Dispersion Relations:
Applications to meson interactions

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Future Directions in Spectroscopy Analysis
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Outline

1. Introduction and Motivation

2. Dispersion Relations: the method

3. Applications
    - Two body: $\pi\pi$ and $K\pi$ form factors
    - Three body: $\eta \to 3\pi, K_{l4}$ decays

Talk by L. Dai

Talk by I. Danilkin, P. Guo

4. Conclusion and Outlook

1. Introduction and Motivation

1.1 Hadronic Physics

- Hadronic Physics: Interactions of quarks at low energy
  - Precise tests of the Standard Model:

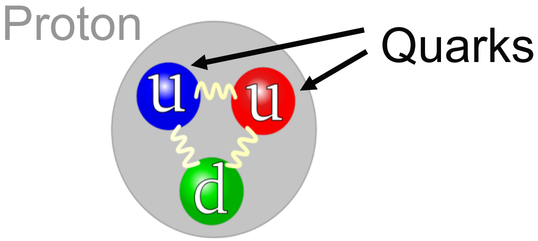
    ⇨ Extraction of V_{us}, α_S, light quark masses…
  - Look for physics beyond the Standard Model: High precision at low energy as a key to new physics?
  - Look for exotics, new hadronic states

1.2 Tools

- Hadronic Physics: Interactions of quarks at low energy
- Low energy (Q <~1 GeV), long distance: *α_S becomes large !*

  ⇨ *Non-perturbative QCD*
- A perturbative expansion in the usual sense fails
- Use of alternative approaches, expansions…: e.g.
  - Effective field theory

    Ex: ChPT for light quarks
  - Dispersion relations
  - Numerical simulations on the lattice



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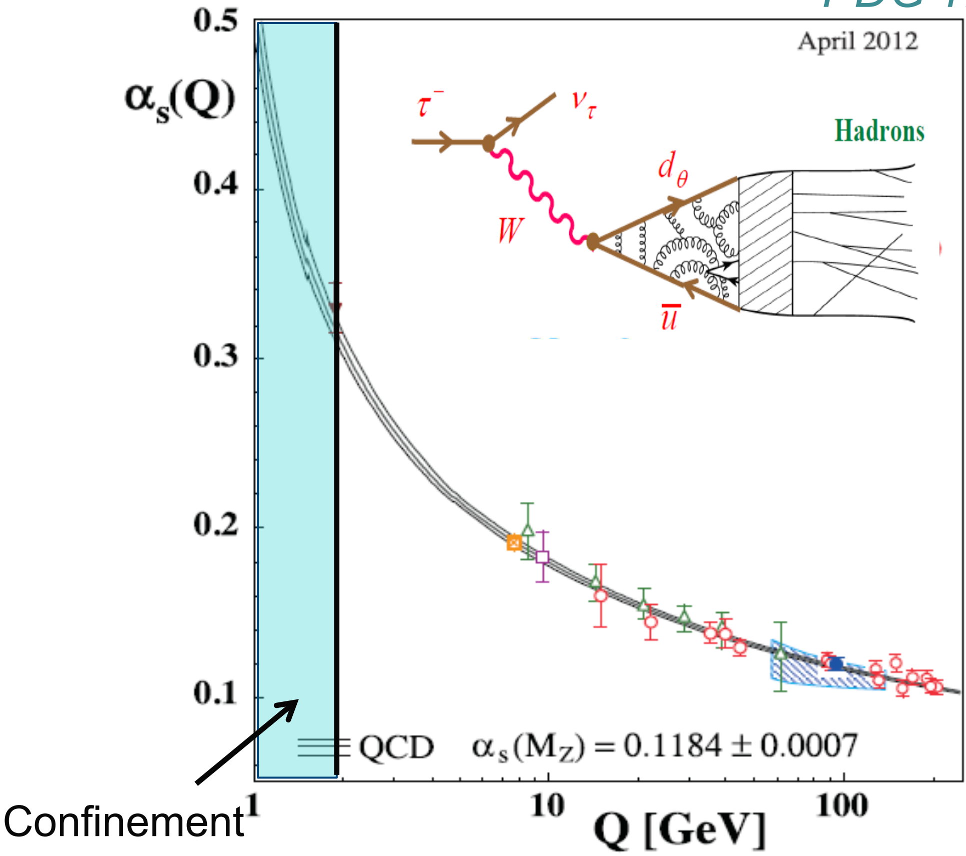


1.3 On the interest of using Dispersion Relations

- If E > 1 GeV: ChPT not valid anymore to describe dynamics of the process

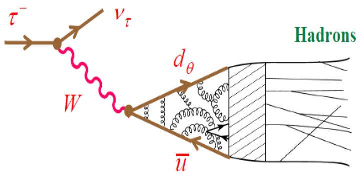
  ⇨ Resonances appear :

  - For ππ: *I=1*: ρ(770), ρ(1450), ρ(1700), …, *I=0*: “σ(~500)”, f_0(980),…
  - For Kπ: *I=1*: K*(892), K*(1410), K*(1680), …, *I=0*: “κ(~800)”, …

- Two-body case: form factor: $\boxed{H_\mu = \langle PP | \left(V_\mu - A_\mu\right) e^{iL_{QCD}} |0\rangle = \left(Lorentz\ \mathbf{struct.}\right)^i_\mu F_i(s)}$

Ex: Kπ form factors:

- $\tau \to K\pi\nu_\tau$:



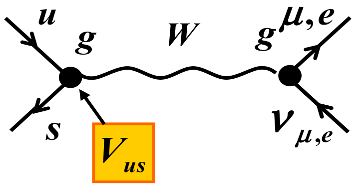
$$\langle K\pi | \bar{s}\gamma_\mu u |0\rangle = \left[\left(p_K - p_\pi\right)_\mu + \frac{\Delta_{K\pi}}{s}\left(p_K + p_\pi\right)_\mu\right] f_+(s) - \frac{\Delta_{K\pi}}{s}\left(p_K + p_\pi\right)_\mu f_0(s)$$

vector scalar

$$s = q^2 = (p_K + p_\pi)^2$$

$$\bar{f}_{0,+}(t) = \frac{f_{0,+}(t)}{f_+(0)}$$

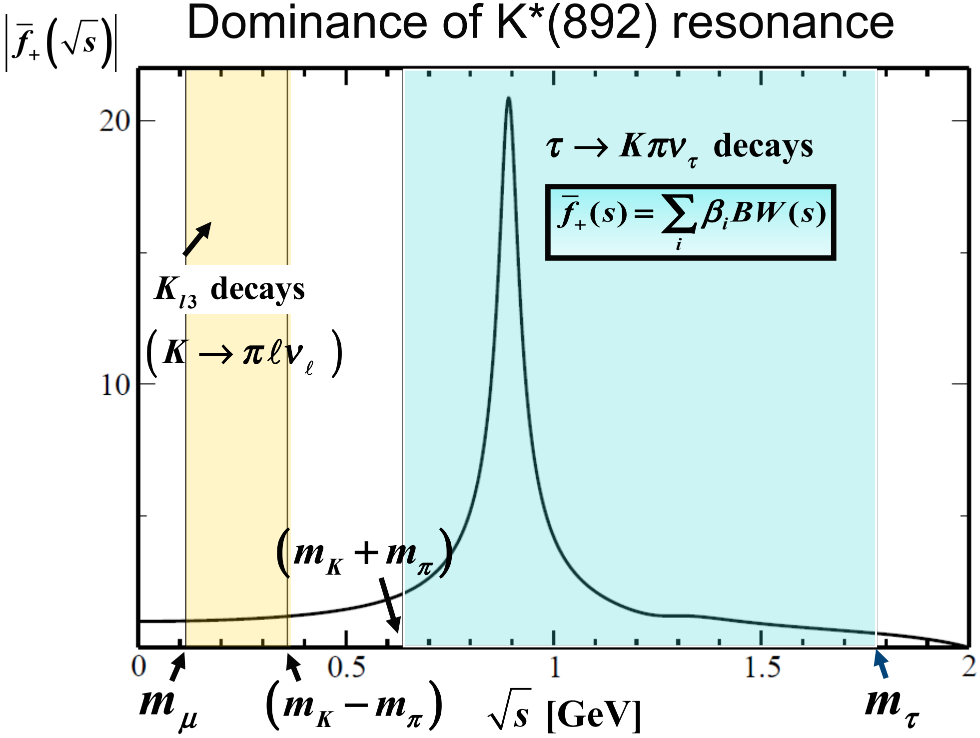
- K_{l3} decays $\left(K \to \pi \ell \bar{\nu}_\ell\right)$:



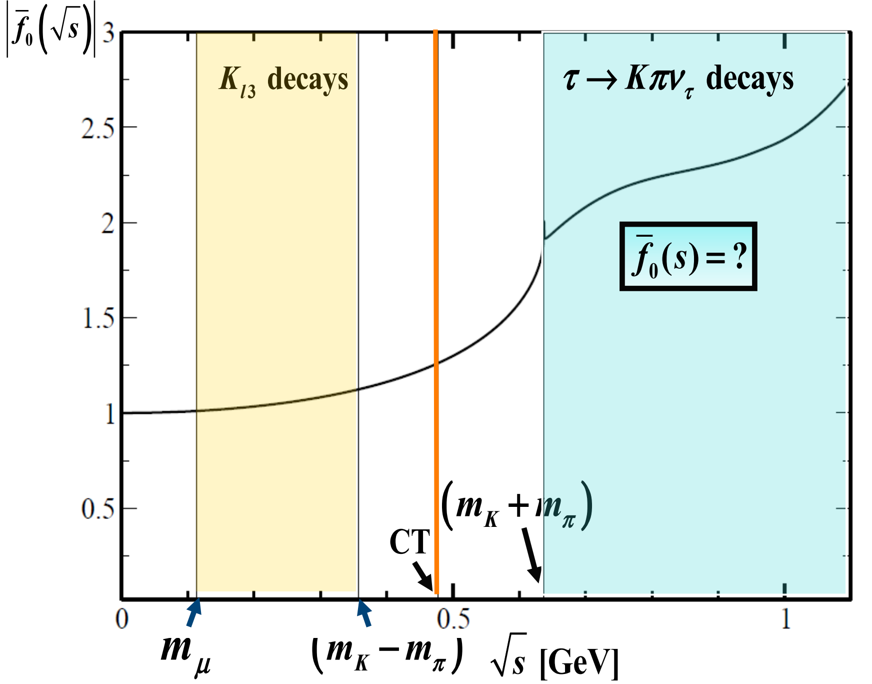
$$\langle \pi(p_\pi) | \bar{s}\gamma_\mu u |K(p_K)\rangle = \left[\left(p_K + p_\pi\right)_\mu - \frac{\Delta_{K\pi}}{t}\left(p_K - p_\pi\right)_\mu\right] f_+(t) + \frac{\Delta_{K\pi}}{t}\left(p_K - p_\pi\right)_\mu f_0(t)$$

1.3 On the interest of using Dispersion Relations

Kπ vector form factor:
Dominance of K*(892) resonance



Kπ scalar form factor:
No obvious dominance of a resonance



- With Dispersion Relations:
  - no need for making assumptions of a dominance of resonances
    ⇨ directly given by the parametrization, phase shifts taken as inputs
  - Parametrization valid in a large range of energy: analyse several processes simultanously where the same quantity: FFs, amplitude appear

1.3 On the interest of using Dispersion Relations

- Allow to take into account large final state interactions

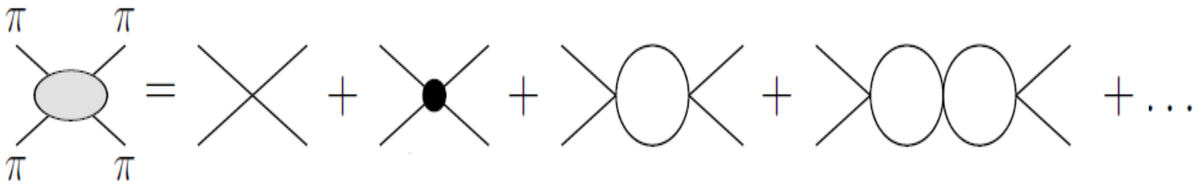
Ex: η → 3π
Slow convergence of the chiral series:

$$\boxed{\Gamma_{\eta\to3\pi} = (\underbrace{66}_{\text{LO}} + \underbrace{94}_{\text{NLO}} + \underbrace{138}_{\text{NNLO}} + ...)\,\text{eV} = (300 \pm 12)\,\text{eV}}$$

PDG'14

LO: *Osborn, Wallace'70*
NLO: *Gasser & Leutwyler'85*
NNLO: *Bijnens & Ghorbani'07*

⇨ Large $\pi\pi$ final state interactions



- Need to use Dispersion Relations to improve on the convergence of ChPT!

1.4 Strategy

- Build a parametrization to analyse the data relying on:
  - Physical properties of the amplitude:
    - Analyticity
    - Unitarity
    - Crossing symmetry

    ⇨ *Dispersion Relations*
  - Statisfies Chiral constraints at low energy
  - Statisfies the asymptotic behaviour dictated by perturbative QCD
- Aim: have the best physically motivated and the more model independent parametrization for the hadronic part of the process under study: amplitude or form factor to analyse the data accurately

  ⇨ More precise extraction of form factors or amplitude

2. Dispersion Relations: the method

2.1 Unitarity

- Two-body case: form factor: $\boxed{H_\mu = \langle PP| \left(V_\mu - A_\mu\right) e^{iL_{QCD}} |0\rangle = \left(Lorentz\ \text{struct.}\right)^i_\mu F_i\left(s\right)}$

$s = \left(p_{P_1} + p_{P_2}\right)^2$

- Unitarity ⇨ the discontinuity of the form factor is known

$$\boxed{\frac{1}{2i} disc\ F_{PP}(s) = \text{Im}\, F_{PP}(s) = \sum_n F_{PP\to n}\left(\text{T}_{n\to PP}\right)^*}$$

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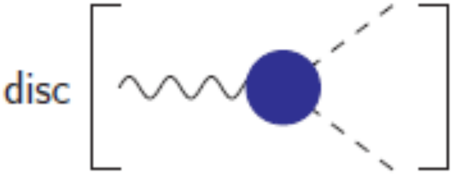

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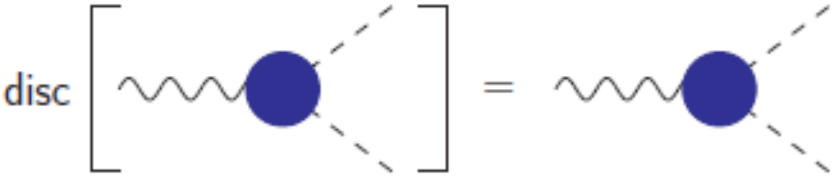

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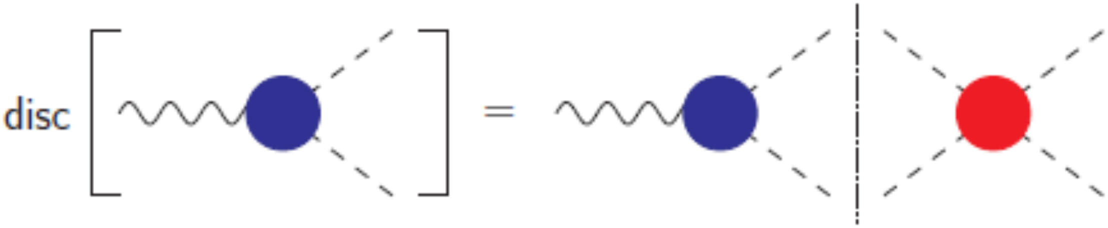

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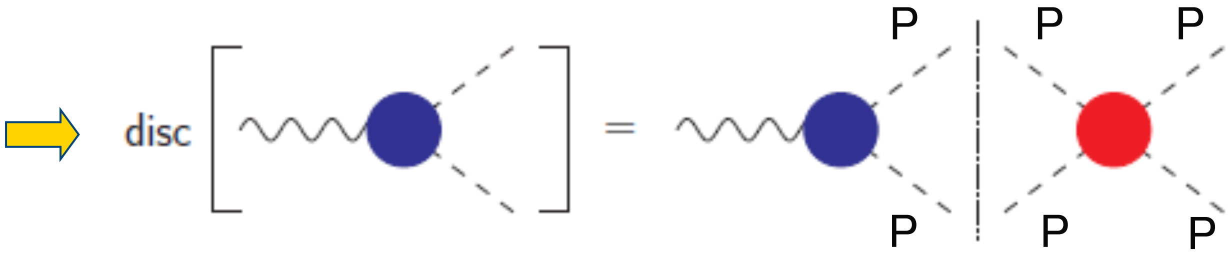
- Only one channel n = PP (elastic region)



$$\boxed{\frac{1}{2i} disc\ F_I(s) = \mathrm{Im}\, F_I(s) = F_I(s)\ \sin\delta_I(s) e^{-i\delta_I(s)}}$$

Watson's theorem

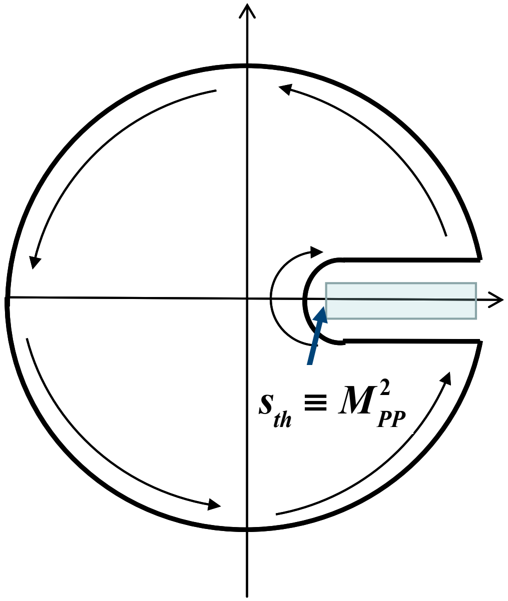
PP scattering phase known from experiment

2.2 Analyticity: Dispersion Relations

- Knowing the discontinuity of F ➡ write a dispersion relation for it

- Cauchy Theorem and Schwarz reflection principle

$$F(s) = \frac{1}{\pi}\oint \frac{F(s')}{s'-s}ds' \quad \Rightarrow \quad \frac{1}{2i\pi}\int_{M_{PP}^2}^{\infty} \frac{disc\left[F(s')\right]}{s'-s-i\varepsilon}ds'$$



- If F does not drop off fast enough for $|s| \to \infty$

  ➡ subtract the DR

$$\boxed{F(s) = P_{n-1}(s) + \frac{s^n}{\pi}\int_{M_{PP}^2}^{\infty}\frac{ds'}{s'^n}\frac{\text{Im}\left[F(s')\right]}{\left(s'-s-i\varepsilon\right)}}$$

$P_{n-1}(s)$ polynomial

2.2 Analyticity: Dispersion Relations

- Solution: Use analyticity to reconstruct the form factor in the entire space

⇨ Omnès representation : $F_I(s) = \underset{\substack{\nearrow \\ \text{polynomial}}}{P_I(s)}\,\underset{\substack{\nwarrow \\ \text{Omnès function}}}{\Omega_I(s)}$

- Omnès function : $\boxed{\Omega_I(s) = \exp\left[\frac{s}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\frac{\delta_I(s')}{s'-s-i\varepsilon}\right]}$

- Polynomial: $P_I(s)$ not known but determined from a matching to experiment or to ChPT at low energy

2.3 Assumptions

- Form factor: $$F_I(s) = \exp\left[\frac{s}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\frac{\phi_I(s')}{s'-s-i\varepsilon}\right]$$

- Up to the first inelastic threshold ($s < s_{in}$) :

  $\phi_I(s) = \delta_I(s)$ elastic phase, known

- In the inelastic region ($s \geq s_{in}$) phase not known except asymptotic behaviour

  $\phi_{+,0}(s) \rightarrow \phi_{+,0\,as}(s) = \pi$ $\left(\bar{f}_{+,0}(s) \rightarrow 1/s\right)$ *[Brodsky&Lepage]*

- Different strategies:
  - Subtract the dispersive integrals to weaken the high-energy contribution not known ⇨ subtraction constants to fit to the data
  - Build a model for the phase and fit to the data: done for the Kpi and pipi vector form factors ⇨ Data from Belle and BaBar on $\tau \rightarrow K\pi\nu_\tau$ or $\tau \rightarrow \pi\pi\nu_\tau$
  - Conformal mapping to include inelasticities, see talk by *I. Danilkin*
  - Perform a coupled channel analysis

3. Applications

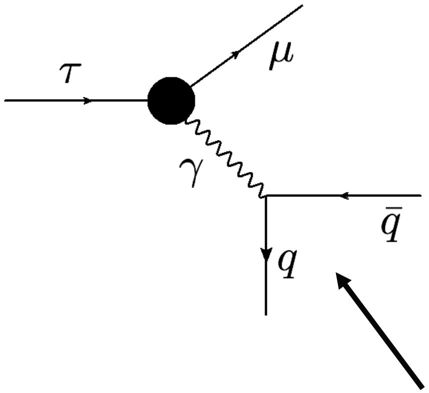
3.1 Application 1: ππ form factors and probing New physics with Tau LFV

- Constrain new physics operators from low energy decays: ex: $\tau \to \mu\pi\pi$

- Effective Lagrangian: $\mathcal{L} = \mathcal{L}_{SM} + \frac{C^{(5)}}{\Lambda} O^{(5)} + \boxed{\sum_i \frac{C_i^{(6)}}{\Lambda^2} O_i^{(6)}} + \ldots$

- Each UV model generates a *specific pattern* of D=6 operators: $\tau \to \mu\pi\pi$ very interesting probe to discriminate them ⇨ For these need to know the FFs!

- 4 form factors :



$$\langle \pi^+(p_{\pi^+})\pi^-(p_{\pi^-}) \left| \tfrac{1}{2}(\bar{u}\gamma^\alpha u - \bar{d}\gamma^\alpha d) \right| 0 \rangle \equiv F_V(s)(p_{\pi^+} - p_{\pi^-})^\alpha$$

3.1 Application 1: ππ form factors and probing New physics with Tau LFV

Celis, Cirigliano, E.P.'14

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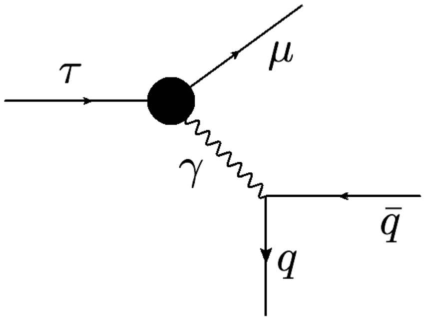
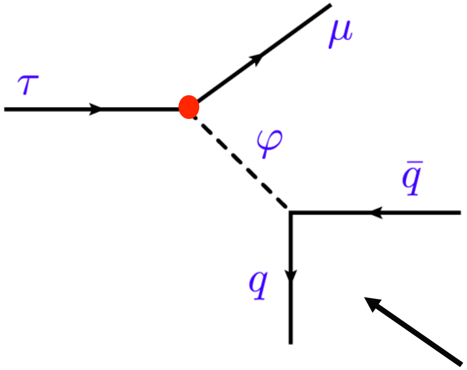
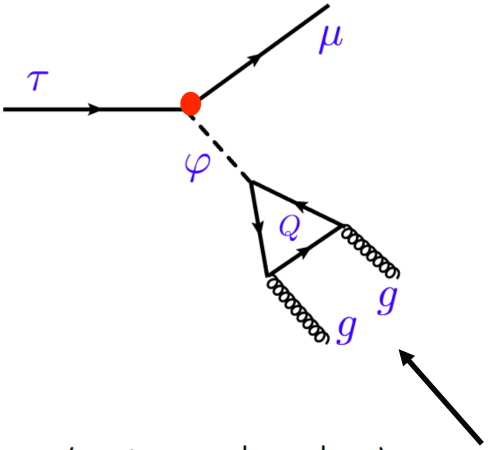






$$\langle \pi^+\pi^- | m_u \bar{u}u + m_d \bar{d}d | 0 \rangle \equiv \Gamma_\pi(s)$$

$$\langle \pi^+\pi^- | \theta^\mu_\mu | 0 \rangle \equiv \theta_\pi(s)$$

$$\langle \pi^+\pi^- | m_s \bar{s}s | 0 \rangle \equiv \Delta_\pi(s)$$

Determination of $F_V(s)$

Celis, Cirigliano, E.P.'14

- Vector form factor
  - Precisely known from experimental measurements

    $e^+e^- \to \pi^+\pi^-$ and $\tau^- \to \pi^0\pi^-\nu_\tau$ (isospin rotation)

  - Theoretically: Dispersive parametrization for $F_V(s)$

Guerrero, Pich'98, Pich, Portolés'08
Gomez, Roig'13

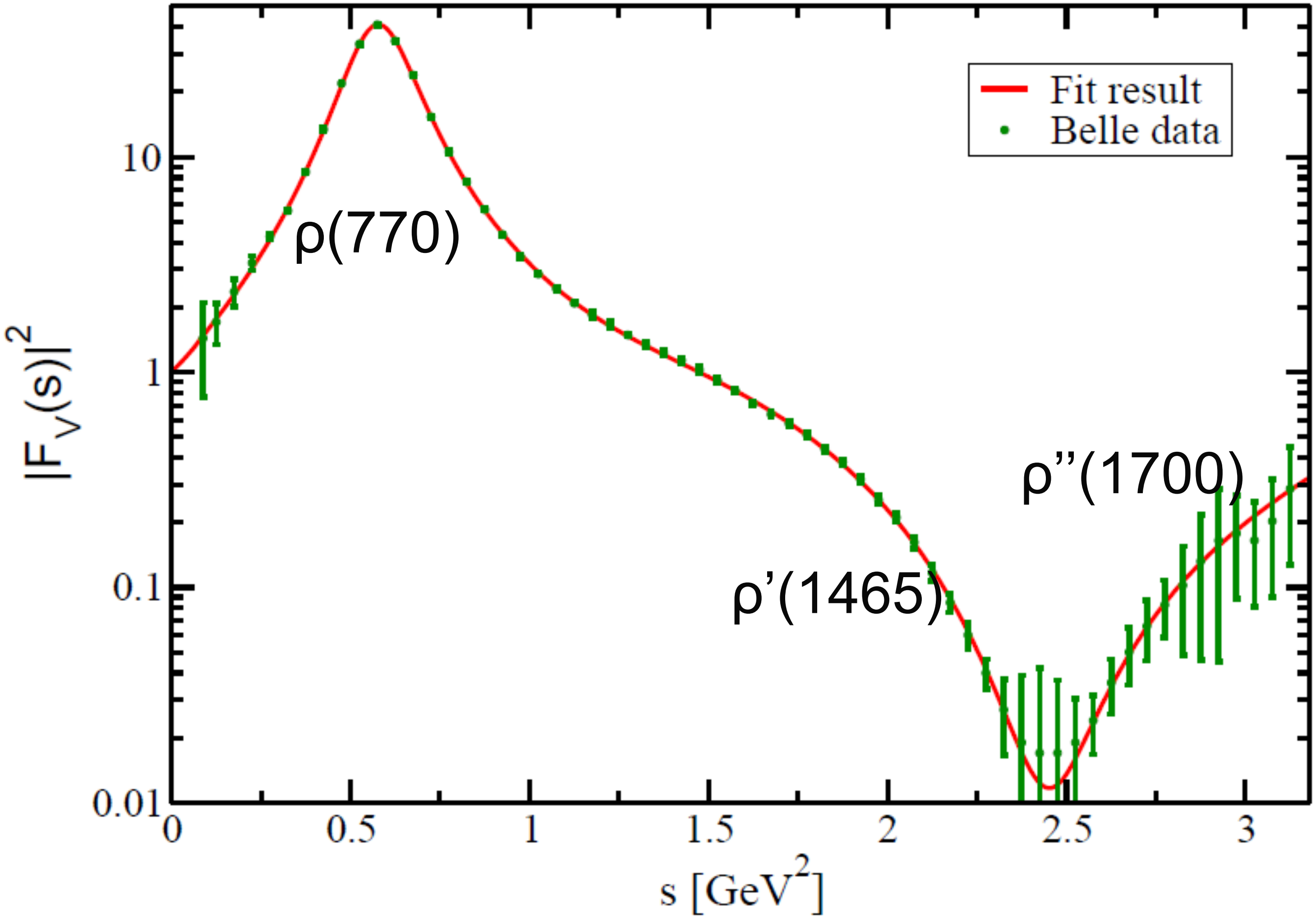
$$\boxed{F_V(s) = \exp\left[\lambda_V' \frac{s}{m_\pi^2} + \frac{1}{2}\left(\lambda_V'' - \lambda_V'^2\right)\left(\frac{s}{m_\pi^2}\right)^2 + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^3}\frac{\phi_V(s')}{\left(s'-s-i\varepsilon\right)}\right]}$$

Extracted from a model including 3 resonances ρ(770), ρ'(1465) and ρ''(1700) fitted to the data

  - Subtraction polynomial + phase determined from a *fit* to the *Belle data* $\tau^- \to \pi^0\pi^-\nu_\tau$

Determination of $F_V(s)$

Celis, Cirigliano, E.P.'14



Determination of $F_V(s)$ thanks to precise measurements from Belle!

Emilie Passemar

Determination of the Kπ FFs: Dispersive representation

Celis, Cirigliano, E.P.’14

- Model for $\phi_V(s)$:

Guerrero, Pich’98, Pich, Portolés’08
Gomez, Roig’13

$$\boxed{\tilde{F}_V(s) = \frac{\tilde{M}_\rho^2 + \left(\alpha' e^{i\phi'} + \alpha'' e^{i\phi''}\right) s}{\tilde{M}_\rho^2 - s + \kappa_\rho \, \mathrm{Re}\left[A_\pi(s) + \frac{1}{2} A_K(s)\right] - i \tilde{M}_\rho \tilde{\Gamma}_\rho(s)} - \frac{\alpha' e^{i\phi'} s}{D(\tilde{M}_{\rho'}, \tilde{\Gamma}_{\rho'})} - \frac{\alpha'' e^{i\phi''} s}{D(\tilde{M}_{\rho''}, \tilde{\Gamma}_{\rho''})}}$$

with $$\boxed{D(\tilde{M}_R, \tilde{\Gamma}_R) = \tilde{M}_R - s + \kappa_R \mathrm{Re} A_\pi(s) - i \tilde{M}_R \tilde{\Gamma}_R(s) \, .}$$

$$\boxed{\tan \phi_V \equiv \tan \delta_{\pi\pi}^P = \frac{\mathrm{Im}\, \tilde{F}_V(s)}{\mathrm{Re}\, \tilde{F}_V(s)}}$$

- Determine the resonance parameters by finding the poles in the complex plane

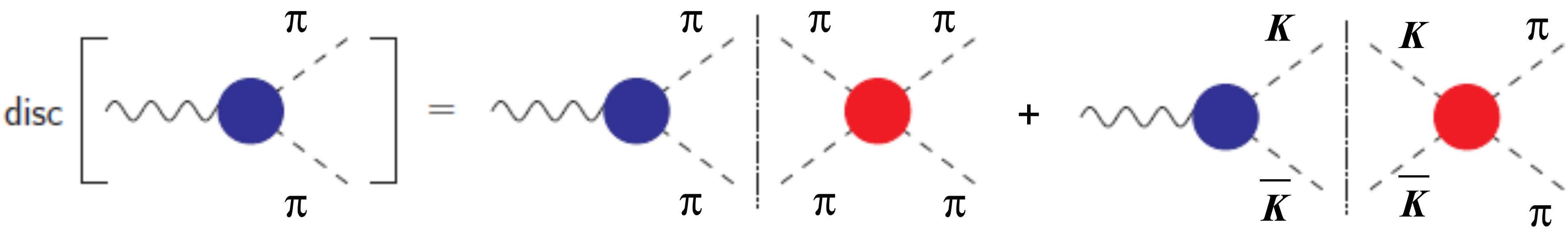
Determination of the form factors : $\Gamma_\pi(s)$, $\Delta_\pi(s)$, $\theta_\pi(s)$

Celis, Cirigliano, E.P.'14

- No experimental data for the other FFs ⇨ *Coupled channel analysis*
  up to √s ~1.4 GeV
  Inputs: I=0, S-wave ππ and KK data

Donoghue, Gasser, Leutwyler'90
Moussallam'99
Daub et al'13

- Unitarity:



⇨

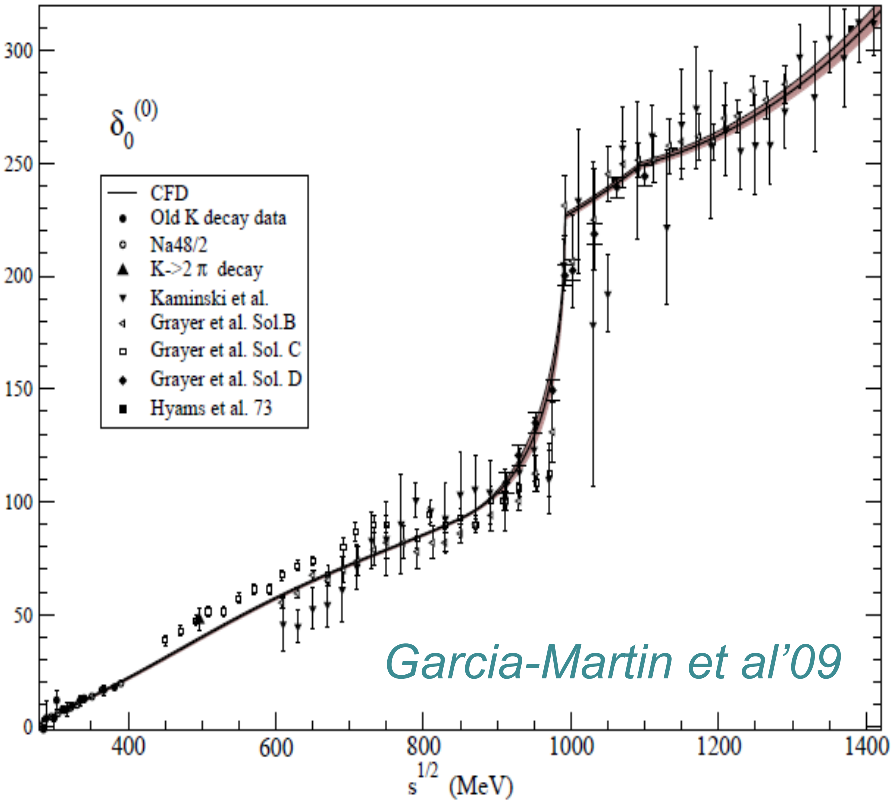
$$\boxed{\mathrm{Im}F_n(s) = \sum_{m=1}^{2} T^*_{nm}(s)\sigma_m(s)F_m(s)}$$

$$n = \pi\pi, K\bar{K}$$

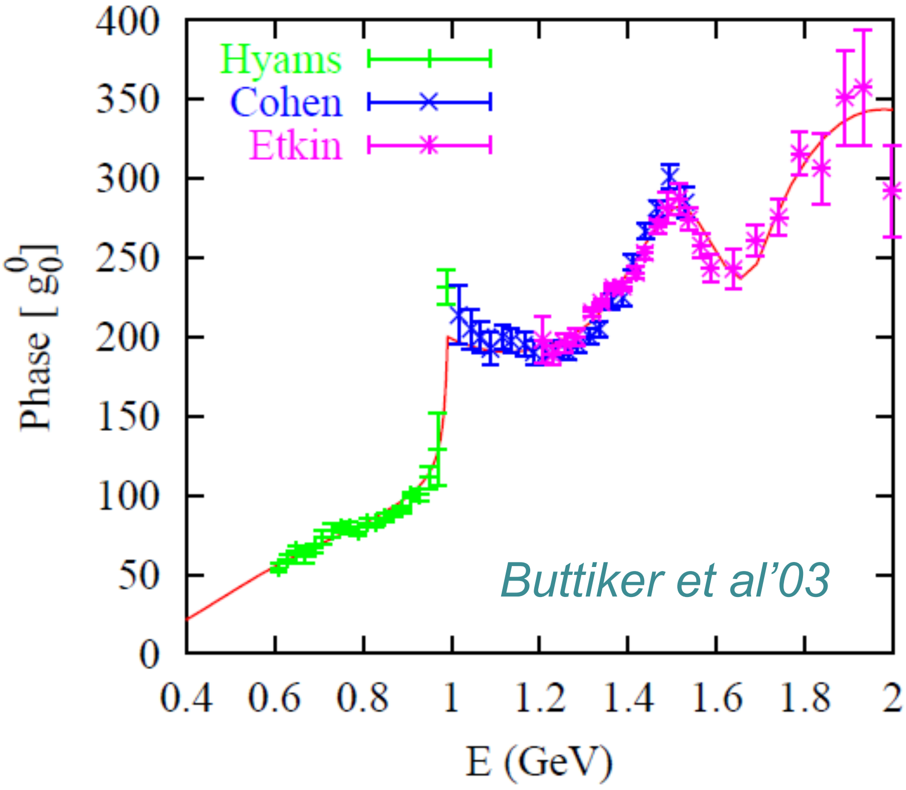
Determination of the form factors : $\Gamma_\pi(s)$, $\Delta_\pi(s)$, $\theta_\pi(s)$

Celis, Cirigliano, E.P.'14

- Inputs : $\pi\pi \to \pi\pi, KK$



Garcia-Martin et al'09



Buttiker et al'03

- A large number of theoretical analyses *Descotes-Genon et al'01, Kaminsky et al'01, Buttiker et al'03, Garcia-Martin et al'09, Colangelo et al.'11* and all agree
- 3 inputs: $\delta_\pi(s)$, $\delta_K(s)$, η from *B. Moussallam* ⇨ *reconstruct T matrix*

Determination of the form factors : $\Gamma_\pi(s)$, $\Delta_\pi(s)$, $\theta_\pi(s)$

Celis, Cirigliano, E.P.'14

- General solution:

$$\boxed{\begin{pmatrix} F_\pi(s) \\ \frac{2}{\sqrt{3}}F_K(s) \end{pmatrix} = \begin{pmatrix} C_1(s) & D_1(s) \\ C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} P_F(s) \\ Q_F(s) \end{pmatrix}}$$

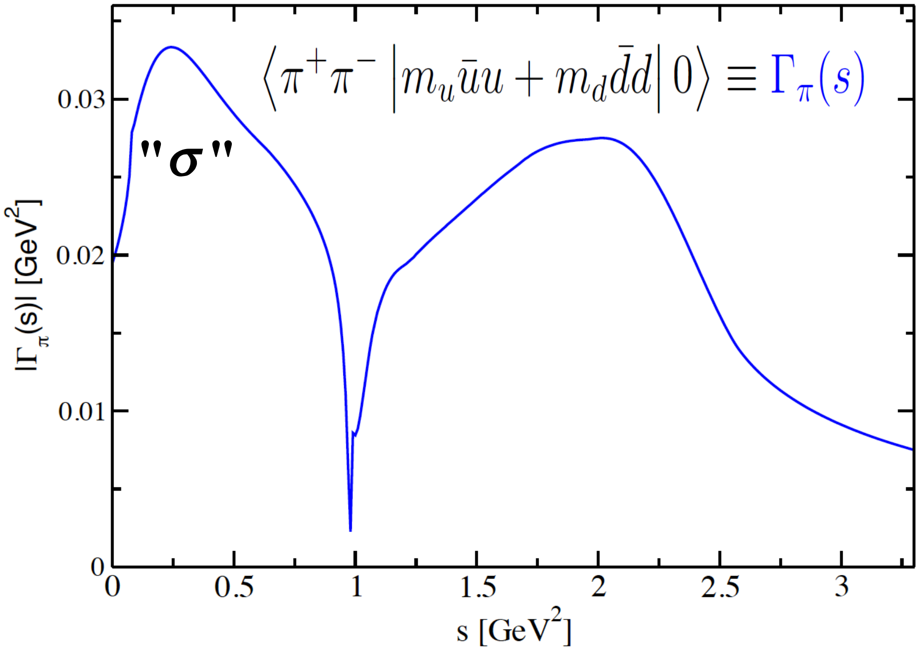
Canonical solution

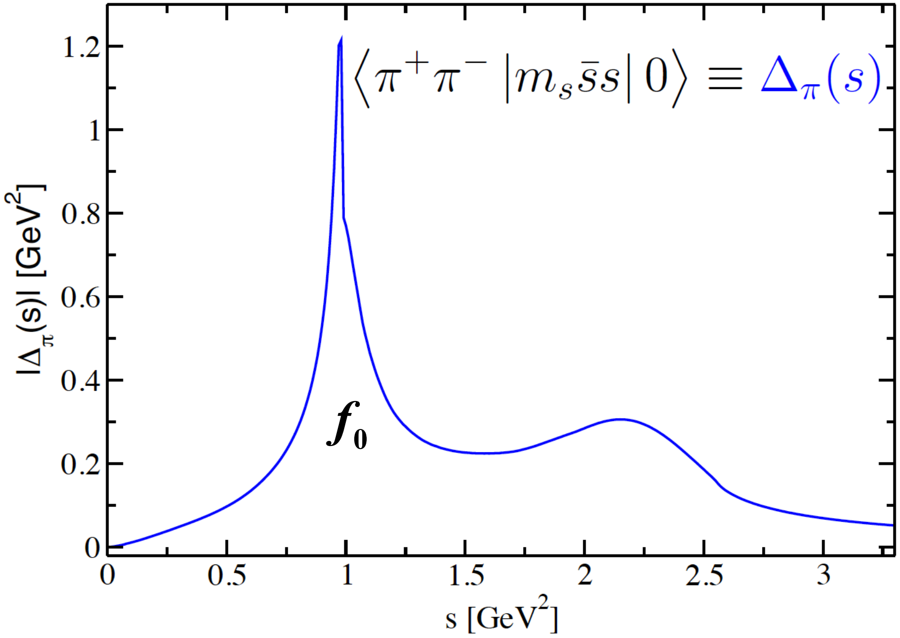
Polynomial determined from a matching to ChPT + lattice

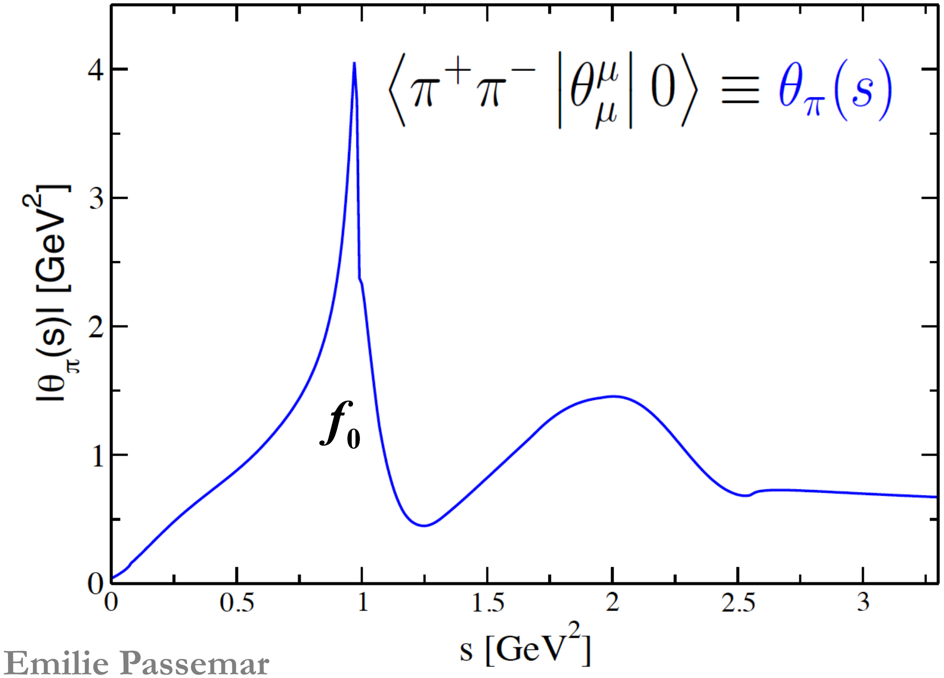
- Canonical solution found by solving the dispersive integral equations iteratively starting with Omnès functions

$$\boldsymbol{X(s) = C(s), D(s)}$$

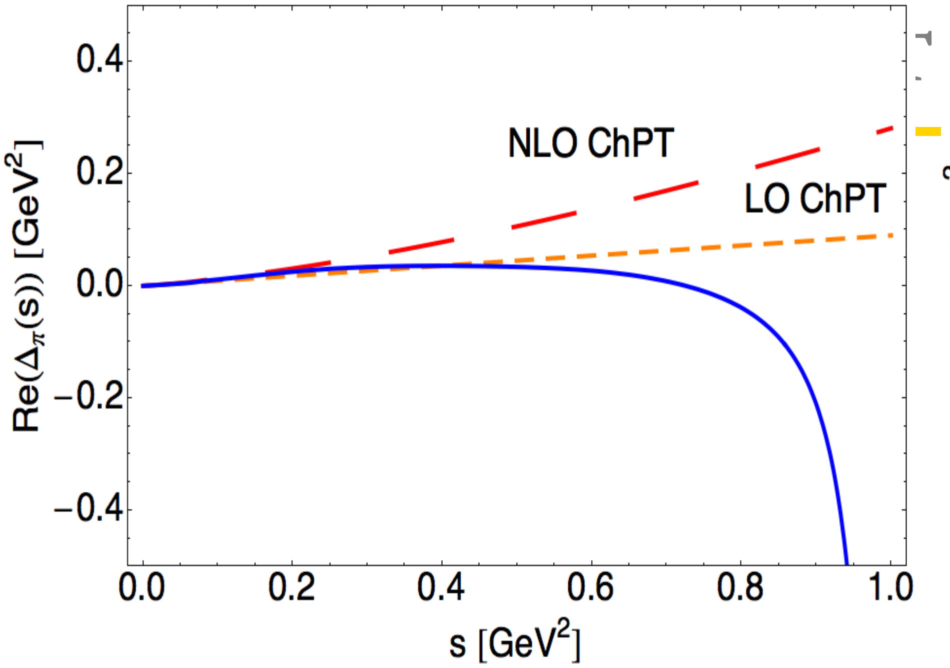
$$\boxed{\mathrm{Im}X_n^{(N+1)}(s) = \sum_{m=1}^{2} \mathrm{Re}\left\{T^*_{nm}\sigma_m(s)X_m^{(N)}\right\}} \longrightarrow \boxed{\mathrm{Re}X_n^{(N+1)}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'-s}\mathrm{Im}X_n^{(N+1)}}$$

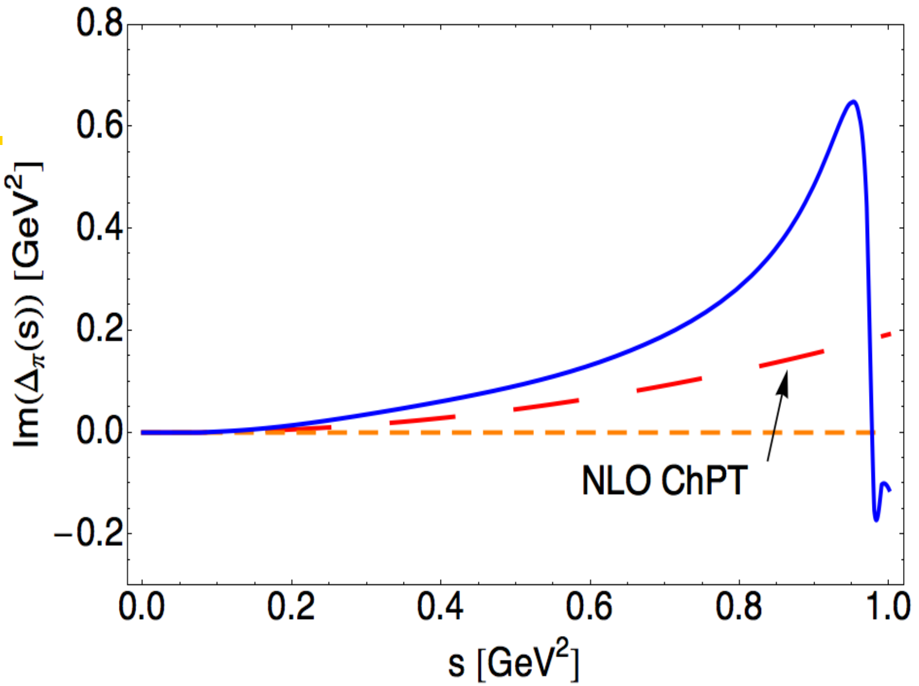


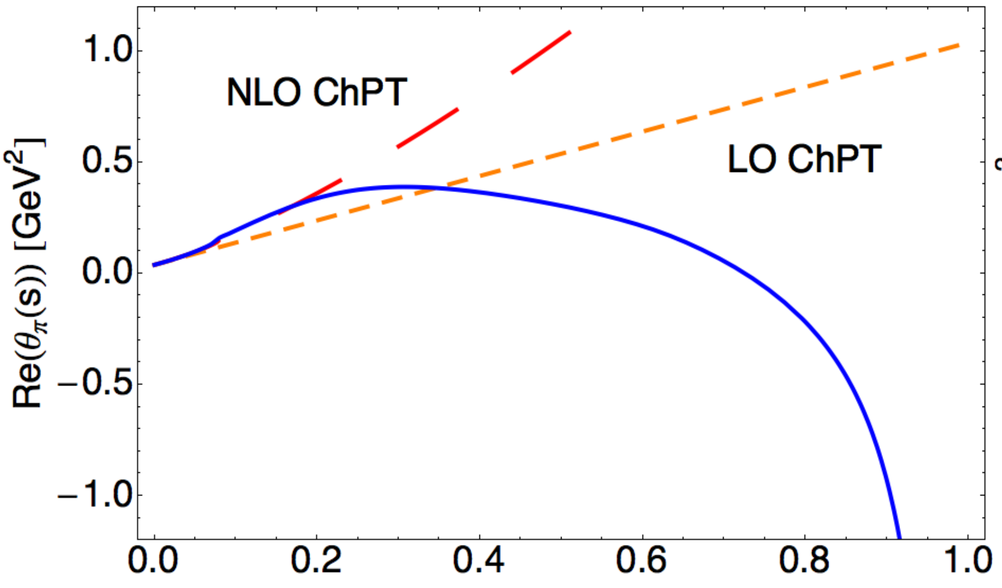


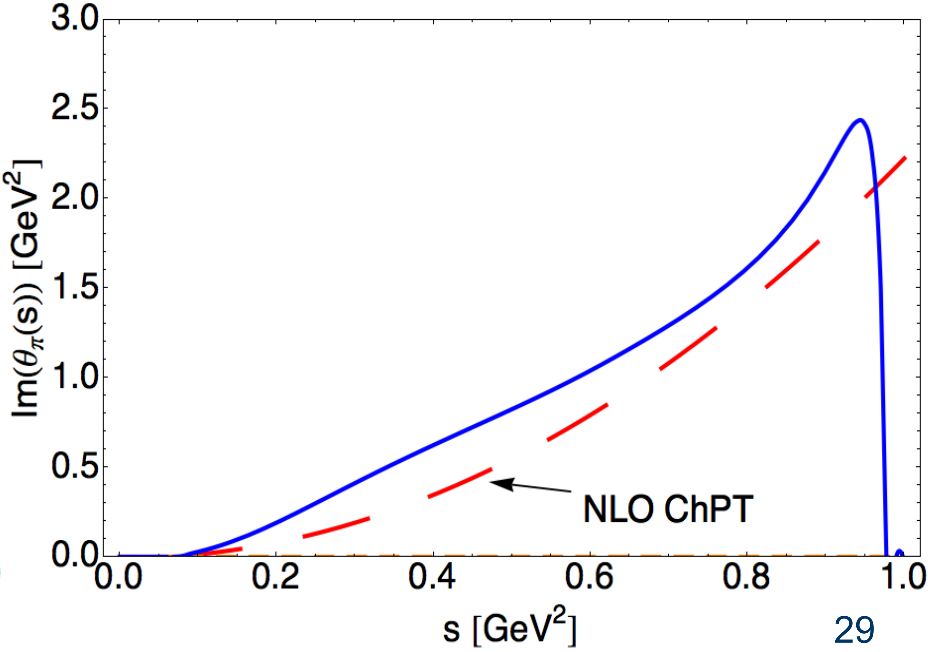


Dispersion relations:
Model-independent method,
based on first principles
that extrapolates ChPT
based on data


NLO ChPT
LO ChPT
$\mathrm{Re}(\Delta_\pi(s))$ [GeV^2]
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0
s [GeV^2]

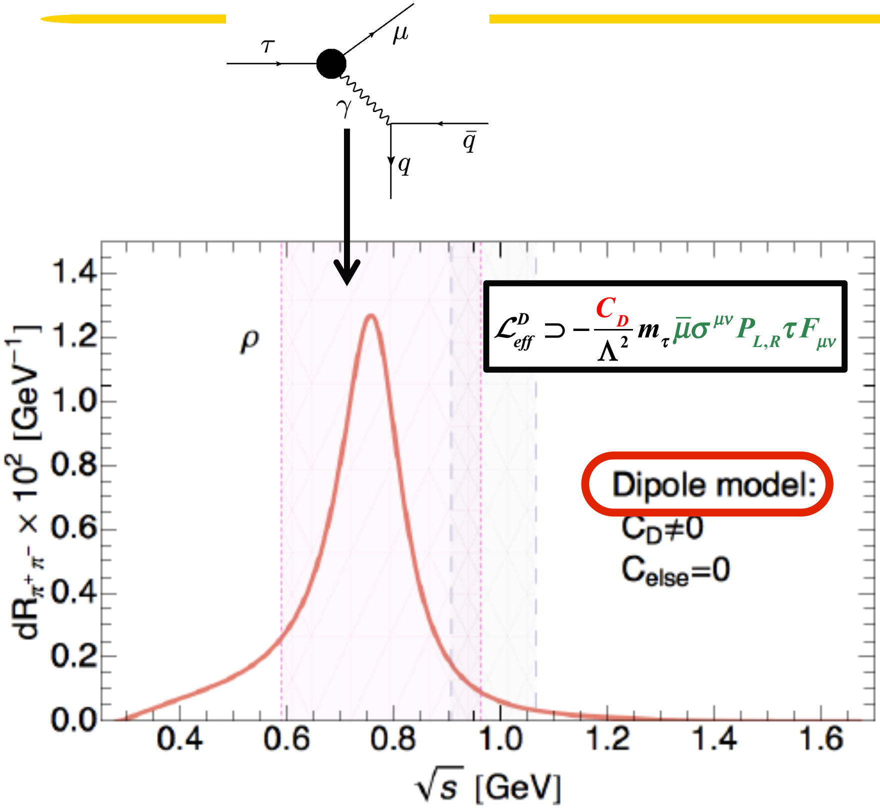

$\mathrm{Im}(\Delta_\pi(s))$ [GeV^2]
0.8
0.6
0.4
0.2
0.0
−0.2
NLO ChPT
0.0
0.2
0.4
0.6
0.8
1.0
s [GeV^2]


NLO ChPT
LO ChPT
$\mathrm{Re}(\theta_\pi(s))$ [GeV^2]
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
s [GeV^2]


$\mathrm{Im}(\theta_\pi(s))$ [GeV^2]
3.0
2.5
2.0
1.5
1.0
0.5
0.0
NLO ChPT
0.0
0.2
0.4
0.6
0.8
1.0
s [GeV^2]

Discriminating power of $\tau \to \mu(e)\pi\pi$ decays

Celis, Cirigliano, E.P.'14

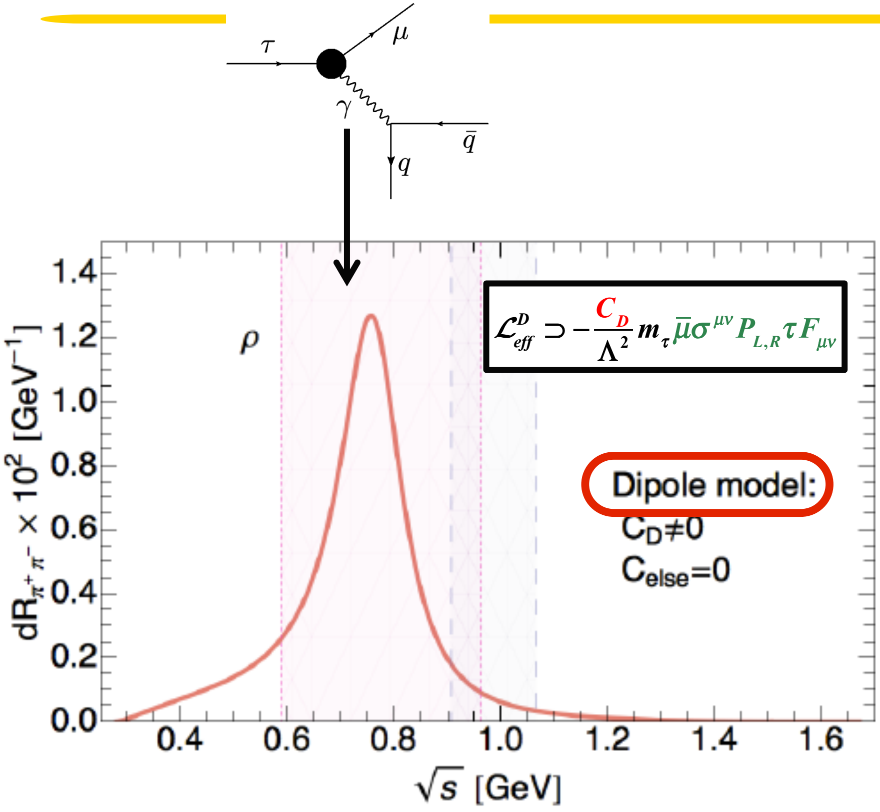


$$\mathcal{L}^{D}_{eff} \supset -\frac{C_D}{\Lambda^2} m_\tau \bar{\mu}\sigma^{\mu\nu} P_{L,R}\tau F_{\mu\nu}$$

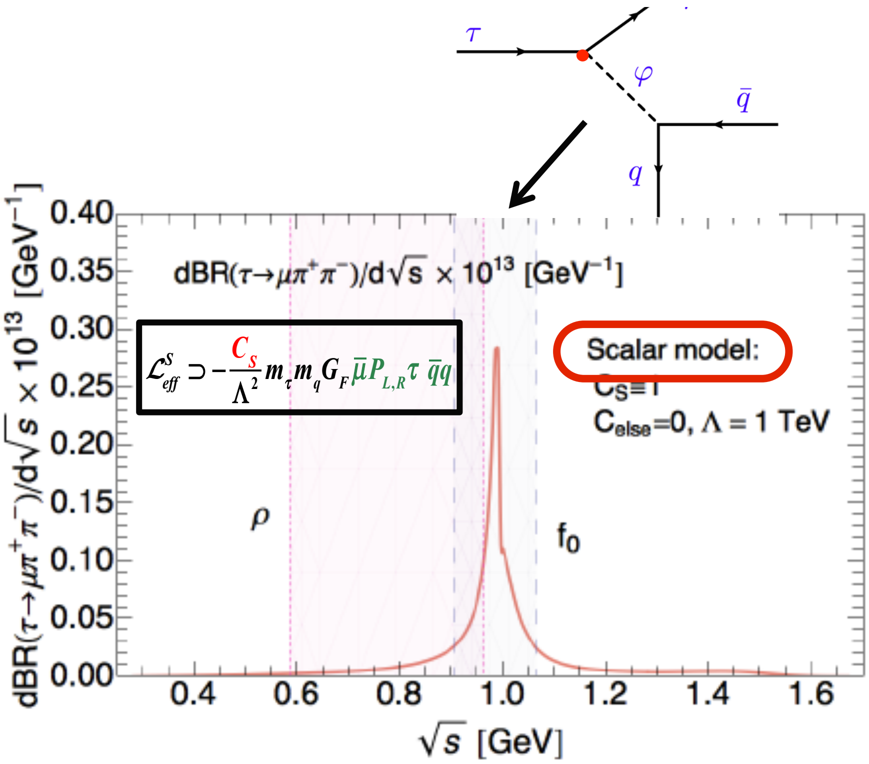
Dipole model:
$C_D \neq 0$
$C_{else} = 0$

Discriminating power of τ→μ(e)ππ decays

Celis, Cirigliano, E.P.'14

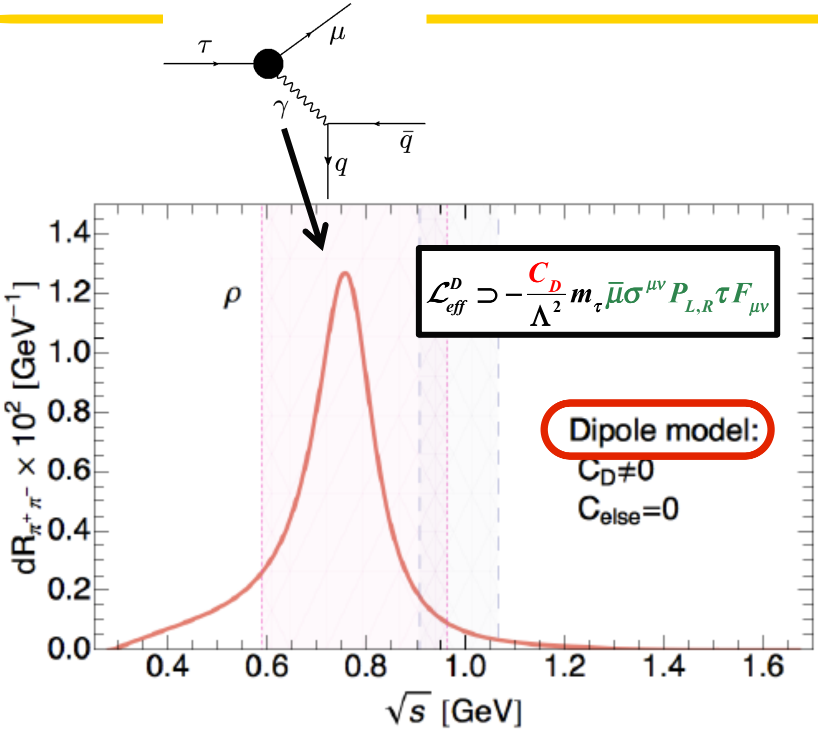


$$\mathcal{L}_{eff}^{D} \supset -\frac{C_D}{\Lambda^2} m_\tau \bar{\mu}\sigma^{\mu\nu} P_{L,R} \tau F_{\mu\nu}$$

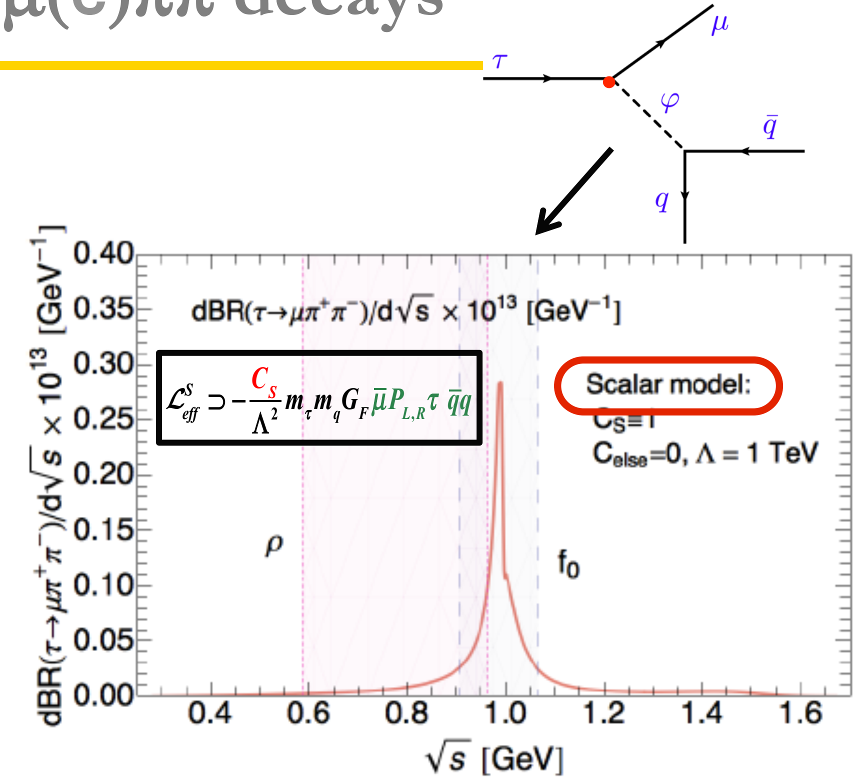


$$\mathcal{L}_{eff}^{S} \supset -\frac{C_S}{\Lambda^2} m_\tau m_q G_F \bar{\mu} P_{L,R} \tau\, \bar{q}q$$

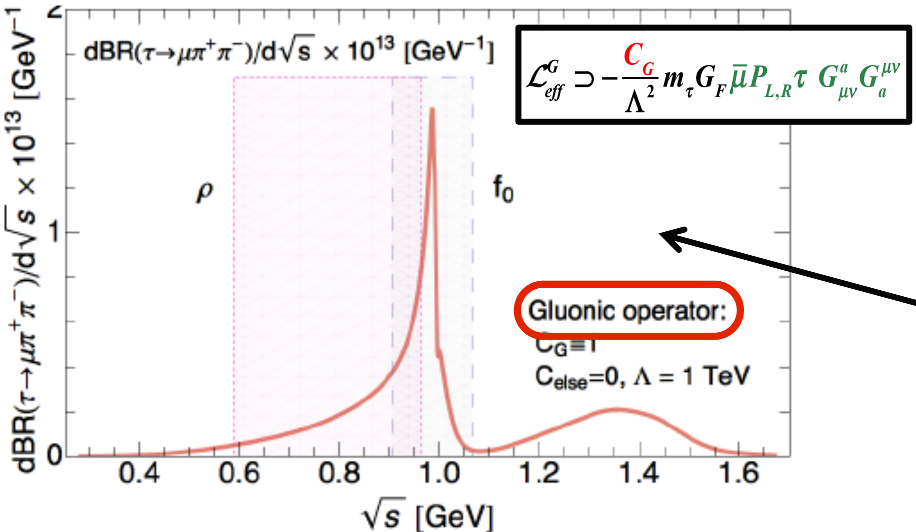
Discriminating power of τ→μ(e)ππ decays



$$\mathcal{L}_{eff}^{D} \supset -\frac{C_D}{\Lambda^2} m_\tau \bar{\mu} \sigma^{\mu\nu} P_{L,R} \tau F_{\mu\nu}$$

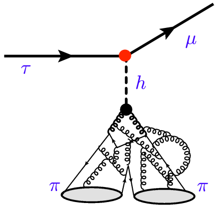


$$\mathcal{L}_{eff}^{S} \supset -\frac{C_S}{\Lambda^2} m_\tau m_q G_F \bar{\mu} P_{L,R} \tau\, \bar{q} q$$



$$\mathcal{L}_{eff}^{G} \supset -\frac{C_G}{\Lambda^2} m_\tau G_F \bar{\mu} P_{L,R} \tau\, G^{a}_{\mu\nu} G_{a}^{\mu\nu}$$

Very different distributions according to the *final hadronic state!*

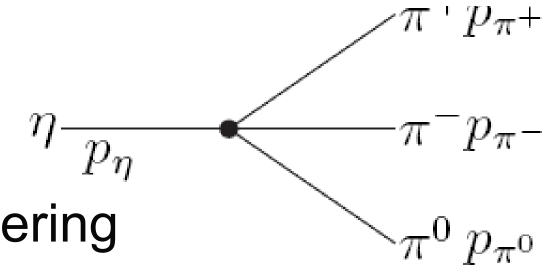


3.2 Application 2: η → 3π and light quark masses

G. Colangelo, S. Lanz, H. Leutwyler , E.P.

- η → 3π : decay forbiden by isospin symmetry

  ⇒ Clean access to ($m_u - m_d$)



- Dispersion relations and 3 body final state rescattering

  ⇒ allow to improve on the ChPT bad convergence

  The amplitude has all the good properties of analyticity + unitarity + crossing symmetry

  Improve on Breit-Wigner models

$$\left(Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right)$$

- $$\boxed{\langle \pi^+\pi^-\pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)}$$

$$\boxed{A(s,t,u) = -\frac{1}{Q^2} \frac{M_K^2}{M_\pi^2} \frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2} M(s,t,u)}$$

⇒

$$\boxed{\Gamma_{\eta\to 3\pi} \propto \int \left| A(s,t,u) \right|^2 \propto Q^{-4}}$$

- Compute the normalized amplitude *M(s,t,u)* with the best accuracy

The Method

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Decomposition of the amplitude as a function of ππ isospin states

$$\boxed{M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - ➢ M_I isospin *I* rescattering in two particles
  - ➢ Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - ➢ Main two body rescattering corrections inside M_I

- Unitary relation for $M_I(s)$:

$$\boxed{disc\, M_I(s) = 2i\, \left(M_I(s) + \hat{M}_I(s)\right)\, \sin\delta_I(s) e^{-i\delta_I(s)} \theta\left(s - 4M_\pi^2\right)}$$

right-hand cut left-hand cut

The Method

G. Colangelo, S. Lanz, H. Leutwyler , E.P.

- Unitary relation for $M_I(s)$:

$$\boxed{disc\ M_I(s) = 2i\left(M_I(s) + \qquad\right)\sin\delta_I(s)e^{-i\delta_I(s)}\theta\left(s-4M_\pi^2\right)}$$

right-hand cut

- Right-hand cut only ⇒ Omnès problem

$$M_I(s) = P_I(s)\ \Omega_I(s) \qquad \left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

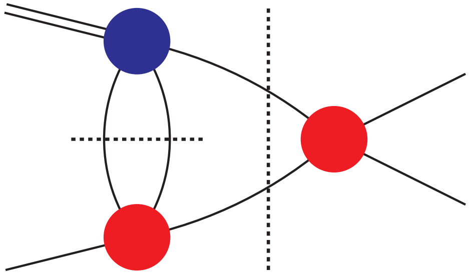
The Method

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Unitary relation for $M_I(s)$:

$$\boxed{\mathrm{disc}\; M_I(s) = 2i\left(M_I(s) + \hat{M}_I(s)\right)\,\sin\delta_I(s)\,e^{-i\delta_I(s)}\,\theta\left(s - 4M_\pi^2\right)}$$

right-hand cut left-hand cut



- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\,\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

Omnès function

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

- $\hat{M}_I(s)$: singularities in the t and u channels, depend on the other $M_I(s)$
  subtract $M_I(s)$ from the partial wave projection of $M(s,t,u)$
  ⇨ Angular averages of the other functions ⇨ Coupled equations

Determination of the Amplitude

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Omnès function

- Solve by iterative procedure:
  Inputs needed : S and P-wave phase shifts of ππ scattering

:

Determination of the Amplitude

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Omnès function

- Solve by iterative procedure:
  Inputs needed : S and P-wave phase shifts of ππ scattering

- Solution depends on *subtraction constants* only
  ⇨ fitted from experimental results

- Normalisation from matching to ChPT

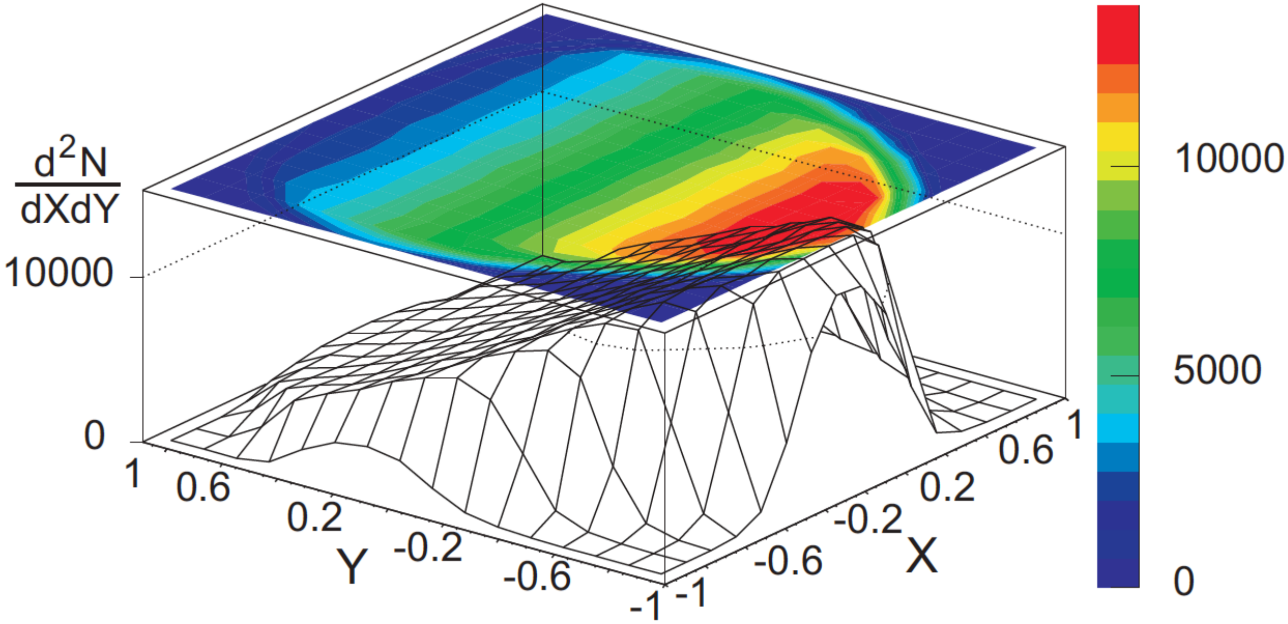
Experimental measurements : Charged channel

- Charged channel measurements with high statistics from *KLOE* and *WASA*

  e.g. *KLOE:* ~1.3 x 10^6 $\eta \rightarrow \pi^+ \pi^- \pi^0$ events from $e^+e^- \rightarrow \varphi \rightarrow \eta \gamma$

$$\boxed{\left|A_c(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)}$$

KLOE'08



$$\boxed{Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1}$$

$$\boxed{X = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)}$$

Experimental measurements : Neutral channel

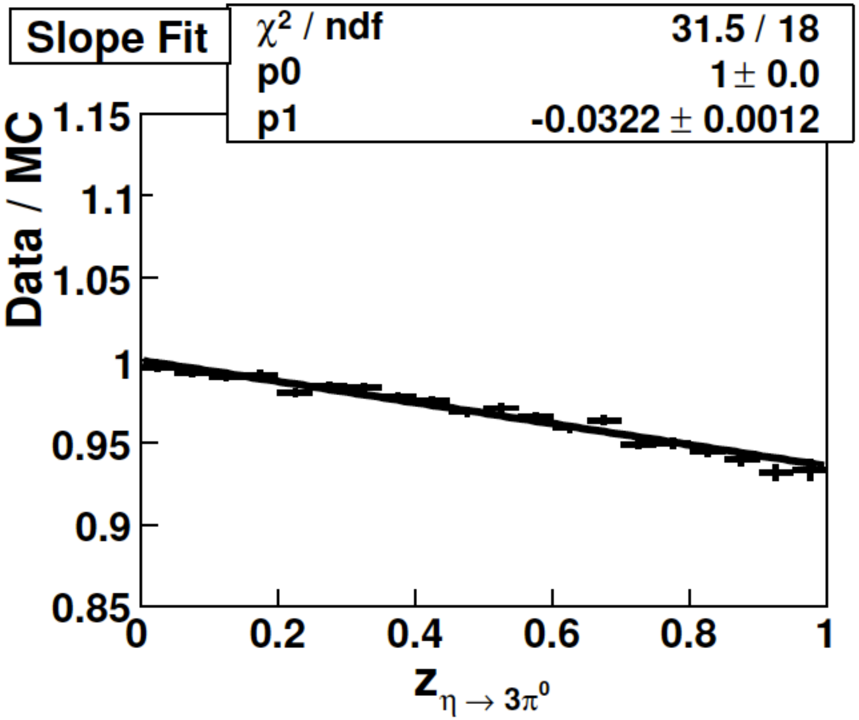
- Neutral channel measurements with high statistics from *MAMI-B*, *MAMI-C* and *WASA* e.g. *MAMI-C:* ~3 x 10^6 $\eta \to 3\pi^0$ events from $\gamma p \to \eta p$

$$\left|A_n(s,t,u)\right|^2 = N\left(1+2\alpha Z+6\beta Y\left(X^2-\frac{Y^2}{3}\right)+2\gamma Z^2\right)$$

$$Z=\frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n}-1\right)^2 = X^2+Y^2$$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

⇨ Extraction of the slope :



MAMI-C'09

$$X=\frac{\sqrt{3}\,(T_+-T_-)}{Q_c}=\frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y=\frac{3T_0}{Q_c}-1=\frac{3}{2M_\eta Q_c}\left(\left(M_\eta-M_{\pi^0}\right)^2-s\right)-1$$

Qualitative results of our analysis

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Determination of Q from the dispersive approach :

$$\boxed{\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2 - M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \ \left|M(s,t,u)\right|^2}$$

$$\Gamma_{\eta\to3\pi} = 300 \pm 12 \text{ eV}$$

PDG'14

$$\left(Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right)$$

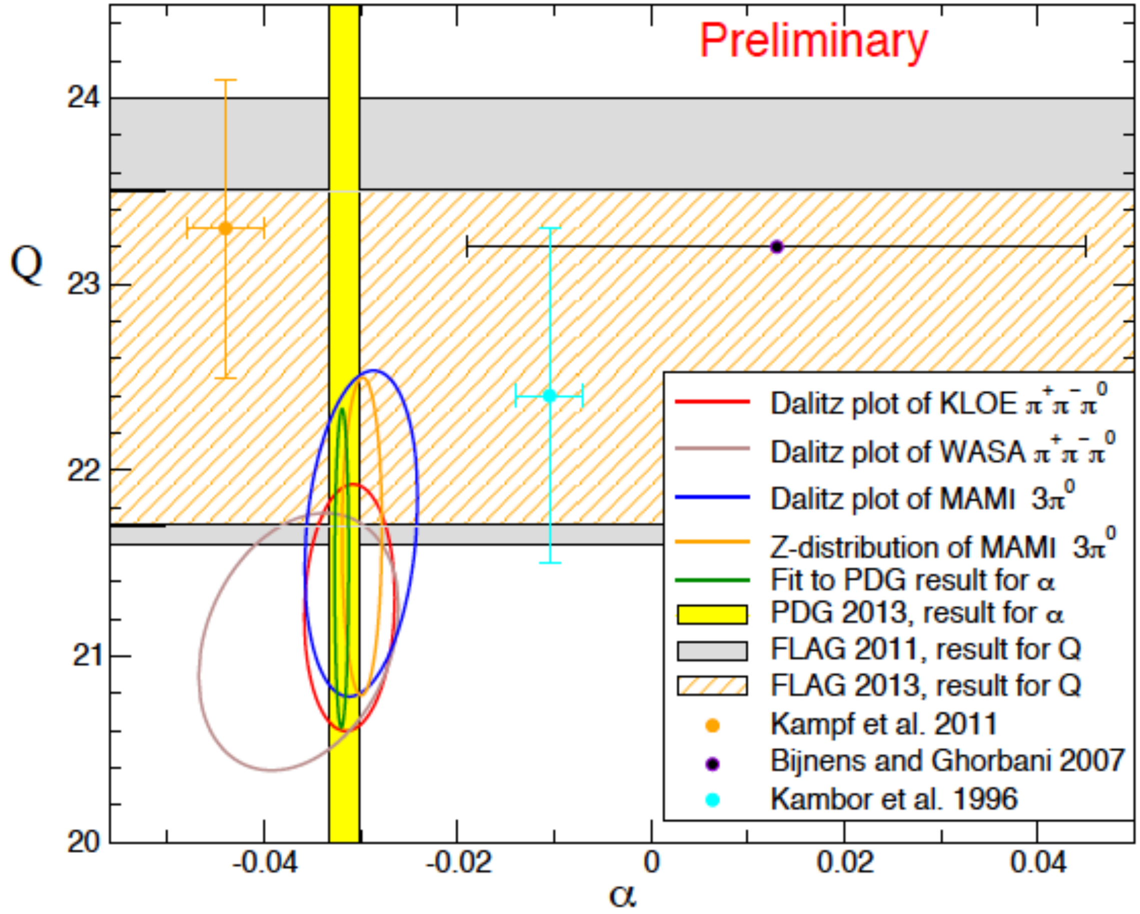
- Determination of α

$$\boxed{\left|A_n\left(s,t,u\right)\right|^2 = N\left(1 + 2\alpha Z\right)}$$

Qualitative results of our analysis

G. Colangelo, S. Lanz, H. Leutwyler , E.P.

- Plot of Q versus α :



NB: Isospin breaking has not been accounted for

From kaon mass spliting :

$$\boxed{Q = 20.7 \pm 1.2}$$

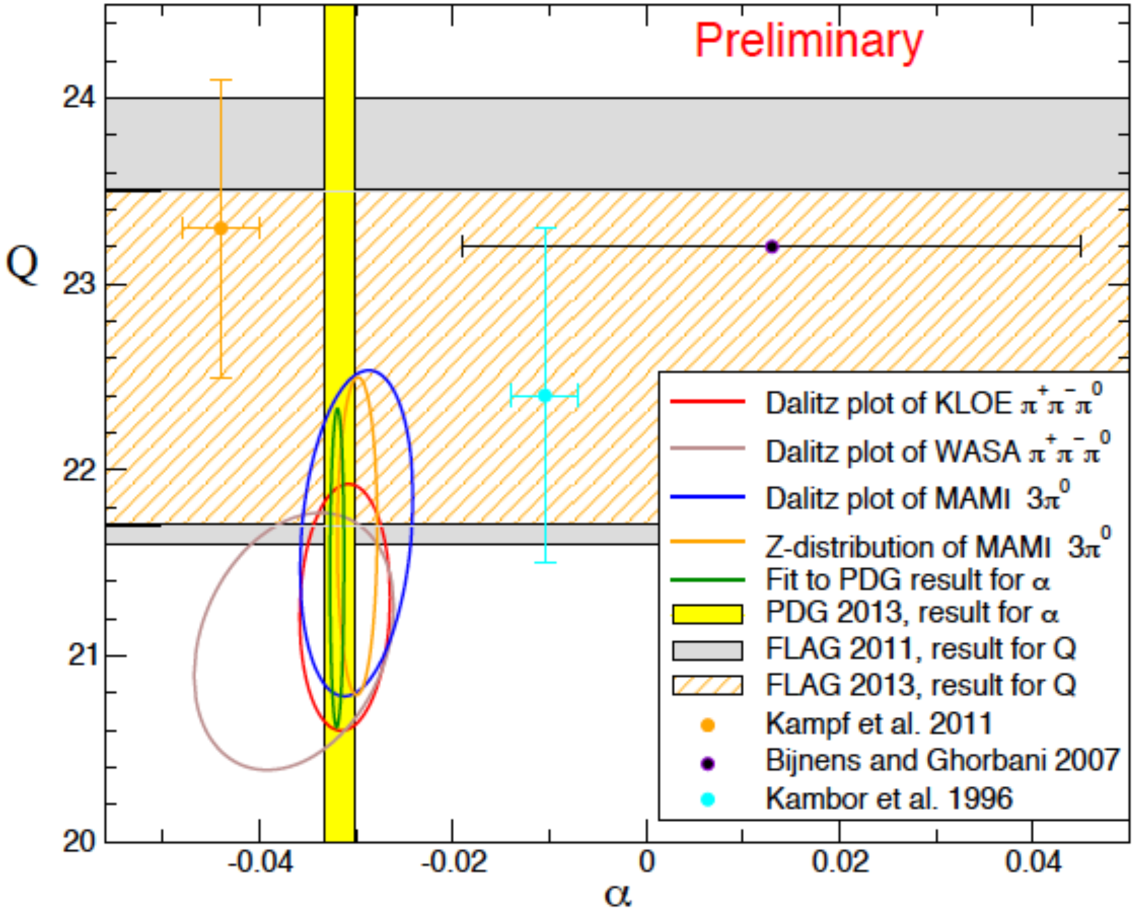
Kastner & Neufeld'08

- All the data give consistent results. The preliminary outcome for Q is intermediate between the lattice result and the one of Kastner and Neufeld.

Qualitative results of our analysis

G. Colangelo, S. Lanz,
H. Leutwyler , E.P.

- Plot of Q versus α :

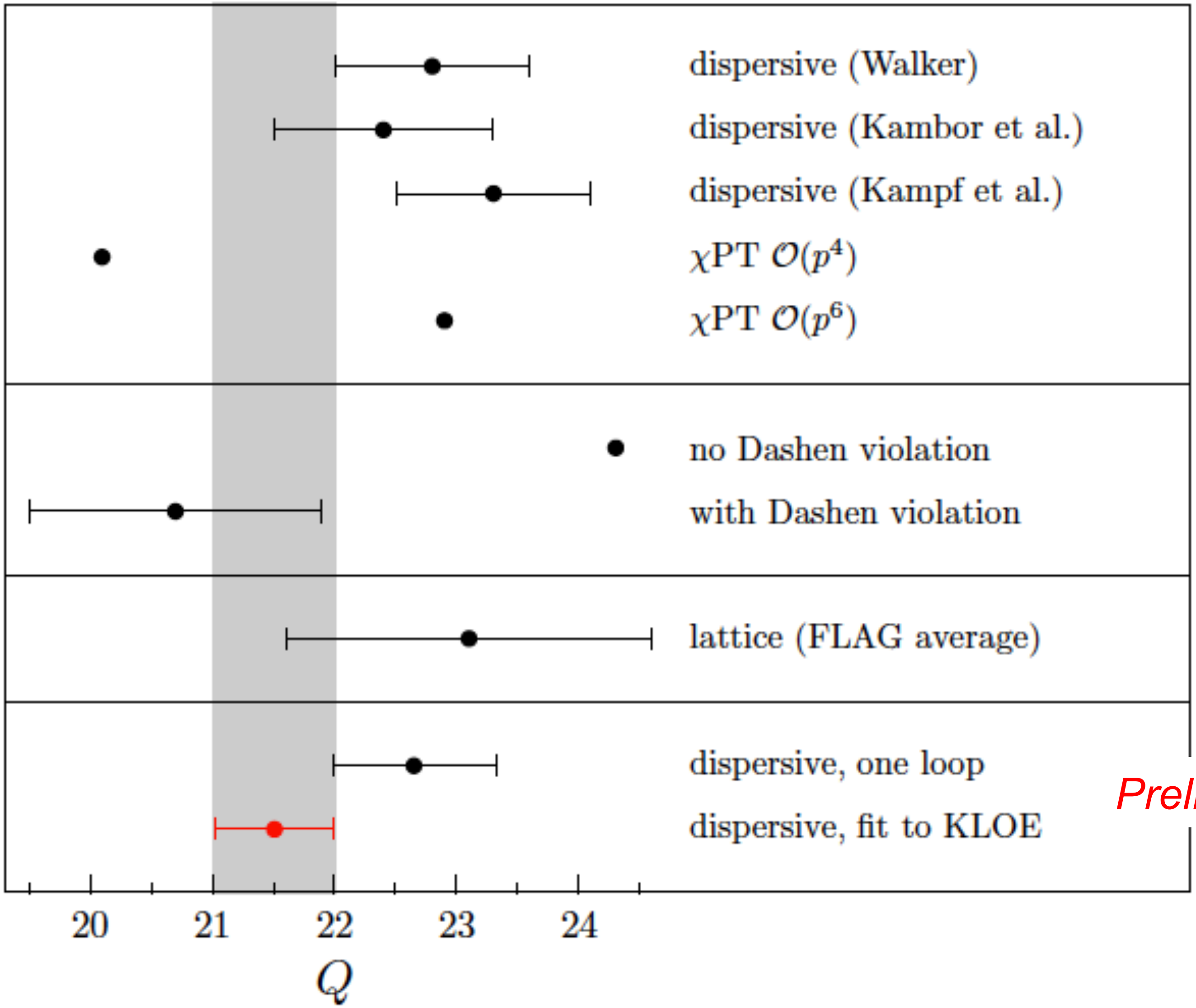


NB: Isospin breaking
has not been accounted for

- All our preliminary results give a negative value for α. In particular the result using KLOE data for $\eta \rightarrow \pi^+ \pi^- \pi^0$ is in perfect agreement with the PDG value!

Comparison of results for Q

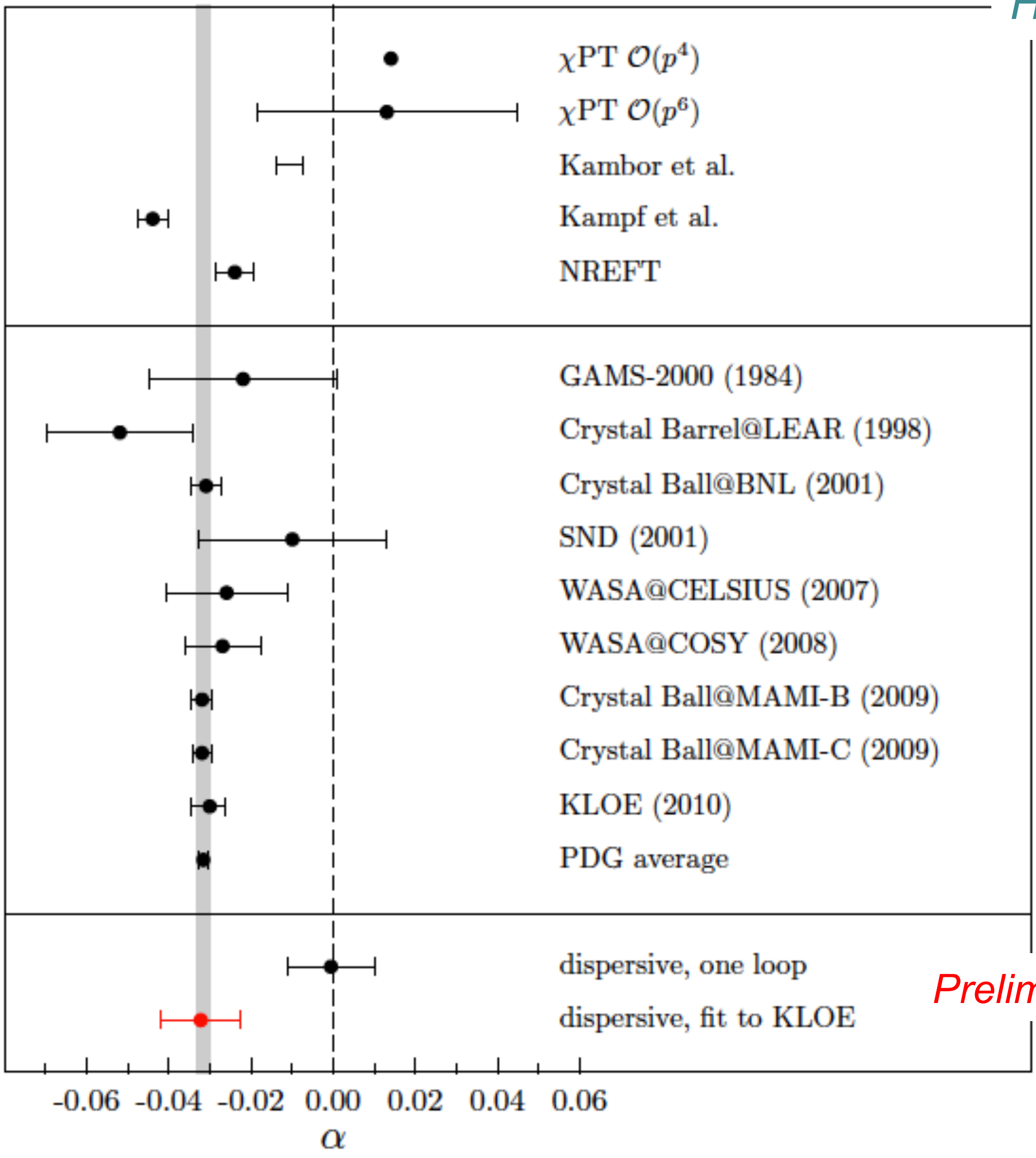
G. Colangelo, S. Lanz, H. Leutwyler , E.P.



Preliminary

Comparison of results for α

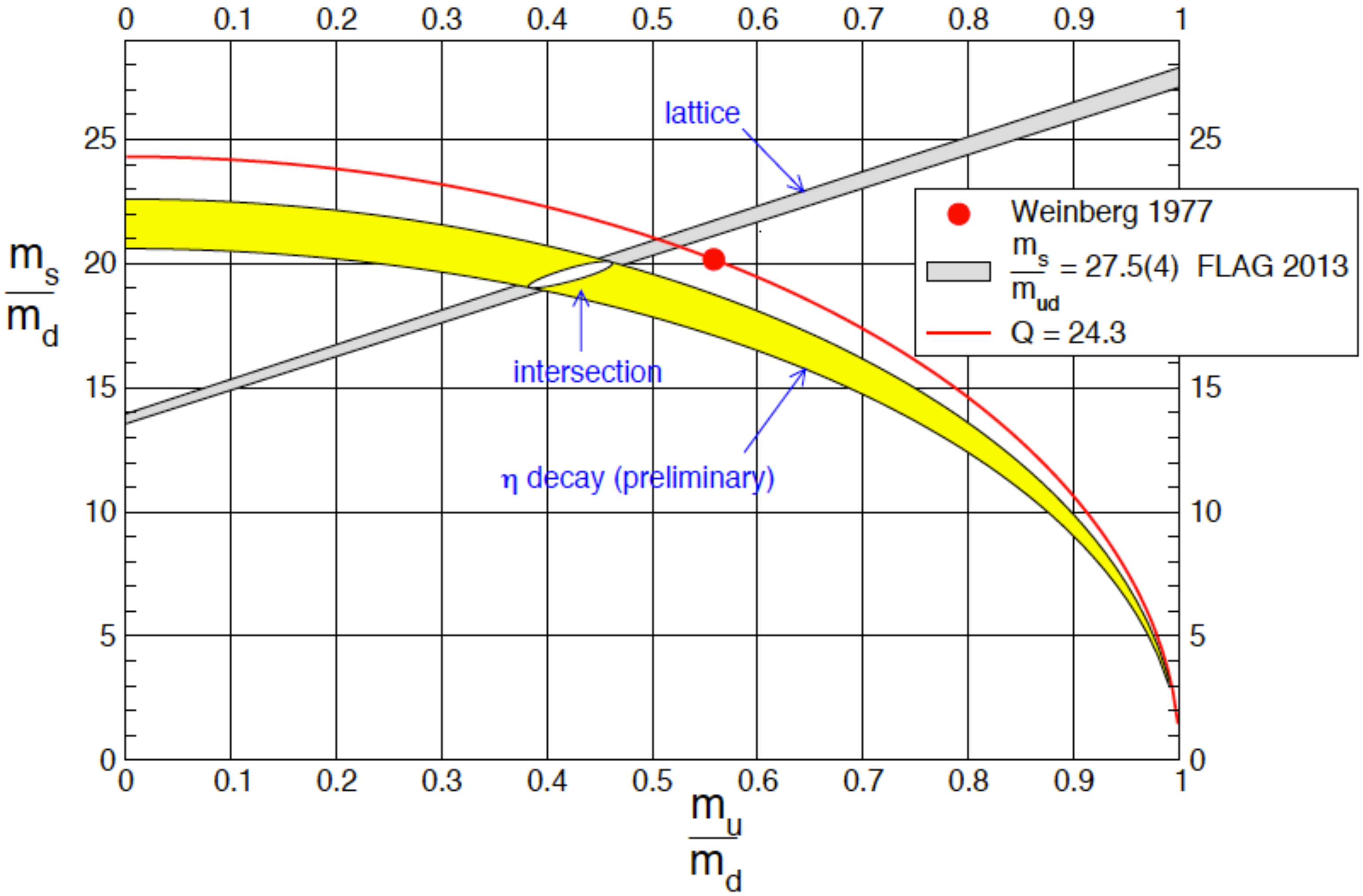
G. Colangelo, S. Lanz, H. Leutwyler , E.P.



Preliminary

Light quark masses

H. Leutwyler



- Smaller values for Q ⇨ smaller values for ms/md and mu/md than LO ChPT

4. Conclusion and outlook

4.1 Conclusion

- Look for exotics, new hadronic states: ⇨ need to know the hadronic background

- In this talk 2 examples :
  - Two body: $\pi\pi$ form factors
  - Three body: $\eta \to 3\pi$ decays

- Use dispersion relations

- Dispersion relations rely on analyticity, unitarity and crossing symmetry

  ⇨ Rigorous treatment of two and three hadronic final state

4.2 Outlook

- For reaching a high level of precision, theoretical challenges : in the dispersion relation
  - include inelasticities
  - Take isospin breaking and electromagnetic corrections into account

⇨ Work in this direction in JPAC

Talk by L. Dai, I. Danilkin, P. Guo, V. Mathieu

- Collaboration with experimentalists to analyse the data efficiently:
  - find the best parametrization to analyse the data
  - take into account systematics etc…

- Apply dispersion relations to other processes:
  - baryons: nucleons, etc
  - heavy mesons: J/Ψ, D, B decays

5. Back-up

Determination of the Kπ FFs: Dispersive representation

- Model for $\phi_V(s)$:

$$\boxed{\tilde{F}_V(s) = \frac{\tilde{M}_\rho^2 + \left(\alpha' e^{i\phi'} + \alpha'' e^{i\phi''}\right) s}{\tilde{M}_\rho^2 - s + \kappa_\rho \, \mathrm{Re}\left[A_\pi(s) + \frac{1}{2} A_K(s)\right] - i\tilde{M}_\rho \tilde{\Gamma}_\rho(s)} - \frac{\alpha' e^{i\phi'} s}{D(\tilde{M}_{\rho'}, \tilde{\Gamma}_{\rho'})} - \frac{\alpha'' e^{i\phi''} s}{D(\tilde{M}_{\rho''}, \tilde{\Gamma}_{\rho''})}}$$

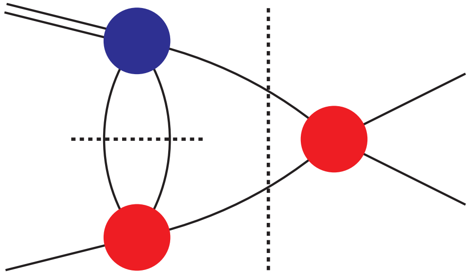
with $$\boxed{D(\tilde{M}_R, \tilde{\Gamma}_R) = \tilde{M}_R - s + \kappa_R \mathrm{Re} A_\pi(s) - i\tilde{M}_R \tilde{\Gamma}_R(s) .}$$

⇨ $$\boxed{\tan \phi_+ \equiv \tan \delta^P_{\pi\pi} = \frac{\mathrm{Im}\, \tilde{f}_+(s)}{\mathrm{Re}\, \tilde{f}_+(s)}}$$

3.1 Application 3: η → 3π and light quark masses

- Unitary relation for $M_I(s)$:

$$\boxed{disc\ M_I(s) = 2i\left(\underbrace{M_I(s)}_{\text{right-hand cut}} + \underbrace{\hat{M}_I(s)}_{\text{left-hand cut}}\right)\sin\delta_I(s)e^{-i\delta_I(s)}\theta\left(s-4M_\pi^2\right)}$$



- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \underbrace{\Omega_I(s)}_{\text{Omnès function}}\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

- Inputs needed : S and P-wave phase shifts of ππ scattering

:

3.1 Application 3: η → 3π and light quark masses

- Unitary relation for $M_I(s)$:

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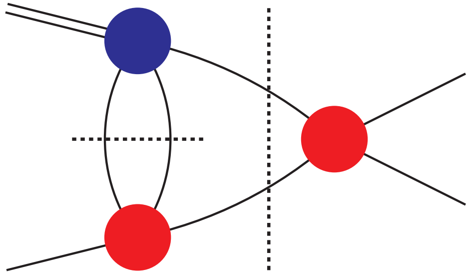
right-hand cut

left-hand cut



- Dispersion relation for the M_I's

$$\boxed{M_I(s)=\Omega_I(s)\left(P_I(s)+\frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

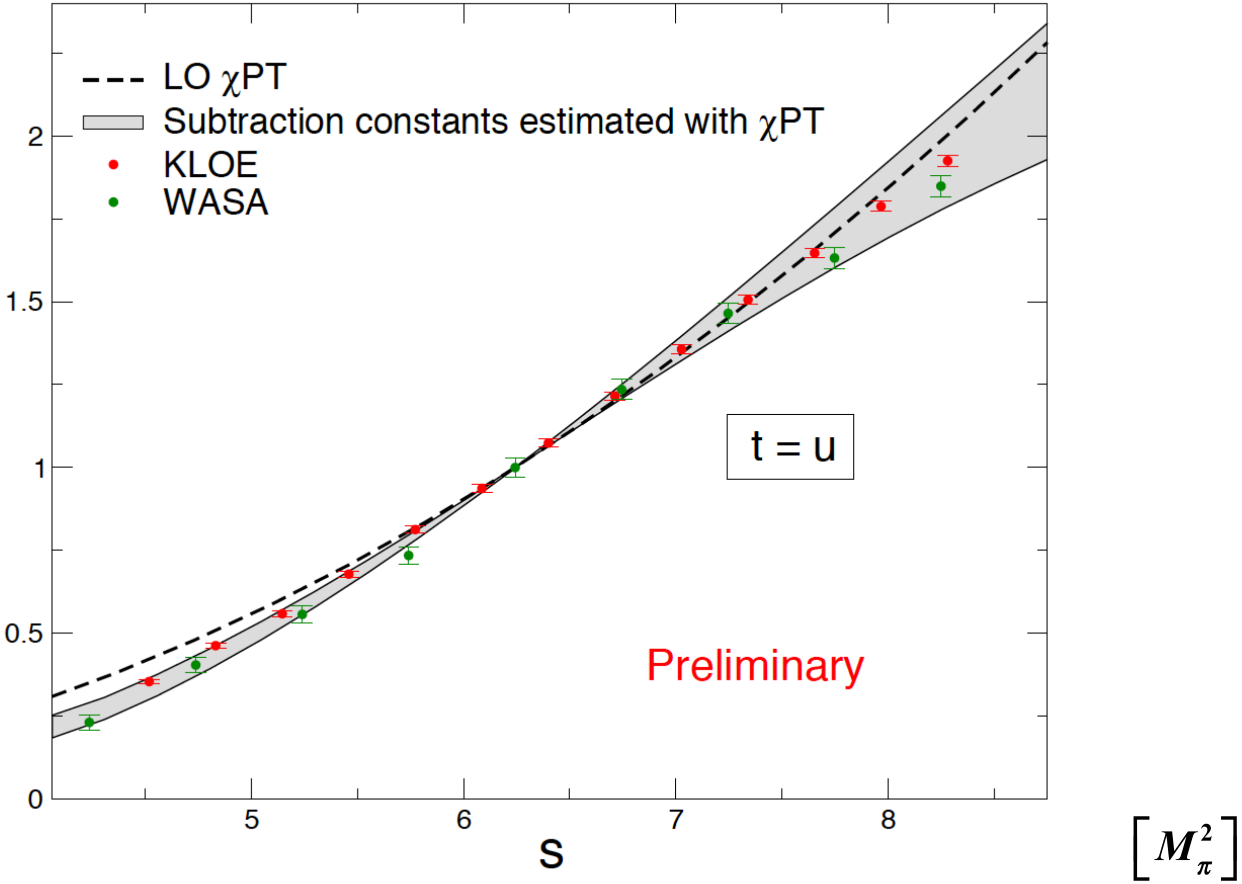
Omnès function

$$\left[\Omega_I(s)=\exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

- Solution depends on *subtraction constants* only ⇨ solve by iterative procedure + match with experiment

3.1 Dalitz plot distribution of $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

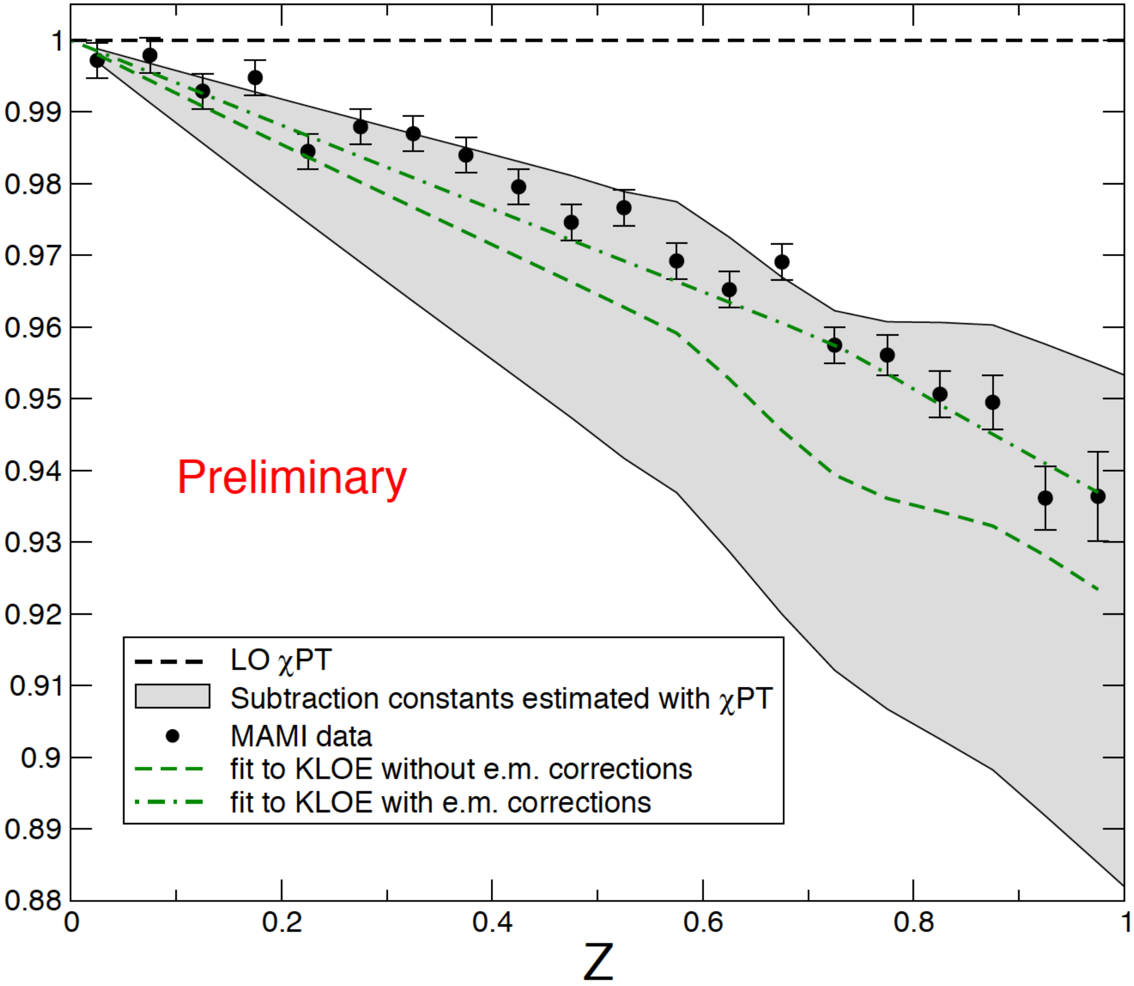
- The amplitude squared along the line t = u :



- Good agreement between theory and experiment
- The theoretical error bars are large ⇨ fit the subtraction constants to the data to reduce the uncertainties

3.2 Z distribution for $\eta \rightarrow \pi^0 \pi^0 \pi^0$ decays

- If one wants to fit the data, at this level of precision the e.m. corrections matter
  ⇨ use the one loop e.m. calculations from *Ditsche, Kubis and Meissner'09*

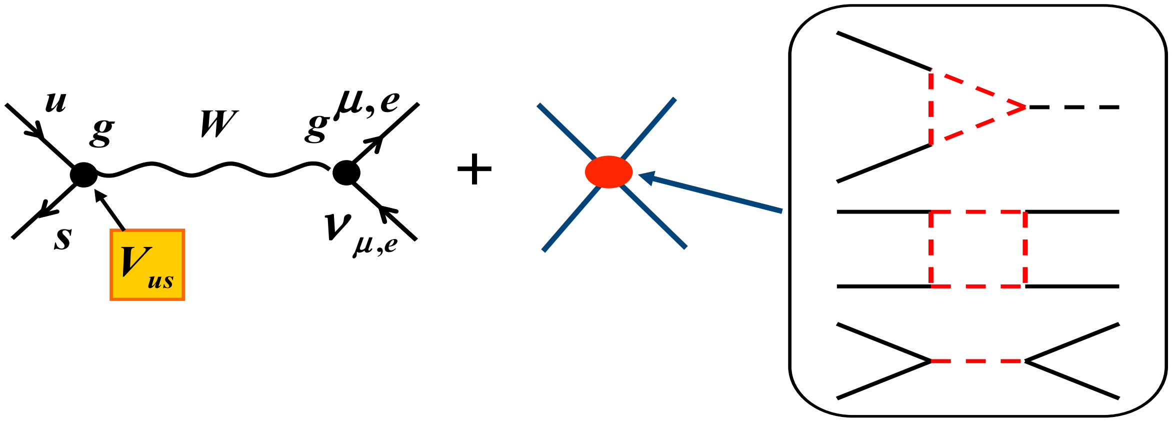


1.1 Hadronic Physics

- Hadronic Physics: Interactions of quarks at low energy
  - Precise tests of the Standard Model:

    ⇨ Extraction of V_{us}, α_S, light quark masses…

  - Look for physics beyond the Standard Model: High precision at low energy as a key to new physics?



SUSY loops
Z’ , Charged Higgs,
Right-Handed
Currents,….

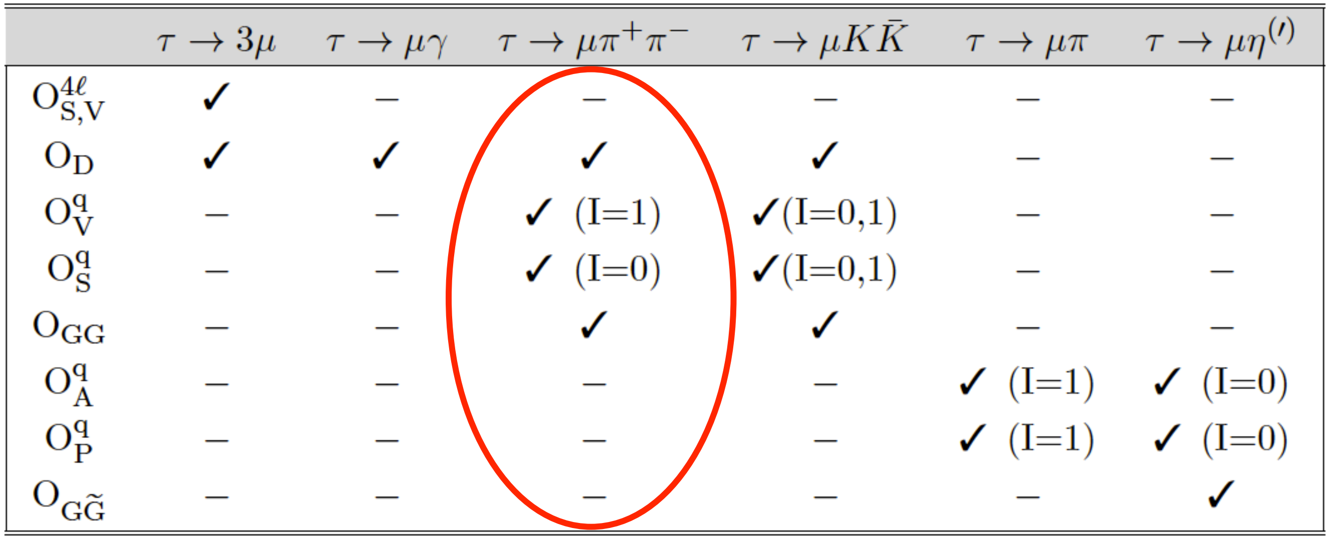
  - Look for exotics, new hadronic states

3.1 Application 1: ππ form factors and probing New physics with Tau LFV

- Constrain new physics operators from low energy decays: ex: $\tau \to \mu\pi\pi$

- Effective Lagrangian: $$\mathcal{L} = \mathcal{L}_{SM} + \frac{C^{(5)}}{\Lambda} O^{(5)} \boxed{+ \sum_i \frac{C_i^{(6)}}{\Lambda^2} O_i^{(6)}} + \dots$$

- Summary table:



	$\tau \to 3\mu$	$\tau \to \mu\gamma$	$\tau \to \mu\pi^+\pi^-$	$\tau \to \mu K\bar{K}$	$\tau \to \mu\pi$	$\tau \to \mu\eta^{(\prime)}$
$\mathrm{O}^{4\ell}_{\mathrm{S,V}}$	✓	–	–	–	–	–
O_{D}	✓	✓	✓	✓	–	–
$\mathrm{O}^{\mathrm{q}}_{\mathrm{V}}$	–	–	✓ (I=1)	✓(I=0,1)	–	–
$\mathrm{O}^{\mathrm{q}}_{\mathrm{S}}$	–	–	✓ (I=0)	✓(I=0,1)	–	–
O_{GG}	–	–	✓	✓	–	–
$\mathrm{O}^{\mathrm{q}}_{\mathrm{A}}$	–	–	–	–	✓ (I=1)	✓ (I=0)
$\mathrm{O}^{\mathrm{q}}_{\mathrm{P}}$	–	–	–	–	✓ (I=1)	✓ (I=0)
$\mathrm{O}_{\mathrm{G\tilde{G}}}$	–	–	–	–	–	✓

- Each UV model generates a *specific pattern* of D=6 operators: $\tau \to \mu\pi\pi$ very interesting probe to discriminate them ⇨ For these need to know the FFs!

τ→μ(e)ππ decays

- τ→μ(e)ππ differential decay rate:

$$\frac{d\Gamma(\tau\to\mu\pi^+\pi^-)}{ds}=\frac{(s-4m_\pi^2)^{1/2}(m_\tau^2-s)^2}{1536\pi^3\,\Lambda^4\,m_\tau\,s^{5/2}}$$

$$\times\left\{3s^2G_F^2|\mathrm{Q_L(s)}|^2-4(4m_\pi^2-s)|F_V(s)|^2\left[4\pi\alpha_{\mathrm{em}}(2m_\tau^2+s)|\mathrm{C_{DL}}|^2\right.\right.$$

$$\left.+\,s(\mathrm{C^d_{VL}}-\mathrm{C^u_{VL}})\left(12\sqrt{\pi\alpha_{\mathrm{em}}}\,\mathrm{C_{DL}}+\frac{(m_\tau^2+2s)}{m_\tau^2}(\mathrm{C^d_{VL}}-\mathrm{C^u_{VL}})\right)\right]$$

$$\left.+\,(\mathrm{L}\to\mathrm{R})\right\}.$$

$$\mathrm{Q_L}(s)\;=\Big(\theta_\pi(s)-\Gamma_\pi(s)-\Delta_\pi(s)\Big)\mathrm{C_{GL}}+\Delta_\pi(s)\,\mathrm{C^s_{SL}}+\Gamma_\pi(s)\;\left(\mathrm{C^u_{SL}}+\mathrm{C^d_{SL}}\right)$$

- 4 form factors to be determined:
  - Vector: $\left\langle \pi^+(p_{\pi^+})\pi^-(p_{\pi^-})\right|\frac{1}{2}(\bar u\gamma^\alpha u-\bar d\gamma^\alpha d)\left|0\right\rangle\equiv F_V(s)(p_{\pi^+}-p_{\pi^-})^\alpha$
  - Scalars: $\left\langle\pi^+\pi^-\right|m_u\bar u u+m_d\bar d d\left|0\right\rangle\equiv\Gamma_\pi(s)$, $\left\langle\pi^+\pi^-\right|m_s\bar s s\left|0\right\rangle\equiv\Delta_\pi(s)$
  - Gluonic: $\left\langle\pi^+\pi^-\right|\theta_\mu^\mu\left|0\right\rangle\equiv\theta_\pi(s)$ with $\boxed{\theta_\mu^\mu=-9\frac{\alpha_s}{8\pi}G^a_{\mu\nu}G_a^{\mu\nu}+\sum_{q=u,d,s}m_q\bar q q}$

- Recent progress in the determination of the form factors using *dispersive techniques*

Daub et al'13, Celis, Cirigliano, E.P.'14

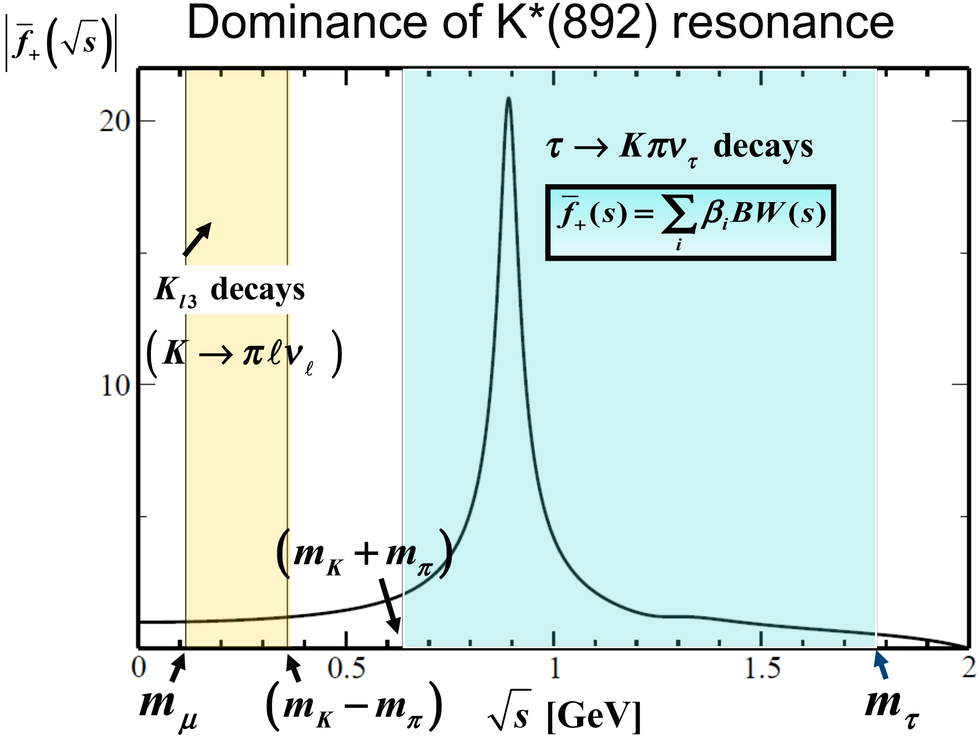
1.3 On the interest of using Dispersion Relations

- If E > 1 GeV: ChPT not valid anymore to describe dynamics of the process

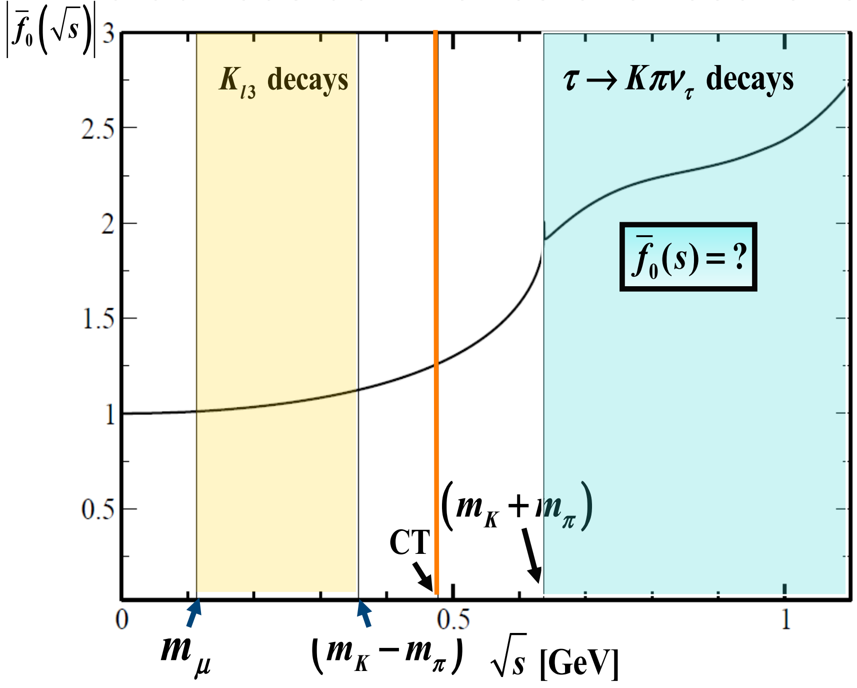
  ⇨ Resonances appear :

  - For ππ: *I=1*: ρ(770), ρ(1450), ρ(1700), …, *I=0*: "σ(~500)", f_0(980),…
  - For Kπ: *I=1*: K*(892), K*(1410), K*(1680), …, *I=0*: "κ(~800)", …

Kπ vector form factor:
Dominance of K*(892) resonance



Kπ scalar form factor:
No obvious dominance of a resonance



1.3 On the interest of using Dispersion Relations

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  - For Kπ: *I=1*: K*(892), K*(1410), K*(1680), …, *I=0*: "κ(~800)", …

- With Dispersion Relation:
  - no need for making assumptions of a dominance of resonances

    ⇒ directly given by the parametrization,

    phase shifts taken as inputs

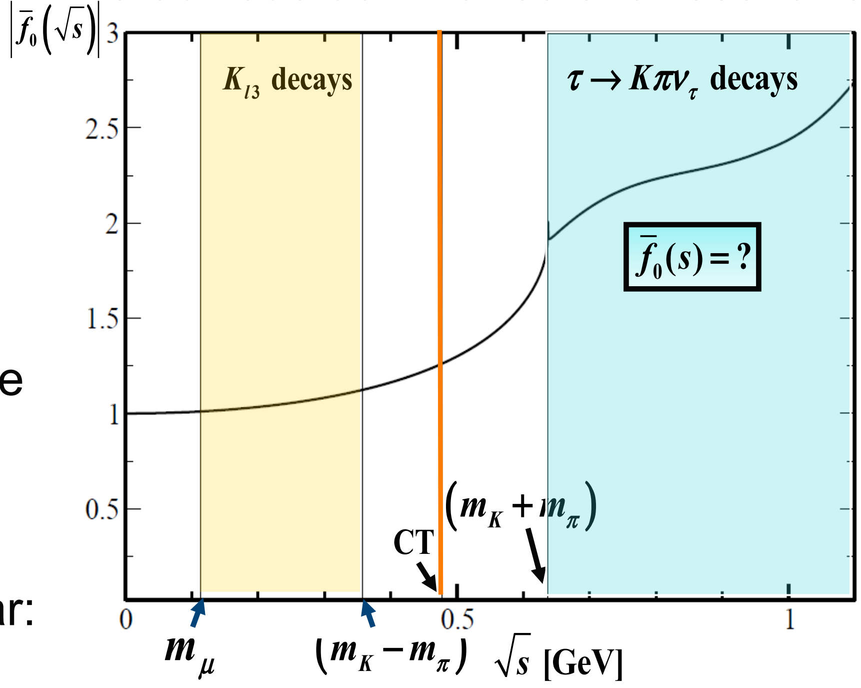
  - Parametrization valid in a large range of energy:

    ⇒ analyse several processes simultanously where the same quantity: FFs, amplitude appear:

    Ex: K_{l3} decays, $\tau \rightarrow K\pi\nu_\tau$

Kπ scalar form factor:

No obvious dominance of a resonance



Extraction of V_{us}

- Decay rate master formula

Antonelli, Cirigliano, Lusiani, E.P.'13

$$\boxed{\Gamma\left(\tau\to\overline{K}\pi\nu_\tau\left[\gamma\right]\right)=\frac{G_F^2 m_\tau^5}{96\pi^3}C_K^2 S_{EW}^{\tau}\left|V_{us}\right|^2\left|f_+^{K^0\pi^-}(0)\right|^2 I_K^{\tau}\left(1+\delta_{\mathrm{EM}}^{K\tau}+\cancel{\tilde{\delta}_{\mathrm{SU(2)}}^{K\pi}}\right)^2}$$

$$BR\left(\tau\to\overline{K^0}\pi^-\nu_\tau\right)=\left(0.416\pm0.008\right)\%$$

Belle'14

$$S_{\mathrm{ew}}=1.0201$$

Marciano & Sirlin'88, Braaten & Li'90, Erler'04

$$\delta_{\mathrm{EM}}^{\bar{K}^0\tau}=\left(-0.15\pm0.2\right)\%$$

$$I_{K^0}^{\tau}=0.50432\pm0.01721$$

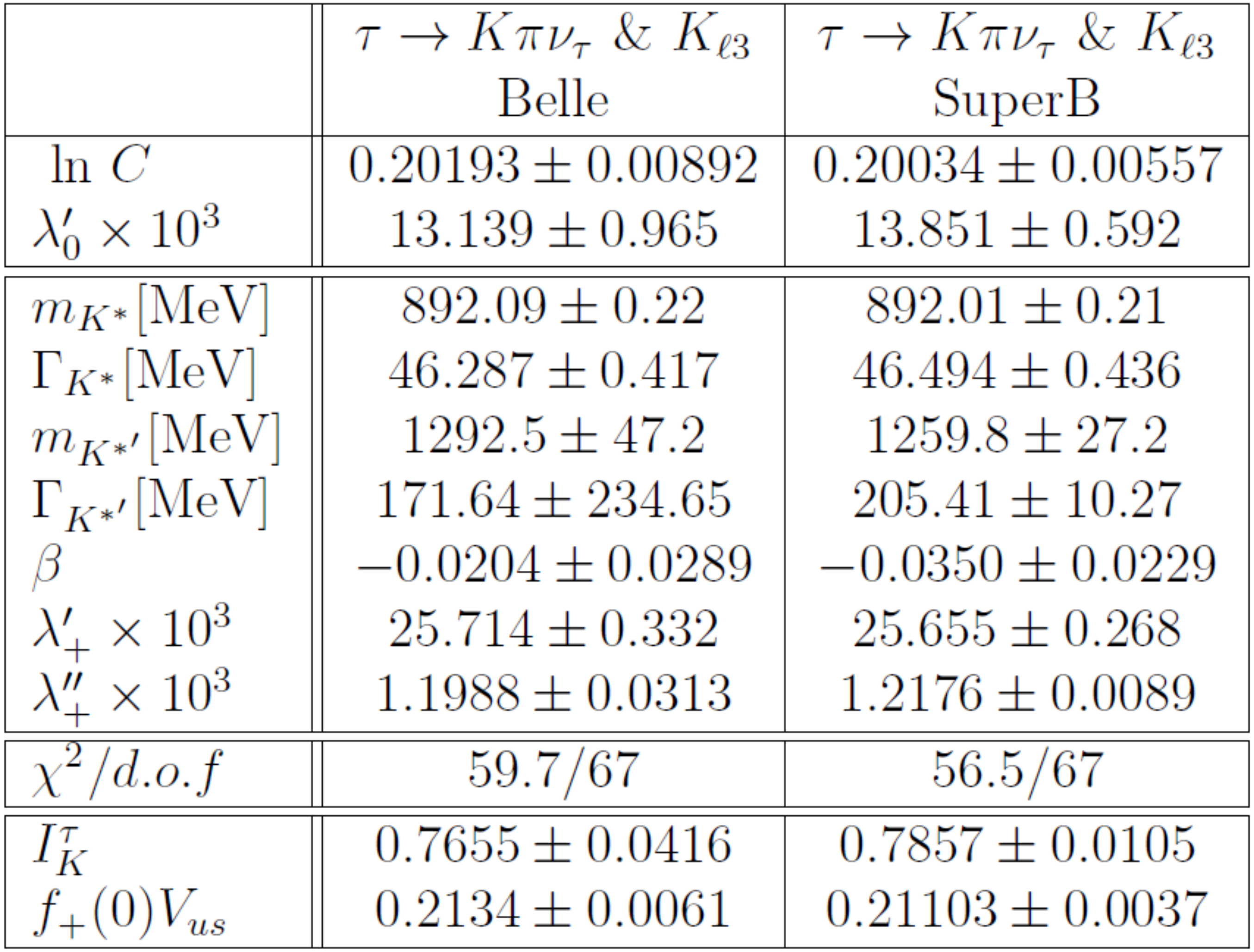
$$f_+\left(0\right)=0.9661\left(32\right)$$

$$\Rightarrow\quad f_+\left(0\right)\left|V_{us}\right|=0.2141\pm0.0014_{I_K}\pm0.0021_{\mathrm{exp}}$$

$$\Rightarrow\quad\left|V_{us}\right|=0.2216\pm0.0027$$

Bernard, Boito, E.P., in progress
Antonelli, Lusiani, E.P. in progress

- Preliminary results :



	$\tau \to K\pi\nu_\tau$ & $K_{\ell 3}$ Belle	$\tau \to K\pi\nu_\tau$ & $K_{\ell 3}$ SuperB
$\ln C$	0.20193 ± 0.00892	0.20034 ± 0.00557
$\lambda'_0 \times 10^3$	13.139 ± 0.965	13.851 ± 0.592
m_{K^*} [MeV]	892.09 ± 0.22	892.01 ± 0.21
Γ_{K^*} [MeV]	46.287 ± 0.417	46.494 ± 0.436
$m_{K^{*\prime}}$ [MeV]	1292.5 ± 47.2	1259.8 ± 27.2
$\Gamma_{K^{*\prime}}$ [MeV]	171.64 ± 234.65	205.41 ± 10.27
β	-0.0204 ± 0.0289	-0.0350 ± 0.0229
$\lambda'_+ \times 10^3$	25.714 ± 0.332	25.655 ± 0.268
$\lambda''_+ \times 10^3$	1.1988 ± 0.0313	1.2176 ± 0.0089
$\chi^2/d.o.f$	$59.7/67$	$56.5/67$
I_K^τ	0.7655 ± 0.0416	0.7857 ± 0.0105
$f_+(0)V_{us}$	0.2134 ± 0.0061	0.21103 ± 0.0037

Very accurate determination of K*(892)!

3.1 Application 1: Kπ form factors and V_{us}

- Master formula for $\tau \to K\pi\nu_\tau$:

$$\boxed{\Gamma\left(\tau\to\overline{K}\pi\nu_\tau\left[\gamma\right]\right)=\frac{G_F^2 m_\tau^5}{96\pi^3}C_K^2 S_{EW}^{\tau}\left|V_{us}\right|^2\left|f_+^{K^0\pi^-}(0)\right|^2 \underset{\uparrow}{I_K^\tau}\left(1+\delta_{\mathrm{EM}}^{K\tau}+\tilde{\delta}_{\mathrm{SU(2)}}^{K\pi}\right)^2}$$

$$\boxed{I_K^\tau=\int ds\; F\left(s,\bar{f}_+(s),\bar{f}_0(s)\right)}$$

Hadronic matrix element: Crossed channel from K → $\pi l\nu_l$

$$\boxed{\langle \mathrm{K}\pi|\ \bar{\mathrm{s}}\gamma_\mu \mathrm{u}\ |0\rangle=\left[\left(p_K-p_\pi\right)_\mu+\frac{\Delta_{K\pi}}{s}\left(p_K+p_\pi\right)_\mu\right]\underset{\substack{\uparrow\\ \text{vector}}}{f_+(s)}-\frac{\Delta_{K\pi}}{s}\left(p_K+p_\pi\right)_\mu \underset{\substack{\uparrow\\ \text{scalar}}}{f_0(s)}}$$

with $s=q^2=(p_K+p_\pi)^2\ ,\quad \bar{f}_{0,+}(t)=\dfrac{f_{0,+}(t)}{f_+(0)}$

⇨ Use a *dispersive parametrization* to combine with K_{l3} analysis

Determination of the Kπ FFs: Dispersive representation

Bernard, Boito, E.P.'11

- $\bar{f}_0(s)$: dispersion relation with 3 subtractions: 2 in s=0 and 1 in s = $(m_K+m_\pi)^2$

Callan-Treiman

$$\bar{f}_0(s) = \exp\left[\frac{s}{\Delta_{K\pi}}\left(\ln C + \left(s-\Delta_{K\pi}\right)\left(\frac{\ln C}{\Delta_{K\pi}} - \frac{\lambda_0'}{m_\pi^2}\right) + \frac{\Delta_{K\pi}s\left(s-\Delta_{K\pi}\right)}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}\frac{ds'}{s'^2}\frac{\phi_0(s')}{\left(s'-\Delta_{K\pi}\right)\left(s'-s-i\varepsilon\right)}\right)\right]$$

- $\bar{f}_+(s)$: dispersion relation with 3 subtractions in s=0

Boito, Escribano, Jamin'09,'10

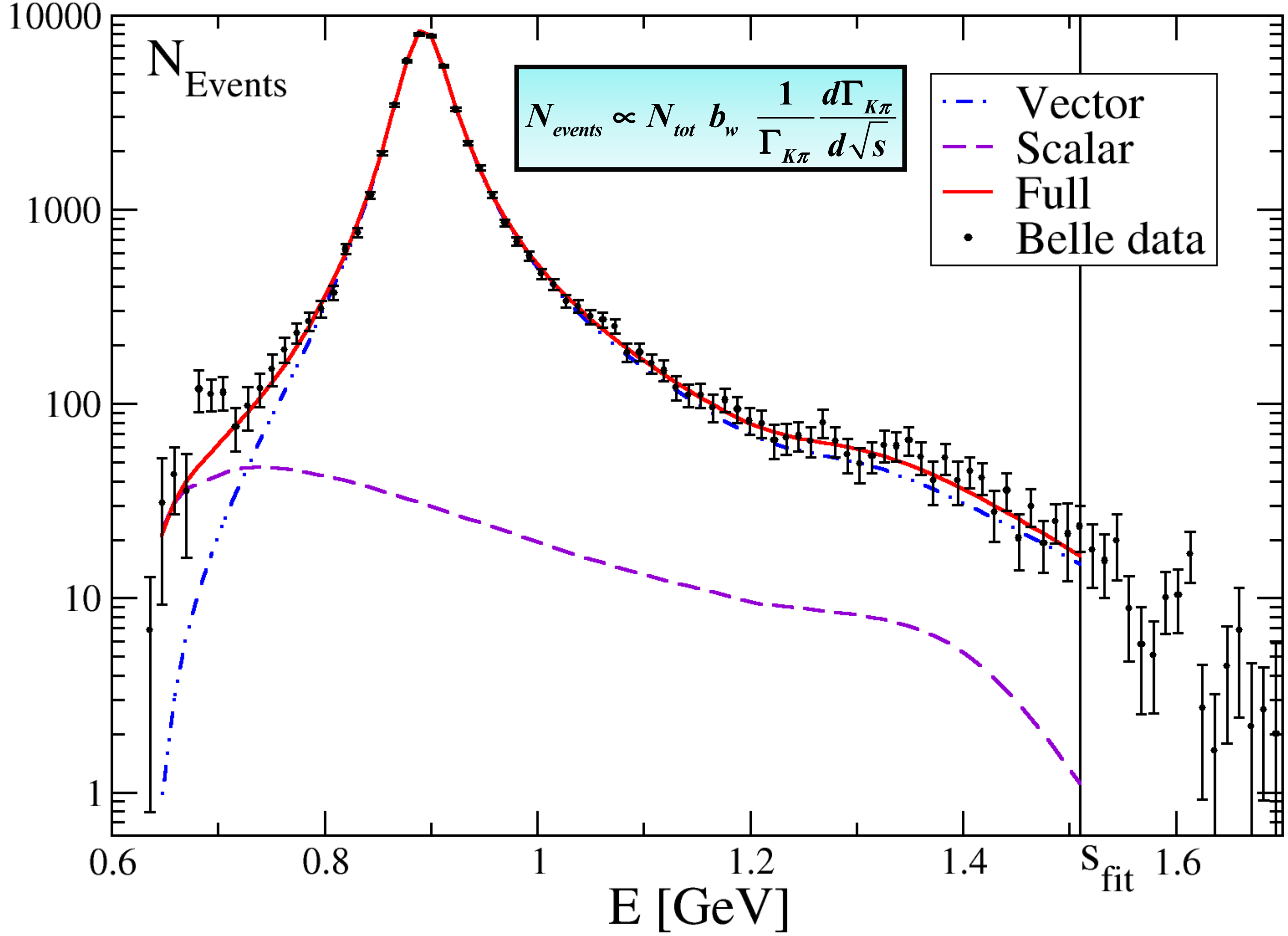
$$\bar{f}_+(s) = \exp\left[\lambda_+'\frac{s}{m_\pi^2} + \frac{1}{2}\left(\lambda_+'' - \lambda_+'^2\right)\left(\frac{s}{m_\pi^2}\right)^2 + \frac{s^3}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}\frac{ds'}{s'^3}\frac{\phi_+(s')}{\left(s'-s-i\varepsilon\right)}\right]$$

Extracted from a model including
2 resonances K*(892) and K*(1414)

Jamin, Pich, Portolés'08

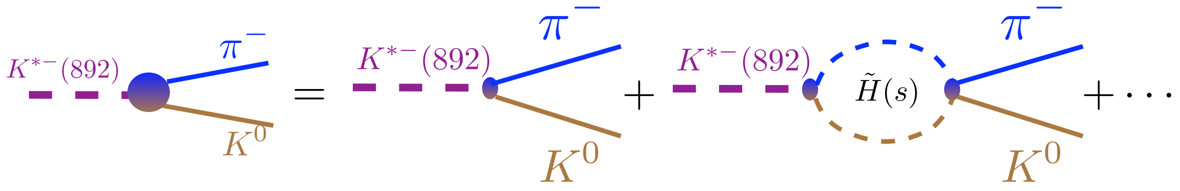
Fit to the $\tau \to K\pi\nu_\tau$ decay data + K_{l3} constraints

Bernard, Boito, E.P.'11



Determination of the Kπ FFs: Dispersive representation

- Model for $\phi_+(s)$:



$$\boxed{\tilde f_+(s) = \left[\frac{m_{K^*}^2 - \kappa_{K^*}\left(\mathrm{Re}\,\tilde H_{K\pi}(0) + \mathrm{Re}\,\tilde H_{K\eta}(0)\right) + \beta s}{D\left(m_{K^*},\Gamma_{K^*}\right)} - \frac{\beta s}{D\left(m_{K^{*\prime}},\Gamma_{K^{*\prime}}\right)} \right]}$$

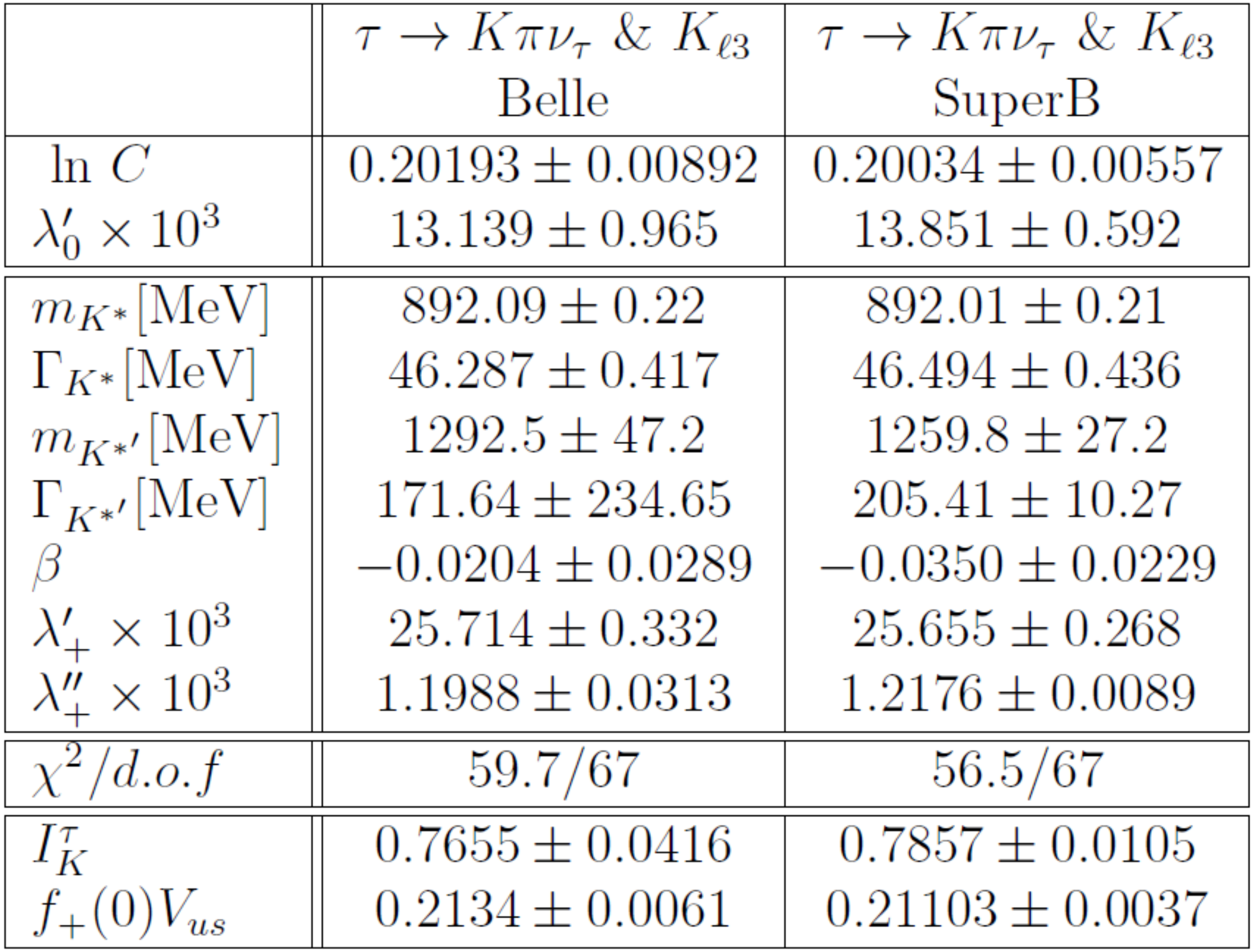
$K^*(892)$ $K^*(1410)$

with $$\boxed{D\left(m_n,\Gamma_n\right) = m_n^2 - s - \kappa_n \sum \mathrm{Re}\,\tilde H - i m_n \Gamma_n(s)}$$

$$\boxed{\tan\phi_+ \equiv \tan\delta_{K\pi}^{P,1/2} = \frac{\mathrm{Im}\,\tilde f_+(s)}{\mathrm{Re}\,\tilde f_+(s)}}$$

Bernard, Boito, E.P., *in progress*
Antonelli, Lusiani, E.P. *in progress*

- Preliminary results :



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$$\boxed{f_+(0) = 0.9661(32)}$$

FLAG'13

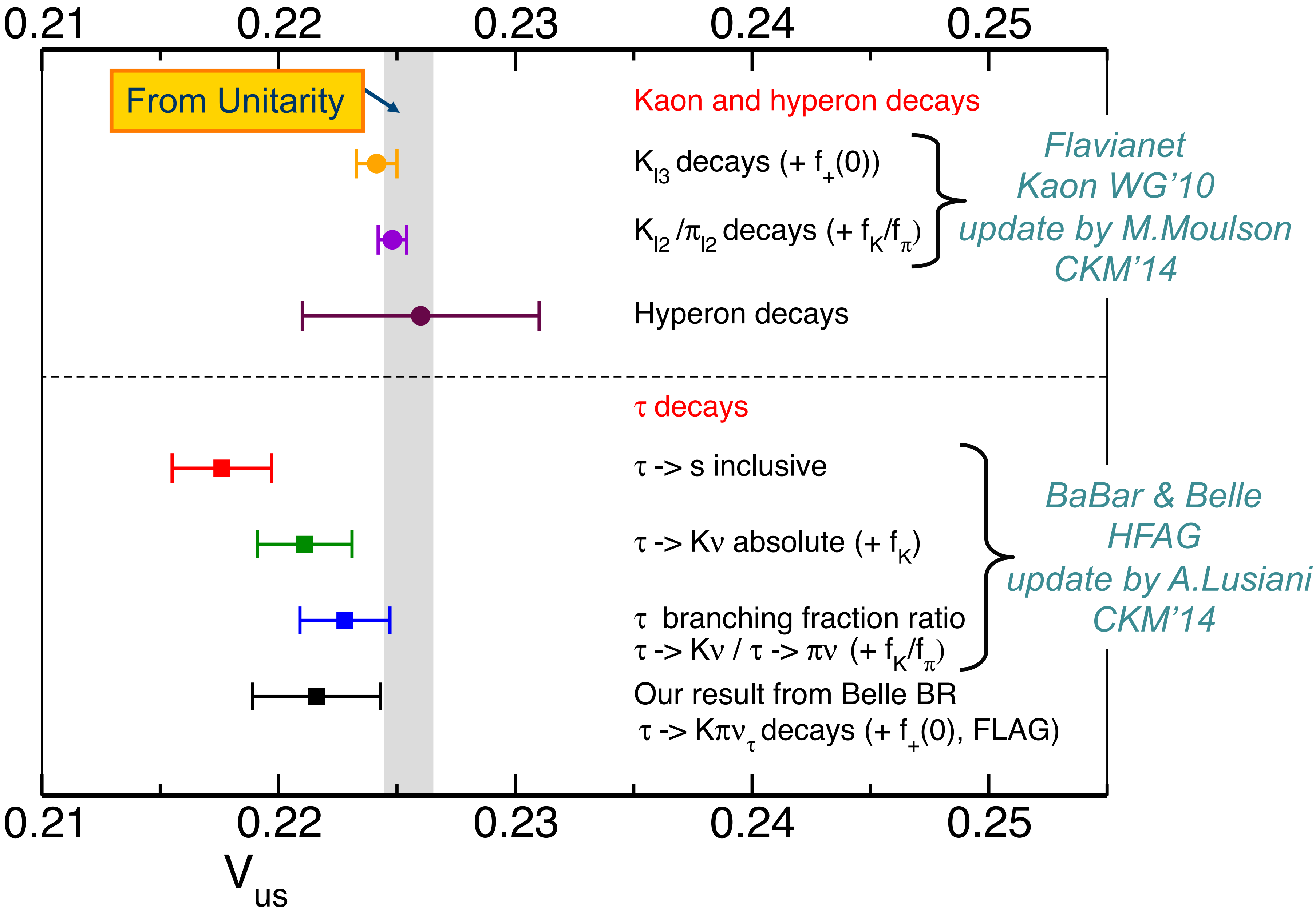
↓

⇨ $$\boxed{f_+(0)\left|V_{us}\right| = 0.2141 \pm 0.0014_{I_K} \pm 0.0021_{\mathrm{exp}}}$$ ⇨ $$\boxed{\left|V_{us}\right| = 0.2216 \pm 0.0027}$$

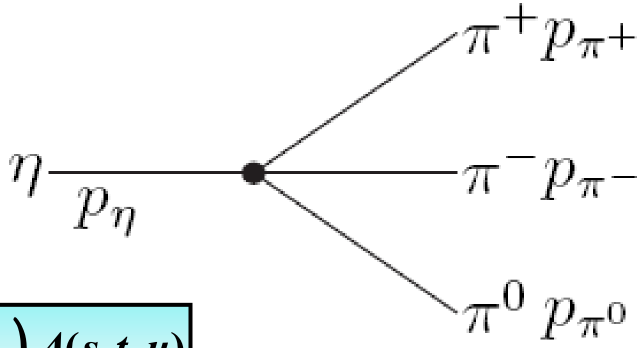
- Result of fit to K_{l3} + $\tau \longrightarrow K\pi\nu_\tau$ and $K\pi$ scattering data including inelasticities in the dispersive FFs

⇨ $$\boxed{f_+(0)\left|V_{us}\right| = 0.2163 \pm 0.0014}$$

Bernard'14


0.21
0.22
0.23
0.24
0.25
From Unitarity
Kaon and hyperon decays
K_{l3} decays (+ $f_+(0)$)
K_{l2}/π_{l2} decays (+ f_K/f_π)
Flavianet
Kaon WG'10
update by M.Moulson
CKM'14
Hyperon decays
τ decays
τ -> s inclusive
τ -> Kν absolute (+ f_K)
τ branching fraction ratio
τ -> Kν / τ -> $\pi\nu$ (+ f_K/f_π)
BaBar & Belle
HFAG
update by A.Lusiani
CKM'14
Our result from Belle BR
τ -> K$\pi\nu_\tau$ decays (+ $f_+(0)$, FLAG)
0.21
0.22
0.23
0.24
0.25
V_{us}

1.1 Definitions



- η decay: $\eta \rightarrow \pi^+ \pi^- \pi^0$

$$\boxed{\langle \pi^+\pi^-\pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)}$$

- Mandelstam variables $s = \left(p_{\pi^+} + p_{\pi^-}\right)^2, \; t = \left(p_{\pi^-} + p_{\pi^0}\right)^2, \; u = \left(p_{\pi^0} + p_{\pi^+}\right)^2$

$$s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

⇒ only two independent variables

- Neutral channel: $\eta \rightarrow \pi^0 \pi^0 \pi^0$:

$$\boxed{\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)}$$

2.5 Subtraction constants

- As we have seen, only Dalitz plots are measured, *unknown normalization!*

$$\boxed{A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)}$$

To determine Q, one needs to know the normalization

⇨ For the normalization one needs to use ChPT

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$

$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

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Only *6 coefficients* are of physical relevance

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$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

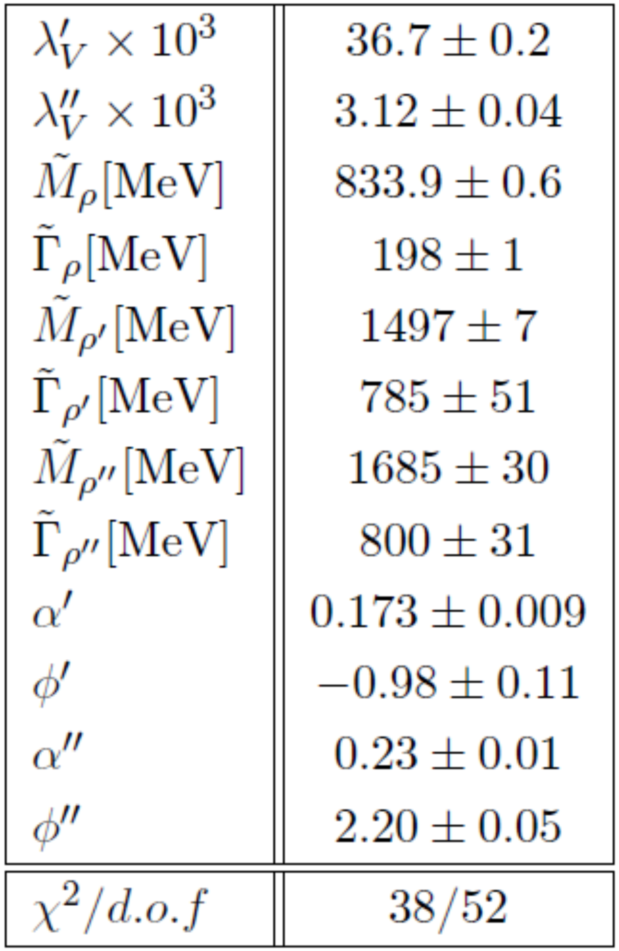
Only *6 coefficients* are of physical relevance

- They are determined from
  - Matching to one loop ChPT ⇒ $\delta_0 = \gamma_1 = 0$
  - Combine ChPT with fit to the data ⇒ δ_0 and γ_1 are determined from the data

- Matching to one loop ChPT : Taylor expand the dispersive M_I
  Subtraction constants ⇔ Taylor coefficients

- Important : Adler zero should be reproduced! ⇒ Can be used to constrain the fit

Results for the fit of the $\pi\,\pi$ vector form factor



$\lambda'_V \times 10^3$	36.7 ± 0.2
$\lambda''_V \times 10^3$	3.12 ± 0.04
$\tilde{M}_\rho$[MeV]	833.9 ± 0.6
$\tilde{\Gamma}_\rho$[MeV]	198 ± 1
$\tilde{M}_{\rho'}$[MeV]	1497 ± 7
$\tilde{\Gamma}_{\rho'}$[MeV]	785 ± 51
$\tilde{M}_{\rho''}$[MeV]	1685 ± 30
$\tilde{\Gamma}_{\rho''}$[MeV]	800 ± 31
α'	0.173 ± 0.009
ϕ'	-0.98 ± 0.11
α''	0.23 ± 0.01
ϕ''	2.20 ± 0.05
$\chi^2/d.o.f$	$38/52$

Details on the parametrization of the phase

- Model for the phase: ⇨ $\boxed{\tan\phi_V = \frac{\text{Im}\,\tilde{F}_V(s)}{\text{Re}\,\tilde{F}_V(s)}}$

Guerrero, Pich'98, Pich, Portolés'08
Gomez, Roig'13

$$\tilde{F}_V(s) = \frac{\tilde{M}_\rho^2 + \left(\alpha' e^{i\phi'} + \alpha'' e^{i\phi''}\right) s}{\tilde{M}_\rho^2 - s + \kappa_\rho \,\text{Re}\left[A_\pi(s) + \frac{1}{2} A_K(s)\right] - i\tilde{M}_\rho \tilde{\Gamma}_\rho(s)} - \frac{\alpha' e^{i\phi'} s}{D(\tilde{M}_{\rho'}, \tilde{\Gamma}_{\rho'})} - \frac{\alpha'' e^{i\phi''} s}{D(\tilde{M}_{\rho''}, \tilde{\Gamma}_{\rho''})}$$

with $D(\tilde{M}_R, \tilde{\Gamma}_R) = \tilde{M}_R - s + \kappa_R \text{Re} A_\pi(s) - i\tilde{M}_R \tilde{\Gamma}_R(s)$

and $\tilde{\Gamma}_R(s) = \tilde{\Gamma}_R \frac{s}{\tilde{M}_R^2} \frac{\left(\sigma_\pi^3(s) + 1/2\ \sigma_K^3(s)\right)}{\left(\sigma_\pi^3(\tilde{M}_R^2) + 1/2\ \sigma_K^3(\tilde{M}_R^2)\right)}$

$$\kappa_R(s) = \frac{\tilde{\Gamma}_R}{\tilde{M}_R} \frac{s}{\pi\left(\sigma_\pi^3(\tilde{M}_R^2) + 1/2\ \sigma_K^3(\tilde{M}_R^2)\right)}$$

Details on the fit

- The minimized quantity:

$$\chi^2 = \sum_{i=1}^{62} \left(\frac{(|F_V(s)|^2)_i^{\rm theo} - (|F_V(s)|^2)_i^{\rm exp}}{\sigma_{(|F_V(s)|^2)_i^{\rm exp}}} \right)^2 + \left(\frac{\lambda'_V - \lambda'^{\,\rm sr}_V}{\sigma_{\lambda'^{\,\rm sr}_V}} \right)^2 + \left(\frac{\alpha_{2v} - \alpha_{2v}^{\rm sr}}{\sigma_{\alpha_{2v}^{\rm sr}}} \right)^2$$

- 2 sum-rules are added such that $F_V(s) \rightarrow 1/s$ *Brodsky & Lepage*

$$\lambda'^{\,\rm sr}_V = \frac{m_\pi^2}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\phi_V(s')}{s'^2}$$

$$(\lambda''_V - \lambda'^2_V)^{\rm sr} = \frac{2m_\pi^4}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\phi_V(s')}{s'^3} \equiv \alpha_{2v}^{\rm sr}$$

Determination of the polynomial

- General solution

$$\begin{pmatrix} F_\pi(s) \\ \frac{2}{\sqrt{3}} F_K(s) \end{pmatrix} = \begin{pmatrix} C_1(s) & D_1(s) \\ C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} P_F(s) \\ Q_F(s) \end{pmatrix}$$

- Fix the polynomial with requiring $F_P(s) \rightarrow 1/s$ (*Brodsky & Lepage*) + ChPT:

Feynman-Hellmann theorem:

$$\Gamma_P(0) = \left(m_u \frac{\partial}{\partial m_u} + m_d \frac{\partial}{\partial m_d} \right) M_P^2$$

$$\Delta_P(0) = \left(m_s \frac{\partial}{\partial m_s} \right) M_P^2$$

- At LO in ChPT:

$$M_{\pi^+}^2 = (m_{\mathrm{u}} + m_{\mathrm{d}})\, B_0 + O(m^2)$$
$$M_{K^+}^2 = (m_{\mathrm{u}} + m_{\mathrm{s}})\, B_0 + O(m^2)$$
$$M_{K^0}^2 = (m_{\mathrm{d}} + m_{\mathrm{s}})\, B_0 + O(m^2)$$

$$\begin{aligned} P_\Gamma(s) &= \Gamma_\pi(0) = M_\pi^2 + \cdots \\ Q_\Gamma(s) &= \frac{2}{\sqrt{3}} \Gamma_K(0) = \frac{1}{\sqrt{3}} M_\pi^2 + \cdots \\ P_\Delta(s) &= \Delta_\pi(0) = 0 + \cdots \\ Q_\Delta(s) &= \frac{2}{\sqrt{3}} \Delta_K(0) = \frac{2}{\sqrt{3}} \left(M_K^2 - \frac{1}{2} M_\pi^2 \right) + \cdots \end{aligned}$$

Determination of the polynomial

- General solution

$$\begin{pmatrix} F_\pi(s) \\ \frac{2}{\sqrt{3}}F_K(s) \end{pmatrix} = \begin{pmatrix} C_1(s) & D_1(s) \\ C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} P_F(s) \\ Q_F(s) \end{pmatrix}$$

- At LO in ChPT:

$$\begin{aligned} M^2_{\pi^+} &= (m_\mathrm{u} + m_\mathrm{d})\, B_0 + O(m^2) \\ M^2_{K^+} &= (m_\mathrm{u} + m_\mathrm{s})\, B_0 + O(m^2) \\ M^2_{K^0} &= (m_\mathrm{d} + m_\mathrm{s})\, B_0 + O(m^2) \end{aligned}$$

⇒

$$\begin{aligned} P_\Gamma(s) &= \Gamma_\pi(0) = M_\pi^2 + \cdots \\ Q_\Gamma(s) &= \frac{2}{\sqrt{3}}\Gamma_K(0) = \frac{1}{\sqrt{3}}M_\pi^2 + \cdots \\ P_\Delta(s) &= \Delta_\pi(0) = 0 + \cdots \\ Q_\Delta(s) &= \frac{2}{\sqrt{3}}\Delta_K(0) = \frac{2}{\sqrt{3}}\left(M_K^2 - \frac{1}{2}M_\pi^2\right) + \cdots \end{aligned}$$

- Problem: large corrections in the case of the kaons!

  ⇒ Use lattice QCD to determine the SU(3) LECs

$$\Gamma_K(0) = (0.5 \pm 0.1)\ M_\pi^2$$

$$\Delta_K(0) = 1^{+0.15}_{-0.05}\,(M_K^2 - 1/2M_\pi^2)$$

Dreiner, Hanart, Kubis, Meissner'13

Bernard, Descotes-Genon, Toucas'12

Determination of the polynomial

- General solution

$$\begin{pmatrix} F_\pi(s) \\ \frac{2}{\sqrt{3}} F_K(s) \end{pmatrix} = \begin{pmatrix} C_1(s) & D_1(s) \\ C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} P_F(s) \\ Q_F(s) \end{pmatrix}$$

- For θ_P enforcing the asymptotic constraint is not consistent with ChPT
  The unsubtracted DR is not saturated by the 2 states

⇨ Relax the constraints and match to ChPT

$$\begin{aligned} P_\theta(s) &= 2M_\pi^2 + \left(\dot{\theta}_\pi - 2M_\pi^2 \dot{C}_1 - \frac{4M_K^2}{\sqrt{3}} \dot{D}_1\right) s \\ Q_\theta(s) &= \frac{4}{\sqrt{3}} M_K^2 + \frac{2}{\sqrt{3}} \left(\dot{\theta}_K - \sqrt{3} M_\pi^2 \dot{C}_2 - 2M_K^2 \dot{D}_2\right) s \end{aligned}$$