Oppositeness in Buildings and Simple Modules for Finite Groups of Lie Type

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Introduction

In this talk we consider the oppositeness relations in a Tits building of a finite group of Lie type from the point of view of representation theory.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots

Parabolic subgroups

- $J \subseteq I$
- $W_J := \langle w_i \mid i \in J \rangle$ *standard parabolic subgroup* of W
- $P_J = BW_JB$, *standard parabolic subgroup* of G

Types and objects of the building

- A *type* simply a subset of I its *cotype* is its complement.
- An object of cotype J is a right coset of P_J in G.

Opposite types

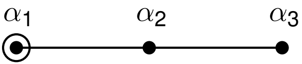
Definition

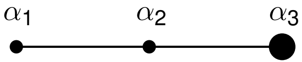
Two types J and K are *opposite* if

$$\{-w_0(\alpha_i) \mid i \in J\} = \{\alpha_j \mid j \in K\},$$

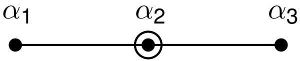
or, equivalently, if

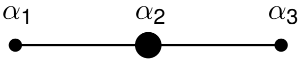
$$\{w_0 w_i w_0 \mid i \in J\} = \{w_i \mid i \in K\}.$$



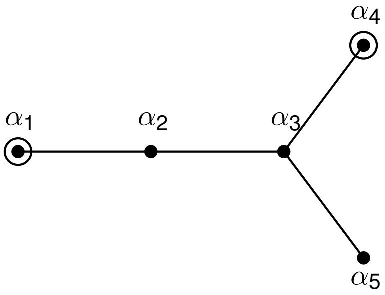


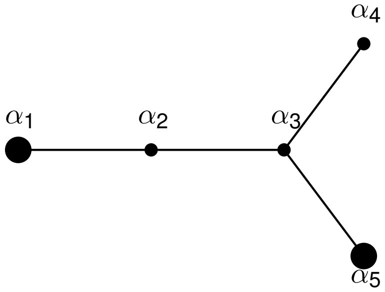
A_3, skew lines in $PG(3, q)$



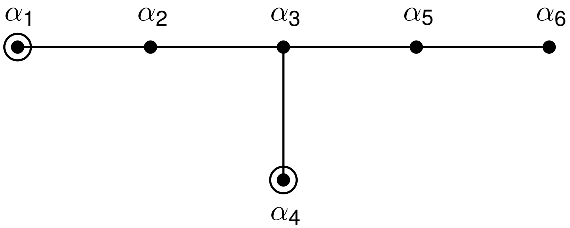


D_5, flags in oriflamme geometry





E_6


α_1
α_2
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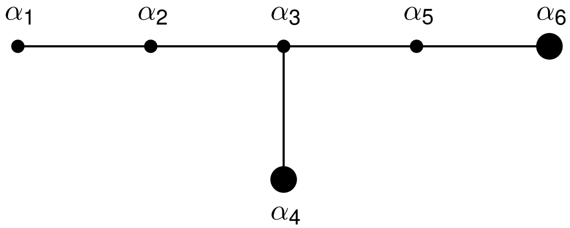
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Opposite objects

Let J and K be fixed opposite types.

Definition

An object $P_J g$ of cotype J is *opposite* an object $P_K h$ of cotype K if $P_K h g^{-1} P_J = P_K w_0 P_J$.
$(\iff P_K h \subseteq P_J w_0 P_K g \iff P_J g \subseteq P_K w_0 P_J h)$.

Example. Type A_ℓ, J=$I \setminus \{i\}$.

- $G \cong \mathrm{SL}(V)$, dim $V = (\ell + 1)$
- Objects of cotype J are i-dimensional subspaces.
- Objects of the opposite cotype $K = I \setminus \{\ell + 1 - i\}$ are $\ell + 1 - i$-dimensional subspaces.
- A subspace of cotype J is opposite one of cotype K if their intersection is the zero subspace.
- A familar special case is when $\ell = 3$ and $i = \ell + 1 - i = 2$. Thinking projectively, the objects are lines in space and the oppositeness relation is skewness.

Example. Opposite Flags, type A_ℓ

- Object of cotype $J = I \setminus \{j_1, \ldots, j_m\}$ is a flag

$$V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_m}$$

  with dim $V_{i_j} = i_j$
- If $V'_{k_1} \supset V'_{k_2} \supset \cdots \supset V'_{k_m}$ is an object of the opposite cotype, then the two flags are opposite iff $V_{i_j} \cap V'_{k_j} = \{0\}$, for $j = 1$, …, m.

Example: Classical modules

- G be of type B_ℓ, C_ℓ, or D_ℓ and $J = I \setminus \{1\}$.
- J is opposite to itself.
- In the B_ℓ case, objects of cotype J can be identified with singular points (one-dimensional subspaces) with respect to a nondegenerate quadratic form in a finite vector space of dimension $2\ell + 1$. singular points are opposite if and only if they are not orthogonal.
- For C_ℓ and D_ℓ the objects of cotype J can be viewed as singular points of a 2ℓ-dimensional vector space with respect to a symplectic symplectic form or a split quadratic form. Two points are opposite if and only if they do not lie on a singular line.

The oppositeness matrix

- The oppositeness graph $\Gamma_{J,K}$ is the bipartite graph whose parts are the sets of objects of cotypes J and K respectively, with two vertices adjacent when the objects are opposite.
- Let $A = A(J, K)$ be the oppositeness matrix for objects of cotypes J and K.
- Then the adjacency matrix of $\Gamma_{J,K}$ is $\begin{bmatrix} 0 & A \\ A' & 0 \end{bmatrix}$, where A' is the transpose of A.

Theorem

(Brouwer, 2009) If G is defined over $\mathbf{F}_q$, then the square of every eigenvalue λ of A is a power of q.

Topics for today

- One can consider other invariants of the incidence matrix A such as its Smith normal form or its p-rank. We'll consider the p-rank.
- We'll show that the p-rank is the dimension of an irreducible p-modular representation of G.
- This follows from a general theorem of Carter and Lusztig (1976).
- Then we'll describe the simple module in terms of its highest weight and discuss methods for computing its character.

Permutation modules on flags

- Let k be a field of characteristic p. Let $\mathcal{F}_J$ denote the space of functions from the set $P_J\backslash G$ of objects of cotype J to k. Then $\mathcal{F}_J$ is a left kG-module by the rule

$$(xf)(P_Jg) = f(P_Jgx), \quad f \in \mathcal{F}_J, \quad g, x \in G.$$

  Let δ_{P_Jg} denote the characteristic function of the object $P_Jg \in P_J\backslash G$. Then $\mathcal{F}_J$ is generated as a kG-module by δ_{P_J}

The oppositeness homomorphism

- The relation of oppositeness defines a kG-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_K h) = \sum_{P_J g \subseteq P_J w_0 P_K h} f(P_J g). \tag{1}$$

- We have

$$\eta(\delta_{P_J g}) = \sum_{P_K h \subseteq P_K w_0 PJg} \delta_{P_K h}.$$

so the characteristic function of an object of cotype J is sent to the sum of the characteristic functions of all objects opposite to it.

Simplicity of oppositeness modules

Theorem

The image of η is a simple module, uniquely characterized by the property that its one-dimensional U-invariant subspace has full stablizer equal to P_J, which acts trivially on it.

This result is essentially a corollary of a more general result of Carter and Lusztig (1976) on the *Iwahori-Hecke Algebra* $\mathrm{End}_{kG}(\mathcal{F}_\emptyset)$. We next describe their result.

The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_\emptyset$.
- For $w \in W$ define $T_w \in \mathrm{End}_k(\mathcal{F})$ by

$$T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

- Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

- One can show that

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- For any reduced expression

$$w_0 = w_{j_k} \cdots w_{j_1}$$

define

$$\Theta_{j_i} = \begin{cases} T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \notin V_J \\ I + T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \in V_J \end{cases} \tag{2}$$

and set

$$\Theta^J_{w_0} = \Theta_{j_k} \Theta_{j_{k-1}} \cdots \Theta_{j_k}.$$

- The definition depends on the choice of reduced expression but it can be seen that different expressions give the same endomorphism up to a nonzero scalar multiple.

Theorem

(Carter,Lusztig) The image $\Theta^J_{w_0}(\mathcal{F})$ *is a simple kG-module. The full stablizer of the one-dimensional subspace of U-fixed points in this module is* P_J *and the action of* P_J *on this one-dimensional subspace is trivial.*

Deduction of Theorem

- the first step is to choose a special expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let
$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1} \tag{3}$$
be a reduced expression for w_J. The above expression can be extended to a reduced expresion
$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1} \tag{4}$$
of w_0. Thus $m = |R_J^+|$ and $k = |R^+|$.
Then
$$w^* = w_{i_k} \cdots w_{i_{m+1}}. \tag{5}$$
is a reduced expression for w^*.
- Use above expression for w_0 to define $\Theta^J_{w_0}$.

- Since w_J sends all positive roots in V_J to negative roots and w_0 sends all roots to positive roots, it is clear that for the first m partial products the new positive root sent to a negative root belongs to V_J, and that the new positive roots for the remaining partial products are the elements of $R^+ \setminus R_J^+$, so do not belong to V_J. Thus we have

$$\Theta_{w_0}^J = T_{w^*}(1 + T_{i_m}) \cdots (1 + T_{i_1}), \tag{6}$$

- Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta_{w_0}^J$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to T_{w^*}.

The projections π_J and π_K

- Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

and π_K defined similarly. It is easily checked that π_J and π_K are *kG*-module homomorphisms and they are surjective since $\pi_J(\delta_B) = \delta_{P_J}$.

- The main step is to compare $\eta\pi_J$ with $\pi_K T_{w^*w}$ for $w \in W_J$. For $f \in \mathcal{F}$, we compute

$$
\begin{aligned}
[\eta(\pi_J(f))](P_K g) &= \sum_{P_J h \subseteq P_J w^{*-1} P_K g} \sum_{Bx \subseteq P_J h} f(Bx) \\
&= \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx).
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
[\pi_K(T_{w^*w}(f))](P_K h) &= \sum_{Bg \subseteq P_K h} (T_{w^*w} f)(Bg) \\
&= \sum_{Bg \subseteq P_K h} \sum_{Bx \subseteq B(w^*w)^{-1} Bg} f(Bx) \\
&= \sum_{Bg \subseteq P_K h} \sum_{Bg \subseteq B(w^*w) Bx} f(Bx) \\
&= q^{\ell(w)} \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx).
\end{aligned}
\tag{8}
$$

- Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_J & \xrightarrow{q^{\ell(w)}\eta} & \mathcal{F}_K \\ \uparrow{\scriptstyle \pi_J} & & \uparrow{\scriptstyle \pi_K} \\ \mathcal{F}_S & \xrightarrow{T_{ww^*}} & \mathcal{F}_S, \end{array} \tag{9}$$

- If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.
- Hence $\pi\Theta^J_{w_0} = \pi T_{w^*} = \eta\pi$.
- Therefore, since $\Theta^J_{w_0}(\mathcal{F})$ is simple $\eta\pi(\mathcal{F}_J) \neq 0$, we see that $\eta\pi(\mathcal{F}_J) \cong \Theta^J_{w_0}(\mathcal{F})$.
- Since π is surjective, we have $\eta(\mathcal{F}_J) \cong \Theta^J_{w_0}(\mathcal{F})$. $\square$

Highest weights of oppositeness modules

- $G = G(q)$ is a Chevalley group of universal type or a twisted subgroup.
- Simple modules are restrictions of certain simple rational modules $L(\lambda)$ of the ambient algebraic group, so we want to identify the highest weight λ_{opp} of the oppositeness modules.

Highest weights of oppositeness modules

- If G is an untwisted group, then the fundamental weights ω_i for the ambient algebraic group are indexed by I.
- $\lambda_{opp} = \sum_{i \in I \setminus J} (q-1)\omega_i$.

Highest weights of oppositeness modules

- There are two cases when G is a twisted group,
- Suppose that all roots of G^* have the same length (${}^2A_\ell$, ${}^2D_\ell$, ${}^3D_4$, ${}^2E_6$). Then G arises from a symmetry ρ of the Dynkin diagram of $G^* = G^*(q^e)$, where e is the order of ρ. Let $I^* = \{1, \ldots, \ell^*\}$ index the fundamental roots of G^*. The index set I for G labels the ρ-orbits on I^*. Let ω_i, $i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in J.
- $\lambda_{opp} = \sum_{i \in I^* \setminus J^*} (q-1)\omega_i$.
- If G is a Suzuki or Ree group, then the untwisted group $G^*(q)$ has two root lengths. Then the set I for G indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots. and for $J \subset I$.
- $\lambda_{opp} = \sum_{i \in I \setminus J} (q-1)\omega_i$.

Extreme cases

- $\lambda_{opp} = (q-1)\tilde{\omega}$, with $\tilde{\omega}$ a sum of fundamental weights.
- We can consider the extreme cases. If $J = K = \emptyset$, then $L(\lambda_{opp}) \cong k$. If $J = K = I$, $L(\lambda_{opp})$ is the *Steinberg module*, of dimension equal to the p-part of $|G|$.

Reduction to prime fields

- If $q = p^t$, then by Steinberg's Tensor Product Theorem,

  $$L((q-1)\tilde{\omega}) \cong L((p-1)\tilde{\omega}) \otimes L((p-1)\tilde{\omega})^{(p)} \otimes \cdots \otimes L((p-1)\tilde{\omega})^{(p^{t-1})} \tag{10}$$

  (Superscripts indicate twisting by powers of Frobenius.)

Reduction to prime fields

Proposition

Let the root system R and opposite types J and K be given and let $A(q) = A(q)_{J,K}$ *denote the oppositeness incidence matrix for objects of cotypes J and K in the building over* $F(q)$*, where* $q = p^t$*. Then* $\operatorname{rank}_p A(q) = (\operatorname{rank}_p A(p))^t$*.*

This reduction of the to the prime case is significant because Weyl modules with highest weight $(p-1)\tilde{\omega}$ are much less complex in structure than those of highest weight $(q-1)\tilde{\omega}$, say.

Jantzen Sum Formula

The Weyl module $V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \mathrm{rad}\, V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

Notation key

- $V(\lambda)$, Weyl module of highest weight λ,
- $L(\lambda)$, its simple quotient.
- ρ is the half-sum of the positive roots
- $v_p(m)$ p-adic valuation of m.
- $\chi(\mu)$, Weyl character; there is a unique weight fo the form $\mu' = w(\mu + \rho) - \rho$ in the region $\{\nu : \langle \nu + \rho, \alpha^\vee \rangle \geq 0, \forall \alpha \in R^+\}$, where $w \in W$. Then $\chi(\mu)$ is the sign$(w) \, \mathrm{Ch} \, V(\mu')$ if μ' is dominant, and zero otherwise.

- The usefulness of the sum formula comes from the fact that the characters of the Weyl modules themselves are given by *Weyl's Character Formula*, so that the right hand side can be computed from p, R and λ.
- The Jantzen sum gives an upper estimate on the composition multiplicities in the radical of the Weyl module $V(\lambda)$ in terms of the composition factors of Weyl modules which have lower highest weights.
- Sometimes, for weights of a special form, it may be that the highest weights of the Weyl characters $\chi(\mu)$ in the Jantzen sum are very few in number or all have a similar form. In such cases, it is possible to deduce the character of $L((p-1)\tilde{\omega})$.

Subspaces: Type A_ℓ, $J = I \setminus \{i\}$

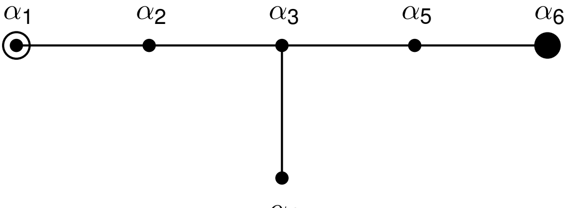
- In this case, the simple modules $L((p-1)\omega_i)$ can be found without reference to Weyl modules.
- $S(i(p-1)) :=$ degree $i(p-1)$ homogeneous component of the truncated polynomial ring $k[x_0, \ldots, x_\ell]/(x_i^p; 0 \leq i \leq \ell)$
- $S(i(p-1))$ is a simple kG-module.
- By highest weights, $S(i(p-1)) \cong L((p-1)\omega_{\ell+1-i})$, for $i = 1, \ldots, \ell$.
- There is also work (Chandler-PS-Xiang (2006), Brouwer (2010), Ducey-PS (2010)) on some cases of the Smith normal form.

Classical modules: Types B_ℓ, C_ℓ D_ℓ, $J = I \setminus \{1\}$

- p-ranks have been computed by Arslan-PS (2009) using Weyl modules.
- The Weyl modules in question are $V((p-1)\omega_1)$.
- For type C_ℓ they are simple.
- For B_ℓ and D_ℓ, use sum formula.
- Method extends to clasical modules of non-split orthogonal groups (type ${}^2D_\ell$) unitary groups (type ${}^2A_\ell$).
- In the twisted cases the relevant Weyl module is $V((p-1)(\omega_1 + \omega_\ell))$.

An E_6 Example

- $G = E_6(q)$, group of isometries of a certain 3-form on a 27-dimensional vector space V. The geometry of this space has been studied in great detail. (Dickson, Aschbacher, Buekenhout-Cohen, Cooperstein, Pasini.)
- Consider the objects of type 1 and the opposite type 6. We can view these, respectively, as the singular points and singular (in a dual sense) hyperplanes of V. A singular point $\langle v \rangle$ is opposite a singular hyperplane H if and only $v \notin H$.



Point-hyperplane oppositeness for $E_6(q)$

- $rank_p A = \dim L((q-1)\omega_1)) = \dim L((p-1)\omega_1))^t$, where $q = p^t$. (Steinberg's tensor product theorem)
- Work out $\dim L((p-1)\omega_1))$ using Weyl modules, Weyl Character formula, Jantzen sum formula).

Repeated use of Jantzen Sum Formula yields an exact sequence

$$
\begin{aligned}
0 \to V((p-11)\omega_1 + 2\omega_2) &\to V((p-10)\omega_1 + \omega_2 + \omega_5) \\
\to V((p-9)\omega_1 + \omega_3 + \omega_6) &\to V((p-8)\omega_1 + \omega_4 + 2\omega_6) \\
\to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) &\to L((p-1)\omega_1) \to 0
\end{aligned}
$$

The dimensions of the $V(\mu)$ are given by Weyl's formula. Hence

$$
\begin{aligned}
\dim L((p-1)\omega_1) = \frac{1}{2^7.3.5.11} & p(p+1)(p+3) \\
& \times (3p^8 - 12p^7 + 39p^6 + 320p^5 \\
& - 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440)
\end{aligned}
$$

2, 27
3, 351
5, 19305
7, 439439
11, 45822672
13, 274187550
17, 5030354043
19, 16937278357
23, 137112098409
29, 1744146121068
31, 3628038332724
37, 25349391871621
41, 78345931447980
43, 132256396016732
47, 351675426454470
53, 1317968719988571
59, 4286665842359706
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71, 32843689463427543
73, 44580694495895104
79, 106281498207828698
83, 182978611275724173
89, 394284508288312914
97, 1016219651834875565

Concluding Remarks

- The oppositeness relations of the building of a finite group of Lie type give rise to simple modules.
- We have considered some basic examples of oppositeness relations and described their associated modules, but the general case remains open.
- The p-rank problem for oppositeness relations has been reduced to groups over the prime field and equivalent to the dimension problem for simple modules for the algebraic group whose highest weights have coefficients $(p-1)$ and 0.
- When $p = 2$ every restricted highest weight is of this form.

- Thank you for your attention!