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# A COMPARATIVE STUDY ON AXIOMATIZED FIRST-ORDER THEORY VRS SET THIORY

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**ABSTRACT**

*In the present study axiomatized first-order theory was compared with set theory. The foundational theory under mathematics is known as Zermelo-Fraenkel set theory (ZF). V. Dan Denl described that ZF contains seven axioms plus two infinite axiom schemes. The theorem of Montague states that a finite axiomatization of ZF is not possible in the language of ZF without assuming extra objects as derived by B. Montague. We formulize the idea of quantification over an n-tuple of variables ranging over sets. In the language of ZF set theory. We derive the infinite schemes SEP and SUB of ZF from FAST. We consistency of the theory for functions f, use the notation f:x→y for ‹x, y› ϵ f.*

**Key words:** Zermelo-Fraenkel, Set theory, Axiom, Family Set Axiom, theorem of Montague.

## INTRODUCTION

The foundational theory under mathematics is known as Zermelo-Fraenkel set theory (ZF). V. Dan Denl described that ZF contains seven axioms plus two infinite axiom scheme, these seven axioms are as follow : 1. Extensionality Axiom (EXT), 2. Empty Set Axiom (EMPTY), 3. Pair Axiom (PAIR), Sumset Axiom (SUM), 4. Powerset Axiom (POW), 5. Infinite Set Axiom (INF), and Axiom of Regularity (REG), the two axiom schemes are the Separation Axiom Scheme (SEP) and the Substitution Axiom (Sub). Hence, ZF is thus an infinitely axiomatized theory; the theorem of Montague states that a finite axiomatization of ZF is not possible in the language of ZF without assuming extra objects as derived by B. Montague. All non-logical axioms of FAST have been introduced. However, since quantification over a family of variables indexed in a set X is a new concept in logic, at least the logical axioms for the elimination of the quantifiers from FAM must be given to derive theorems.

ZF, several theories have been introduced in placed by be finitely axiomatized. Structure but do not give a complete description Von Neumann-Godel- Bernays set theory (NGB) and Cantor-Von Neumann set theory (CVN). F.A. Mulber either depart from the ontology of ZF or use non-constructive axioms. A constructive axiom is an axiom that, when certain things are given states the existence of a uniquely determined other set.

NGB uses classes and CVN states that there is a set including all sets. We show that, “everything is a set” of ZF, a finite axiomatization of set theory is possible with a new constructive axiom which uses a new concept of universal quantification.

**1.Formalization of universal quantification:**

We formulize the idea of quantification over an n-tuple of variables ranging over sets. In the language of ZF set theory

$$Ɐu_1, u_2, \ldots\ldots, u_n\Phi \quad \ldots\ldots (1.1)$$

Stands for any n-tuple of sets $u_1, u_2, \ldots\ldots, u_n\Phi$. (1.1) is called the language of ZF as the postulate of meaning

$$Ɐu_1, u_2, \ldots\ldots, u_n\Phi \leftrightarrow Ɐ u_1\, Ɐu_2 \ldots Ɐu_n\, \Phi \quad \ldots (1.2)$$

Hence, view quantification over an n-tuple of variables $u_1, u_2, \ldots\ldots, u_n$ as quantification over a family of variables $u_i$ indexed in finite set {1, 2,……,n}, denoted by $(u_i)_{i\in\{1,2,\ldots,n\}}$. The point is, however, that in the language of ZF it is not possible to quantify over an infinite family, because the right- hand side of (1.2) has to be a finite formula. We remove this restriction, and to generalize the idea formalized in (1.1) into quantification over a family of variables indexed in an arbitrary set X. Hence, for a given set X, an expression like

$$Ɐ(u_i)_{i\in x}\Phi \quad \ldots (1.3)$$

Is quantification for any family of sets $u_i$ indexed in X, Φ. However, expression (1.2) is not expressible, since the set X may be infinite: the quantification over a family of sets $u_i$ indexed in a set X is thus a primitive concept, which has no equivalent in the language of ZF.

We thus conclude that the quantification over a family of variables $Ɐ(u_i)_{i\in x\ldots}$ does not entail a departure of the ontology of ZF. Let us consider quantification (1.3) without assuming new objects. In ZF which is similar; e.g. in PAIR- when formulized as an expression of the type $Ɐx_1, x_2\psi$ using definition (1.2) quantification over a two-tuple as a new object in itself.

**Axiomatic structure of finite group:**

The universe of discourse is the universe of sets whose all terms are sets. In the formal language, we add the following definition in the syntax of the language of ZF:

i. For every constant x, we have variables $a_x$, $b_x$,….. ranging over sets; here the subscript x is the label;
ii. We have generic variables $a_i$, $b_j$...... of which the subscripts are the label; by an interpretation of the label I as a constant x, a generic variable $u_i$ becomes a variable $u_x$ ranging over sets as above.
iii.If Φ is a formula, $u_i$ is a generic variable, and X is a term, then $Ɐ(u_i)_{i\in x}\Phi$ is a formula.

The formula Ʉ$(u_i)_{i€x}$Φ is taken for any family of sets $u_i$ indexed in X, Φ. Quantification Ʉ$(u_i)_{i€x}$…… is a simultaneous quantification over all those variables $u_x$ of which the label is a constant of the set X.

Reinterpretation of axioms of finitely axiomatisable set theory in the language of set theory. In the axiomatic structure of FAST, the first eleven axioms are the following theorems of ZF.

(i) Axiom of Extensionality (EXT): Two sets X and Y are identical if they have the same elements.

(ii) Empty Set Axiom (EMPTY): there exists a set X=Ǿ who has no elements.

(iii) Sum Set Axiom (SUM): For every set X there exists a set Y=SX made up of the elements of the elements of X.

(iv) Powerset Axiom (POW): For every set X there is a set Y =P(X) made up of the subsets of X.

(v) Infinite Set Axiom (INF): There exists a set that has the empty set as element, as well as the successor {x} of each element x.

(vi) Axiom of Regulatory (REG): Every nonempty set X contains an element Y that has no elements in common with X.

(vii) Difference Set Axiom (DIFF): For every pair of sets X and Y there is a set Z=X-Y such that the elements of Z are precisely those elements of X that do not occur in Y.

(viii) Product Set Axiom (PROD): For any two nonempty sets X and Y there is a set Z=X × Y made up of all ordered two-tuples. A two-tuple ‹x, y› is a set:{x,{x,y}}. ‹x,y› of which x is an element of X and y an element of Y.

(ix) Function Space Axom (FUN): For any two nonempty sets X and Y there is a set Z = YX made up of all functions. A function $f$ from X to Y is a subset of X×Y with precisely one two-tuple ‹x, y› for each x϶ X. $f$ from X to Y.

(x) Image Set Axiom (IM): For any function $f$ on a function f on set X is a set $f$ made up of precisely one two-tuple ‹x, y› for every element x϶ X. Every element g of a function space Y X is thus a function on X a set X, there is a set Z= $f$ [X] made up of precisely those elements y, for which there is an element x϶ X such that ‹x, y› ϵ $f$.

(xi) Reverse Image Set Axiom (REV): For any function $f$ on a set X and for any element y, there is a set Z = $f^1$ (y) made up of precisely those x€ X such that ‹x, y› ϵ $f$.

### A. Axiom

**Family Set Axiom (FAM):** $\forall X \forall (u_i)_{i\in x} \exists Z \forall y(y \in Z \leftrightarrow \exists_i \in X(y = u_i))$

Implies that for any nonempty set for any family of sets $u_i$ indexed in X there is a set Z made up of precisely the family of sets $u_i$. Hence, of EXT the set Z is unque and is denoted by $Z = \{u_i | _I \in X\}$. Let us suppose the constant (set) X, constructed a uniquely determined set $u_x$ for every constant $x \in X$.

We have constructed a family of sets $(u_i)_{i\in x}$. then we have not et constructed the set that contains this family of sets indexed in X, $(u_i)_{i\in x}$, as elements. But FAM then guarantees that this set exists – regardless how this family of sets is $(u_i)_{i\in x}$ constructed!

### B. Axiom

**First Elimination Axiom (ELI):**

$\forall X \forall (u_i)_{i\in x} \exists Z \forall y(y \in Z \leftrightarrow \exists_i \in X(y = u_i)) \leftrightarrow \forall (u_i)_{i\in x} \exists Z \forall y(y \in Z \leftrightarrow \exists_i \in X(y = u_i))$ for any constant X.

### C. Axiom

**Second Elimination Axiom (EL2):**

$\forall (u_i)_{i\in x} \exists Z \forall y(y \in Z \leftrightarrow \exists_i \in X(y = u_i)) \quad \exists Z \Rightarrow y(y \in Z \leftrightarrow \exists_i \in X(y = u_i))$ for any family of constants $(u_i)_{i\in x}$.

PAIR of ZF follows from of FAM by universal quantifier elimination $EL_1$ the following expression, which is a theorem of FAST, follows from axiom A for the case X = {1,2}:

$$\forall (u_i)_{i\in x\{1,2\}} \exists Z \forall y(y \in Z \leftrightarrow \exists_i \in \{1,2\}(y = u_i)) \quad \ldots.(1.4)$$

So, suppose that we have constructed two sets, $X_1$ and $X_2$. In other words: suppose we have constructed a family of sets $(X_i)_{i\in\{1,2\}}$. Then we do not have them in a set yet. None of the first eleven axioms of FAST can put these $X_i$'s in a set; however, on account of logical axiom $EL_2$. We now derive from theorem (1.4).

$$\exists Z \forall y(y \in Z \leftrightarrow \exists_i \in \{1,2\}(y = u_i)) \quad \ldots.(1.5)$$

Of course, formula (1.5) is equivalent to $\exists Z \forall y(y \in Z \leftrightarrow y = X_1 v y = X_2)$, so theorem (1.4) is then equivalent to PAIR of ZF: $\forall u_1 u_2 \exists Z \forall y(y \in Z \leftrightarrow y = u_1 \text{ v } y = u_2)$. This concludes the axiomatic introduction of FAST. We proceed with its discussion.

### Discussion and conclusions:

We derive the infinite schemes SEP and SUB of ZF from FAST. We consistency of the theory for functions $f$, use the notation $f$:x→y for ‹x, y› $\in f$.

**References:**


Bernay, P., 1968. Axiomatic Set Theory, *Dover Publication inc.,* Mineola.

Bernay, P., 1954. A system of axiomatic Set Theory, *J.S.L.* Vol. 13, 19-48.

Dalen D-van, H.C. Doets, 1978. Axiomatic and applied, *Pergaman Press Oxford.*

Harbeck K., 1998. Axiomatic foundation for non standard analysis fund math, 1-19.

Harbeck K., 1978. Axiomatic foundation for non standard analysis fund math. Vol. 98, 1-19.

Nasso, M. D., 1997. ZFC an axiomatic approach to non standard methods, *CR Acad SC, Paris*. 324, serial No.9, 963-967.


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