (c) is now replaced with the stronger condition: (c''') $u \in \mathcal{L}_{\text{loc}}^p(\Omega)$ for some $p > 0$.

3. ON QUASI-NEARLY SUBHARMONICITY OF SEPARATELY QUASI-NEARLY SUBHARMONIC FUNCTIONS

3.1 A counterpart to Armitage's and Gardiner's result

Next a counterpart to the cited result of Armitage and Gardiner [1, Theorem 1, p. 256] for quasi-nearly subharmonic functions. We will present our result with a complete proof elsewhere. We mention here only that the method of proof is more or less a straightforward and technical, though by no means easy, modification of Armitage's and Gardiner's argument [1, proof of Proposition 2, pp. 257-259, proof of Theorem 1, pp. 258-259] and of Domar's argument [5, Lemma 1, pp. 431-432 and 430]. Our result gives a slight generalization, at least seemingly and formally, also for the classical situation of separately subharmonic functions.

3.2 The result

Theorem. ([22, Theorem 4.2]) *Let Ω be a domain in \mathbb{R}^{m+n} , $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \rightarrow [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied:*

- (a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is K -quasi-nearly subharmonic.

- (b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is K -quasi-nearly subharmonic.

- (c) *There are increasing functions $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ and $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that*

- (c1) *the inverse functions φ^{-1} and ψ^{-1} are defined on $[\inf\{\varphi(s_1 - s_0), \psi(s_1 - s_0)\}, +\infty)$,*

- (c2) *$2K(\psi^{-1} \circ \varphi)(s - s_0) \leq (\psi^{-1} \circ \varphi)(s)$ for all $s \geq s_1$,*

- (c3) *the function*

$$[s_1 + 1, +\infty) \ni s \mapsto \frac{(\psi^{-1} \circ \varphi)(s + 1)}{(\psi^{-1} \circ \varphi)(s)} \in \mathbb{R}$$

is bounded,

- (c4) $\int_{s_0+1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty,$

- (c5) $\psi \circ u^+ \in \mathcal{L}_{\text{loc}}^1(\Omega).$

Then u is quasi-nearly subharmonic in Ω .

Corollary 1. *Let Ω be a domain in \mathbb{R}^{m+n} , $m \geq n \geq 2$, and let $K \geq 1$. Let $u : \Omega \rightarrow [-\infty, +\infty)$ be a Lebesgue measurable function. Suppose that the following conditions are satisfied:*

- (a) *For each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is K -quasi-nearly subharmonic.

- (b) *For each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is K -quasi-nearly subharmonic.

- (c) *There is a strictly increasing function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ such that*

- (c1) $\int_{s_0+1}^{+\infty} \frac{s^{(n-1)/(m-1)}}{\varphi(s-s_0)^{1/(m-1)}} ds < +\infty,$

- (c2) $\varphi(\log^+ u^+) \in \mathcal{L}_{\text{loc}}^1(\Omega).$

Then u is quasi-nearly subharmonic in Ω .

Though the next corollary does not contain Armitage's and Gardiner's result, it nevertheless improves the cited previous results of Lelong, of Avanissian, of Arsove and of ours, and as such it might be of some interest. As a matter of fact, it is already a corollary of our previous result [21, Theorem 3.1], but it can be considered as a consequence of the present Theorem, too.

Corollary 2. ([21, Corollary 3.3]) *Let Ω be a domain in \mathbb{R}^{m+n} , $m, n \geq 2$. Let $u : \Omega \rightarrow [-\infty, +\infty)$ be such that*

- (a) *for each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is nearly subharmonic, and, for almost every $y \in \mathbb{R}^n$, subharmonic,

- (b) *for each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is upper semicontinuous, and, for almost every $x \in \mathbb{R}^m$, subharmonic,

- (c) *there exists a non-constant permissible function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\psi \circ u^+ \in \mathcal{L}_{\text{loc}}^1(\Omega)$.*

Then u is subharmonic in Ω .

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