COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 11, Number 1, 1-15, 2003

Holomorphic approximation in Fréchet spaces

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We prove a Runge type approximation result in a class of Fréchet spaces that includes the space s of rapidly decreasing sequences.

1. Introduction.

In this paper we shall prove a Runge type approximation result in a class of Fréchet spaces that includes the space s of rapidly decreasing sequences. The space s is special among Fréchet spaces. For one thing, spaces frequently occurring in geometry, such as smooth functions on a closed manifold, are isomorphic to it; it also has certain universality properties. As demonstrated in [L4], approximations are a key ingredient to cohomology vanishing in Banach spaces. Similarly, in [L5] we shall use the results of this paper to study analytic cohomology in Fréchet spaces.

The class of spaces we shall consider are certain generalized sequence spaces. Let Γ be a set and $p : \mathbb{R} \times \Gamma \to (0, \infty)$ a function such that $\log p(\cdot, \gamma)$ is even and convex for every $\gamma \in \Gamma$. If $x : \Gamma \to \mathbb{C}$ and $\theta \in \mathbb{R}$, define

$$\|x\|_\theta = \sum_\gamma p(\theta, \gamma)|x(\gamma)| \leq \infty, \tag{1.1}$$

and put

$$X = \{x : \Gamma \to \mathbb{C} \mid \|x\|_\theta < \infty \text{ for every } \theta\}. \tag{1.2}$$

The norms $\| \ \|_\theta$, $\theta \in \mathbb{R}$, endow X with the structure of a complete locally convex topological vector space. Convexity of $\log p$ implies that the norms $\| \ \|_\theta$ for $\theta \in \mathbb{Z}$ would induce the same topology, hence X is a Fréchet space. If $\theta, r \in \mathbb{R}$, $r > 0$, put $B_\theta(r) = \{x \in X : \|x\|_\theta < r\}$. With these assumptions and notation we shall prove

Theorem 1.1. *Given $0 < r < R$, $\theta \in \mathbb{R}$, any holomorphic function $f : B_\theta(R) \to \mathbb{C}$ can be approximated by a holomorphic $g : X \to \mathbb{C}$, uniformly on $B_\theta(r)$.*

[1]Research partially supported by an NSF grant.

When p is independent of θ, $X \approx l^1(\Gamma)$, a case covered in [L1]. When $\Gamma = \mathbb{N}$ and $p(\theta,\gamma) = \gamma^{|\theta|}$, the space X becomes s. Certain weights p define spaces that are isomorphic to complemented subspaces of s, such as $p(\theta,\gamma) = 2^{\gamma|\theta|}$, with the corresponding $X \approx \mathcal{O}(\mathbb{C})$. Beyond these, we do not know of weights that would lead to spaces X that occur in other contexts as well. Still, Theorem 1.1 is formulated in the given generality, rather than just for $X = s$, because this formulation brings out the features of s that matter for approximation. Thus nuclearity or separability are irrelevant. On the other hand, convexity of $\log p$ has to do with the existence of a so called dominant norm—all $\| \ \|_\theta$ will be such in the case at hand,—a condition that is necessary for approximation, see section 6. (The assumption that p is even simplifies the exposition but could be done away with.) We do not know how important it is that the norms in (1.1) are isomorphic to l^1 norms. In Banach spaces approximation theorems are available for much more general norms, see [L2].

It is remarkable that Theorem 1.1 depends on a certain convexity assumption. The importance of convexity to complex analysis has been a main theme in the twentieth century. Theorem 1.1 and the related condition on dominant norms reinforce this idea. There is a difference, though: while previously geometric convexity (pseudoconvexity, plurisubharmonicity) was the issue, here we deal with a convexity property of the topology.

For the purposes of [L5] Theorem 1.1 above is too special. In sections 4, 5 we shall formulate and prove the generalizations that will be needed in [L5]. For the moment we content ourselves with sketching a proof in the situation of Theorem 1.1; it is an extension of the proof in [L1].

Thus, one first expands f in a monomial series

$$f(x) = \sum_k a_k x^k = \sum_k a_k \prod_{\gamma\in\Gamma} x(\gamma)^{k(\gamma)}, \quad a_k \in \mathbb{C}, \tag{1.3}$$

where k runs through multiindices, i.e. maps $k : \Gamma \to \mathbb{N} \cup \{0\}$ with finite support. One shows that the series in (1.3) converges to f uniformly on compact subsets of $B_\theta(R)$, and proves sharp estimates for the coefficients a_k ("Cauchy-Hadamard formula"). These estimates are expressed in terms of a certain semigroup Σ of compact operators in X. It is in establishing various properties of Σ (Theorem 2.5) that convexity of $\log p$ is needed. The proof is concluded by showing that with a carefully chosen family of positive numbers ω_k the function $g(x) = \sum_{|a_k|\geq\omega_k} a_k x^k$ has the required properties.

We shall assume the reader has some familiarity with basic complex analysis in finite and infinite dimensions. [H, D, No] are good references for much more than what we need here. We shall write $\mathcal{O}(M; M')$ for the family of holomorphic maps between complex manifolds M, M', and $\mathcal{O}(M)$ when $M' = \mathbb{C}$.

2. Topology.

We start by introducing notation and terminology, partly following [L2]. Thus, V will always denote a sequentially complete, locally convex topological vector space over $\mathbb{C}$, whose topology is given by a family Ψ of seminorms. Suppose v, $v_j \in V$, for j belonging to some index set J. We write $\sum v_j = v$ to mean that for any $\psi \in \Psi$ and $\epsilon > 0$ there is a finite $J_0 \supset J$ such that $\psi(v - \sum_{J_1} v_j) < \epsilon$ whenever $J_1 \supset J_0$ is finite. We say that a series $\sum v_j$ is normally convergent if $\sum \psi(v_j) < \infty$ for all $\psi \in \Psi$. If only countably many v_j differ from zero then normal convergence of $\sum v_j$ implies $\sum v_j = v$ for some $v \in V$. Suppose S is an arbitrary set and $f_j : S \to V$, $j \in J$. We say that $\sum f_j$ converges normally on S if $\sum \sup_S \psi(f_j) < \infty$ for all $\psi \in \Psi$, and that $\sum f_j = f : S \to V$ uniformly if for every $\psi \in \Psi$ and $\epsilon > 0$ there is a finite $J_0 \subset J$ such that $\sup_S \psi(f - \sum_{J_1} f_j) < \epsilon$ whenever $J_1 \supset J_0$ is finite. Pointwise and normal convergence on S together imply uniform convergence on S. If S is a topological space, the f_j are continuous, and $f = \sum f_j$ converges uniformly on S then f is also continuous. Similarly, if S is an open subset of a locally convex space, the f_j are holomorphic, and $f = \sum f_j$ converges uniformly on S then f is holomorphic.

If Γ, p, $\| \, \|_\theta$, X are as in the introduction, we shall refer to X as a simple space, a term we plan to restrict to this paper. For technical reasons we shall have to deal with l^∞ sums of finitely many simple spaces as well, which we call semisimple. Thus a semisimple space X is of the form

$$X = \left\{ x : \Gamma \to \mathbb{C} \mid \max_{1 \le j \le n} \sum_{\gamma \in \Gamma^j} p(\theta, \gamma) |x(\gamma)| = \|x\|_\theta < \infty \right\},$$

where Γ^j, $1 \le j \le n$, partition Γ, and $p : \mathbb{R} \times \Gamma \to (0, \infty)$ is such that $\log p(\cdot, \gamma)$ is even and convex for all $\gamma \in \Gamma$. The spaces $X_j = \{x \in X : \operatorname{supp} x \subset \Gamma^j\}$ with the inherited norms $\| \, \|_\theta$ are simple, and $X = \bigoplus X_j$ (l^∞ sum). In what follows, X will always denote a semisimple space, and Γ, Γ^j, p, $\| \, \|_\theta$ will be as above. We write $B_\theta(r) = \{x \in X : \|x\|_\theta < r\}$.

Let $I \subset \mathbb{R}$ be an interval. A function $u : I \to [0, \infty)$ is called log-convex if $u(\alpha\theta + (1-\alpha)\theta') \le u(\theta)^\alpha u(\theta')^{1-\alpha}$ for any θ, $\theta' \in I$, $\alpha \in (0,1)$. That is, $\log u$ (allowed to take the value $-\infty$) is to be convex. If such a u vanishes on I then it must be identically 0 on int I.

Lemma 2.1. *Suppose u_j are log-convex on I and $c_j \in [0,\infty)$ for j in some index set J. If $u = \sum c_j u_j$ converges then it is log-convex.*

Proof. This is certainly not new. Assuming J is finite, with an arbitrary collection ξ of numbers $\xi_j > 0$, $u_\xi = \prod_j (c_j u_j/\xi_j)^{\xi_j}$ is log-convex. Since $u = \sup_\xi u_\xi^{1/\sum \xi_j}$ by the inequality between the arithmetic and geometric means, u is indeed log-convex. By passing to the limit the lemma is obtained for arbitrary J.

Proposition 2.2. *If θ, $\theta' \in \mathbb{R}$, $\alpha \in [0,1]$, and $x \in X$ then*

$$\|x\|_{\alpha\theta+(1-\alpha)\theta'} \le \|x\|_\theta^\alpha \|x\|_{\theta'}^{1-\alpha}, \tag{2.1}$$

i.e. $\|x\|_\theta$ is a log-convex function of θ. Also, it is increasing for $\theta \ge 0$.

Proof. This follows from Lemma 2.1 because the maximum of log-convex functions is also log-convex; and because even, convex functions increase on the positive half line.

(2.1) implies that X is a DN space, i.e. it has a dominant norm, see [V1].

Proposition 2.3. *The function $\nu(\theta, x) = \|x\|_\theta$ is continuous on $\mathbb{R}\times X$.*

Proof. Proposition 2.2 implies ν is continuous for fixed x. In addition, if $|\theta| \le a$

$$\big|\|x\|_\theta - \|y\|_\theta\big| \le \|x-y\|_\theta \le \|x-y\|_a, \qquad \text{hence}$$
$$\nu(\theta,x) - \nu(\tau,y) = (\|x\|_\theta - \|y\|_\theta) + (\|y\|_\theta - \|y\|_\tau) \to 0$$

as $(\theta,x) \to (\tau,y)$.

Recall that a set $S \subset X$ is bounded if $\sup_{x\in S}\|x\|_\theta < \infty$ for all θ.

Proposition 2.4. *A set $K \subset X$ has compact closure if and only if it is bounded and for every θ the series*

$$\sum_\gamma p(\theta,\gamma)|x(\gamma)| \tag{2.2}$$

converges uniformly for $x \in K$.

Proof. The series (2.2) converges for $x \in X$, and its partial sums are uniformly equicontinuous:

$$\Big|\sum_{\gamma\in\Gamma_0} p(\theta,\gamma)|x(\gamma)| - \sum_{\gamma\in\Gamma_0} p(\theta,\gamma)|y(\gamma)|\Big| \le n\|x-y\|_\theta,$$

$\Gamma_0 \subset \Gamma$ finite. If K has compact closure, the Arzelà-Ascoli theorem implies (2.2) converges uniformly; also K must be bounded.

Conversely, assume K is bounded and (2.2) converges uniformly. In a complete metric space such as X, having compact closure is equivalent to being totally bounded. To see this latter property, fix θ and $\epsilon > 0$, choose a finite $\Gamma_0 \subset \Gamma$ such that

$$\sum_{\Gamma \setminus \Gamma_0} p(\theta, \gamma)|x(\gamma)| < \frac{\epsilon}{2} \quad \text{for } x \in K.$$

Denote the characteristic function of Γ_0 by $\chi : \Gamma \to \{0, 1\}$, and introduce the projection $\pi(x) = \chi x$. Since $\pi(K)$ is a bounded set in a finite dimensional space, it can be covered by finitely many balls $B_\theta(\epsilon/2) + x_i$, $i = 1, \dots, m$. It follows that K is covered by the balls $B_\theta(\epsilon) + x_i$, so that it is indeed totally bounded.

Now consider a bounded function $\sigma : \Gamma \to [0, \infty)$. Multiplication by σ induces a continuous linear operator $x \mapsto \sigma x$ in X, and thus we obtain an action of the multiplicative semigroup of bounded functions $\sigma : \Gamma \to [0, \infty)$ on X. If $A \subset X$, let σA denote its image under σ, and consider

$$\Sigma = \{\sigma : \Gamma \to [0, 1) \mid \sigma B_\theta(1) \text{ has compact closure for some } \theta\}. \tag{2.3}$$

Theorem 2.5. *(a) If $\sigma \in \Sigma$ and $\epsilon > 0$ then $\Gamma_0 = \{\gamma \in \Gamma : \sigma(\gamma) > \epsilon\}$ is finite;*

(b) Σ is closed under multiplication;

(c) if $\sigma \in \Sigma$ and α is a positive number then $\sigma^\alpha \in \Sigma$;

(d) if $\sigma \in \Sigma$ then $\sigma B_\theta(R)$ has compact closure in $B_\theta(R)$ for all $\theta \in \mathbb{R}$, $R > 0$;

(e) conversely, for any compact $L \subset B_\theta(R)$ there are $\sigma \in \Sigma$ and a compact $L_1 \subset B_\theta(R)$ such that $L = \sigma L_1$.

Proof. It will suffice to give the proof under the assumption that X is simple.

(a) With $Y = \{x \in X : \text{ supp } x \subset \Gamma_0\}$, the set $Y \cap \sigma B_\theta(1)$ is a neighborhood of $0 \in Y$. If it has compact closure then Y must be finite dimensional, i.e. Γ_0 is finite.

(b) follows simply because the continuous image of a compact set is also compact.

(c) If $\alpha \geq 1$ then $\sigma^\alpha(\gamma) \leq \sigma(\gamma)$ and the claim is obvious. Otherwise let $\sigma B_\theta(1)$ have compact closure, and with arbitrary $\Gamma_1 \subset \Gamma$, $x \in X$, $\tau \in \mathbb{R}$, and variable $\alpha \in [0,1]$ put

$$u(\alpha) = \sum_{\gamma \in \Gamma \setminus \Gamma_1} p(\theta + \alpha\tau, \gamma)\sigma(\gamma)^\alpha |x(\gamma)|; \tag{2.4}$$

we use the convention $0^0 = 1$. Lemma 2.1 implies u is log-convex, whence

$$u(\alpha) \leq u(0)^{1-\alpha} u(1)^\alpha \leq u(1)^\alpha, \tag{2.5}$$

provided $x \in B_\theta(1)$. Use this first with $\Gamma_1 = \emptyset$, to obtain

$$\|\sigma^\alpha x\|_{\theta+\alpha\tau} \leq \|\sigma x\|_{\theta+\tau}^\alpha.$$

Thus $\| \ \|_{\theta+\alpha\tau}$ is bounded on $\sigma^\alpha B_\theta(1)$. Since τ is arbitrary, this means $\sigma^\alpha B_\theta(1)$ is bounded, provided $\alpha \in (0,1]$. To show $\sigma^\alpha B_\theta(1)$ has compact closure, we use Proposition 2.4. Fix $\epsilon > 0$, choose a finite $\Gamma_1 \subset \Gamma$ such that

$$\sum_{\Gamma \setminus \Gamma_1} p(\theta + \tau, \gamma)\sigma(\gamma)|x(\gamma)| < \epsilon, \quad x \in B_\theta(1).$$

By (2.5)

$$\sum_{\Gamma \setminus \Gamma_1} p(\theta + \alpha\tau, \gamma)\sigma(\gamma)^\alpha |x(\gamma)| < \epsilon^\alpha, \quad x \in B_\theta(1),$$

so that $\sum_\Gamma p(\theta + \alpha\tau, \gamma)|y(\gamma)|$ converges uniformly for $y \in \sigma^\alpha B_\theta(1)$. This being true for all $\tau \in \mathbb{R}$, $\sigma^\alpha B_\theta(1)$ has compact closure by Proposition 2.4.

(d) If $\sigma \in \Sigma$, then by part (c) $\sigma^{1/2} B_\omega(1)$ has compact closure for some ω. Fix $\theta \in \mathbb{R}$, let $\tau = 2(\omega - \theta)$ and $c = \sup_{y \neq 0} \|\sigma^{1/2} y\|_{\theta+\tau} / \|y\|_\omega < \infty$. With $\Gamma_1 = \emptyset$ and u as in (2.4) we have $u(1/2) \leq u(0)u(1)/u(1/2)$, or

$$\|\sigma^{1/2} x\|_\omega \leq \|x\|_\theta \|\sigma x\|_{\theta+\tau} / \|\sigma^{1/2} x\|_\omega \leq c\|x\|_\theta.$$

In other words, $\sigma^{1/2} B_\theta(R) \subset B_\omega(cR)$, and so $\sigma B_\theta(R) \subset \sigma^{1/2} B_\omega(cR)$ has compact closure. This closure is contained in $B_\theta(R)$, because $\sup \sigma < 1$ by part (a), and (d) indeed holds.

(e) We can assume $\theta \geq 0$. Fix $q \in (\sup_L(\| \ \|_\theta / R)^{1/2}, 1)$, and construct a sequence $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma$, each Γ_m finite, so that

$$\sum_{\Gamma \setminus \Gamma_m} p(\tau, \gamma)|x(\gamma)| \leq 4^{-m}(1-q)^m q^2 R, \qquad x \in L, \ \tau = \theta + m. \tag{2.6}$$

Since the left hand side increases with $\tau \geq 0$, the same inequality holds if $|\tau| \leq \theta + m$. (2.6) implies $x(\gamma) = 0$ if $x \in L$, $\gamma \notin \bigcup_m \Gamma_m = \Gamma^*$. Define

$$\sigma(\gamma) = \begin{cases} 2^{-m}q, & \text{for } \gamma \in \Gamma_{m+1} \setminus \Gamma_m \\ 0, & \text{for } \gamma \notin \Gamma^*, \end{cases}$$

and a closed set $L_1 = \{y \in X : \sigma y \in L, y(\gamma) = 0 \text{ if } \gamma \notin \Gamma^*\}$. Obviously $\sigma L_1 = L$. Since for any $y \in L_1$, $N = 0, 1, \dots, |\tau| \leq \theta + N$, and $x = \sigma y$

$$\begin{aligned} \sum_{\Gamma \setminus \Gamma_N} p(\tau, \gamma)|y(\gamma)| &\leq \sum_{m \geq N} \sum_{\Gamma_{m+1} \setminus \Gamma_m} 2^m q^{-1} p(\tau, \gamma)|x(\gamma)| \\ &\leq \sum_{m \geq N} 2^{-m}(1-q)^m qR < 2^{-N}R, \end{aligned}$$

$L_1 \subset B_\theta(R)$ is compact by Proposition 2.4.

3. Monomial expansions.

Now we bring in the torus

$$T = \{t : \Gamma \to \mathbb{R}/\mathbb{Z}\}; \tag{3.1}$$

with the product topology T is a compact Abelian group. In this section we shall consider certain complex manifolds Ω on which T acts by biholomorphisms. The action will induce the expansion of functions $f \in \mathcal{O}(\Omega)$ in so called monomial series, and we shall study convergence properties of such series.

Some more notation and terminology. A function $k : \Gamma \to \mathbb{N} \cup \{0\}$ with finite support will be called a multiindex. In this paper k will always stand for a multiindex, $\#k$ for the cardinality of its support, and $|k|$ for $\sum_\gamma k(\gamma)$. If $t \in T$ we write $k \cdot t$ for $\sum_\gamma k(\gamma)t(\gamma) \in \mathbb{R}/\mathbb{Z}$; if $z : \Gamma \to \mathbb{C}$ we write z^k for $\prod_\gamma z(\gamma)^{k(\gamma)} \in \mathbb{C}$. The Haar measure on T of total mass 1 will be denoted dt. With X a semisimple space we define an action $\overline{\rho}$ of T on X by

$$(\overline{\rho}_t x)(\gamma) = e^{2\pi i t(\gamma)} x(\gamma), \quad t \in T, \ x \in X. \tag{3.2}$$

This is a continuous action, isometric for all norms $\| \ \|_\theta$.

If D is a topological space, $s : D \to (0, \infty)$ and $\theta : D \to \mathbb{R}$ are continuous functions, define

$$D_\theta(s) = \{(\xi, x) \in D \times X : \ \|x\|_{\theta(\xi)} < s(\xi)\}. \tag{3.3}$$

From now on we assume D is a complex manifold, locally biholomorphic to a Fréchet space. Proposition 2.3 implies $\Omega = D_\theta(s)$ is open in $D \times X$, hence it is a complex manifold.

For $\xi \in D$ and k a multiindex define

$$(3.4)\qquad \begin{aligned} M_k(\xi) &= \sup_{\|x\|_{\theta(\xi)}<s(\xi)} |x^k|, \qquad \text{and} \\ \Delta_{\theta,s}(q,\xi,x) &= \sum_k |q|^{\#k}|x^k|/M_k(\xi), \qquad q\in\mathbb{C}, \quad (\xi,x)\in\Omega. \end{aligned}$$

Clearly, for any monomial cx^k

$$(3.5)\qquad |c| = \sup_{\|x\|_{\theta(\xi)}<s(\xi)} |cx^k|/M_k(\xi).$$

Proposition 3.1. *(a) The series in (3.4) converges uniformly on compact subsets of $\mathbb{C}\times\Omega$, and $\Delta_{\theta,s}$ is continuous.*

(b) Given $\mu\in(0,1)$ there is a $q\neq 0$ such that $\Delta_{\theta,s}(q,\cdot)$ is bounded on $D_\theta(\mu s)$.

Proof. If $X = \bigoplus X_j$ is the decomposition of X into simple spaces, corresponding to a partition $\Gamma = \bigcup \Gamma_j$, and $\Delta^{(j)}_{\theta,s}$ are the associated functions, one easily checks that

$$\Delta_{\theta,s}(q,\xi,x) = \prod_j \Delta^{(j)}_{\theta,s}(q,\xi,x|\Gamma_j).$$

Therefore it suffices to prove the Proposition for X simple, which we shall henceforward assume. Define a continuous map $A : D\times X \to l^1(\Gamma)$ by $A(\xi,x)(\gamma) = p(\theta(\xi),\gamma)x(\gamma)/s(\xi)$. The image of $\{\xi\}\times B_{\theta(\xi)}(s(\xi))$ is dense in the unit ball B of $l^1(\Gamma)$. Hence $M_k = \sup_{y\in B}|y^k| = \sup_{\|x\|_{\theta(\xi)}<s(\xi)}|A(\xi,x)^k|$, and setting

$$\Delta(q,y) = \sum_k |q|^{\#k}|y^k|/M_k, \qquad y\in B,$$

we have $\Delta_{\theta,s}(q,z) = \Delta(q,Az)$ by (3.5). The proposition thus follows from the corresponding results in $l^1(\Gamma)$, see [L1, Théorème 2.1] and [L3, Lemma 4.1].

If V is a sequentially complete locally convex space whose topology is given by a family Ψ of seminorms, functions $f\in\mathcal{O}(\Omega;V)$ can be analyzed using the torus action

$$\rho_t(\xi,x) = (\xi,\overline{\rho}_t x), \qquad t\in T,\ (\xi,x)\in\Omega.$$

We expand f in a (partial) monomial series

$$f \sim \sum_k f_k, \quad f_k = \int_T e^{-2\pi i k \cdot t} \rho_t^* f \, dt, \tag{3.6}$$

with $f_k \in \mathcal{O}(\Omega; V)$. By restricting to the dense subset $\Omega_0 \subset \Omega$ of pairs (ξ, x) with supp x finite, one finds that $f_k(\xi, x) = a_k(\xi) x^k$, where $a_k \in \mathcal{O}(D; V)$. Convergence of finite dimensional Taylor series implies that $\sum f_k = f$ on Ω_0. We shall call f_k the monomial components of f.

Theorem 3.2. *(a) The monomial components of $f \in \mathcal{O}(\Omega; V)$ satisfy*

$$\sup_k \sigma^k \sup_{K_\theta(s)} \psi(f_k) < \infty, \quad \sigma \in \Sigma, \ \psi \in \Psi, \ K \subset D \text{ compact}, \tag{3.7}$$

and the series $\sum f_k$ converges to f, uniformly on compact subsets of Ω.

(b) Conversely, if $a_k \in \mathcal{O}(D; V)$ and $f_k(\xi, x) = a_k(\xi) x^k$ satisfy (3.7) then $\sum f_k$ converges to some $h \in \mathcal{O}(\Omega; V)$, uniformly on compact subsets of Ω. The monomial expansion of h is $\sum f_k$.

Proof. (b) Any compact subset of Ω can be covered with finitely many compact sets of form $K \times L \subset \Omega$, $K \subset D$, $L \subset X$, such that $\sup_K |\theta| = \tau < \infty$ and

$$\sup_{x \in L} \|x\|_\tau < \inf_K s = S. \tag{3.8}$$

Therefore, to prove uniform convergence on compacts it suffices to attend to compact sets of the above form. By (3.8) $L \subset B_\tau(S)$, and Proposition 2.2 implies $K \times B_\tau(S) \subset \Omega$. Using Theorem 2.5(e) choose $\sigma \in \Sigma$ and $L_1 \subset B_\tau(S)$ compact so that $\sigma L_1 = L$; thus $K \times L_1 \subset \Omega$. With arbitrary $\psi \in \Psi$, $\xi \in K$, $x \in L$, and $y \in L_1$ such that $\sigma y = x$ we have

$$\psi(f_k(\xi, x)) = \sigma^k \psi(f_k(\xi, y)) = \sigma^k \sup_{\|z\|_{\theta(\xi)} < s(\xi)} \psi(f_k(\xi, z)) |y^k| / M_k(\xi), \tag{3.9}$$

by (3.5). Since by Proposition 3.1 $\sum |y^k|/M_k(\xi)$ converges uniformly for $(\xi, y) \in K \times L_1$, (3.7) and (3.9) imply that $\sum \psi(f_k)$, hence also $\sum f_k$ converges, uniformly on $K \times L$. Thus $\sum f_k$ converges uniformly on compacts, its sum h is easily seen to be holomorphic, see e.g. [L3, Propositions 2.1, 2.2] that carry over to our current set up; and upon evaluating the integrals in (3.6) with $f = h$ one verifies that the monomial expansion of h is $\sum f_k$.

(a) To prove (3.7) let $S = \max_K s$ and $\alpha = \sup_\Gamma \sigma < 1$, cf. Theorem 2.5(a). By Theorem 2.5(d) the set

$$P = \{(\xi, \sigma x) : (\xi, x) \in K_\theta(s)\} \subset K \times \sigma B_0(S)$$

has compact closure. Indeed $\overline{P} \subset \Omega$: for given $(\xi, y) \in \overline{P}$, let $(\xi, y) = \lim_n(\xi_n, \sigma x_n)$ with $(\xi_n, x_n) \in K_\theta(s)$. Then $\|y\|_{\theta(\xi)} = \lim_n \|\sigma x_n\|_{\theta(\xi_n)} \leq \lim_n \alpha s(\xi_n) < s(\xi)$ shows $(\xi, y) \in \Omega$.

Therefore $\sup_P \psi(f) = c < \infty$, so that by (3.6) $c \geq \sup_P \psi(f_k) = \sigma^k \sup_{K_\theta(s)} \psi(f_k)$, and (3.7) follows. Now (b) implies the convergence of $\sum f_k$. Since its sum, a holomorphic function, agrees with f on Ω_0, the two must also agree on Ω.

4. Approximation.

In this section we are going to prove our main result. Let X be semisimple, Σ and V as before.

Theorem 4.1. *Let D be a (finite dimensional) Stein manifold, $D' \subset D$ open, $K \subset D'$ an $\mathcal{O}(D)$ convex compact. Let furthermore $r, R : D \to (0, \infty)$ and $\theta : D \to \mathbb{R}$ be continuous, $r < R$. With notation as in (3.3) suppose $f \in \mathcal{O}(D'_\theta(R); V)$, $\varphi \in \Psi$, and $\epsilon > 0$. Then there exists $g \in \mathcal{O}(D \times X; V)$ such that $\varphi(f - g) < \epsilon$ on $K_\theta(r)$.*

The proof will be based on Theorem 3.2. To facilitate calculations with estimates like (3.7) we start by formulating

Lemma 4.2. *(a) Suppose $\mathcal{K}$ is a set of multiindices, and for each $k \in \mathcal{K}$ we are given numbers c_k, $d_k \geq 0$ such that*

$$\sup_{k \in \mathcal{K}} \sigma^k c_k < \infty, \qquad \sup_{k \in \mathcal{K}} \sigma^k d_k < \infty$$

for all $\sigma \in \Sigma$. If α, $\beta \in (0, \infty)$ then

$$\sup_{k \in \mathcal{K}} \sigma^k c_k^\alpha d_k^\beta < \infty, \qquad \sigma \in \Sigma. \tag{4.1}$$

(b) If $Q > 0$, $\sigma \in \Sigma$ then $\sup_k \sigma^k Q^{\#k} < \infty$.

Proof. (a) If $\sigma \in \Sigma$ then $\sigma_1 = \sigma^{1/(2\alpha)}$, $\sigma_2 = \sigma^{1/(2\beta)} \in \Sigma$ by Theorem 2.5(c). Since $\sigma^k c_k^\alpha d_k^\beta = (\sigma_1^k c_k)^\alpha (\sigma_2^k d_k)^\beta$, (4.1) follows. (b) We can assume $Q > 1$.

Given $\sigma \in \Sigma$, the set $\Gamma_0 = \{\gamma \in \Gamma : \sigma(\gamma) \geq 1/Q\}$ is finite according to Theorem 2.5(a). Define $\sigma_1 : \Gamma \to [0,1)$ by

$$\sigma_1(\gamma) = \begin{cases} \sigma(\gamma), & \text{if } \gamma \in \Gamma_0 \\ Q\sigma(\gamma), & \text{if } \gamma \notin \Gamma_0. \end{cases}$$

Then $\sup_k \sigma^k Q^{\#k} \leq \sup_k \sigma_1^k Q^{|\Gamma_0|} \leq Q^{|\Gamma_0|}$.

Proposition 4.3. *With assumptions and notation as in Theorem 4.1, let $G \subset\subset D'$ be an open neighborhood of K. There exists $h \in \mathcal{O}(G \times X; V)$ such that $\varphi(f - h) < \epsilon$ on $K_\theta(r)$.*

Proof. Let $f = \sum f_k$ be the monomial expansion of f. Fix a $b \in (1, \infty)$ such that $b^2 r < R$ on G. With $Q > 1$, $\epsilon_1 > 0$ to be determined later, define

$$\mathcal{K} = \{k : \sup_{G_\theta(R)} \varphi(f_k) > Q^{-\#k} b^{|k|} \epsilon_1\} \tag{4.2}$$

and

$$h = \sum_{k \in \mathcal{K}} f_k. \tag{4.3}$$

First we prove that $h \in \mathcal{O}(G \times X; V)$. This would follow if we could show that (4.3) converges uniformly on compact subsets of $G_\theta(\kappa R)$, for any constant $\kappa > 1$. By Theorem 3.2(b) it therefore suffices to prove

$$\sup_{k \in \mathcal{K}} \sigma^k \sup_{G_\theta(\kappa R)} \psi(f_k) < \infty, \quad \sigma \in \Sigma, \ \psi \in \Psi. \tag{4.4}$$

To verify (4.4) we can assume $\psi \geq \varphi$ (otherwise replace ψ by $\psi + \varphi$). We have

$$\begin{aligned} &\sup_{k \in \mathcal{K}} \sigma^k c_k < \infty, \qquad c_k = Q^{-\#k} b^{|k|} / \sup_{G_\theta(R)} \psi(f_k), \\ &\sup_{k} \sigma^k d_k < \infty, \qquad d_k = Q^{\#k} \sup_{G_\theta(R)} \psi(f_k), \end{aligned} \tag{4.5}$$

by (4.2) resp. Theorem 3.2(a) and Lemma 4.2(b). Choose $\alpha > 0$ and $\beta = \alpha + 1$ so that $b^\alpha = \kappa$. Then for $k \in \mathcal{K}$

$$c_k^\alpha d_k^\beta = Q^{\#k} (b^\alpha)^{|k|} \sup_{G_\theta(R)} \psi(f_k) \geq \sup_{G_\theta(\kappa R)} \psi(f_k),$$

so that by Lemma 4.2(a), (4.5) implies (4.4), and h is indeed holomorphic on $G \times X$.

Next we estimate $f - h$ on $G_\theta(r)$. Let $s = R/b$ and $M_k(\xi)$ as in (3.4), then for $(\xi, x) \in G_\theta(s)$ (3.5) implies

$$\varphi(f_k(\xi, x)) \le |x^k| \sup_{G_\theta(s)} \varphi(f_k)/M_k(\xi). \tag{4.6}$$

When $k \notin \mathcal{K}$

$$\sup_{G_\theta(s)} \varphi(f_k) = b^{-|k|} \sup_{G_\theta(R)} \varphi(f_k) \le Q^{-\#k}\epsilon_1 \tag{4.7}$$

by (4.2), so that putting (4.3), (4.6), and (4.7) together

$$\varphi(f(\xi, x) - h(\xi, x)) \le \sum_{k \notin \mathcal{K}} \varphi(f_k(\xi, x)) \le \epsilon_1 \sum_k Q^{-\#k}|x^k|/M_k(\xi)$$

for $(\xi, x) \in G_\theta(s)$. Fix Q so large that $\Delta_{\theta,s}(1/Q, \cdot)$ is bounded on $D_\theta(s/b)$, cf. Proposition 3.1(b), then

$$\sup_{G_\theta(r)} \varphi(f - h) \le \epsilon_1 \sup_{D_\theta(s/b)} \Delta_{\theta,s}(1/Q, \cdot) < \epsilon,$$

provided ϵ_1 is small enough.

In particular, when $D = K(= D' = G)$ is a singleton, we obtain Theorem 1.1 for all semisimple X, even for vector valued f.

Proof of Theorem 4.1. Let $P = \max_K r$, then $K_\theta(r) \subset K \times B_0(P)$. In light of Proposition 4.3 all we need to prove is that if $G \subset D$ is an open neighborhood of K, and $h \in \mathcal{O}(G \times X; V)$ then there is a $g \in \mathcal{O}(D \times X; V)$ such that $\varphi(h - g) < \epsilon$ on $K \times B_0(P)$. Assume first that $D = \mathbb{C}^N$ and K is a polydisc centered at 0. Now $\mathbb{C}^N \oplus X$ is also semisimple, and with an appropriate choice of norms $\| \ \|'_\tau$ on it, $K \times \overline{B_0(P)}$ will be a ball $\{(\xi, x) \in \mathbb{C}^N \oplus X : \|(\xi, x)\|'_0 \le P\}$. Hence we can apply Theorem 1.1, by now proved for semisimple spaces and V valued functions, to obtain g as desired. The case of general D, G, K can be reduced to the special case of polydiscs, as explained in the proof of [L1, Théorème 1.1].

5. Approximation, bis.

For application in [L5] we shall need approximations in open sets that are defined in terms of a metric rather than a family of norms. First we shall discuss one of the many ways to metrize a Fréchet space.

Proposition 5.1. *Suppose that the topology of a Fréchet space X is induced by a family of norms $\| \quad \|_\theta$, $\theta \in (0,\infty)$, and $\|x\|_\theta$ is an increasing continuous function of θ. Define*

$$\|x\| = \inf\{\alpha \in (0,\infty) : \|x\|_{1/\alpha} \le \alpha\}, \qquad x \in X. \tag{5.1}$$

Then $\| \ \|$ is a pseudonorm that induces the topology of X, and $\|x\| < R$ is equivalent to $\|x\|_{1/R} < R$, $R \in (0,\infty)$.

That $\| \ \|$ is a pseudonorm means $\|x\| \ge 0$, with equality if and only if $x = 0$; $\|\lambda x\| \le \|x\|$ if $|\lambda| \le 1$; and $\|x+y\| \le \|x\| + \|y\|$.

Proof. Of the three axioms of pseudonorms we shall only verify the triangle inequality. By the intermediate value theorem, for any $x \ne 0$ there is a unique α such that $\|x\|_{1/\alpha} = \alpha$; and this $\alpha = \|x\|$. Suppose $y \ne 0$, and put $\beta = \|y\|$. Then

$$\|x+y\|_{1/(\alpha+\beta)} \le \|x\|_{1/(\alpha+\beta)} + \|y\|_{1/(\alpha+\beta)} \le \|x\|_{1/\alpha} + \|y\|_{1/\beta} = \alpha + \beta.$$

Hence, by (5.1), $\|x+y\| \le \alpha+\beta = \|x\| + \|y\|$. Since this inequality is trivial when x or $y = 0$, $\| \ \|$ is indeed a pseudonorm.

Next, if $\|x\| = \alpha < R$ then $\|x\|_{1/R} \le \|x\|_{1/\alpha} = \alpha < R$; and conversely, if $\|x\|_{1/R} < R$ then (5.1) (and continuity of $\|x\|_\theta$) implies $\|x\| < R$. From this it also follows that the balls $\{x \in X : \|x\| < R\}$, $R > 0$ form a neighborhood basis of $0 \in X$ as claimed.

Now return to our semisimple space X, and endow it with the pseudonorm (5.1). If D is a topological space, and $R : D \to (0,\infty)$ is continuous, let

$$D(R) = \{(\xi, x) \in D \times X : \|x\| < R(\xi)\}.$$

Theorem 5.2. *Let D be a Stein manifold, $D' \subset D$ open, $K \subset D'$ an $\mathcal{O}(D)$ convex compact. Let furthermore r, $R : D \to (0,\infty)$ be continuous, $r < R$. Given $f \in \mathcal{O}(D'(R); V)$, $\varphi \in \Psi$, and $\epsilon > 0$, there exists $g \in \mathcal{O}(D \times X; V)$ such that $\varphi(f-g) < \epsilon$ on $K(r)$.*

Proof. Since $D(R) = D_\theta(R)$ with $\theta = 1/R$, and $K(r) = K_{1/r}(r) \subset K_\theta(r)$ according to Proposition 5.1, the theorem follows from Theorem 4.1.

6. A necessary condition.

In this last section we want to point out that even for a weak version of our approximation theorems to hold in a Fréchet space X it is necessary that X have a dominant norm. This condition, also called (DN), requires that the topology of X be induced by norms $\| \ \|_n$, $n = 1, 2, \dots$ satisfying $\|x\|_n^2 \le \|x\|_1 \|x\|_{n+1}$. DN spaces were introduced by Vogt, and their relevance to complex analysis was further explored by himself, Dineen, Meise, and others, see [D, V1, V2, MV]. Our theorem below easily follows from the analysis of Meise and Vogt.

Theorem 6.1. *Suppose that in a Fréchet space X any neighborhood U of $0 \in X$ contains another neighborhood W of 0 such that holomorphic functions on U can be approximated by entire functions, uniformly on W. Then X has a dominant norm.*

Proof. Suppose X fails to have a dominant norm, and write $X = \mathbb{C} \oplus Y$, with Y a closed subspace. In [V1, MV] Meise and Vogt construct a sequence $h_j \in \mathcal{O}(Y)$ such that there is no $f \in \mathcal{O}(\mathbb{C} \oplus Y)$ with

$$f(j, \cdot) = h_j, \qquad j \in \mathbb{N}. \tag{6.1}$$

This will imply that for the sets $U_R = \{(\eta, y) \in \mathbb{C} \oplus Y : |\eta| < R\}$ there is no W as in the theorem.

Indeed, suppose there is a corresponding W for U_1. Then RW will do for $U_R = RU_1$. Let $\omega \in \mathcal{O}(\mathbb{C})$ have simple zeros at each $n \in \mathbb{N}$, and no other zero. Extend ω to $\mathbb{C} \oplus Y$, constant in the Y variable. Inductively construct $f_n \in \mathcal{O}(\mathbb{C} \oplus Y)$, $n \in \mathbb{N}$, such that

$$f_n(j, \cdot) = h_j, \qquad j = 1, 2, \dots, n,$$

and $|f_n - f_{n+1}| < 2^{-n}$ on nW. This can be done as follows. Take $f_1(\eta, y) = h_1(y)$; if f_n has been found, construct—by Lagrange interpolation—an $l \in \mathcal{O}(\mathbb{C} \oplus Y)$ such that $l(j, \cdot) = h_j$, $j \le n + 1$. Thus $(l - f_n)/\omega$ is holomorphic on U_n, and there is a $g \in \mathcal{O}(\mathbb{C} \oplus Y)$ such that

$$\left| \frac{(l - f_n)}{\omega} - g \right| < \frac{1}{2^n \sup_{nW} |\omega|}$$

on nW. Hence $f_{n+1} = l - \omega g$ is as required.

Now the properties of f_n imply that $f = \lim f_n \in \mathcal{O}(\mathbb{C} \oplus Y)$ satisfies (6.1), after all. This contradiction proves the theorem.

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Received November 16, 2000.