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On the Modelling of Thermo-Unelastic Periodic Composites: Microlocal Parameter Theory

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Summary

In the paper it is shown how to formulate certain homogenized models of thermo-elastic-unelastic periodic composites under large strains and large temperature gradients. The models obtained describe local stresses and heat fluxes in every material component in term of averaged displacement and temperature fields and certain extra unknowns called microlocal parameters.

1. Introduction

The modelling of composites is based on various averaged mathematical descriptions of nonhomogeneous material structures representing what are called the effective theories of composites, cf. [4]—[14]. The aim of the paper is to propose a certain general effective theory of thermo-elastic-unelastic composites with fine periodic structure under large strains and large temperature gradients. The idea of the method constitutes a generalization of the nonstandard approach to the homogenization of thermo-elastic composites given in [3].

Throughout the paper indices i, j as well as K, L and α, β run over 1, 2, 3; summation convention holds. Indices A, B and a, b run over $1, \ldots, M$ and $1, \ldots, m$, respectively, while index E runs over $1, \ldots, N$ (summation convention with respect to a, b, E holds if otherwise stated). For an arbitrary differentiable function $\varphi(\boldsymbol{X}, t)$, $\boldsymbol{X} = (X^\alpha)$, we define $\varphi_{,\alpha} \equiv \partial\varphi/\partial X^\alpha$ and $\dot{\varphi} \equiv \partial\varphi/\partial t$.

2. Exact Equations of Thermo-Elastic-Unelastic Composites

Let $(\boldsymbol{y}, t) \in R^3 \times R$, $\boldsymbol{y} \equiv (y^i)$, be the inertial coordinates in the Galilean space-time and $\mathscr{B}$ stands for a regular region in R^3 occupied at $t = t_0$ by the body under consideration in its natural state. Setting $x^K = \delta_i^{\ K} y^i$, $\boldsymbol{y} \in \mathscr{B}$, we define the rectilinear coordinates in $\mathscr{B}$. In $\mathscr{B}$ we also introduce the curvilinear coordinates by means of the known smooth mapping $x = \varkappa(X)$ with $X = (X^\alpha) \in \Omega$. Once for all we assume that $(X, t) \in \bar{\Omega} \times [t_0, t_f]$ are our independent variables. The position vectors and absolute temperatures of material points will be denoted by $\chi(X, t)$ and $\Theta(X, t)$, respectively. By $\varrho_\varkappa(X)$, $T_\varkappa = (T_\varkappa^{KL}(X, t))$, $h_\varkappa = (h_\varkappa^K(X, t))$ we denote the mass density, the second Piola-Kirchhoff stress tensor and the heat flux vector, respectively, related to $\mathscr{B}$. Moreover, let $b = (b^K(X, t))$, $\alpha(X, t)$, $\varepsilon(X, t)$, $\mathbf{s}(X, t)$, $\sigma(X, t)$ stand for the body forces, heat absorption, internal energy, boundary tractions and boundary heat supply, respectively. Define

$$\begin{gathered} J(X) \equiv \det \nabla\varkappa(X), \qquad A(X) \equiv (\nabla\varkappa(X))^{-1} = (A^\alpha{}_K(X)), \\ S^{k\alpha}(X, t) \equiv J(X)\, \chi^k_{,\beta}(X, t)\, A_K{}^\beta(X) T_\varkappa{}^{KL}(X, t) A_L{}^\alpha(X), \\ h^\alpha(X, t) \equiv J(X)\, A_K{}^\alpha(X)\, h_\varkappa{}^K(X, t), \qquad \varrho(X) \equiv J(X)\, \varrho_\varkappa(X), \end{gathered} \tag{2.1}$$

and assume that Γ, Π are the known parts of $\partial\Omega$. Under the forementioned denotations we shall postulate the law of motion and the heat conduction equation in the integral (weak) form given below which has to hold for all test functions $v_k \in C^1(\bar{\Omega})$, $\zeta \in C^1(\bar{\Omega})$ such that $v \mid \partial\Omega \setminus \bar{\Gamma} = 0$, $\zeta \mid \partial\Omega \setminus \bar{\Pi} = 0$:

$$\begin{aligned} \int_\Omega S^{k\alpha}(X, t)\, v_{k,\alpha}(X)\, dV &= \int_\Omega \varrho(X)\, [b^k(X, t) - \ddot{\chi}^k(X, t)]\, v_k(X)\, dV \\ &\quad + \int_\Gamma s^k(X, t)\, v_k(X)\, dA(X), \\ \int_\Omega h^\alpha(X, t)\, \zeta_{,\alpha}(X)\, dV &= \int_\Omega \big[\varrho(X)\, (\alpha(X, t) - \dot{\varepsilon}(X, t)) \\ &\quad + S^{k\alpha}(X, t)\, \dot{\chi}_{k,\alpha}(X, t)\big]\, \zeta(X)\, dV \\ &\quad + \int_\Pi \sigma(X, t)\, \zeta(X)\, dA(X), \qquad t \in [t_0, t_f], \end{aligned} \tag{2.2}$$

where $dV \equiv dX^1\, dX^2\, dX^3$ and $dA(X)$ is an element of $\partial\Omega$ at X.

Now we introduce the Lagrangian strain tensor $L \equiv 0.5(\nabla\chi^T\, \nabla\chi - \nabla\varkappa^T\, \nabla\varkappa$ and define the strain tensor $B(X, t) \equiv A^T(X)\, L(X, t)\, A(X)$ related to $\mathscr{B}$. We also introduce the temperature gradient $g(X, t)$ related to $\mathscr{B}$ by means of $g(X, t) \equiv A^T(X)\, \nabla\Theta$. We shall assume that the body under consideration is made of M homogeneous materials. Hence there is known the decomposition $\bar{\Omega} = \bigcup \bar{\Omega}_A$, $A = 1, \ldots, M$, where $\Omega_A \cap \Omega_B = \emptyset$ for every $A \neq B$ and where every Ω_A is a finite set of disjointed regular regions (for some A we may deal with one but multiconnected region Ω_A) in R^3, such that every $\varkappa(\Omega_A)$ is occupied in the natural state $\mathscr{B}$ by the A-th material component. All material properties related to $\mathscr{B}$ are assumed to be constant in every $\mathscr{B}_A \equiv \varkappa(\Omega_A)$. We also assume that every material component represents the elastic-unelastic material described with the aid of the internal state variables $V = (V^1, \ldots, V^S) \in R^S$ (cf. [21], where the full list of references and the detailed discussion of the constitutive relations can be found).

Hence for $A = 1, \ldots, M$ the following constitutive relations are assumed to be known:

$$\begin{aligned}
\boldsymbol{T}_\kappa(\boldsymbol{X}, t) &= \hat{\boldsymbol{T}}_\kappa^A\big(\boldsymbol{B}(\boldsymbol{X}, t), \Theta(\boldsymbol{X}, t), \boldsymbol{V}(\boldsymbol{X}, t)\big), \qquad \varrho_\kappa(\boldsymbol{X}) = \varrho_\kappa^A, \\
\varepsilon(\boldsymbol{X}, t) &= \hat{\varepsilon}^A\big(\boldsymbol{B}(\boldsymbol{X}, t), \Theta(\boldsymbol{X}, t), \boldsymbol{V}(\boldsymbol{X}, t)\big), \\
\boldsymbol{h}_\kappa(\boldsymbol{X}, t) &= \hat{\boldsymbol{h}}_\kappa^A\big(\boldsymbol{B}(\boldsymbol{X}, t), \Theta(\boldsymbol{X}, t), \boldsymbol{g}(\boldsymbol{X}, t), \boldsymbol{V}(\boldsymbol{X}, t)\big), \\
\dot{\boldsymbol{V}}(\boldsymbol{X}, t) &= \lambda_A \delta_A(f_A, f_A')\, \boldsymbol{G}_A\big(\boldsymbol{T}_\kappa(\boldsymbol{X}, t), \Theta(\boldsymbol{X}, t), \boldsymbol{V}(\boldsymbol{X}, t)\big); \\
&\qquad \boldsymbol{X} \in \Omega_A, \qquad t \in [t_0, t_f],
\end{aligned} \tag{2.3}$$

where

$$f_A \equiv f_A\big(\boldsymbol{T}_\kappa(\boldsymbol{X}, t), \Theta(\boldsymbol{X}, t), \boldsymbol{V}(\boldsymbol{X}, t)\big) \in R, \qquad \delta_A \equiv \delta_A(f_A, f_A') \in \{0, 1\},$$

$$f_A' \equiv \operatorname{tr}\left(\frac{\partial f_A}{\partial \boldsymbol{T}_\kappa} \dot{\boldsymbol{T}}_\kappa\right) + \frac{\partial f_A}{\partial \Theta} \dot{\Theta},$$

and where $\lambda_A \geqq 1$ if the material properties are independent of the time scaling and $\lambda_A = 1$ if otherwise; at the same time $\delta_A(f_A, f_A') = 0$ if and only if there is no dissipation.

In the paper we restrict ourselves to the composites with so called Δ-periodic material structure, where $\Delta = 0.5(-\bar{X}^1, \bar{X}^1) \times 0.5(-\bar{X}^2, \bar{X}^2) \times 0.5(-\bar{X}^3, \bar{X}^3)$ with $\bar{X}^\alpha$ as the triple of the known positive numbers. It means that $\boldsymbol{Y}_0 + \Delta \subset \Omega$ for some $\boldsymbol{Y}_0 \in \Omega$ and there exists the decomposition $\bar{\Delta} = \cup \bar{\Delta}_A$, $A = 1, \ldots, M$, of $\bar{\Delta}$ into M disjointed open sets Δ_A such that

$$\Omega_A = \{\boldsymbol{X} \in \Omega : \boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}, \boldsymbol{Y} \in \Lambda, \boldsymbol{Z} \in \Delta_A\}, \qquad A = 1, \ldots, M, \tag{2.4}$$

where $\Lambda \equiv \{\boldsymbol{Y} \in R^3 : Y^1 = \nu_1 \bar{X}^1,\ Y^2 = \nu_2 \bar{X}^2,\ Y^3 = \nu_3 \bar{X}^3;\ \nu_\alpha = 0, \pm 1, \pm 2, \ldots;\ \alpha = 1, 2, 3\}$. In the sequel we shall treat Eqs. (2.1)—(2.3) as the exact governing equations of Δ-periodic thermo-elastic-unelastic composites. The boundary-value problem of finding functions $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ on the basis of the forementioned equations (and the pertinent initial conditions) will be denoted by $\mathcal{P}$.

The Δ-periodic composites met in engineering problems comprise a very big number of periodicity cells; hence the form of every part Ω_A of the region Ω is very complicated. That is why the exact theory of composites cannot be successfully applied to engineering problems. However, the exact theory of Δ-periodic composites will be used below as the starting point for the formulation of a certain effective theory of the composites. The proposed passage from the exact to the effective theory will be called the nonstandard homogenization method due to the fact that it takes into account some concepts of the nonstandard analysis [1]. The idea of the method is based on the heuristic assumption that a body with a sufficiently fine periodic material structure can be modelled by a hypothetical body having the "infinitely small" periodicity cells; the dimensions of such cells have to be described by the infinitely small numbers well defined within the structure of the nonstandard analysis.

3. Nonstandard Homogenization Method

3.1 Fine Periodicity Assumption

Let be known the Δ-periodic composite governed by Eqs. (2.1)—(2.4), and let ε, $\varepsilon \in (0, 1]$, be a parameter. Setting

$$\Omega_A^{\varepsilon} \equiv \{\boldsymbol{X} \in \Omega : \boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z},\ \boldsymbol{Y} \in \varepsilon\Lambda,\ \boldsymbol{Z} \in \varepsilon\Delta_A\}, \qquad A = 1, \ldots, M,$$

we shall introduce, from the purely formal point of view, a certain $\varepsilon\Delta$-periodic material structure by assuming that every Ω_A in Eqs. (2.3) is replaced by Ω_A^{ε} for some $\varepsilon \in (0, 1]$. Let us also assume that all external agents $\boldsymbol{b}(\cdot)$, $\alpha(\cdot)$, $\mathbf{s}(\cdot)$, $\sigma(\cdot)$ as well as the mapping $\varkappa$ remain unchanged for every $\varepsilon \in (0, 1]$. For the fixed initial conditions on this way we can formulate the one-parameter family $\mathscr{P}_{\varepsilon}$, $\varepsilon \in (0, 1]$, of problems related to $\varepsilon\Delta$-periodic composites (obviously $\mathscr{P}_1 = \mathscr{P}$). The basic unknowns in the problem $\mathscr{P}_{\varepsilon}$ will be denoted by $\chi^{\varepsilon}(\cdot)$, $\Theta^{\varepsilon}(\cdot)$, $\boldsymbol{V}^{\varepsilon}(\cdot)$. It must be emphasized that in every physical situation we deal with the problem $\mathscr{P}$; the family $\mathscr{P}_{\varepsilon}$, $\varepsilon \in (0, 1]$ of problems has a purely formal meaning and has been introduced in order to define what can be called the "fine periodic structure". Namely, the Δ-periodic structure in the problem $\mathscr{P}$ will be called "fine" if the solution $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ to the problem $\mathscr{P}$ can be approximated by the pertinent solution $\chi^{\varepsilon}(\cdot)$, $\Theta^{\varepsilon}(\cdot)$, $\boldsymbol{V}^{\varepsilon}(\cdot)$ to the problem $\mathscr{P}_{\varepsilon}$ for every $1/\varepsilon \in \mathrm{N}$. In the sequel we shall deal only with Δ-periodic composites of fine periodic structures and hence we introduce the following

Fine Periodicity Assumption. The solution $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ to the problem $\mathscr{P}$ under consideration can be approximated by the pertinent solution $\chi^{\varepsilon}(\cdot)$, $\Theta^{\varepsilon}(\cdot)$, $\boldsymbol{V}^{\varepsilon}(\cdot)$ to an arbitrary problem $\mathscr{P}_{\varepsilon}$, $1/\varepsilon \in \mathrm{N}$, such that the approximation formulae[1]

$$\begin{aligned}
&\chi(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \chi^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t), && \mathscr{D}\chi(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\chi^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t),\\
&\Theta(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \Theta^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t), && \mathscr{D}\Theta(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\Theta^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t),\\
&\boldsymbol{V}(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \boldsymbol{V}^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t), && \mathscr{D}\boldsymbol{V}(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\boldsymbol{V}^{\varepsilon}(\boldsymbol{Y} + \boldsymbol{Z}\varepsilon, t),
\end{aligned} \tag{3.1}$$

hold for $\boldsymbol{Y} \in \Lambda$, $\boldsymbol{Z} \in \Delta$ and $\boldsymbol{Y} + \boldsymbol{Z} \in \Omega_0$, where Ω_0 is a subset of Ω which can be treated as a certain "approximation" of Ω.

It has to be emphasized that the conditions (3.1) cannot de directly verified since the solutions $\chi^{\varepsilon}(\cdot)$, $\Theta^{\nu}(\cdot)$, $\boldsymbol{V}^{\varepsilon}(\cdot)$ to the problems $\mathscr{P}_{\varepsilon}$, $\varepsilon \in (0, 1]$, are not known a priori. Nevertheless, we shall tacitly assume that in the problems under con-

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[1] Symbol $\mathscr{D}\chi(\cdot)$ stands for all material, time or mixed derivatives of $\chi(\cdot)$ which occur in the problem $\mathscr{P}$; similarly we define $\mathscr{D}\Theta(\cdot)$, $\mathscr{D}\chi^{\varepsilon}(\cdot)$ etc. The symbol $\sim$ stands for "can be approximated by"; this approximation has to be sufficient from the point of view of the possible engineering applications of the theory under consideration.

sideration the Δ-periodic structure is sufficiently "fine", i.e. that the fine periodicity assumption holds.

The first step in our line of approach will be based on the passage from the problem $\mathcal{P}$ to a certain problem $\mathcal{P}_\delta$, where δ is an arbitrary but fixed infinitely small positive number. Since there are no infinitely small (and infinitely large) numbers among standard notions of analysis, we have to formulate the problem $\mathcal{P}_\delta$ as the nonstandard analysis problem, [1], [2]. Then the mathematical consequence of the fine periodicity assumption is (via so called transfer principle) that the solution $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ to the problem $\mathcal{P}$ can be approximated by the pertinent solution $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$ to the (nonstandard) problem $\mathcal{P}_\delta$. All mathematical entities in problems $\mathcal{P}_\varepsilon$, $\varepsilon \in (0, 1]$, which are independent of ε, such as Ω, Π, Γ, $\boldsymbol{b}(\cdot)$, $\boldsymbol{T}_\varkappa{}^A(\cdot)$, ..., have to be represented by the pertinent standard entities, [1], such as $^*\Omega$, $^*\Pi$, $^*\Gamma$, $^*\boldsymbol{b}(\cdot)$, $^*\boldsymbol{T}_\kappa{}^A(\cdot)$, ... Hence the nonstandard problem $\mathcal{P}_\delta$ will be governed by the conditions

$$
\begin{aligned}
\int\limits_{*\Omega} \operatorname{tr}[\boldsymbol{S}^\delta(\boldsymbol{X}, t)\, \nabla \boldsymbol{v}(\boldsymbol{X})]\, dV &= \int\limits_{*\Omega} \varrho^\delta(\boldsymbol{X})\, [{}^*\boldsymbol{b}(\boldsymbol{X}, t) - \ddot{\chi}^\delta(\boldsymbol{X}, t)] \cdot \boldsymbol{v}(\boldsymbol{X})\, dV \\
&\quad + \int\limits_{*\Gamma} {}^*\mathbf{s}(\boldsymbol{X}, t) \cdot \boldsymbol{v}(\boldsymbol{X})\, dA(\boldsymbol{X}), \\
\int\limits_{*\Omega} \boldsymbol{h}^\delta(\boldsymbol{X}, t) \cdot \nabla \zeta(\boldsymbol{X})\, dV &= \int\limits_{*\Omega} \{\varrho^\delta(\boldsymbol{X})\, [{}^*\alpha(\boldsymbol{X}, t) - \dot{\varepsilon}^\delta(\boldsymbol{X}, t)] \\
&\quad + \operatorname{tr}[\boldsymbol{S}^\delta(\boldsymbol{X}, t)\, \nabla \dot{\chi}^\delta(\boldsymbol{X}, t)]\}\, \zeta(\boldsymbol{X})\, dV \\
&\quad + \int\limits_{*\Pi} {}^*\sigma(\boldsymbol{X}, t)\, \zeta(\boldsymbol{X})\, dA(\boldsymbol{X}), \qquad t \in {}^*[t_0, t_f],
\end{aligned} \tag{3.2}
$$

which have to hold for every $\boldsymbol{v}(\cdot) \in {}^*[C^1(\bar{\Omega})^3]$, $\zeta(\cdot) \in {}^*C^1(\bar{\Omega})$, such that $\boldsymbol{v} \mid {}^*(\partial\Omega \setminus \bar{\Gamma}) = \boldsymbol{0}$ and $\zeta \mid {}^*(\partial\Omega \setminus \bar{\Pi}) = 0$, and by the constitutive relations

$$
\boldsymbol{T}_\kappa{}^\delta(\boldsymbol{X}, t) = {}^*\hat{\boldsymbol{T}}_\kappa{}^A\big(\boldsymbol{B}^\delta(\boldsymbol{X}, t), \Theta^\delta(\boldsymbol{X}, t), \boldsymbol{V}^\delta(\boldsymbol{X}, t)\big), \qquad \varrho_\kappa{}^\delta(\boldsymbol{X}) = \varrho_\kappa{}^A, \tag{3.3.1, 2}
$$

$$
\varepsilon^\delta(\boldsymbol{X}, t) = {}^*\hat{\varepsilon}^A\big(\boldsymbol{B}^\delta(\boldsymbol{X}, t), \Theta^\delta(\boldsymbol{X}, t), \boldsymbol{V}^\delta(\boldsymbol{X}, t)\big), \tag{3.3.3}
$$

$$
\boldsymbol{h}_\kappa{}^\delta(\boldsymbol{X}, t) = {}^*\hat{\boldsymbol{h}}_\kappa{}^A\big(\boldsymbol{B}^\delta(\boldsymbol{X}, t), \Theta^\delta(\boldsymbol{X}, t), g^\delta(\boldsymbol{X}, t), \boldsymbol{V}^\delta(\boldsymbol{X}, t)\big), \tag{3.3.4}
$$

$$
\begin{gathered}
\dot{\boldsymbol{V}}^\delta(\boldsymbol{X}, t) = \lambda_A \delta_A(f_A, f_A')\, {}^*\boldsymbol{G}_A\big(\boldsymbol{T}_\kappa{}^\delta(\boldsymbol{X}, t), \Theta^\delta(\boldsymbol{X}, t), \boldsymbol{V}^\delta(\boldsymbol{X}, t)\big); \\
\boldsymbol{X} \in \Omega_A{}^\delta, \qquad t \in {}^*[t_0, t_f]
\end{gathered} \tag{3.3.5}
$$

where

$$
\Omega_A{}^\delta \equiv \{\boldsymbol{X} \in {}^*\Omega\colon\ \boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z},\ \boldsymbol{Y} \in \delta{}^*\Lambda, \boldsymbol{Z} \in \delta{}^*\Delta_A\}, \qquad A = 1, \ldots, M,
$$

as well as by the formulae

$$
\begin{aligned}
\boldsymbol{S}^\delta(\boldsymbol{X}, t) &= {}^*J(\boldsymbol{X})\, \nabla \chi^\delta(\boldsymbol{X}, t)\, {}^*\boldsymbol{A}(\boldsymbol{X})\, \boldsymbol{T}_\kappa{}^\delta(\boldsymbol{X}, t)\, {}^*\boldsymbol{A}^T(\boldsymbol{X}), \\
\boldsymbol{h}^\delta(\boldsymbol{X}, t) &= {}^*J(\boldsymbol{X})\, {}^*\boldsymbol{A}(\boldsymbol{X})\, \boldsymbol{h}_\kappa{}^\delta(\boldsymbol{X}, t), \qquad \varrho^\delta(\boldsymbol{X}) = {}^*J(\boldsymbol{X})\, \varrho_\kappa{}^\delta(\boldsymbol{X}),
\end{aligned} \tag{3.4}
$$

where we have denoted

$$\boldsymbol{B}^\delta(\boldsymbol{X}, t) \equiv {}^*\boldsymbol{A}^T(\boldsymbol{X})\, \boldsymbol{L}^\delta(\boldsymbol{X}, t)\, {}^*\boldsymbol{A}(\boldsymbol{X}),$$

$$\boldsymbol{L}^\delta(\boldsymbol{X}, t) \equiv \frac{1}{2}\, [(\nabla\chi^\delta)^T\, \nabla\chi^\delta - (\nabla{}^*\varkappa)^T\, \nabla{}^*\varkappa]\, (\boldsymbol{X}, t),$$

$$\boldsymbol{g}^\delta(\boldsymbol{X}, t) \equiv {}^*\boldsymbol{A}^T(\boldsymbol{X})\, \nabla\Theta^\delta(\boldsymbol{X}, t).$$

At the same time from (3.1) we obtain now

$$\begin{aligned} &{}^*\chi(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \chi^\delta(\boldsymbol{V} + \boldsymbol{Z}\delta, t), && \mathscr{D}{}^*\chi(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\chi^\delta(\boldsymbol{Y} + \boldsymbol{Z}\delta, t), \\ &{}^*\Theta(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \Theta^\delta(\boldsymbol{Y} + \boldsymbol{Z}\delta, t), && \mathscr{D}{}^*\Theta(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\Theta^\delta(\boldsymbol{Y} + \boldsymbol{Z}\delta, t), \\ &{}^*\boldsymbol{V}(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \boldsymbol{V}^\delta(\boldsymbol{Y} + \boldsymbol{Z}\delta, t), && \mathscr{D}{}^*\boldsymbol{V}(\boldsymbol{Y} + \boldsymbol{Z}, t) \sim \mathscr{D}\boldsymbol{V}^\delta(\boldsymbol{Y} + \boldsymbol{Z}\delta, t), \end{aligned} \tag{3.5}$$

for $\boldsymbol{Y} \in {}^*\Lambda$, $\boldsymbol{Z} \in {}^*\Delta$ and $\boldsymbol{Y} + \boldsymbol{Z} \in {}^*\Omega_0$, $1/\delta \in {}^*N - N$.

Summing up we conclude that on the basis of the fine periodicity assumption the problem $\mathscr{P}$ can be approximated by the nonstandard problem $\mathscr{P}_\delta$, where δ is an arbitrary but fixed infinitely small positive number. It means, roughly speaking, that the continuous body with the fine Δ-periodic structure can be "approximated" by a body with the infinitely small periodicity cells. Such a body can be defined exclusively within the framework of the nonstandard analysis and that is why we refer our approach to as a nonstandard homogenization method.

3.2 Microlocal Approximation Assumption

The second step in our line of approach will be based on the passage from the nonstandard problem $\mathscr{P}_\delta$ to a certain (also nonstandard) problem $\tilde{\mathscr{P}}_\delta$ by applying the known method of internal constraints, [22], [23]. To this aid we shall replace Eq. (3.3.5) (i.e. so called evolutional equation) by its weak form given by the condition

$$\sum_{A=1}^{M} \int_{\Omega_A^\delta} \left[\dot{\boldsymbol{V}}^\delta(\boldsymbol{X}, t) - \lambda_A\delta_A(f_A, f_A')\, {}^*\boldsymbol{G}_A\big(\boldsymbol{T}_\kappa{}^\delta(\boldsymbol{X}, t), \Theta^\delta(\boldsymbol{X}, t), \boldsymbol{V}^\delta(\boldsymbol{X}, t)\big)\right] \circ \boldsymbol{U}(\boldsymbol{X})\, dV = 0 \tag{3.6}$$

which has to hold for every $\boldsymbol{U}(\cdot) \in {}^*[C(\bar{\Omega})]^S$, and where $\circ$ stands for a scalar product in R^S (S is the number of the internal state variables, cf. Section 2). In the problem $\tilde{\mathscr{P}}_\delta$ we look for the approximate solution to the problem $\mathscr{P}_\delta$ which is assumed to belong to the special class of functions. In order to specify this class we introduce the sequence $l_a(\cdot)$, $a = 1, \ldots, m$, of the known linear independent real-valued Δ-periodic functions (defined on R^3), having the piecewise continuous first order derivatives such that

$$\int_\Delta \nabla l_a(X)\, dV = 0, \qquad a = 1, \ldots, m.$$

For the sake of simplicity we shall also assume that there exists the decomposition $\bar{\Delta} = \bigcup \bar{\Delta}^E$, $E = 1, \ldots, N$, of Δ into N, $N \geq M$, disjointed regular regions Δ^E such that the functions $l_a(\cdot)$ are linear in every Δ^E. Thus we can define the system of $N \times m$ vectors $\Lambda_a{}^E$ in R^3, setting

$$\Lambda_a{}^E \equiv (\Lambda_{a\alpha}^E), \qquad \Lambda_{a\alpha}^E \equiv l_{a,\alpha}(\boldsymbol{X}) \quad \text{for} \quad \boldsymbol{X} \in \Delta^E, \qquad \alpha = 1, 2, 3.$$

Every Δ^E will be called the finite element of Δ. We shall also assume that every finite element Δ^E is a subset of a certain part Δ_A of Δ. Introducing the (non-standard) sets

$$\Omega^{E\delta} \equiv \{\boldsymbol{X} \in {}^*\Omega\colon \boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}, \boldsymbol{Y} \in \delta^*\Lambda, \boldsymbol{Z} \in \delta^*\Delta^E\}, \qquad E = 1, \ldots, N,$$

we define functions $\mu_{E\delta}(\cdot)$ by means of

$$\mu_{E\delta}(\boldsymbol{X}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{X} \in \Omega^{E\delta}, \\ 0 & \text{if} \quad \boldsymbol{X} \in {}^*\Omega \setminus \Omega^{E\delta}, \end{cases}$$

as the characteristic functions of $\Omega^{E\delta}$. The meaning of objects introduced above will be explained in the sequel.

Now we can formulate the second heuristic assumption of the proposed approach which will be referred to as

Microlocal Approximation Assumption. The approximate solution[2] $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$ to the nonstandard problem $\mathcal{P}_\delta$ can be found in the class of functions given by

$$\begin{aligned} \chi^\delta(\boldsymbol{X}, t) &= {}^*\boldsymbol{p}(\boldsymbol{X}, t) + \delta^* l_a(\boldsymbol{X}/\delta)\, {}^*\boldsymbol{q}^a(\boldsymbol{X}, t), \\ \Theta^\delta(\boldsymbol{X}, t) &= {}^*\vartheta(\boldsymbol{X}, t) + \delta^* l_a(\boldsymbol{X}/\delta)\, {}^*\pi^a(\boldsymbol{X}, t), \\ \boldsymbol{V}^\delta(\boldsymbol{X}, t) &= \mu_{E\delta}(\boldsymbol{X})\, {}^*\boldsymbol{W}^E(\boldsymbol{X}, t); \qquad \boldsymbol{X} \in {}^*\Omega, \qquad t \in {}^*[t_0, t_f], \end{aligned} \tag{3.7}$$

where $l_a(\cdot)$, $\mu_{E\delta}(\cdot)$ are the known functions introduced above and $\boldsymbol{p}(\cdot)$, $\boldsymbol{q}^a(\cdot)$, $\vartheta(\cdot)$, $\pi^a(\cdot)$, $\boldsymbol{W}^E(\cdot)$ are sufficiently regular unknown functions defined almost everywhere on $\Omega \times [t_0, t_f]$ with values in R^3, R^3, R_+, R, R^S, respectively.

Now the problem $\tilde{\mathcal{P}}_\delta$ (constituting a certain approximation of the problem $\mathcal{P}_\delta$) can be stated as follows: find the functions $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$ in the class of functions given by Eqs. (3.7), such that the conditions (3.2), (3.6) hold for every

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{X}) &= {}^*\boldsymbol{v}^0(\boldsymbol{X}) + \delta^* l_a(\boldsymbol{X}/\delta)\, {}^*\boldsymbol{v}^a(\boldsymbol{X}), \\ \zeta(\boldsymbol{X}) &= {}^*\zeta^0(\boldsymbol{X}) + \delta^* l_a(\boldsymbol{X}/\delta)\, {}^*\zeta^a(\boldsymbol{X}), \\ \boldsymbol{U}(\boldsymbol{X}) &= \mu_{E\delta}(\boldsymbol{X})\, {}^*\boldsymbol{U}^E(\boldsymbol{X}); \qquad \boldsymbol{X} \in {}^*\bar{\Omega}, \end{aligned} \tag{3.8}$$

[2] The approximate and exact solutions to $\mathcal{P}_\delta$ are denoted here by the same symbols $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$.

where $\boldsymbol{v}^0(\cdot)$, $\boldsymbol{v}^a(\cdot) \in C^1(\bar{\Omega})^3$, $\zeta^0(\cdot)$, $\zeta^a(\cdot) \in C^1(\bar{\Omega})$, $\boldsymbol{U}^E(\cdot) \in C(\bar{\Omega})^S$ are arbitrary functions satisfying $\boldsymbol{v}^0/\partial\Omega \setminus \bar{\Gamma} = \boldsymbol{0}$, $\zeta^0 \mid \partial\Omega \setminus \bar{\Pi} = 0$, and where $\boldsymbol{S}^\delta(\cdot)$, $\varrho^\delta(\cdot)$, $\boldsymbol{h}^\delta(\cdot)$, $\varepsilon^\delta(\cdot)$ are expressed by means of Eqs. (3.4), (3.3). Moreover, the standard parts $\boldsymbol{p}(\cdot)$, $\vartheta(\cdot)$ of unknown functions $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, respectively, are assumed to satisfy the boundary and initial conditions similar to those for functions $\chi(\cdot)$, $\Theta(\cdot)$ in the problem $\mathscr{P}$. At last, the functions $\boldsymbol{W}^E(\cdot)$ are assumed to satisfy the initial conditions similar to those for the functions $\boldsymbol{V}/\Omega_A$, where $\Delta_A \supset \Delta^E$, in the problem $\mathscr{P}$.

The microlocal approximation assumption, which makes it possible to pass from the problem $\mathscr{P}_\delta$ to the problem $\tilde{\mathscr{P}}_\delta$, constitutes the second heuristic assumption of the proposed method of modelling. The postulated a priori functions $l_a(\cdot)$ in Eq. (3.7) are called the shape functions since their role is similar to that of the shape functions in the well known finite element method. The new unknown functions $\boldsymbol{p}(\cdot)$ and $\vartheta(\cdot)$ will be called macrodeformations and macrotemperatures, respectively. The unknown functions $\boldsymbol{W}^E(\cdot)$ will be referred to as microlocal state variables; they are related to the pertinent components of the composite (if $\Delta^E \subset \Delta_A$ then $\boldsymbol{W}^E(\cdot)$ is related to the A-th material component). For the particulars the reader is referred to [3], [16], where some examples of Eqs. (3.7) are given. The main role play here the unknown functions $\boldsymbol{q}^a(\cdot)$, $\pi^a(\cdot)$ which describe the deformational and thermal effects due to the jump nonhomogeneity of the material structure of the composite and are called the kinematic and thermal microlocal parameters, respectively.

If the microlocal approximation (3.7) is properly choosen then the solution $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$ to the problem $\tilde{\mathscr{P}}_\delta$ should constitute a good approximation of the solution to the problem $\mathscr{P}_\delta$. Hence, via the formulae (3.5), the solution to the problem $\tilde{\mathscr{P}}_\delta$ can be also treated as a certain approximation of the solution $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ to the primary problem $\mathscr{P}$. Combining Eqs. (3.7) with the formulae (3.5) and setting

$$\mu_E(\boldsymbol{X}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{X} \in \Omega^E, \\ 0 & \text{if} \quad \boldsymbol{X} \in \Omega \setminus \Omega^E, \end{cases}$$

where

$$\Omega^E \equiv \{\boldsymbol{X} \in \Omega\colon \boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}, \boldsymbol{Y} \in \Lambda, \boldsymbol{Z} \in \Delta^E\}, \qquad E = 1, \dots, N,$$

we arrive at the following important approximation formulae (summation convention holds!)

$$\begin{aligned} &\chi(\boldsymbol{X}, t) \sim \boldsymbol{p}(\boldsymbol{X}, t), \\ &\nabla\chi(\boldsymbol{X}, t) \sim \nabla\boldsymbol{p}(\boldsymbol{X}, t) + \mu_E(\boldsymbol{X})\, \Lambda_a{}^E \boldsymbol{q}^a(\boldsymbol{X}, t), \\ &\Theta(\boldsymbol{X}, t) \sim \vartheta(\boldsymbol{X}, t), \\ &\nabla\Theta(\boldsymbol{X}, t) \sim \nabla\vartheta(\boldsymbol{X}, t) + \mu_E(\boldsymbol{X})\, \Lambda_a{}^E \pi^a(\boldsymbol{X}, t), \\ &\boldsymbol{V}(\boldsymbol{X}, t) \sim \mu_E(\boldsymbol{X})\, \boldsymbol{W}^E(\boldsymbol{X}, t). \end{aligned} \tag{3.9}$$

The time derivatives of $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ and their material gradients (provided that they exist) also have to be approximated on the basis of the formulae (3.9). Thus we conclude that under the fine periodicity and microlocal approximation assumptions it is possible to evaluate the solution of the problem $\mathscr{P}$ in terms of macrodeformations $\boldsymbol{p}(\cdot)$, macrotemperatures $\vartheta(\cdot)$, microlocal parameters $\boldsymbol{q}^a(\cdot)$, $\pi^a(\cdot)$, $a = 1, \ldots, m$, and microlocal state variables $\boldsymbol{W}^E(\cdot)$, $E = 1, \ldots, N$.

3.3 Nonstandard Homogenization Statement

The heuristic foundations of the nonstandard homogenization approach proposed here are represented by the fine periodicity and microlocal approximation assumptions. This approach is also based on the mathematical fact that the (nonstandard) problem $\tilde{\mathscr{P}}_\delta$ of finding functions $\chi^\delta(\cdot)$, $\Theta^\delta(\cdot)$, $\boldsymbol{V}^\delta(\cdot)$ (in the class of functions (3.7)) can be reduced to the standard problem $\tilde{\mathscr{P}}$ for functions $\boldsymbol{p}(\cdot)$, $\vartheta(\cdot)$, $\boldsymbol{q}^a(\cdot)$, $\pi^a(\cdot)$, $\boldsymbol{W}^E(\cdot)$. Moreover, the governing equations of the problem $\tilde{\mathscr{P}}$ represent, roughly speaking, a certain "homogenized" material continuum and hence they constitute an effective theory of the composites under consideration.

In order to formulate the standard problem $\tilde{\mathscr{P}}$ we have to introduce some new mathematical entities which will be defined exclusively in term of the notions previously introduced. Firstly, we define the following strain measures

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{X}, t) \equiv \frac{1}{2}\,[\nabla \boldsymbol{p}^T(\boldsymbol{X}, t)\, \nabla \boldsymbol{p}(\boldsymbol{X}, t) - \nabla \boldsymbol{\varkappa}^T(\boldsymbol{X})\, \nabla \boldsymbol{\varkappa}(\boldsymbol{X})],\\
&\boldsymbol{D}^a(\boldsymbol{X}, t) \equiv \nabla \boldsymbol{p}^T(\boldsymbol{X}, t)\, \boldsymbol{q}^a(\boldsymbol{X}, t), \qquad \boldsymbol{D}(\boldsymbol{X}, t) \equiv \big(D_\alpha{}^a(\boldsymbol{X}, t)\big), \qquad (3.10)\\
&\boldsymbol{Q}^{ab}(\boldsymbol{X}, t) \equiv \frac{1}{2}\, \boldsymbol{q}^a(\boldsymbol{X}, t) \cdot \boldsymbol{q}^b(\boldsymbol{X}, t), \qquad \boldsymbol{Q}(\boldsymbol{X}, t) \equiv \big(Q^{ab}(\boldsymbol{X}, t)\big).
\end{aligned}$$

Setting

$$\begin{aligned}
\boldsymbol{B}^E(\boldsymbol{X}, t) &\equiv \boldsymbol{A}^T(\boldsymbol{X})\, \boldsymbol{E}(\boldsymbol{X}, t)\, \boldsymbol{A}(\boldsymbol{X}) + \boldsymbol{A}^T(\boldsymbol{X})\, \boldsymbol{D}^a(\boldsymbol{X}, t) \otimes \boldsymbol{A}^T(\boldsymbol{X})\, \Lambda_a{}^E\\
&\quad + \boldsymbol{A}^T(\boldsymbol{X})\, \Lambda_a{}^E \otimes \boldsymbol{A}^T(\boldsymbol{X})\, \Lambda_b{}^E Q^{ab}(\boldsymbol{X}, t),\\
\boldsymbol{g}^E(\boldsymbol{X}, t) &\equiv \boldsymbol{A}^T(\boldsymbol{X})\, \nabla\vartheta(\boldsymbol{X}, t) + \boldsymbol{A}^T(\boldsymbol{X})\, \Lambda_a{}^E \pi^a(\boldsymbol{X}, t),
\end{aligned}$$

it can be shown that $\boldsymbol{B}^\delta \simeq {}^*\boldsymbol{B}^E$ and $\boldsymbol{g}^\delta \simeq {}^*\boldsymbol{g}^E$ for $\boldsymbol{X} \in \Omega^{E\delta}$ (symbol $\simeq$ stands for "is infinitely close to", cf. [1]). Secondly, introducing the symbol

$$\alpha_E{}^A = \alpha_A{}^E \equiv \begin{cases} 1 & \text{if } \Delta^E \subset \Delta_A \\ 0 & \text{if otherwise,} \end{cases}$$

and taking into account the definitions of $\boldsymbol{B}^E$ and $\boldsymbol{g}^E$, we shall define the following constitutive functions

$$\begin{aligned} \tilde{\boldsymbol{T}}_\kappa{}^E(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{Q}, \vartheta, \boldsymbol{W}^E; \boldsymbol{A}) &\equiv \alpha_A{}^E \hat{\boldsymbol{T}}_\kappa{}^A(\boldsymbol{B}^E, \vartheta, \boldsymbol{W}^E), \\ \tilde{\varepsilon}^E(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{Q}, \vartheta, \boldsymbol{W}^E; \boldsymbol{A}) &\equiv \alpha_A{}^E \hat{\varepsilon}^A(\boldsymbol{B}^E, \vartheta, \boldsymbol{W}^E), \\ \tilde{\boldsymbol{h}}_\kappa{}^E(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{Q}, \vartheta, \nabla\vartheta, \boldsymbol{\pi}, \boldsymbol{W}^E; \boldsymbol{A}) &\equiv \alpha_A{}^E \hat{\boldsymbol{h}}_\kappa{}^A(\boldsymbol{B}^E, \vartheta, \boldsymbol{g}^E, \boldsymbol{W}^E), \end{aligned} \tag{3.11}$$

$\varrho_\kappa{}^E \equiv \alpha_A{}^E \varrho_\kappa{}^A$; summation with respect to A holds! At last we introduce the stress and heat-flux measures

$$\begin{aligned} \boldsymbol{S}^E(\boldsymbol{X}, t) &\equiv J(\boldsymbol{X}) \left[\nabla \boldsymbol{p}(\boldsymbol{X}, t) + \Lambda_a{}^E \otimes \boldsymbol{q}^a(\boldsymbol{X}, t)\right] \boldsymbol{A}(\boldsymbol{X})\, \boldsymbol{T}_\kappa{}^E(\boldsymbol{X}, t)\, \boldsymbol{A}^T(\boldsymbol{X}), \\ \boldsymbol{h}^E(\boldsymbol{X}, t) &\equiv J(\boldsymbol{X})\, \boldsymbol{A}(\boldsymbol{X})\, \boldsymbol{h}_\kappa{}^E(\boldsymbol{X}, t), \\ \varrho^E(\boldsymbol{X}) &\equiv J(\boldsymbol{X})\, \varrho_\kappa{}^E, \end{aligned} \tag{3.12}$$

and using the extra denotation $\nu_E \equiv \operatorname{vol} \Delta^E / \operatorname{vol} \Delta$, we also define the following fields

$$\begin{aligned} \boldsymbol{S}_0(\boldsymbol{X}, t) &\equiv \sum_{E=1}^N \nu_E \boldsymbol{S}^E(\boldsymbol{X}, t), & \boldsymbol{S}_a(\boldsymbol{X}, t) &\equiv \sum_{E=1}^N \nu_E \boldsymbol{S}^E(\boldsymbol{X}, t)\, \Lambda_a{}^E, \\ \boldsymbol{h}_0(\boldsymbol{X}, t) &\equiv \sum_{E=1}^N \nu_E \boldsymbol{h}^E(\boldsymbol{X}, t), & \boldsymbol{h}_a(\boldsymbol{X}, t) &\equiv \sum_{E=1}^N \nu_E \boldsymbol{h}^E(\boldsymbol{X}, t)\, \Lambda_a{}^E, \\ \varrho_0(\boldsymbol{X}) &\equiv \sum_{E=1}^N \nu_E \varrho^E(\boldsymbol{X}), & \gamma(\boldsymbol{X}, t) &\equiv \sum_{E=1}^N \nu_E \varepsilon^E(\boldsymbol{X}, t)\, \varrho_\kappa{}^E / \varrho_\kappa(\boldsymbol{X}). \end{aligned} \tag{3.13}$$

Using the definitions introduced above we shall formulate the basic statement of the nonstandard homogenization approach:

Nonstandard Homogenization Statement. Functions $\boldsymbol{p}(\cdot)$, $\vartheta(\cdot)$, $\boldsymbol{q}^a(\cdot)$, $\pi^a(\cdot)$, $a = 1, \ldots, m$, $\boldsymbol{W}^E(\cdot)$, $E = 1, \ldots, N$, in the formulae (3.7) which satisfy the governing relations of the problem $\tilde{\mathscr{P}}_\delta$ have also to satisfy (almost everywhere) in $\Omega \times [t_0, t_f]$ the constitutive relations

$$\begin{aligned} \boldsymbol{T}_\kappa{}^E(\boldsymbol{X}, t) &= \tilde{\boldsymbol{T}}_\kappa{}^E\big(\boldsymbol{E}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t), \boldsymbol{Q}(\boldsymbol{X}, t), \vartheta(\boldsymbol{X}, t), \boldsymbol{W}^E(\boldsymbol{X}, t); \boldsymbol{A}(\boldsymbol{X})\big), \\ \varepsilon^E(\boldsymbol{X}, t) &= \tilde{\varepsilon}^E\big(\boldsymbol{E}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t), \boldsymbol{Q}(\boldsymbol{X}, t), \vartheta(\boldsymbol{X}, t), \boldsymbol{W}^E(\boldsymbol{X}, t); \boldsymbol{A}(\boldsymbol{X})\big), \\ \boldsymbol{h}_\kappa{}^E(\boldsymbol{X}, t) &= \tilde{\boldsymbol{h}}_\kappa{}^E\big(\boldsymbol{E}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t), \boldsymbol{Q}(\boldsymbol{X}, t), \vartheta(\boldsymbol{X}, t), \\ &\qquad \nabla\vartheta(\boldsymbol{X}, t), \boldsymbol{\pi}(\boldsymbol{X}, t), \boldsymbol{W}^E(\boldsymbol{X}, t); \boldsymbol{A}(\boldsymbol{X})\big), \\ \dot{\boldsymbol{W}}^E(\boldsymbol{X}, t) &= \lambda_E \delta_E(f_E, f_E')\, \boldsymbol{G}_E\big(\boldsymbol{T}_\kappa{}^E(\boldsymbol{X}, t), \vartheta(\boldsymbol{X}, t), \boldsymbol{W}^E(\boldsymbol{X}, t)\big) \end{aligned} \tag{3.14}$$

where $\delta_E(\cdot) \equiv \alpha_E{}^A \delta_A(\cdot)$, $\lambda_E(\cdot) \equiv \alpha_E{}^A \lambda_A(\cdot)$, $G_E(\cdot) \equiv \alpha_E{}^A G_A(\cdot)$ and

$$f_E \equiv \alpha_E{}^A f_A(\boldsymbol{T}_\kappa{}^E, \vartheta, \boldsymbol{W}^E), \qquad f_E' \equiv \operatorname{tr}\left(\frac{\partial f_E}{\partial \boldsymbol{T}_\kappa{}^E}\, \dot{\boldsymbol{T}}_\kappa{}^E\right) + \frac{\partial f_E}{\partial \vartheta}\, \dot{\vartheta},$$

as well as the following field equations

$$
\begin{gathered}
\operatorname{Div} \boldsymbol{S}_0(\boldsymbol{X}, t) + \varrho_0(\boldsymbol{X})\, \boldsymbol{b}(\boldsymbol{X}, t) = \varrho_0 \ddot{\boldsymbol{p}}(\boldsymbol{X}, t), \\
\boldsymbol{S}_a(\boldsymbol{X}, t) = \boldsymbol{0}, \\
\operatorname{Div} \boldsymbol{h}_0(\boldsymbol{X}, t) + \varrho_0(\boldsymbol{X}) \left[\alpha(\boldsymbol{X}, t) - \dot{\gamma}(\boldsymbol{X}, t)\right] + \operatorname{tr}\left[\boldsymbol{S}_0(\boldsymbol{X}, t)\, \nabla \dot{\boldsymbol{p}}(\boldsymbol{X}, t)\right] = 0, \\
h_a(\boldsymbol{X}, t) = 0; \qquad a = 1, \ldots, n,
\end{gathered}
\tag{3.15}
$$

where the denotations (3.13), (3.12) have to be taken into account. At the same time the arguments $\boldsymbol{E}(\boldsymbol{X}, t)$, $\boldsymbol{D}(\boldsymbol{X}, t)$, $\boldsymbol{Q}(\boldsymbol{X}, t)$ of the constitutive functions in Eq. (3.14) are related to the basic kinematical unknowns $\boldsymbol{p}(\cdot)$, $\boldsymbol{q}^a(\cdot)$ by means of Eq. (3.10). Moreover, the boundary conditions

$$
\begin{aligned}
\boldsymbol{S}_0(\boldsymbol{X}, t)\, \boldsymbol{n}(\boldsymbol{X}) &= \boldsymbol{s}(\boldsymbol{X}, t), \qquad \boldsymbol{X} \in \Gamma, \\
\boldsymbol{h}_0(\boldsymbol{X}, t) \cdot \boldsymbol{n}(\boldsymbol{X}) &= \sigma(\boldsymbol{X}, t), \qquad \boldsymbol{X} \in \Pi,
\end{aligned}
\tag{3.16}
$$

where $\boldsymbol{n}(\boldsymbol{X})$ is the unit outward normal to $\partial\Omega$ at $\boldsymbol{X}$, together with the pertinent boundary and initial conditions for $\boldsymbol{p}(\cdot)$, $\vartheta(\cdot)$, $\boldsymbol{W}^E(\cdot)$ implied by the problem $\tilde{\mathscr{P}}_\delta$ have to hold.

The homogenization statement constitutes the crucial point of the nonstandard homogenization approach to the thermo-elastic-unelastic composites with the fine periodic material structure. Equations (3.10)—(3.16) can be treated as the governing equations of a certain effective theory of the composites under consideration; this theory will be referred to as the microlocal parameter theory. In particular $\boldsymbol{S}_0(\boldsymbol{X}, t)$, $\boldsymbol{h}_0(\boldsymbol{X}, t)$, $\varrho_0(\boldsymbol{X})$ will be called the mean stress, the mean heat flux and the mean mass density (related to Ω). Similarly, $\boldsymbol{S}^E(\boldsymbol{X}, t)$, $\boldsymbol{h}^E(\boldsymbol{X}, t)$, $\varrho^E(\boldsymbol{X})$ can be called the partial stress, the partial heat flux and the partial mass density (also related to Ω but describing the behavior of pertinent material components). The proof of the homogenization statement can be shown via the direct calculations with the aid of the known theorems of the nonstandard analysis. To do this we have to substitute the right hand side of Eqs. (3.7), (3.8) into the variational conditions (3.2), (3.6). Introducing the internal fine partition of the standard region $^*\Omega$ and using the procedure similar to that applied in [3], we arrive at the standard variational conditions which under the denotations (3.10)—(3.13) and under the known regularity conditions lead to Eqs. (3.15), (3.16) as well as to the constitutive Eq. (3.14).

4. General Conclusions

The line of approach leading from the problem $\mathscr{P}$ of the "exact" theory of composites (i.e. the theory of composites as nonhomogeneous bodies) to the problem $\tilde{\mathscr{P}}$ of the effective microlocal parameter theory and then to the approximate solution to the problem $\mathscr{P}$, is realized by the following procedure:

(i) We start from the formulation of the problem $\mathcal{P}$ by assuming that (for every A-th material component) there are known the constitutive relations (2.3) and the external agents $\boldsymbol{b}(\cdot)$, $\alpha(\cdot)$, $\mathbf{s}(\cdot)$, $\sigma(\cdot)$ in Eq. (2.2). We also assume that the boundary and initial conditions for the unknown functions $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ are prescribed.

(ii) We introduce the decomposition of the basic periodicity cell Δ into N non-intersecting regions Δ^E, $E = 1, \ldots, N$ (basic finite elements) bearing in mind that every Δ^E comprises only one material component. After that we introduce the Δ-periodic shape functions $l_a(\cdot)$, $a = 1, \ldots, m$. The finite elements and the shape functions are similar to those of the finite element method but are introduced only in an arbitrary but fixed periodicity cell.

(iii) Using Eqs. (3.11)—(3.13) we formulate the governing relations (3.10), (3.14)—(3.16) of the microlocal parameter theory and the problem $\tilde{\mathcal{P}}$ within this theory.

(iv) After obtaining the solution $\boldsymbol{p}(\cdot)$, $\vartheta(\cdot)$, $\boldsymbol{q}^a(\cdot)$, $\pi^a(\cdot)$, $a = 1, \ldots, m$, $\boldsymbol{W}^E(\cdot)$, $E = 1, \ldots, N$ to the problem $\tilde{\mathcal{P}}$ we evaluate the approximate solution $\chi(\cdot)$, $\Theta(\cdot)$, $\boldsymbol{V}(\cdot)$ to the primary problem $\mathcal{P}$ by means of the approximation formulae (3.9).

It has to be emphasized that this line of approach leads to the reliable approximation only if the Δ-periodic structure of the composite under consideration is sufficiently fine, i.e., if the fine periodicity assumption is justified in the problem we deal with. Also the microlocal approximation assumption (i.e., the choice of finite elements Δ^E and shape functions $l_a(\cdot)$) has to be motivated by the possible distributions of the deformation and temperature gradients within the periodicity cells.

5. Final Remarks

The effective theories of thermo-elastic-unelastic periodic composites have been derived here from the "exact" theories by the use of some concepts and theorems of the nonstandard analysis. However, the derived microlocal parameter theories do not involve any nonstandard analysis notions. The main features of the approach outlined in Section 4 can be listed as follows:

(i) The approach proposed in the paper is very general, i.e., it can be applied to different thermo-elastic-unelastic composites (such as elastic-plastic with hardening or elastic/viscoplastic) under large strains and large temperature gradients. Hence the obtained relations constitute a good starting point for the specification of many special theories of unelastic periodic composites.

(ii) The analytical formulation of problems within microlocal parameter theories is not much more complicated than the pertinent formulations of problems for homogeneous materials since the system of equations for microlocal parameters is algebraic. Moreover, the solution to the boundary value problem on the basic periodicity cell (needed in the known asymptotic homogenization methods) is not required here.

(iii) The adaptive refinement of the obtained approximate solutions is possible by the passage to more suitable microlocal parameter approximations; we can use here an approach similar to that of the known adaptive finite element method.

The examples of some special theories based on the nonstandard homogenization approach were derived independently in [3], [16]—[19] for thermoelastic periodic composites. The general approach to the microlocal parameter theories for unelastic periodic composites, detailed in this paper, was applied to the elastic-plastic composites with the kinematic strain hardening in [20].

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