5th lesson

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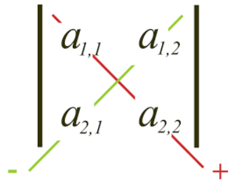
Theory

Proposition 1.

- *Determinant* of a matrix $\mathbf{A}$ of type 2×2 is

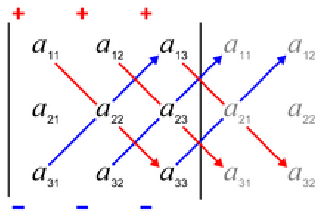
$$\det \mathbf{A} = a_{1,1} \cdot a_{2,2} - a_{1,2} \cdot a_{2,1}.$$



Source 1: http://umv.science.upjs.sk/madaras/MZIa/MZIa2011_4en.pdf

Determinant of a matrix $\mathbf{A}$ of type 3×3 is

$$\det \mathbf{A} = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,2}a_{2,1}a_{3,3} - a_{1,1}a_{2,3}a_{3,2}.$$



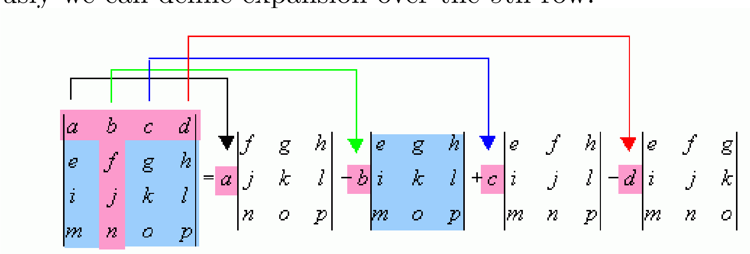
Source 2: https://de.wikipedia.org/wiki/Regel_von_Sarrus

Determinant of a matrix $\mathbf{A}$ of type $n \times n$ is

$$\det \mathbf{A} = a_{r,1}(-1)^{r+1} \det \mathbf{A}_{r,1} + a_{r,2}(-1)^{r+2} \det \mathbf{A}_{r,2} + \cdots + a_{r,n}(-1)^{r+n} \det \mathbf{A}_{r,n},$$

where the symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

Analogously we can define expansion over the sth row.



Source 3: http://mathcentral.uregina.ca/QQ/database/QQ.09.06/h/suud1.html

Theorem 2. Let $j, n \in \mathbb{N}$, $j \le n$, and the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(n \times n)$ coincide at each row except for the jth row. Let the jth row of $\boldsymbol{A}$ be equal to the sum of the jth rows of $\boldsymbol{B}$ and $\boldsymbol{C}$. Then $\det \boldsymbol{A} = \det \boldsymbol{B} + \det \boldsymbol{C}$.

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 + v_1 & \dots & u_n + v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 & \dots & u_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ v_1 & \dots & v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Theorem 3 (determinant and transformations)**.** Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

1. If the matrix $\boldsymbol{A}'$ is created from the matrix $\boldsymbol{A}$ by multiplying one row in $\boldsymbol{A}$ by a real number μ, then $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$.

2. If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by interchanging two rows in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the first type), then $\det \boldsymbol{A}' = -\det \boldsymbol{A}$.

3. If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by adding a μ-multiple of a row in $\boldsymbol{A}$ to another row in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the third type), then $\det \boldsymbol{A}' = \det \boldsymbol{A}$.

4. If $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by applying a transformation, then $\det \boldsymbol{A} \neq 0$ if and only if $\det \boldsymbol{A}' \neq 0$.

Theorem 4 (determinant and invertibility)**.** Let $\boldsymbol{A} \in M(n \times n)$. Then $\boldsymbol{A}$ is invertible if and only if $\det \boldsymbol{A} \neq 0$.

Theorem 5 (determinant of a product)**.** Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$. Then $\det \boldsymbol{AB} = \det \boldsymbol{A} \cdot \det \boldsymbol{B}$.

Theorem 6 (determinant of a transpose)**.** Let $\boldsymbol{A} \in M(n \times n)$. Then $\det \boldsymbol{A}^T = \det \boldsymbol{A}$.

Theorem 7 (Cramer's rule)**.** Let $\boldsymbol{A} \in M(n \times n)$ be an invertible matrix, $\vec{b} \in M(n \times 1)$, $\vec{x} \in M(n \times 1)$, and $\boldsymbol{A}\vec{x} = \vec{b}$. Then

$$x_j = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{vmatrix}}{\det \boldsymbol{A}}$$

for $j = 1, \dots, n$.

Exercises

1. Find the determinant

(a) $\begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix}$ (d) $\begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix}$

2. Find the determinant

(a) $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{vmatrix}$ (c) $\begin{vmatrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 4 & 5 \\ -3 & -2 & 1 \end{vmatrix}$

3. Find the determinant

(a) $\begin{vmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{vmatrix}$

4. Find the determinant $\begin{vmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{vmatrix}$

5. Explore the following properties (Choose suitable matrices (2x2 is enough), give some hypotesis and verify.

   (a) $\det 3A =?$

   (b) $\det AB =?$

   (c) $\det A^{-1} =?$

   (d) $\det A^T =?$

   (e) $\det(A+B) =?$

   (f) det of a matrix with two interchanged rows?

   (g) det of a matrix with two interchanged columns?

6. Apply operations and then find the determinant

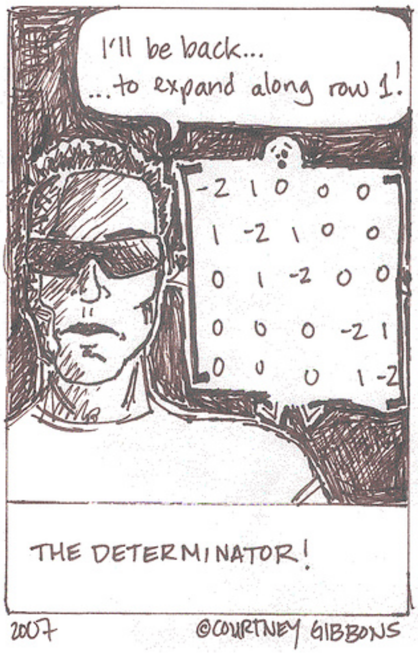
(a) $\begin{vmatrix} 3 & 7 & -9 \\ 6 & 1 & 4 \\ -9 & 5 & 2 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 5 & -2 & -4 \\ 2 & 4 & -2 & 8 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{vmatrix}$

7. Use Cramer rule to solve systems

(a) $\begin{aligned} 5x - 4y &= 2 \\ 6x - 5y &= 1 \end{aligned}$

(b) $\begin{aligned} x + 2y - z &= -4 \\ x + 4y - 2z &= -6 \\ 2x + 3y + z &= 3 \end{aligned}$



Source 4: https://mathsci2.appstate.edu/ sjg/class/2240/2007-03-30-determinator.jpg