Nonlinear Approximation

- An Idiot Abroad

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KAAS Colloquium

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1. *informally* discuss some *basic* ideas from nonlinear approximation and their applications in computation;
2. vaguely describe the results of [1], with *minimal* use of black magic from theory of function spaces

[1] M. Lind & P. Petrushev, *Nonlinear nonnested spline approximation* submitted, in revision

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Still, the subjects are closely connected.

Notation

$\Omega \subseteq \mathbb{R}^d$ domain, $p \in (0, \infty]$.

$L^p(\Omega)$ - space of functions such that

$$\|f\|_p := \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < \infty$$

or

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty.$$

$H_0^1(\Omega)$ - space of functions f such that $f = 0$ on $\partial\Omega$ and

$$\|f\|_{H_0^1(\Omega)} := \|f\|_2 + \||\nabla f|\|_2 < \infty$$

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(approximation scheme)
Error of best approximation of $f \in X$

$$E_n(f)_X = E_n(f, \mathcal{F})_X = \inf_{g \in X_n} \|f - g\|_X$$

Note $E_n(f)_X$ decreasing sequence of real numbers

Example: Weierstrass' theorem

Example: $X = C(0,1)$ with norm $\|\cdot\|_X = \|\cdot\|_\infty$

$$X_n = \mathcal{P}_n = \{\text{polynomials of degree} \leq n\}$$

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Weierstrass' theorem:

$$\lim_{n \to \infty} E_n(f)_\infty = 0$$

One main question

$E_n(f)_X$ encodes quality of approximation - central object.

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Question

relationship between

$$\text{intrinsic properties of } f \quad \leftrightsquigarrow \quad \text{behaviour of } E_n(f)_X?$$

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Ideally, direct and inverse results should match (i.e. $\square = \square$); not always the case

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$aX_j = X_j \quad (\forall a \in \mathbb{R})$ and

$$X_n + X_n = \{x + y : x, y \in X_n\} \subset X_{cn}$$

some fixed $c \in \mathbb{N}$ (bounded nonlinearity)

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As before,

$$E_n(f)_X = E_n(f, \mathcal{F})_X = \inf_{g \in X_n} \|f - g\|_X$$

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Compression: approximate a signal having $\sharp(\mathit{spectrum}) = M$ by using $N << M$ frequencies.

Example: Free knot spline approximation

Set of points ("knots")

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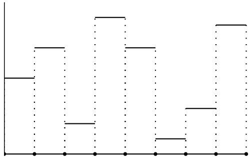
$S_0(\mathcal{T})$=all step functions (uniform step $1/n$)



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Spline manifolds

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What is the point?! Point is that partitions are allowed to adapt to target function $\Rightarrow$ better approximating power

A comparison

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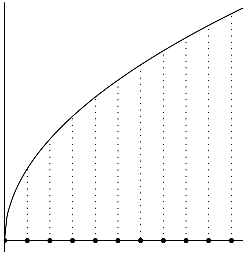
Case $\alpha = 1/2$, $n = 10$



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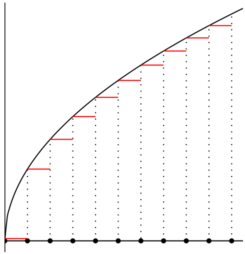

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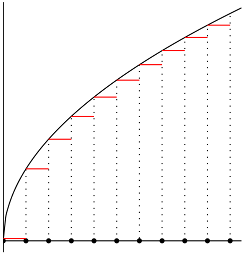

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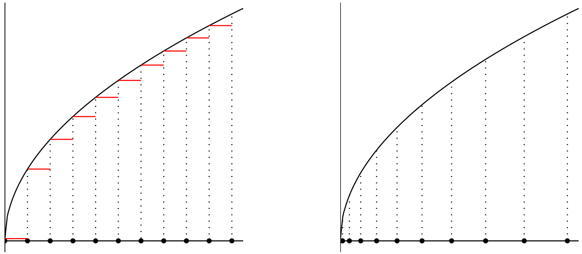

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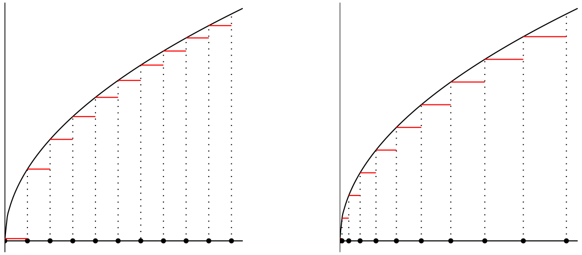

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Error rates for approximation of $f(x) = x^\alpha$ (0th order spline, L^∞-norm):

$$E_n^L(f)_\infty \asymp \frac{1}{n^\alpha} \quad \text{and} \quad E_n^{NL}(f)_\infty = \frac{1}{n}$$

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General Optimal rate attained for wider class of functions.

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Extended example (applications)

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How to choose V?

Nonlinear approximation and computations

- $\mathcal{T}$=triangulation of Ω, i.e. $\Omega \approx \bigcup_{\Delta \in \mathcal{T}} \Delta$
- $\mathcal{V}$=vertices of triangles of $\mathcal{T}$

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Courant elements: for each $P \in \mathcal{V}$, define a continuous function φ_P by

1. $\varphi_P(P) = 1$;
2. $\varphi_P(Q) = 0$ for $Q \in \mathcal{V} \setminus \{P\}$;
3. the restriction of φ_P to each $\Delta \in \mathcal{T}$ is affine

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$$V := S_1^0(\mathcal{T}) = \mathrm{span}\{\varphi_P : P \in \mathcal{V}\},$$

1st degree splines (restrictions to Δ's have degree≤ 1) with smoothness 0 (i.e. continuous).

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A priori estimate: u exact solution to $(\star)$

$$\|u - u_h\|_{H_0^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

if $\partial\Omega$ smooth.

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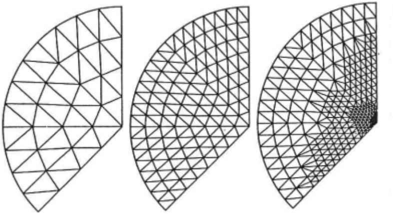

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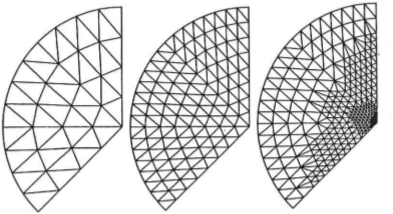

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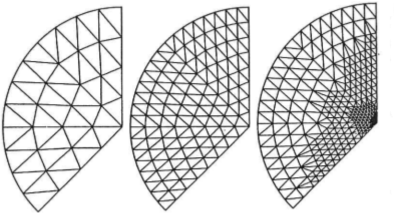

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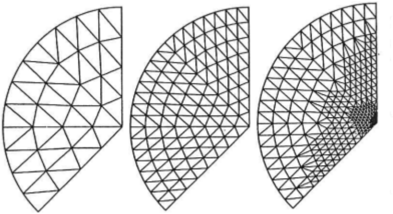

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Substitute: n=number of triangles

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Theorem (Binev, Dahmen, DeVore '04)

Let u be the solution to $(\star)$. If u can be approximated (nonlinearly) by 1st order continuous splines with rate

$$E_n(u)_{H_0^1(\Omega)} = \mathcal{O}(n^{-\gamma})$$

Nonlinear approximation and computations: AFEM

Adaptive schemes 'seems' reasonable, but rigorous derivation?

For any Galerkin solution U (S_1^0-spline on n triangles)

$$E_n(u)_{H_0^1(\Omega)} \leq \|u - U\|_{H_0^1(\Omega)}$$

but $\|u - U\|_{H_0^1(\Omega)}$ may be much larger

Theorem (Binev, Dahmen, DeVore '04)

Let u be the solution to $(\star)$. If u can be approximated (nonlinearly) by 1st order continuous splines with rate

$$E_n(u)_{H_0^1(\Omega)} = \mathcal{O}(n^{-\gamma})$$

then there is an explicit adaptive algorithm that in $\mathcal{O}(n)$ steps constructs a triangulation $\mathcal{T}_n$ with $\sharp\mathcal{T}_n = \mathcal{O}(n)$ and a Galerkin solution $u_n \in S_1^0(\mathcal{T}_n)$ s.t.

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Moral of the story: nontrivial computational information obtained from rate of approximation.

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$B^s_{\tau,\tau}$ closely related to nonlinear approximation in $L^p(\Omega)$ $(\Omega \subset \mathbb{R}^d)$ when

$$\frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}$$

(Critical line; Sobolev embedding theorem $B^s_{\tau,\tau} \hookrightarrow L^p(\Omega)$.)

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Notation $\mathcal{S}(n, 1, 0)$: set of continuous functions S on Ω such that there exists a 'triangulation' $\mathcal{T} = \{\Delta\}$ with

$$S|_\Delta \quad \text{is affine} \quad \text{and} \quad \sharp\mathcal{T} \le n$$

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$S_1 \in B^s_{\tau,\tau}$ 'simple' function, S_2 'complex' function;
If error $\|S_2 - S_1\|_p = \mathcal{O}(n^{-s/d})$, then

$$|S_2|_{B^s_{\tau,\tau}} \leq |S_1|_{B^s_{\tau,\tau}} + cn^{s/d} \times n^{-s/d} < \infty.$$